

MAX-PLANCK-INSTITUT FÜR INFORMATIK

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Ondrej Šýkora Imrich Vrťo

MPI-I-91-125

November 1991



Im Stadtwald
66123 Saarbrücken
Germany

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“Das diesem Bericht zugrunde liegende Vorhaben wurde mit Mitteln des Bundesministers für Forschung und Technologie (Betreuungskennzeichen ITR 9102) gefördert. Die Verantwortung für den Inhalt dieser Veröffentlichung liegt beim Autor.”

EDGE SEPARATORS FOR GRAPHS OF BOUNDED GENUS WITH APPLICATIONS

Ondrej Sýkora and Imrich Vrto^{*†}

Max Planck Institute for Computer Science
Im Stadtwald, D-W-6600 Saarbrücken, FRG

Abstract

We prove that every n -vertex graph of genus g and maximal degree k has an edge separator of size $O(\sqrt{gkn})$. The upper bound is best possible to within a constant factor. This extends known results on planar graphs and similar results about vertex separators. We apply the edge separator to the isoperimetric number problem, graph embeddings and lower bounds for crossing numbers.

1 Introduction

Many divide and conquer algorithms on graphs are based on finding a small set of vertices or edges whose removal divides the graph roughly in half. Applications include VLSI layouts [14], Gaussian elimination [15] and graph embeddings [16].

Formally, a class of graphs has $f(n)$ vertex (edge) separator if every n -vertex graph in the class has a vertex (edge) cutset of size $f(n)$ that divides the graph in two parts having no more than $2n/3$ vertices. Lipton and Tarjan [17] proved that planar graphs have $O(\sqrt{n})$ vertex separator. The genus of a graph is the minimum number of handles that must be added to a sphere so that the graph can be imbedded in the resulting sphere with no crossing edges. Djidjev [6] and Gilbert, Hutchinson and Tarjan [9] proposed algorithms that find an $O(\sqrt{gn})$ vertex separator for graphs of genus g . Further generalization was done in [20, 23] and recently in [19]. Miller [18] and Diks et al. [4] showed that every n -vertex planar graph of maximal degree k has an $O(\sqrt{kn})$ edge separator. Extensions of these results can be found in [8].

In this paper we prove that any n -vertex graph of positive genus g and maximal degree k has an $O(\sqrt{gkn})$ edge separator. This bound is best possible to within a constant factor. The separator can be found in $O(g + n)$ time provided that we start with an imbedding of the graph in its genus surface. We apply the edge separator to the isoperimetric problem, to efficient embeddings of graphs of genus g into various classes of graphs including trees, meshes and hypercubes and to showing lower bounds on crossing numbers of K_n , $K_{m,n}$ and Q_n drawn on surfaces of genus g .

^{*}The research of both authors was supported by Alexander von Humboldt Foundation, Bonn, Germany

[†]The authors were on leave from Institute for Informatics, Slovak Academy of Sciences, Dúbravská 9, 84235 Bratislava, CSFR

2 Separation of graphs of genus g

2.1 Upper bound

We prove a stronger "weighted" version of the edge separator theorem mentioned in introduction. Before proving it we state some notions and an important lemma.

Let $G = (V, E)$ be an n -vertex graph of genus $g > 0$ and maximal degree k whose vertices have nonnegative weights summing to 1 such that no weight exceed $2/3$. Let us denote the sum of weights of vertices belonging to a set X as $\text{weight}(X)$. Let $|E| = m$.

Lemma 2.1 [7] *If G has a breadth first search spanning tree of radius r and rooted in a vertex t then there is a partition of V into three sets A, B, C such that no edge joins a vertex in A with a vertex in B , $\text{weight}(A) \leq 2/3$, $\text{weight}(B) \leq 2/3$ and $|C| \leq (4g + 2)r + 1$, where $t \in C$.*

Theorem 2.1 *There is a partition of V into sets A, B and a set of edges D such that $\text{weight}(A) \leq 2/3$, $\text{weight}(B) \leq 2/3$, $|D| \leq 5\sqrt{3gkn}$ and every edge between A and B belongs to D .*

Proof: Let G be connected and t be an arbitrary vertex of G . Suppose that G has a breadth first search spanning tree of radius r rooted in t . According to Lemma 2.1 there is a set of vertices C separating G such that $|C| \leq (4g + 2)r + 1$.

Assume that $r < \sqrt{m/((4g + 2)k)}$. Let D be the set of all edges incident to the set C . Hence

$$|D| < \left(\sqrt{\frac{m}{(4g + 2)k}} (4g + 2) + 1 \right) k < (\sqrt{6} + 1) \sqrt{gkm}.$$

By distributing properly the set C between A and B we receive the desired partition.

If $r \geq \sqrt{m/((4g + 2)k)}$ then we set $s = \sqrt{m/((4g + 2)k)}$. We can assume $s > 1$, otherwise $m \leq (4g + 2)k$ and we have a trivial edge separator of size $m = \sqrt{m}\sqrt{m} \leq \sqrt{6gkm}$. Partition the vertices of G into levels U_0, U_1, \dots, U_r according to their distance from t . Define $L_j = \{(u, v) \mid u \in U_{i-1}, v \in U_i, i = j \bmod s, i = 1, 2, 3, \dots, r\}$. As $|\bigcup_{j=0}^{s-1} L_j| \leq m$ and $|L_i \cap L_j| = 0$, for $i \neq j$, there exists s_0 such that $|L_{s_0}| \leq m/s$. By removing the edges of L_{s_0} the graph G is partitioned into connected components $G_i = (V_i, E_i), i = 1, 2, 3, \dots$

If $\text{weight}(V_i) \leq 2/3$ for all i then we easily combine the desired partition A, B . Set $D = L_{s_0}$. Thus

$$|D| \leq \frac{m}{s} \leq \frac{m}{\sqrt{m/((4g + 2)k)}} = \sqrt{(4g + 2)km} \leq \sqrt{6gkm}.$$

Let $\text{weight}(V_i) > 2/3$ for some i . Let $h(l)$ be the highest (lowest) level of vertices in V_i . Delete the vertices of G at levels $> h$. Shrink the vertices of G at levels $< l$ into the root t . The result is a graph H_i of genus $\leq g$, $|V_i| + 1$ vertices and radius s . Apply Lemma 2.1 to the graph H_i . We obtain a partition of H_i into sets A_i, B_i and C_i such that

$$\text{weight}(A_i) \leq \frac{2}{3}(\text{weight}(V_i) + \text{weight}(t)), \quad \text{weight}(B_i) \leq \frac{2}{3}(\text{weight}(V_i) + \text{weight}(t))$$

and

$$|C_i| \leq (4g + 2)s + 1.$$

Delete t from H_i . Let D_i denote the set of edges incident to C_i . Removing D_i from G_i we partition V_i into two sets having weights $\leq 2/3$. Thus we have divided G into components whose weights do not exceed $2/3$. One can easily combine the components into the desired partition A and B . The total number of deleted edges is

$$|D| = |L_{s_0}| + |D_i| \leq \frac{m}{s} + (4g + 2)ks = 2\sqrt{(4g + 2)km} \leq 2\sqrt{6gkm}.$$

Finally, suppose $g \leq n/48$. Then

$$|D| \leq 2\sqrt{6gkm} \leq 2\sqrt{6gk(3n+6g)} \leq \sqrt{75gkn}.$$

If $g \geq n/48$ then we have a trivial separator of size

$$m = \sqrt{m}\sqrt{m} \leq \sqrt{3n+6g}\sqrt{\frac{kn}{2}} \leq \sqrt{75gkn}.$$

In case that G is not connected we apply the above procedure to the component with the greatest weight. \square

Our proof can be directly transformed to an algorithm for finding the edge separator. Provided that we start with an imbedding of G in its genus surface the time complexity is $O(m) = O(g+n)$ because finding both the set L_{s_0} and the vertex cut from Lemma 2.1 requires $O(g+n)$ time [7].

For some applications it can be useful to have an edge cut that divides the graph into two parts whose numbers of vertices differ at most by 1. Such edge cuts are called bisectors.

Corollary 2.1 *Any n -vertex graph of genus g and maximal degree k has a bisector of size $48\sqrt{gkn}$.*

The proof follows the method used in Corollary 3 of [17].

2.2 Lower bound

In this section we prove that the bound in Theorem 2.1 is tight to within a constant factor whenever $gk = O(n)$. We show this for the unweighted version of Theorem 2.1, i.e. all vertices have the same weight. We essentially use the following lemma.

Lemma 2.2 [9] *There exists a constant α such that for infinitely many $g, n_0, g < n_0$, there is a regular graph G_0 with n_0 vertices, genus g and of degree 6 whose every vertex cut dividing G_0 into parts having $\leq 3/4$ vertices has size $\geq \alpha\sqrt{gn_0}$.*

Corollary 2.2 *Every edge cut that divides G_0 into parts having $\leq 3n/4$ vertices has size at least $\alpha\sqrt{gn_0}$.*

Proof: Let the claim be false. Hence there is an edge cut D of G_0 of size $|D| < \alpha\sqrt{gn_0}$ that partitions the vertices of G_0 in A and B , $|A|, |B| \leq 3n/4$. Let $|A| \leq |B|$. Delete all vertices that are incident to edges from D and belong to B . We have constructed a vertex cut of size $< \alpha\sqrt{gn_0}$ that divides G_0 into parts having $\leq 3n/4$ vertices. \square

Theorem 2.2 *For $k = 0 \pmod 6$ and infinitely many g, n , $gk < 2n$ there is a graph G of n vertices, genus g and maximal degree k such that every edge separator of G has size $\Omega(\sqrt{gkn})$.*

Proof: Consider the graph G_0 from Lemma 2.2. Replace each edge of G_0 by $k/6$ new parallel edges. Put one new vertex on each new edge. We get a graph $G = (V, E)$ of genus g , maximal degree k and with n vertices where $n = n_0k/2 + n_0$. Consider a minimal edge separator of G , i.e. we have a partition of V in A, B such that $|A| \leq 2n/3, |B| \leq 2n/3$ and every edge between A and B belongs to a set D . Our aim is to prove that $|D| = \Omega(\sqrt{gkn})$. Let $V_0 \subset V$ denote the set of vertices that correspond to the vertices of G_0 . Denote $A_1 = A \cap V_0, B_1 = B \cap V_0$. It holds $|A_1| + |B_1| = n_0$. Assume $|A_1| \leq |B_1|$. We distinguish two cases:

1. $|A_1| \geq n_0/4$. Then $|B_1| \leq 3n_0/4n$. According to Corollary 2.2, G contains at least $\alpha\sqrt{gn_0}$ tuples (u, v) such that $u \in A, v \in B$ and u and v are joined by $k/6$ paths of length 2. From each such path at least one edge must belong to D . Hence

$$|D| \geq \alpha \frac{k}{6} \sqrt{gn_0} = \alpha \frac{k}{6} \sqrt{\frac{2gn}{k+2}} \geq \frac{\alpha\sqrt{6}}{12} \sqrt{gkn}.$$

2. $|A_1| < n_0/4$. We estimate the number of vertices in $A - A_1$ that have a neighbour in B_1 . Each such vertex together with the neighbour defines an edge that must belong to D . Hence

$$\begin{aligned} |D| &\geq |A - A_1| - |\{\text{vertices of } A - A_1 \text{ that have both neighbours in } A_1\}| \geq \\ &\geq \frac{n}{3} - |A_1| - |A_1| \frac{k}{2} > \frac{n}{3} - \frac{n_0}{4} \left(1 + \frac{k}{2}\right) = \frac{n}{12} \geq \frac{\sqrt{2}}{24} \sqrt{gkn}. \quad \square \end{aligned}$$

3 Applications

In this section we apply the edge separator to the isoperimetric problem, to graph embeddings and to finding lower bounds for crossing numbers of complete, bipartite and hypercube graphs drawn on a surface of genus g .

3.1 Isoperimetric number

The isoperimetric number $i(G)$ of a graph $G = (V, E)$ is defined as

$$i(G) = \min \left\{ \frac{|\delta(X)|}{|X|} : X \subset V, 1 \leq |X| \leq \frac{|V|}{2} \right\},$$

where $\delta(X)$ is a set of edges having one edge in X and the other in $V - X$. The quantity $i(G)$ is a discrete analog of the well-known Cheeger [2] isoperimetric constant measuring the minimal possible ratio between the size of the surface and the volume of a geometric figure. Isoperimetric numbers for important graphs are computed in [21]. Boshier [1] proved that if G is an n -vertex graph of genus g and maximal degree k then

$$i(G) \leq \frac{3(g+2)k}{\sqrt{n/2} - 3(g+2)},$$

for $n > 18(g+2)^2$. Our edge separator immediately implies the following improvement.

Theorem 3.1

$$i(G) \leq 15\sqrt{\frac{3gk}{n}}.$$

Proof: Let A, B and D be the sets from the unweighted version of Theorem 2.1. Clearly

$$i(G) = \min_{X \subset V} \frac{|\delta(X)|}{|X|} \leq \frac{|D|}{\min\{|A|, |B|\}} \leq 15\sqrt{\frac{3gk}{n}}. \quad \square$$

3.2 Graph embeddings

Many computational problems can be mathematically formulated as the graph embedding, e.g. representing some kind of data structure by another data structure [16], simulation of interconnection networks of parallel computers [22] and laying out circuits in standard format [14].

Let $G_1 = (V_1, E_1), G_2 = (V_2, E_2)$ be graphs such that $|V_1| \leq |V_2|$. An embedding of G_1 into G_2 is a couple of mappings (ϕ, ψ) satisfying

$$\phi : V_1 \rightarrow V_2 \text{ is an injection}$$

$$\psi : E_1 \rightarrow \{\text{set of all simple paths in } G_2\}$$

such that if $(u, v) \in E_1$ then $\psi((u, v))$ is a path between $\phi(u)$ and $\phi(v)$.

We shall study the following measures of the quality of the embedding

$$\text{adil}(\phi, \psi) = \sum_{e \in E_1} \frac{|\psi(e)|}{|E_1|},$$

where $|\psi(e)|$ denotes the length of the path $\psi(e)$.

$$\text{exp}(\phi, \psi) = \frac{|V_2|}{|V_1|}$$

$$\text{cg}(\phi, \psi) = \max_{e \in E_2} |\{f \in E_1 : e \text{ belongs to } \psi(f)\}|.$$

The above measures are called the average dilation, expansion and congestion.

Lipton and Tarjan [16] pointed out that edge separators can be used to embed graphs with a small average dilation. Applying their method Diks et al. [4, 5] proved the following results on graph embeddings.

Theorem 3.2 *Every n -vertex planar graph of maximal degree k can be embedded in a path, 2-dimensional mesh, d -dimensional mesh ($d \geq 3$), complete binary tree and a hypercube with average dilations $\Theta(\sqrt{kn}), O(\sqrt{k} \log(n/k)), \Theta(d\sqrt[k]{k}), \Theta(\log k), O(\log k)$ and expansion $O(1)$.*

Using the same method we can extend this result as follows.

Theorem 3.3 *Every n -vertex graph of genus g and maximal degree k can be embedded in a path, 2-dimensional mesh, d -dimensional mesh ($d > 2$), complete binary tree and a hypercube with average dilations: $O(\sqrt{gkn}), O(\sqrt{gk} \log(n/k)), O(d\sqrt{g}\sqrt[k]{k}), O(\sqrt{g} \log k)$ and $O(\sqrt{g} \log k)$, and expansion $O(1)$.*

For the first average dilation we show an optimal lower bound. We use a method of [3].

Lemma 3.1 *Let $|V_1| = n$ and G_2 be an n -vertex path. Let $f_p(n)$ be the size of the minimal edge cut that divides G_1 into two parts having exactly p and $n - p$ vertices. Then*

$$\text{adil}(\phi, \psi) \geq \frac{1}{|E_1|} \sum_{i=1}^{n/2} f_i(n).$$

Theorem 3.4 *Any embedding (ϕ, ψ) of the graph G constructed in Theorem 2.2 into the n -vertex path requires*

$$\text{adil}(\phi, \psi) = \Omega(\sqrt{gkn}).$$

Proof: Let β be the constant behind the Ω in Theorem 2.2. Then setting $G_1 = G$ in Lemma 3.1 we get

$$\text{adil}(\phi, \psi) \geq \frac{1}{|E|} \sum_{i=1}^{n/2} f_i(n) \geq \frac{1}{|E|} \sum_{i=n/3}^{n/2} f_i(n) \geq \left(\frac{n}{2} - \frac{n}{3}\right) \beta \sqrt{gkn} \frac{k+2}{2kn} = \Omega(\sqrt{gkn}). \quad \square$$

In case G_2 is a path, the minimal congestion is usually called cutwidth. This notion has applications in VLSI design. Yannakakis [24] stated an open problem to find a good approximation to the cutwidth of planar graphs. This was partially solved in [4]. We extend the result for graphs of positive geni.

Theorem 3.5 *Any n -vertex graph of positive genus g and maximal degree k has a cutwidth of $O(\sqrt{gkn})$. This bound can not be improved in general.*

Proof: By breaking the graph recursively into roughly equal parts using the edge separator and embedding the parts into subpaths one can easily estimate the cutwidth by

$$\sqrt{75kg}(\sqrt{n} + \sqrt{\frac{2}{3}n} + \sqrt{\left(\frac{2}{3}\right)^2 n} + \dots) = O(\sqrt{gkn}).$$

Because the cutwidth of a graph is not smaller than the size of its minimal edge separator the graph from Theorem 2.2 has the cutwidth at least $\Omega(\sqrt{gkn})$. \square

3.3 Lower bounds on crossing numbers

In this subsection we apply Theorem 2.1 to showing lower bounds for the crossing numbers of complete, bipartite and hypercube graphs drawn in an orientable surface of genus g .

The orientable surface S_g of genus g is obtained from a sphere by adding g handles. The crossing number $\text{cr}_g(G)$ of a graph G is defined as the least number of crossings when G is drawn in S_g . Very little is known on $\text{cr}_g(G)$. In [10] and [11] it is proved that

$$\text{cr}_1(K_n) = \Theta(n^4), \quad \text{cr}_1(K_{m,n}) = \Theta(m^2 n^2).$$

Kainen[12] showed that

$$\text{cr}_g(Q_n) = \Theta(\gamma - g),$$

for $g \geq \gamma - 2^{n-4}$, where Q_n denotes the n -dimensional hypercube graph and γ its genus.

The following theorem which describes a lower bound methods for finding crossing numbers, was originally proved for planar graphs [13]. Our extension to graphs of genus g is straightforward.

Theorem 3.6 *Let $G = (V, E)$ be a graph. Let $\text{mec}(G)$ denote the size of the minimal edge cut that divides G in two parts having $\leq 2 |V|/3$ vertices. Suppose that the class of graphs of genus g and of maximal degree k has an $f_{g,k}(n)$ edge separator. Then*

$$\text{cr}_g(G) \geq f_{g,k}^{-1}(\text{mec}(G)) - |V|,$$

where $f_{g,k}^{-1}$ is the inverse function to $f_{g,k}$.

Corollary 3.1

$$\text{cr}_g(G) \geq \frac{\text{mec}^2(G)}{75gk} - |V|. \quad (1)$$

Proof: From Theorem 2.1 we have $f_{g,k}(n) = \sqrt{75gkn}$. \square

Theorem 3.7

$$cr_g(K_n) > \frac{n^4}{6075g} - \frac{n^3}{2}.$$

Proof: Define a new graph H_n as follows. Consider a drawing of K_n in S_g with minimal number of crossings. Let v be an arbitrary vertex of K_n . Let $u_0, u_1, u_2, \dots, u_{n-2}$ be its neighbours. In S_g find a region homeomorphic to an open disc that contains v and no crossings. Denote $p = \lfloor n/2 \rfloor$. Place new vertices $u_{i1}, u_{i2}, \dots, u_{ip}$ on the edge (u_i, v) so that they lie in the region, for $i = 0, 1, 2, \dots, n-2$. Add edges $(u_{i,j}, u_{(i+1) \bmod (n-1), j})$ for $i = 0, 1, 2, \dots, n-2$ and $j = 1, 2, 3, \dots, p$. Delete the vertex v . The resulting graph H_n has $n(n-1)p$ vertices, genus g and degree 4. Clearly it holds

$$cr_g(K_n) \geq cr_g(H_n) \tag{2}$$

It remains to find $mec(H_n)$. In what follows, we show that

$$mec(H_n) \geq \frac{2}{9}n^2. \tag{3}$$

Setting $G = H_n$ and substituting 2, 3 into 1 we obtain the desired result. \square

Lemma 3.2

$$mec(H_n) \geq \frac{2}{9}n^2.$$

Proof: We use a method of Leighton [13]. Recall the definition of the embedding and the congestion. Leighton proved that

$$mec(G_2) \geq \frac{mec(G_1)}{cg(\phi, \psi)}.$$

Let $2K_{n(n-1)p}$ denote the complete graph on $n(n-1)p$ vertices whose each edge is replaced by two new parallel edges. Set $G_2 = H_n, G_1 = 2K_{n(n-1)p}$. We construct an embedding (ϕ, ψ) of G_1 in G_2 such that

$$cg(\phi, \psi) \leq 2(n-1)^2p^2.$$

Noting that

$$mec(2K_{n(n-1)p}) \geq \frac{4}{9}n^2(n-1)^2p^2$$

we immediately receive the lower bound for $mec(H_n)$.

We shall construct 2 paths between any two vertices of H_n so that the congestion be as small as possible. Let us call the graph induced by the vertices $u_{ij}, i = 0, 1, 2, \dots, n-2, j = 1, 2, 3, \dots, p$, a cobweb.

Let u_{ij} and u_{rs} belong to the same cobweb. Suppose $j \geq s$.

If $i = r$ then we join u_{ij} and u_{rs} by two shortest (identical) paths.

If $i \neq r$ then we join u_{ij} with u_{rs} by two paths $u_{ij}, u_{i+1,j}, \dots, u_{rj}$ and $u_{ij}, u_{i-1,j}, \dots, u_{rj}$ and prolong these paths to u_{rs} as above.

Let two vertices of H_n belong to different cobwebs. Let (x, y) be an edge joining these cobwebs. Join $x(y)$ to the vertex belonging to the same cobweb as $x(y)$ by two paths as above. Connect the paths by adding twice the edge (x, y) . Simple counting analysis shows that the edges between the cobwebs are used by the most number of the paths which gives

$$cg(\phi, \psi) \leq 2(n-1)^2p^2. \quad \square$$

Using the same approach as above we can prove:

Corollary 3.2

$$cr_g(K_{m,n}) > \frac{m^2n^2}{1200g} - \frac{mn(m+n)}{2}, \quad cr_g(Q_n) > \frac{4^n}{1500g} - n^22^{n-1}.$$

4 Conclusions

Our paper leaves several open questions: e.g. improving and completing the upper and lower bounds in Theorems 2.1, 2.2 and 3.3. The most interesting seems to be the problem of drawing of K_n , $K_{m,n}$ and Q_n in the surface of orientable genus g . We can find drawings that have $O(\sqrt{g})$ -times more crossings than the proved lower bounds.

Acknowledgment. The authors thank to Professor Kurt Mehlhorn, Max Planck Institut für Informatik, Saarbrücken for his support. Many thanks also go to Hristo Djidjev for helpful discussions.

References

- [1] D. O. Boshier, Enlarging properties of graphs, Ph.D. Thesis, Royal Holloway and Bedford New Colledge, University of London, 1987.
- [2] J. Cheeger, A lower bound for the smallest eigenvalue of the Laplacian, in: *Problems in Analysis*, (Princeton University Press, 1970) 159-199.
- [3] R. A. De Millo, S. C. Eisenstat and R. J. Lipton, Preserving average proximity in arrays, *Communications of the ACM* 21 (1978) 228-230.
- [4] K. Diks, H. N. Djidjev, O. Sýkora and I. Vrto, Edge separators for planar graphs and their applications, in: *Proceedings of the 13-th Symposium on Mathematical Foundations of Computer Science* (LNCS 324, Springer Verlag, 1988) 280-290.
- [5] K. Diks, H. N. Djidjev, O. Sýkora and I. Vrto, Edge separators for planar graphs with applications, *Journal of Algorithms*, to appear.
- [6] H. N. Djidjev, A linear algorithms for partitioning graphs of fixed genus, *SERDICA Bulgaricae Mathematicae Publications* 11 (1985) 369-387.
- [7] H. N. Djidjev, A separator theorem, *Comptes Rendus de l'Academie Bulgare des Sciences* 34 (1981) 643-645.
- [8] H. Gazit and G. L. Miller, Planar separators and Euclidean norm, in: *Proceedings of Algorithms, International Symposium SIGAL'90* (1990) 338-347.
- [9] J. R. Gilbert, J. Hutchinson and R. E. Tarjan, A separator theorem for graphs of bounded genus, *Journal of Algorithms* 5 (1984) 391-401.
- [10] R. K. Guy, T. Jenkins and J. Schaer, The toroidal crossing number of the complete graph, *Journal of Combinatorial Theory* 4 (1968) 376-390.
- [11] R. K. Guy and T. Jenkins, The toroidal crossing number of the $K_{m,n}$, *Journal of Combinatorial Theory* 6 (1969) 235-250.
- [12] P. C. Kainen, On the stable crossing number of cubes, in: *Proceedings of the American Mathematical Society* 36 (1972) 55-62.
- [13] F. T. Leighton, New lower bound techniques for VLSI, in: *Proceedings of the 22-nd Annual IEEE Symposium on Foundations of Computer Science* (1981) 1-12.

- [14] C. E. Leiserson, Area-efficient graph layouts (for VLSI), in: *Proceedings of the 21-st Annual IEEE Symposium on Foundations of Computer Science* (1980) 270-280.
- [15] R. J. Lipton, D. J. Rose and R. E. Tarjan, Generalized nested dissection, *SIAM Journal on Numerical Analysis* **16** (1979) 346-358.
- [16] R. J. Lipton and R. E. Tarjan, Application of a planar separator theorem, *SIAM Journal on Computing* **9** (1980) 615-627.
- [17] R. J. Lipton and R. E. Tarjan, A separator theorem for planar graphs, *SIAM Journal on Applied Mathematics* **36** (1979) 177-189.
- [18] G. L. Miller, Finding small simple cycle separators for 2-connected planar graphs, *Journal of Computer and System Science* **32** (1986) 265-279.
- [19] G. L. Miller, S. H. Teng and S. A. Vavasis, A unified geometric approach to graph separators, in: *Proceedings of the 32-nd Annual Symposium on Foundations of Computer Science* (1991) 538-547.
- [20] G. L. Miller and W. Thurston, W., Separators in two and three dimensions, in: *Proceedings of the 22-nd Annual ACM Symposium on Theory of Computing* (1990) 300-309.
- [21] B. Mohar, Isoperimetric numbers of graphs, *Journal of Combinatorial Theory, B* **47** (1989) 274-291.
- [22] B. Monien and I. H. Sudborough, Comparing interconnection networks, in: *Proceedings of the 13-th Symposium on Mathematical Foundations of Computer Science* (LNCS 324, Springer Verlag, 1988) 138-153.
- [23] A. Noga, P. Seymour and R. Thomas, A separator theorem for graphs with an excluded minor and its applications, in: *Proceedings of the 22-nd Annual ACM Symposium on Theory of Computing* (1990) 203-299.
- [24] M. Yannakakis, Linear and book embedding of graphs, in: *Proceedings of the Aegean Workshop on Computing* (LNCS 227, Springer Verlag, 1986) 229-240.

