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Univariate polynomial solutions of algebraic difference equations ☆

O. Shkaravska [a], M. van Eekelen [b,c]

[a] *Max Planck Institute for Psycholinguistics, Nijmegen, Netherlands*
[b] *Institute for Computing and Information Sciences, Radboud University Nijmegen, Netherlands*
[c] *School of Computer Science, Open University of the Netherlands, Netherlands*

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ABSTRACT

Contrary to linear difference equations, there is no general theory of difference equations of the form $G(P(x-\tau_1),\dots,P(x-\tau_s)) + G_0(x) = 0$, with $\tau_i \in \mathbb{K}$, $G(x_1,\dots,x_s) \in \mathbb{K}[x_1,\dots,x_s]$ of total degree $D \geqslant 2$ and $G_0(x) \in \mathbb{K}[x]$, where $\mathbb{K}$ is a field of characteristic zero. This article concerns the following problem: given τ_i, G and G_0, find an upper bound on the degree d of a polynomial solution $P(x)$, if it exists. In the presented approach the problem is reduced to constructing a univariate polynomial for which d is a root. The authors formulate a sufficient condition under which such a polynomial exists. Using this condition, they give an effective bound on d, for instance, for all difference equations of the form $G(P(x-a), P(x-a-1), P(x-a-2)) + G_0(x) = 0$ with quadratic G, and all difference equations of the form $G(P(x), P(x-\tau)) + G_0(x) = 0$ with G having an arbitrary degree.

1. Introduction

This article considers polynomial solutions of difference equations of the form

$$G\big(P(x-\tau_1),\dots,P(x-\tau_s)\big) + G_0(x) = 0 \tag{1}$$

where $G(x_1,\dots,x_s) \in \mathbb{K}[x_1,\dots,x_s]$ is a polynomial of *total degree* $D \geqslant 2$ in s variables, $\mathbb{K}$ is a field of characteristic zero, $G_0(x) \in \mathbb{K}[x]$ and $\tau_i \in \mathbb{K}$ are pairwise different and ordered so that $\tau_1 < \dots < \tau_s$.

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E-mail addresses: olhsha@mpi.nl (O. Shkaravska), marko@cs.ru.nl (M. van Eekelen).

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The aim is to find a bound on the degree d of polynomial solutions $P(x)$ if such solutions and a bound exist.

It is worth to note that there are difference equations which are solvable by a polynomial of any degree (therefore, no bound exists), e.g.:

$$\begin{aligned} &P(x)P(x-2)P(x-3) - 2P(x-1)^2P(x-3) + P(x-1)P(x-2)^2 \\ &\quad + P(x)P(x-1)P(x-3) - 2P(x)P(x-2)^2 + P(x-1)^2P(x-2) = 0. \end{aligned} \tag{2}$$

It is solved by any factorial power $g_n(x) = (x+a)(x+a-1)\dots(x+a-(n-1))$. The proof resembles the technique for differential equations from the article van den Essen (1992) and can be found in the technical report Shkaravska and van Eekelen (2010). Moreover, the statement can be checked by a direct substitution using a computer algebra system.

In the present article the equations of form (1) are called *algebraic difference equations with constant coefficients*. The terminology "with constant coefficients" is used because one considers polynomials $G(x_1, \dots, x_s)$ with coefficients which are independent of x. The authors believe that extending the proposed method to difference equations where the coefficients of $x_1^{i_1} \cdots x_s^{i_s}$ depend on x will require only some technical adjustments. However it is left to future work because the results even for constant coefficients require technically involved computations.

Notation

The present article involves reasoning about symbolic vectors, products of powers and indexed polynomials whose coefficients are polynomials as well. Therefore technical overhead in formal reasoning is inevitable. The following list of the most frequently used notation, which can be used as a general reference, should help to handle this overhead:

Notation	Denoted object
$\mathbf{r}$	the vector $(\rho_1, \dots, \rho_d) \in \overline{\mathbb{K}}^d$ of the roots of $P(x) \in \mathbb{K}[x]$, where $\overline{\mathbb{K}}$ is the algebraic closure of $\mathbb{K}$
T	the set $\{(\tau_{k_1}, \dots, \tau_{k_D}) \mid 1 \leqslant k_1 \leqslant \dots \leqslant k_D \leqslant s\}$ of all the ordered vectors with entries from the ordered set $\{\tau_1, \dots, \tau_s\}$
$\mathbf{t}$	a vector $(t_1, \dots, t_D)$ that ranges over T
$\mathbf{u}_\ell$ and $\mathbf{v}_\ell$	vectors $(u_1, \dots, u_\ell)$ and $(v_1, \dots, v_\ell)$ respectively
$\mathbf{i}_\ell$ and $\mathbf{j}_\ell$	vectors $(i_1, \dots, i_\ell)$ and $(j_1, \dots, j_\ell)$ respectively
$\mathbf{v}_\ell^{\mathbf{i}_\ell}$ and $\mathbf{u}_\ell^{\mathbf{j}_\ell}$	monomials $v_1^{i_1} \cdots v_\ell^{i_\ell}$ and $u_1^{j_1} \cdots u_\ell^{j_\ell}$ respectively
$p_\ell(y_1, \dots, y_m)$	the power-sum symmetric polynomial $y_1^\ell + \dots + y_m^\ell$
$\mathbf{p}_\ell(y_1, \dots, y_m)$	the vector $(p_1(y_1, \dots, y_m) \dots, p_\ell(y_1, \dots, y_m))$
$\mathbf{0}_\ell$	the ℓ-dimensional null-vector $(0, \dots, 0)$

The computations supporting the presented results are mainly computer-aided. This means that reading some formulæ might not be easy. Moreover, this explains why such results could not appear a few decades ago or earlier: the field of computer algebra was not developed enough.

The approach in a nutshell and the outline of the paper

Let d denote the degree of a solution $P(x) \in \mathbb{K}[x]$ of Eq. (1). Our aim is to construct a *degree polynomial* for Eq. (1), that is a univariate polynomial for which d is a root. Degree polynomials for linear recurrence relations with polynomial coefficients are defined, e.g., in the book Petkovšek et al. (1996).

The approach presented in this article is based on equating the corresponding coefficients on the right- and left-hand side of an identity between two polynomials. This approach is applied not to Eq. (1), but to the equivalent Eq. (3) below:

$$G_D\big(P(x-\tau_1), \dots, P(x-\tau_s)\big) = -G_{<D}\big(P(x-\tau_1), \dots, P(x-\tau_s)\big) - G_0(x) \tag{3}$$

where $G(x_1, \ldots, x_s)$ is represented as the sum $G_D(x_1, \ldots, x_s) + G_{<D}(x_1, \ldots, x_s)$ with G_D being the homogeneous part with total degree D and $G_{<D}$ containing the terms of G with total degrees $< D$.[1]

Without loss of generality one can assume that $d(D-1) > \deg(G_0)$, otherwise clearly we have a bound $d \leqslant \deg(G_0)/(D-1)$. Then the degree w.r.t. x on the right-hand side of Eq. (3) is at most $d(D-1)$. The degree w.r.t. x of the left-hand side is at most dD. All coefficients of $x^{dD}, x^{dD-1}, \ldots, x^{d(D-1)+1}$ on the left-hand side must vanish because $dD > d(D-1)$. In Section 2 we give a necessary set-up and show that these coefficients can be expressed in terms of the *power-sum symmetric polynomials* evaluated at the roots $\mathbf{r}$ of $P(x)$. Note that for $P(x) \in \mathbb{K}[x]$ the values $p_\ell(\mathbf{r})$ are in $\mathbb{K}$ even if there are roots in $\overline{\mathbb{K}} \setminus \mathbb{K}$. One constructs polynomials $S_\ell(u_0, (u_1, \ldots, u_\ell))$ such that the coefficient of $x^{dD-\ell}$ on the l.h.s. of Eq. (3) is equal to $S_\ell(d, (p_1(\mathbf{r}), \ldots, p_\ell(\mathbf{r})))$. In general, S_ℓ cannot be taken as degree polynomials, because they depend on $\ell + 1$ variables.

In this article we analyse some cases when the variables $u_1, \ldots, u_\ell$ can be eliminated from a certain equation $S_\ell(u_0, (u_1, \ldots, u_\ell)) = 0$, so that the degree polynomial $Q_0(u_0)$ is equal to $S_\ell(u_0, \mathbf{0}_\ell)$. The *framework lemma* in Section 2 gives a sufficient condition for such an elimination to be possible. In Sections 3 and 4, respectively, we consider two independent cases for which the conditions of the framework lemma hold and therefore the degree d can be bounded:

- let $\mathcal{L}$ denote the set $\{\ell \mid S_\ell(u_0, \mathbf{0}_\ell)$ is not everywhere zero$\}$; if $\mathcal{L} \neq \emptyset$ and $L := \min(\mathcal{L}) \leqslant 5$ then either $d \leqslant \max\{L, \deg(G_0)/(D-1)\}$, or d is a root of $S_L(u_0, \mathbf{0}_L)$ (see Theorem 4 and the example in Section 5),
- $d \leqslant \max\{D, \deg(G_0)/(D-1)\}$ for all difference equations of the form $G(P(x), P(x-\tau)) + G_0(x) = 0$ (see Theorem 5).

In Section 6 we sum up the results and outline future work. Technical details of the proofs can be found in the Appendix or the technical report Shkaravska and van Eekelen (2010). The proofs are supported by calculations in Maple (download `nonlindifeq.tar.gz`, available on the site http://resourceanalysis.cs.ru.nl under the item *Technical reports*).

Related work

The bound $d \leqslant D$ for $G(P(x), P(x-\tau)) = 0$ with vanishing $G_0(x)$ resembles the result $d = D$ for ordinary difference equations of the form $G(P(x), P(x-1)) = 0$ where the polynomial $G(x_1, x_2)$ is irreducible in rational field extension and D is the total degree of G, see Feng et al. (2008). The latter gives the *precise* degree of a polynomial solution for an irreducible polynomial G whereas we give just an upper bound. However, we do not demand irreducibility of G. Since G is the product of its irreducible factors, applying the result of the article Feng et al. (2008) for each of them gives $d \leqslant D$.

In the article Tang et al. (2010) the authors investigate the global behaviour of solutions of nonlinear difference equations of the form $x_{n+1} = (\alpha + x_n)/(A + Bx_n + x_{n-k})$, where $n \geqslant 0$, the parameters are positive real numbers and the initial conditions $x_{-k}, \ldots, x_0$ are non-negative real numbers, $k \geqslant 2$. One of the results is that every solution is bounded from above and from below by positive constants. In Öcalan (2009) one gives necessary and sufficient conditions for the oscillation of solutions x_n of nonlinear difference equations of the form $x_{n+1} - x_n + \sum_{i=1}^{m} p_i f_i(x_{n-k_i}) = 0$ where $k_i \in \{\ldots, -2, -1\}$ and $p_i < 0$ for $1 \leqslant i \leqslant m$. Moreover, the result is generalised to equations with non-constant coefficients, p_{in}.

A bound on the degree of polynomial solutions of linear homogeneous recurrence relations with polynomial coefficients $P(n) = G(n, P(n-1), \ldots, P(n-s))$ is obtained in the article Abramov (1989). It is done via a degree polynomial. In the article Mezzarobba and Salvy (2010) a similar problem is considered for complex polynomials, satisfying linear recurrence relations with rational–polynomial coefficients. The authors constructively define a real sequence that dominates the absolute value of the complex polynomial sequence. In Borcea et al. (2011) one gives the asymptotic ratio $\lim_{n\to\infty} \frac{f_{n+1}(\mathbf{x})}{f_n(\mathbf{x})}$

[1] Subsequently, we say that *a monomial* in the variables $x_1, \ldots, x_n$ is the product of powers of x_i. It has the form $x_1^{i_1} \ldots x_n^{i_n}$. A *term* is a product of powers multiplied by a constant.

for $f_n(\mathbf{x})$ satisfying a linear recurrence equation of the form $f_{n+k}(\mathbf{x}) + \sum_{i=1}^{k} \phi_{i,n}(\mathbf{x}) f_{n+k-i}(\mathbf{x}) = 0$, with $n \geqslant k-1$.

2. Coefficients of x in $G_D(P(x-\tau_1), \ldots, P(x-\tau_s))$ as symmetric polynomials

We consider a multivariate polynomial $G_D(x_1, \ldots, x_s) = \sum_{i_1+\cdots+i_s=D} a_{i_1,\ldots,i_s} x_1^{i_1} \cdots x_s^{i_s}$ of total degree D. Re-index the coefficients $a_{i_1,\ldots,i_s}$ of $G_D(x_1, \ldots, x_s)$ in such a way that, for instance, the coefficient $a_{2,0}$ of $x_1^2 = x_1 x_1$ becomes $\alpha_{(\tau_1,\tau_1)}$ and the coefficient $a_{1,1}$ of $x_1 x_2$ becomes $\alpha_{(\tau_1,\tau_2)}$. Consider another example: take $G_5(x_1, x_2, x_3)$ of degree $D = 5$ with $s = 3$. Its term $a_{2,3,0} x_1^2 x_2^3$ is represented as $\alpha_{(\tau_1,\tau_1,\tau_2,\tau_2,\tau_2)} x_1 x_1 x_2 x_2 x_2$. In general, the reindexation $I : \{(i_1, \ldots, i_s) \mid i_1 + \ldots + i_s = D,\ i_j \in \mathbb{N}\} \to T = \{(\tau_{k_1}, \ldots, \tau_{k_D}) \mid 1 \leqslant k_1 \leqslant \ldots \leqslant k_D \leqslant s\}$ maps $(i_1, \ldots, i_s)$ to $\mathbf{t} = (t_1, \ldots, t_D) = (\tau_1^{(i_1)}, \tau_2^{(i_2)}, \ldots, \tau_s^{(i_s)})$, where $\tau^{(i)}$ denotes τ repeated i times. Clearly, I is a bijection since the τ_i are pairwise distinct.

With this reindexation we write

$$G_D(x_1, \ldots, x_s) = \sum_{\mathbf{t}=(\tau_{k_1},\ldots,\tau_{k_D}) \in T} \alpha_{\mathbf{t}} x_{k_1} \ldots x_{k_D}.$$

For instance, for $D = 2$, $s = 3$ one has

$$T = \{(\tau_1, \tau_1), (\tau_1, \tau_2), (\tau_1, \tau_3), (\tau_2, \tau_2), (\tau_2, \tau_3), (\tau_3, \tau_3)\}$$

and for the polynomial $G_2(x_1, x_2, x_3) = x_1^2 - 2x_1 x_2 + x_3^2$ the reindexation yields $\alpha_{(\tau_1,\tau_1)} = 1$, $\alpha_{(\tau_1,\tau_2)} = -2$, $\alpha_{(\tau_3,\tau_3)} = 1$ and $\alpha_{(\tau_1,\tau_3)} = \alpha_{(\tau_2,\tau_2)} = \alpha_{(\tau_2,\tau_3)} = 0$.

Let the polynomial P be represented via its roots: $P(x) = a_d(x-\rho_1)\ldots(x-\rho_d)$. The product $P(x-t_1)\ldots P(x-t_D)$ is equal to $a_d^D \prod_{i=1}^{D} \prod_{j=1}^{d} (x - t_i - \rho_j)$. For this product one wants to find the coefficients $\varepsilon_\ell(\mathbf{t}, \mathbf{r})$ of $x^{Dd-\ell}$, where $0 \leqslant \ell \leqslant d-1$. The sums $(t_i + \rho_j)$, where $1 \leqslant i \leqslant D$, $1 \leqslant j \leqslant d$ are obviously the (only) roots of the polynomial $\prod_{i=1}^{D} \prod_{j=1}^{d} (x - t_i - \rho_j)$. Therefore, its coefficients $\varepsilon_\ell(\mathbf{t}, \mathbf{r})$ are represented via *the elementary symmetric polynomials* $e_\ell(y_1, \ldots, y_{dD}) := \sum_{1 \leqslant i_1 < i_2 \ldots < i_\ell \leqslant dD} y_{i_1} \ldots y_{i_\ell}$ and $e_0(y_1, \ldots, y_{dD}) := 1$ (Macdonald, 1979) in the standard way:

$$\varepsilon_\ell(\mathbf{t}, \mathbf{r}) = (-1)^\ell e_\ell(t_1 + \rho_1, \ldots, t_i + \rho_j, \ldots, t_D + \rho_d). \tag{4}$$

Lemma 1. *If a polynomial P of degree d solves Eq. (3) and $d > \ell$ for some $\ell \geqslant 0$ then the roots $\mathbf{r}$ of $P(x)$ must satisfy the identity*

$$\sum_{\mathbf{t} \in T} \varepsilon_\ell(\mathbf{t}, \mathbf{r}) \alpha_{\mathbf{t}} = 0. \tag{5}$$

Proof. Due to $d > \ell$ one has that $dD - \ell > d(D-1)$. Since P solves Eq. (3), the coefficients $a_d^D \sum_{\mathbf{t} \in T} \varepsilon_\ell(\mathbf{t}, \mathbf{r}) \alpha_{\mathbf{t}}$ of $x^{dD-\ell}$ on the l.h.s. of Eq. (3) must vanish. Having $a_d \neq 0$, one obtains identity (5). □

Lemma 1 does not give direct information about d, since each $\varepsilon_\ell(\mathbf{t}, \mathbf{r})$ depends on d implicitly: d is the dimension of $\mathbf{r}$. To obtain an explicit equation for d from Eq. (5), employ power-sum symmetric polynomials and the Newton–Girard formulæ (Macdonald, 1979):

$$e_\ell(y_1, \ldots, y_m) = (1/\ell) \sum_{\kappa=1}^{\ell} (-1)^{\kappa-1} e_{\ell-\kappa}(y_1, \ldots, y_m) p_\kappa(y_1, \ldots, y_m).$$

One can easily check by the definition of p_κ and the binomial formula, that

$$p_\kappa(t_1 + \rho_1, \ldots, t_i + \rho_j, \ldots, t_D + \rho_d) = \sum_{i=1}^{D} \sum_{j=1}^{d} (t_i + \rho_j)^\kappa = \sum_{\lambda=0}^{\kappa} \binom{\kappa}{\lambda} p_{\kappa-\lambda}(\mathbf{t}) p_\lambda(\mathbf{r}). \tag{6}$$

Substitute $(t_1+\rho_1,\dots,t_i+\rho_j,\dots,t_D+\rho_d)$ for $(y_1,\dots,y_{dD})$ in the Newton–Girard formulæ with $m=dD$ and combine them with identity (6). This yields an inductively defined family of functions $E_\ell(v_0,\mathbf{v}_\ell,u_0,\mathbf{u}_\ell)$:

Definition 1.

$$E_0\big(v_0,(),u_0,()\big):=1,$$

$$E_\ell(v_0,\mathbf{v}_\ell,u_0,\mathbf{u}_\ell):=-(1/\ell)\sum_{\kappa=1}^{\ell}E_{\ell-\kappa}(v_0,\mathbf{v}_{\ell-\kappa},u_0,\mathbf{u}_{\ell-\kappa})\left(\sum_{\lambda=0}^{\kappa}\binom{\kappa}{\lambda}v_{\kappa-\lambda}u_\lambda\right).$$

For instance, $E_1(v_0,\mathbf{v}_1,u_0,\mathbf{u}_1)=-v_1u_0-v_0u_1$. Now we can make the following statement.

Lemma 2. *For all $\ell\geqslant 0$ the following identity holds*:

$$\varepsilon_\ell(\mathbf{t},\mathbf{r})=E_\ell\big(D,\mathbf{p}_\ell(\mathbf{t}),d,\mathbf{p}_\ell(\mathbf{r})\big). \tag{7}$$

Proof. We prove the lemma by induction on ℓ using the Newton–Girard formulæ on the induction step.

For $\ell=0$ one obtains $\varepsilon_0(\mathbf{t},\mathbf{r})=1=E_0(D,\mathbf{p}_0(\mathbf{t}),d,\mathbf{p}_0(\mathbf{r}))$ immediately by the definitions.

For $\ell>0$, combining identity (4) with the Newton–Girard formulæ, where $(y_1,\dots,y_m)$ is replaced by $(t_1+\rho_1,\dots,t_i+\rho_j,\dots,t_D+\rho_d)$, one obtains

$$(-1)^\ell\varepsilon_\ell(\mathbf{t},\mathbf{r})=(1/\ell)\sum_{\kappa=1}^{\ell}(-1)^{\kappa-1}(-1)^{\ell-\kappa}\varepsilon_{\ell-\kappa}(\mathbf{t},\mathbf{r})p_\kappa(t_1+\rho_1,\dots,t_i+\rho_j,\dots,t_D+\rho_d).$$

From this and identity (6) it follows that

$$\varepsilon_\ell(\mathbf{t},\mathbf{r})=-(1/\ell)\sum_{\kappa=1}^{\ell}\varepsilon_{\ell-\kappa}(\mathbf{t},\mathbf{r})\sum_{\lambda=0}^{\kappa}\binom{\kappa}{\lambda}p_{\kappa-\lambda}(\mathbf{t})p_\lambda(\mathbf{r}). \tag{8}$$

Using the induction assumption $\varepsilon_{\ell-\kappa}(\mathbf{t},\mathbf{r})=E_{\ell-\kappa}(D,\mathbf{p}_{\ell-\kappa}(\mathbf{t}),d,\mathbf{p}_{\ell-\kappa}(\mathbf{r}))$ one easily obtains $\varepsilon_\ell(\mathbf{t},\mathbf{r})=E_\ell(D,\mathbf{p}_\ell(\mathbf{t}),d,\mathbf{p}_\ell(\mathbf{r}))$ by the definition. The lemma is proven. □

Using the functions E_ℓ, one can symbolically compute $\varepsilon_\ell(\mathbf{t},\mathbf{r})$ for any $\ell>0$. For instance, $\varepsilon_1(\mathbf{t},\mathbf{r})=-dp_1(\mathbf{t})-Dp_1(\mathbf{r})$.

Now we are ready to combine Definition 1 and Lemma 1. This is expressed via the following definition and lemma.

Definition 2. $S_\ell(u_0,\mathbf{u}_\ell):=\sum_{\mathbf{t}\in T}E_\ell(D,\mathbf{p}_\ell(\mathbf{t}),u_0,\mathbf{u}_\ell)\alpha_\mathbf{t}$.

Lemma 3. *If a polynomial P of degree d solves Eq. (3) and $d>\ell$ for some $\ell\geqslant 0$ then $S_\ell(d,\mathbf{p}_\ell(\mathbf{r}))=0$.*

Proof. By Lemma 2 and the definition of S_ℓ one has $\sum_{\mathbf{t}\in T}\varepsilon_\ell(\mathbf{t},\mathbf{r})\alpha_\mathbf{t}=S_\ell(d,\mathbf{p}_\ell(\mathbf{r}))$. By Lemma 1 one obtains the identity $S_\ell(d,\mathbf{p}_\ell(\mathbf{r}))=0$. □

Yet, from the point of view of bounding the degree d, Lemma 3 is too general. We will figure out the cases when for some non-negative integer number $L\geqslant 0$ the identities $S_\ell(d,\mathbf{p}_\ell(\mathbf{r}))=0$, with $0\leqslant\ell\leqslant L$, yield a non-zero univariate polynomial $Q(u_0)$ such that $Q(d)=0$. To be more precise, we are looking for L such that $Q(u_0)=S_L(u_0,(0,\dots,0))$. For this we have a closer look at the functions $E_\ell(v_0,\mathbf{v}_\ell,u_0,\mathbf{u}_\ell)$. These functions are obviously polynomials in $v_0,\mathbf{v}_\ell,u_0,\mathbf{u}_\ell$. The total degree w.r.t. $v_0,\dots,v_\ell$ and $u_0,\dots,u_\ell$ is ℓ, however one can prove a more precise connection between the powers of the v- and u-variables:

Lemma 4. *For any term with the monomial part $v_0^{i_0} \dots v_\ell^{i_\ell} u_0^{j_0} \dots u_\ell^{j_\ell}$ that occurs in $E_\ell(v_0, \mathbf{v}_\ell, u_0, \mathbf{u}_\ell)$ the following equation holds:* $\sum_{\kappa=0}^{\ell} \kappa(i_\kappa + j_\kappa) = \ell$.

The proof follows by induction on ℓ. This property is used when one wants to give the complete list of all coefficients of the powers of the variables $\mathbf{u}_\ell$, when $E_\ell(v_0, \mathbf{v}_\ell, u_0, \mathbf{u}_\ell)$ is considered as a polynomial in $\mathbf{u}_\ell$.

Definition 3. $A_{\mathbf{i}_\ell}(v_0, \mathbf{v}_\ell, u_0)$ denotes the coefficient of $\mathbf{u}_\ell^{\mathbf{i}_\ell}$ in $E_\ell(v_0, \mathbf{v}_\ell, u_0, \mathbf{u}_\ell)$, that is $E_\ell(v_0, \mathbf{v}_\ell, u_0, \mathbf{u}_\ell) = \sum_{\mathbf{i}_\ell} A_{\mathbf{i}_\ell}(v_0, \mathbf{v}_\ell, u_0) \cdot \mathbf{u}_\ell^{\mathbf{i}_\ell}$.

Using Definition 3 it is easy to obtain the representation of $S_\ell(u_0, \mathbf{u}_\ell)$ as a polynomial in $\mathbf{u}_\ell$:

$$S_\ell(u_0, \mathbf{u}_\ell) = \sum_{\mathbf{i}_\ell} \left(\sum_{\mathbf{t} \in T} \alpha_{\mathbf{t}} A_{\mathbf{i}_\ell}\big(D, \mathbf{p}_\ell(\mathbf{t}), u_0\big) \right) \cdot \mathbf{u}_\ell^{\mathbf{i}_\ell}. \tag{9}$$

In its turn, each of the $A_{\mathbf{i}_\ell}(v_0, \mathbf{v}_\ell, u_0)$ is a polynomial in u_0, where the corresponding coefficients of u_0^μ are denoted by $B_{\mathbf{i}_\ell,\mu}(v_0, \mathbf{v}_\ell)$. As one will see now, the coefficients $B_{\mathbf{0}_\ell,\mu}$ play a special role.

Lemma 5. *Let $L > 0$ be such that for any $0 \leqslant \ell \leqslant L-1$ the polynomial $S_\ell(u_0, \mathbf{0}_\ell)$ is everywhere zero, and moreover, for $\mathbf{i}_L \neq \mathbf{0}_L$ and for all $\mu \leqslant \ell$ there exist polynomials $H_{\mathbf{i}_L,\ell,\mu}(D, u_0)$, such that*

$$A_{\mathbf{i}_L}\big(D, \mathbf{p}_L(\mathbf{t}), u_0\big) = \sum_{\ell=0}^{L-1} \sum_{\mu=0}^{\ell} H_{\mathbf{i}_L,\ell,\mu}(D, u_0) B_{\mathbf{0}_\ell,\mu}\big(D, \mathbf{p}_l(\mathbf{t})\big). \tag{10}$$

Then $S_L(u_0, \mathbf{u}_L) = S_L(u_0, \mathbf{0}_L)$ for all u_0.

Proof. Since for any $0 \leqslant \ell \leqslant L-1$ the polynomial $S_\ell(u_0, \mathbf{0}_\ell)$ is everywhere zero, it follows that for any $0 \leqslant \mu \leqslant \ell$ the coefficient of u_0^μ in $S_\ell(u_0, \mathbf{0}_\ell) = \sum_{\mathbf{t} \in T} \alpha_{\mathbf{t}} A_{\mathbf{0}_\ell}(D, \mathbf{p}_\ell(\mathbf{t}), u_0) = \sum_{\mathbf{t} \in T} \alpha_{\mathbf{t}} \sum_{\mu=0}^{\ell} B_{\mathbf{0}_\ell,\mu}(D, \mathbf{p}_\ell(\mathbf{t})) u_0^\mu$ vanishes:

$$\sum_{\mathbf{t} \in T} \alpha_{\mathbf{t}} B_{\mathbf{0}_\ell,\mu}\big(D, \mathbf{p}_\ell(\mathbf{t})\big) = 0 \quad \text{for all } 0 \leqslant \mu \leqslant \ell. \tag{11}$$

Now, plugging identity (10) into identity (9), we obtain that for any $\mathbf{i}_L \neq \mathbf{0}_L$ the coefficient of $\mathbf{u}_L^{\mathbf{i}_L}$ in $S_L(u_0, \mathbf{u}_L)$ vanishes. Indeed, it is equal to

$$\begin{aligned}
&\sum_{\mathbf{t} \in T} \alpha_{\mathbf{t}} \sum_{\ell=0}^{L-1} \sum_{\mu=0}^{\ell} H_{\mathbf{i}_L,\ell,\mu}(D, u_0) B_{\mathbf{0}_\ell,\mu}\big(D, \mathbf{p}_\ell(\mathbf{t})\big) \\
&\quad = \sum_{\ell=0}^{L-1} \sum_{\mu=0}^{\ell} H_{\mathbf{i}_L,\ell,\mu}(D, u_0) \sum_{\mathbf{t} \in T} \alpha_{\mathbf{t}} B_{\mathbf{0}_\ell,\mu}\big(D, \mathbf{p}_\ell(\mathbf{t})\big) \\
&\quad \overset{\text{identity (11)}}{=} \sum_{\ell=0}^{L-1} \sum_{\mu=0}^{\ell} H_{\mathbf{i}_L,\ell,\mu}(D, u_0) \cdot 0 = 0.
\end{aligned}$$

Therefore, $S_L(u_0, \mathbf{u}_\ell) = S_L(u_0, \mathbf{0}_\ell)$. □

Lemma 6 *(Framework). Let the set $\mathcal{L} := \{\ell \mid S_\ell(u_0, \mathbf{0}_\ell)$ is not everywhere zero$\}$ be non-empty and $L = \min(\mathcal{L})$. Moreover, let for all $\mathbf{i}_L \neq \mathbf{0}_L$ and for all $\mu \leqslant \ell < L$ there exist polynomials $H_{\mathbf{i}_L,\ell,\mu}(D, u_0)$ such that identities (10) hold. Then either $d \leqslant \max\{L, \deg(G_0)/(D-1)\}$, or d is a root of $Q(u_0) := S_L(u_0, \mathbf{0}_L)$.*

Proof. If $d > L$ and $d > \deg(G_0)/(D-1)$, then $dD - L > d(D-1) > \deg(G_0)$ which implies that the coefficient of x^{dD-L} on the l.h.s. of Eq. (3) must vanish. We apply Lemma 3 to obtain $S_L(d, \mathbf{p}_L(\mathbf{r})) = 0$. Next, we apply Lemma 5 and obtain $S_L(u_0, \mathbf{u}_\ell) = S_L(u_0, \mathbf{0}_\ell)$ for all u_0. From this and the condition $S_L(d, \mathbf{p}_L(\mathbf{r})) = 0$, it follows that $Q(d) = 0$. $\square$

3. Existence of a degree polynomial for $0 \leqslant L \leqslant 5$

It turned out that a property stronger than identity (10) holds for $E(v_0, \mathbf{v}_\ell, u_0, \mathbf{u}_\ell)$, where $1 \leqslant \ell \leqslant 5$. It is stated in the following lemma.

Lemma 7. *For all $1 \leqslant L \leqslant 5$, for all $\mathbf{i}_L \neq \mathbf{0}_L$ and for all $\mu \leqslant \ell < L$ there exist polynomials $H_{\mathbf{i}_L,\ell,\mu}(v_0, u_0)$ such that $A_{\mathbf{i}_L}(v_0, \mathbf{v}_L, u_0) = \sum_{\ell=0}^{L-1} \sum_{\mu=0}^{\ell} H_{\mathbf{i}_L,\ell,\mu}(v_0, u_0) B_{\mathbf{0}_\ell,\mu}(v_0, \mathbf{v}_\ell)$.*[2]

Proof. The coefficients $A_{\mathbf{i}_L}(v_0, \mathbf{v}_L, u_0)$, $B_{\mathbf{0}_\ell,\mu}(v_0, \mathbf{v}_\ell)$ and $H_{\mathbf{i}_L,\ell,\mu}(v_0, u_0)$ are computed symbolically for all $0 \leqslant L \leqslant 5$, $\mu \leqslant \ell < L$ in the script `lemma-7.mw` (see Section 1 for the url). Linear algebra suffices to obtain $H_{\mathbf{i}_L,\ell,\mu}(v_0, u_0)$.[3] Fix some $1 \leqslant L \leqslant 5$. Think of $A_{\mathbf{i}_L}(v_0, \mathbf{v}_L, u_0)$, and $B_{\mathbf{0}_\ell,\mu}(v_0, \mathbf{v}_\ell)$ as polynomials in $v_1, \ldots, v_L$ only, but with coefficients which belong to the field $\mathbb{K}(u_0, v_0)$ of rational functions in u_0, v_0 over the field of constants $\mathbb{K}$. Let $H_{\mathbf{i}_L,\ell,\mu}(v_0, u_0)$ belong to $\mathbb{K}(u_0, v_0)$. Make the list of all the monomials in $v_1, \ldots, v_L$ of degree $\leqslant L$. Represent each of $A_{\mathbf{i}_L}(v_0, \mathbf{v}_L, u_0)$ and $B_{\mathbf{0}_\ell,\mu}(v_0, \mathbf{v}_\ell)$ by the corresponding vectors $F = (F_{\mathbf{j}_L})$ and $F_{\ell,\mu} = (F_{\mathbf{j}_L,\ell,\mu})$ of their coefficients w.r.t. $\mathbf{v}_L^{\mathbf{j}_L}$, with $\mu \leqslant \ell < L$. Direct computations show that the linear system $M\mathbf{H} = F$ is solvable over $\mathbb{K}(u_0, v_0)$, where M is the matrix with columns $F_{\ell,\mu}$, and $\mathbf{j}_L$ ranges over rows. Moreover, the entries of the solution $\mathbf{H} = (H_{\mathbf{i}_L,\ell,\mu}(v_0, u_0))$ are polynomials for $L \leqslant 5$.

The polynomials $B_{\mathbf{0}_\ell,\mu}(v_0, \mathbf{v}_\ell)$ and the corresponding coefficients $H_{\mathbf{i}_L,\ell,\mu}(v_0, u_0)$ are given in Appendix A.1. $\square$

The main theorem below gives an effective bound on d in the case when there exists $0 \leqslant L \leqslant 5$ such that $S_L(u_0, \mathbf{0}_L)$ is not everywhere zero.

Theorem 4. *If the set $\mathcal{L} = \{\ell \mid S_\ell(u_0, \mathbf{0}_\ell)$ is not everywhere zero$\}$ is not empty and, moreover, $L := \min(\mathcal{L}) \leqslant 5$, then either $d \leqslant \max\{L, \deg(G_0)/(D-1)\}$, or d must be among the non-negative integer roots of $S_L(u_0, \mathbf{0}_L)$.*

Proof. The condition $L \leqslant 5$ together with Lemma 7 yields the conditions of the framework lemma. Applying it straightforwardly gives the conclusion of the theorem. $\square$

Corollary 1. *For any difference equation (3) with $D = 2$ and $\tau_i = a + i - 1$ where $i = 1, 2, 3$, there is $0 \leqslant L \leqslant 5$ such that $S_L(u_0, \mathbf{0}_L)$ is not everywhere zero. Therefore, the degree d of a polynomial solution P either does not exceed $\max\{L, \deg(G_0)/(D-1)\}$, or must be among the non-negative integer roots of the polynomial $S_L(u_0, \mathbf{0}_L)$.*

Proof. Without lost of generality one can consider only the case $a = 0$. Indeed, if $G(P(x-a), P(x-a-1), P(x-a-2)) + G_0(x) = 0$ has a polynomial solution $P(x)$ then $G(F(x), F(x-1), F(x-2)) + G_0(x) = 0$ has the polynomial solution $F(x) = P(x-a)$, of the same degree.

Now, for $a = 0$ assume the opposite: for all $0 \leqslant \ell \leqslant 5$ the polynomials $S_\ell(u_0, \mathbf{0}_\ell)$ for $G(P(x), P(x-1), P(x-2)) + G_0(x) = 0$ are everywhere zero. We show that in this case G_D is reduced to the zero polynomial. With $D = 2$ and $\tau_i = i$, where $i = 0, 1, 2$, one has $T = \{(i_1, i_2) \mid 0 \leqslant i_1 \leqslant i_2 \leqslant 2\}$. Compute the concrete values of $B_{\mathbf{0}_\ell,\mu}(D, \mathbf{p}_\ell(\mathbf{t}))$ for all $\mathbf{t} \in T$, $1 \leqslant \ell \leqslant 5$, $1 \leqslant \mu \leqslant \ell$, and

[2] This identity is stronger than identity (10) because it holds *for all* v_0 and $\mathbf{v}_\ell$, not only for $v_0 := D$ and $\mathbf{v}_\ell := \mathbf{p}_\ell(\mathbf{t})$.

[3] This observation belongs to M. Petkovšek. Originally we used the procedure Groebner[NormalForm] to perform the division of the polynomial $A_{\mathbf{i}_L}(v_0, \mathbf{v}_L, u_0)$ in $\mathbf{v}_L$ by the polynomials from the set $\{B_{(),0} = 1\} \cup \{B_{\mathbf{0}_\ell,\mu}(v_0, \mathbf{v}_l)\}_{1 \leqslant \ell < L,\ 1 \leqslant \mu \leqslant l}$. Note that $B_{\mathbf{0}_\ell,0}(v_0, \mathbf{v}_\ell) = 0$ for $\ell > 0$.

$B_{(),0} = 1$. These values form the matrix of the over-determined linear system of 16 equations $\sum_{\mathbf{t}\in T} B_{\mathbf{0}_\ell,\mu}(D, \mathbf{p}_\ell(\mathbf{t}))\alpha_{\mathbf{t}} = 0$ for 6 variables $\alpha_{\mathbf{t}}$ (see Appendix A.2 for more detail). This system has only the zero solution $\bar{\alpha} = \mathbf{0}_6$ which means that the polynomial G_D is everywhere zero, which contradicts the condition $D = 2$. □

In the same way one proves the similar statement for difference equations with $D = 3$ and $\tau_i = a + i - 1$, where $i = 1, 2$.

Now consider what happens if the conditions of Theorem 4 do not hold, that is, for all $0 \leqslant \ell \leqslant 5$ the polynomials $S_\ell(u_0, \mathbf{0}_\ell)$ are everywhere zero. The next lemma shows that then, in general, $S_6(u_0, \mathbf{u}_6)$ and $S_6(u_0, \mathbf{0}_6)$ do not have to be equal as polynomials and therefore $S_6(u_0, \mathbf{0}_6)$ cannot be taken as a degree polynomial.

Lemma 8. *If the polynomial $S_\ell(u_0, \mathbf{0}_\ell)$ is everywhere zero for any $0 \leqslant \ell \leqslant 5$, then*

$$S_6(u_0, \mathbf{u}_6) = S_6(u_0, \mathbf{0}_6) + (1/8)\big(u_1^2 - u_2 u_0\big) \sum_{\mathbf{t}\in T} p_2^2(\mathbf{t})\alpha_{\mathbf{t}}. \tag{12}$$

Proof. The computations of $H_{\mathbf{i}_6,\ell,\mu}$ are performed as in the proof of Lemma 7. The coefficients $H_{\mathbf{i}_6,\ell,\mu}$ for $A_{\mathbf{i}_6}(v_0, \mathbf{v}_6, u_0)$ can be found using linear algebra, except those for $A_{(0,1,0,0,0,0)}$ and $A_{(2,0,0,0,0,0)}$. As in the proof of Lemma 5, all $\sum_{\mathbf{t}\in T} B_{\mathbf{0}_\ell,\mu}(D, \mathbf{p}_\ell(\mathbf{t}))\alpha_{\mathbf{t}}$ vanish. Therefore the sums $\sum_{\mathbf{t}\in T} A_{\mathbf{i}_6}(D, \mathbf{p}_6(\mathbf{t}), u_0)\alpha_{\mathbf{t}}$ vanish as well, if $\mathbf{i}_6 \neq (0, 1, 0, 0, 0, 0)$ and $\mathbf{i}_6 \neq (2, 0, 0, 0, 0, 0)$.

The linear systems of the form $M\mathbf{H} = F$ for $A_{(0,1,0,0,0,0)}$ and $A_{(2,0,0,0,0,0)}$ are not solvable over $\mathbb{K}(u_0, v_0)$. However, replacing $B_{\mathbf{0}_2,1}(v_1, v_2) = -v_2/2$ with v_2^2 in the list of polynomials $\{B_{(),0} = 1\} \cup \{B_{\mathbf{0}_\ell,\mu}(v_0, \mathbf{v}_2)\}_{1\leqslant\ell\leqslant5, 1\leqslant\mu\leqslant\ell}$ one can obtain the alternative systems $M'\mathbf{H}' = F$ which are solvable for $A_{(0,1,0,0,0,0)}(v_0, \mathbf{v}_6, u_0)$ and $A_{(2,0,0,0,0,0)}(v_0, \mathbf{v}_6, u_0)$ over $\mathbb{K}(u_0, v_0)$. The coefficients of v_2^2 are $-(1/8)u_0$ and $1/8$ respectively. Identity (12) follows from these identities and the definition of $S_\ell(u_0, \mathbf{u}_\ell)$. Check `lemma-8.mw` from `nonlindifeq.tar.gz` archive mentioned in the Introduction, where all $H_{\mathbf{i}_6,\ell,\mu}$ and $H'_{\mathbf{i}_6,\ell,\mu}$, for the original $M\mathbf{H} = F$ and the alternative $M'\mathbf{H}' = F$ systems, respectively, are computed. □

Therefore, if the polynomials $S_\ell(u_0, \mathbf{0}_\ell)$ are everywhere zero for all $0 \leqslant \ell \leqslant 5$ then the proposed approach, in general, does not give a bound on d.

However, $D = 2$ gives a special case where the framework lemma is applicable for $\ell = 6$.

Corollary 2. *For all difference equations with $D = 2$, if $S_\ell(u_0, \mathbf{0}_\ell)$ is everywhere zero for $0 \leqslant \ell \leqslant 5$ then $S_6(u_0, \mathbf{u}_6) = S_6(u_0, \mathbf{0}_6)$. From this it follows that if $S_6(u_0, \mathbf{0}_6)$ is not everywhere zero, then either $d \leqslant \max\{6, \deg(G_0)\}$, or d is one of the positive integer roots of $S_6(u_0, \mathbf{0}_6)$ if they exist.*

The proof can be found in the technical report Shkaravska and van Eekelen (2010).

4. Difference equations with a single shift

Consider difference equations of the form

$$G\big(P(x), P(x - \tau)\big) + G_0(x) = 0. \tag{13}$$

In this equation $\tau_1 = 0$, $\tau_2 = \tau$. For the sake of convenience denote $(0, \ldots, 0, \tau, \ldots, \tau)$, where τ occurs m times, by $\mathbf{t}(m)$. The aim is to prove

Theorem 5. *The degree of a polynomial solution P of Eq. (13), if it exists, is $d \leqslant \max\{D, \deg(G_0)/(D - 1)\}$.*

To show that the conditions of the framework lemma are satisfied consider a few facts about p_ℓ and $S_\ell(u_0, \mathbf{u}_\ell)$ for Eq. (13). First of all, it is easy to see that $p_\ell(\mathbf{t}(m)) = 0^\ell + \cdots + 0^\ell + \tau^\ell + \cdots + \tau^\ell = m\tau^\ell$. Second, from this and identity (6) it follows that $p_\ell(\mathbf{t}(m) + \mathbf{r}) = m \sum_{\lambda=0}^{\ell} \binom{\ell}{\lambda} \tau^{\ell-\lambda} p_\lambda(\mathbf{r})$, where $\ell \geqslant 1$. Third, by the definition of E_ℓ for $\ell \geqslant 1$ one obtains

$$E_\ell\big(D,\mathbf{p}_\ell\big(\mathbf{t}(m)\big),u_0,\mathbf{u}_\ell\big)=-(m/\ell)\sum_{\kappa=1}^{\ell}E_{\ell-\kappa}\big(D,\mathbf{p}_{\ell-\kappa}\big(\mathbf{t}(m)\big),u_0,\mathbf{u}_{\ell-\kappa}\big)\sum_{\lambda=0}^{\kappa}\binom{\kappa}{\lambda}\tau^{k-\lambda}u_\lambda. \tag{14}$$

From this one obtains the following recurrent formulæ:

$$A_{\mathbf{0}_\ell}\big(D,\mathbf{p}_\ell\big(\mathbf{t}(m)\big),u_0\big)=(-m/\ell)\sum_{\kappa=1}^{\ell}u_0\tau^\kappa A_{\mathbf{0}_{\ell-\kappa}}\big(D,\mathbf{p}_{\ell-\kappa}\big(\mathbf{t}(m)\big),u_0\big),$$

$$B_{\mathbf{0}_\ell,\mu}\big(D,\mathbf{p}_\ell\big(\mathbf{t}(m)\big)\big)=(-m/\ell)\sum_{\kappa=1}^{\ell}\tau^\kappa B_{\mathbf{0}_{\ell-\kappa},\mu-1}\big(D,\mathbf{p}_{\ell-\kappa}\big(\mathbf{t}(m)\big)\big). \tag{15}$$

We refine the upper limit on κ of the sum in the identities for $B_{\mathbf{0}_\ell,\mu}$ in the following way. From the definition of $B_{\mathbf{0}_{\ell-\kappa},\mu-1}$, due to $0\leqslant\mu-1\leqslant\ell-\kappa$, we obtain $\kappa\leqslant\ell-\mu+1$.

Lemma 9. *For $\ell=0$, $\mu=0$, and for all $\ell>0$, $1\leqslant\mu\leqslant\ell$ there exist constants $C_{\ell,\mu}>0$ such that*

$$B_{\mathbf{0}_\ell,\mu}\big(D,\mathbf{p}_\ell\big(\mathbf{t}(m)\big)\big)=(-1)^\mu C_{\ell,\mu}m^\mu\tau^\ell. \tag{16}$$

Proof. For $\ell=0$, $\mu=0$ we have $B_{(),0}=1$, and therefore $C_{0,0}=1$. For $\ell>0$ we prove the lemma by induction on ℓ. To begin with, for $\ell=1$ one has $B_{\mathbf{0}_1,1}(v_0,v_1)=-v_1$. Therefore $B_{\mathbf{0}_1,1}(D,p_1(\mathbf{t}(m)))=-m\tau=(-1)^1m^1\tau^1$, so $C_{1,1}=1$.

Now, fix some $\ell>1$. Use the recurrent formula for $B_{\mathbf{0}_\ell,\mu}$:

$$B_{\mathbf{0}_\ell,\mu}\big(D,\mathbf{p}_\ell\big(\mathbf{t}(m)\big)\big)=-(m/\ell)\sum_{\kappa=1}^{\ell-\mu+1}\tau^\kappa B_{\mathbf{0}_{\ell-\kappa},\mu-1}\big(D,\mathbf{p}_{\ell-\kappa}\big(\mathbf{t}(m)\big)\big). \tag{17}$$

By the induction assumption the statement of the lemma holds for all $\ell'<\ell$. This implies that $B_{\mathbf{0}_\ell,\mu}(D,\mathbf{p}_\ell(\mathbf{t}(m)))$ is equal to

$$\begin{aligned}&-(m/\ell)\sum_{\kappa=1}^{\ell-\mu+1}\tau^\kappa\tau^{\ell-\kappa}(-1)^{\mu-1}m^{\mu-1}C_{\ell-\kappa,\mu-1}\\&\quad=\tau^\ell m^{1+\mu-1}(-1)^{1+\mu-1}(1/\ell)\sum_{\kappa=1}^{\ell-\mu+1}C_{\ell-\kappa,\mu-1}.\end{aligned} \tag{18}$$

From this it follows that $C_{\ell,\mu}=(1/\ell)\sum_{\kappa=1}^{\ell-\mu+1}C_{\ell-\kappa,\mu-1}>0$. For $\mu=1$ we note that $C_{\ell-\kappa,\mu-1}=C_{0,0}=1$ if $\kappa=\ell$, and $C_{\ell-\kappa,\mu-1}=C_{\ell-\kappa,0}=0$ if $\kappa<\ell$. $\square$

Now we can prove Theorem 5.

Proof of Theorem 5. We show that the conditions of the framework lemma hold. To begin with, we show that there exists $0\leqslant L\leqslant D$ such that the polynomial $S_L(u_0,\mathbf{0}_L)$ is not everywhere zero. Assume the opposite: $S_\ell(u_0,\mathbf{0}_\ell)$ are everywhere zero for all $0\leqslant\ell\leqslant D$. This implies that the corresponding coefficients of u_0^μ in $S_\ell(u_0,\mathbf{0}_\ell)$ must be all zeros. Hence by Lemma 9 with $\mu:=\ell$ it follows that $\sum_{m=0}^{D}(-1)^\ell C_{\ell,\ell}\tau^\ell m^\ell\alpha_{\mathbf{t}(m)}=0$ which due to $\tau\neq0$ and $C_{\ell,\ell}>0$ implies $\sum_{m=0}^{D}m^\ell\alpha_{\mathbf{t}(m)}=0$ for all $0\leqslant\ell\leqslant D$. That is, one gets a system of $D+1$ linear equations for $D+1$ variables x_m. The matrix of this system is of rank $D+1$ because its determinant is equal to the $D\times D$ Vandermonde determinant. Therefore, the system has only the zero solution $\alpha_{\mathbf{t}(m)}$ which contradicts the fact that G is of degree D. Therefore there exists $S_L(u_0,\mathbf{0}_L)$ which is not everywhere zero. W.l.o.g. assume that for all $0\leqslant\ell\leqslant L-1$ the polynomials $S_\ell(u_0,\mathbf{0}_\ell)$ are everywhere zero.

If $L = 0$ then $S_0(u_0) \neq 0$. Comparing the left- and right-hand sides of the corresponding equation of the form (3) yields $dD \leqslant \max\{\deg(G_0), d(D-1)\}$. If $\deg(G_0) < d(D-1)$ then $dD \leqslant d(D-1)$ which is impossible. Therefore $\deg(G_0) \geqslant d(D-1)$ which implies $d \leqslant \deg(G_0)/(D-1) \leqslant \max\{D, \deg(G_0)/(D-1)\}$.

Now we consider the case $L \geqslant 1$ in more detail. The function $A_{\mathbf{i}_L}(D, \mathbf{p}_L(\mathbf{t}(m)), u_0)$ can be seen as a polynomial in m because $p_\ell(\mathbf{t}(m)) = m\tau^\ell$. Let $T^{D,\tau}_{\mathbf{i}_{L,\mu}}(u_0)$ denote its coefficients of m^μ. Since $A_{\mathbf{i}_L}(D, \mathbf{p}_L(\mathbf{t}(m)), u_0)$ is a linear combination of m^μ, it is a linear combination of $B_{\mathbf{0}_\mu,\mu}(D, \mathbf{p}_\mu(\mathbf{t}(m))) = (-1)^\mu m^\mu C_{\mu,\mu}\tau^\mu$ as well, with the coefficients $H^\tau_{\mathbf{i}_L,\mu,\mu}(D, u_0) = (-1)^\mu T^{D,\tau}_{\mathbf{i}_L,\mu}(u_0)/(C_{\mu,\mu}\tau^\mu)$ and $H^\tau_{\mathbf{i}_L,\mu,\mu'}(D, u_0) = 0$ for $\mu' \neq \mu$.

Assume that $d > \max\{L, \deg(G_0)/(D-1)\} \geqslant 0$. By the framework lemma one obtains that d is a root of $S_L(u_0, \mathbf{0}_L)$. Since $S_L(u_0, \mathbf{0}_L) = \sum_{m=0}^D \alpha_{\mathbf{t}(m)} A_{\mathbf{0}_L}(D, \mathbf{p}_L(\mathbf{t}(m)), u_0) = \sum_{\mu=0}^L u_0^\mu \sum_{m=0}^D \alpha_{\mathbf{t}(m)} B_{\mathbf{0}_L,\mu}(D, \mathbf{p}_L(\mathbf{t}(m)))$, by Lemma 9 one obtains that $S_L(u_0, \mathbf{0}_L) = \sum_{\mu=0}^L u_0^\mu (-1)^\mu \tau^L C_{L,\mu} \sum_{m=0}^D m^\mu \alpha_{\mathbf{t}(m)}$. Consider the sums $\sum_{m=0}^D m^\mu \alpha_{\mathbf{t}(m)}$ for $0 \leqslant \mu \leqslant L-1$. Since the polynomials $S_\mu(u_0, \mathbf{0}_\mu)$ are everywhere zero for all $0 \leqslant \mu \leqslant L-1$ one gets $\sum_{m=0}^D (-1)^\mu C_{\mu,\mu}\tau^\mu m^\mu \alpha_{\mathbf{t}(m)} = 0$, which implies that $\sum_{m=0}^D m^\mu \alpha_{\mathbf{t}(m)} = 0$ for $0 \leqslant \mu \leqslant L-1$. Therefore, $S_L(u_0, \mathbf{0}_L) = u_0^L (-1)^L \tau^L C_{L,L} \sum_{m=0}^D m^L \alpha_{\mathbf{t}(m)} = 0$. Since $S_L(u_0, \mathbf{0}_L)$ is not everywhere zero in u_0 the inequation $\sum_{m=0}^D m^L \alpha_{\mathbf{t}(m)} \neq 0$ holds. From this it follows that $S_L(d, \mathbf{0}_L) = 0$ implies $d = 0$, which contradicts the assumption $d > 0$.

Therefore, $d \leqslant \max\{L, \deg(G_0)/(D-1)\} \leqslant \max\{D, \deg(G_0)/(D-1)\}$. □

5. Example

The equation $P(x) = P^2(x-1) - 2P(x-1)P(x-2) + 3P(x-1)P(x-3) - 2P^2(x-2) - 17P(x-1) + 29x^2 - 45x + 51$ has a polynomial solution of degree $d = 3$ which is a root of the degree polynomial. Here $D = 2$ and $\deg(G_0)/(D-1) = 2$. To find the degree polynomial it is enough to calculate $S_0(u_0, ())$, $S_1(u_0, (0))$ and $S_2(u_0, (0, 0))$. Use the definition

$$S_\ell(u_0, \mathbf{0}_\ell) = \sum_{\mathbf{t} \in T} A_{\mathbf{0}_\ell}\big(D, \mathbf{p}_\ell(\mathbf{t}), u_0\big)\alpha_{\mathbf{t}}.$$

A direct calculation yields $A_{()}(v_0, (), u_0, ()) = 1$, next $A_{(0)}(v_0, (v_1), u_0) = -v_1 u_0$ and $A_{(0,0)}(v_0, \mathbf{v}_2, u_0) = (1/2)v_1^2 u_0^2 - (1/2)v_2 u_0$ (see Appendix A.1). Compute the values $p_\ell(\mathbf{t})$ (for non-vanishing $\alpha_{\mathbf{t}}$):

$\mathbf{t}$	$p_1(\mathbf{t})$	$p_1^2(\mathbf{t})$	$p_2(\mathbf{t})$	$\alpha_{\mathbf{t}}$
$(1,1)$	$1+1=2$	4	$1^2+1^2=2$	1
$(1,2)$	$1+2=3$	9	$1^2+2^2=5$	-2
$(1,3)$	$1+3=4$	16	$1^2+3^2=10$	3
$(2,2)$	$2+2=4$	16	$2^2+2^2=8$	-2

As one can see from the equation, the coefficients $\alpha_{\mathbf{t}}$ for $\mathbf{t}$ that are not mentioned in the table vanish. Now, by the substitutions $v_\ell := p_\ell(\mathbf{t})$ one obtains

$$S_0\big(u_0, ()\big) = \sum_{\mathbf{t} \in T} \alpha_{\mathbf{t}} = 1 - 2 + 3 - 2 = 0 \quad \text{for all values of } u_0,$$

$$S_1\big(u_0, (0)\big) = u_0 \sum_{\mathbf{t} \in T} p_1(\mathbf{t})\alpha_{\mathbf{t}} = u_0(1 \cdot 2 - 2 \cdot 3 + 3 \cdot 4 - 2 \cdot 4) = 0 \quad \text{for all values of } u_0,$$

$$\begin{aligned} S_2\big(u_0, (0,0)\big) &= u_0(1/2)\bigg(\sum_{\mathbf{t} \in T}\big(u_0 p_1^2(\mathbf{t}) - p_2(\mathbf{t})\big)\alpha_{\mathbf{t}}\bigg) \\ &= (u_0/2)\big(u_0 \cdot (1 \cdot 4 - 2 \cdot 9 + 3 \cdot 16 - 2 \cdot 16) \\ &\quad - (1 \cdot 2 - 2 \cdot 5 + 3 \cdot 10 - 2 \cdot 8)\big) = u_0(u_0 - 3). \end{aligned}$$

So, here $L = 2$. From this it follows, that if the difference equation has a polynomial solution of degree $d > L$ then for this degree it must hold $d = 3$. It is easy to check that there is a solution $P(x) = x^3 + x^2 + x + 1$ for the equation.

6. Conclusions and outlook

The present article concerns polynomial solutions $P(x)$ of difference equations of the form $G(P(x-\tau_1), \ldots, P(x-\tau_s)) + G_0(x) = 0$, where $G(x_1, \ldots, x_s)$ is a known polynomial of degree $D \geqslant 2$ and G_0 is a known polynomial in x. The authors address the cases when one can bound the degree d of a polynomial solution P if such a solution exists. For the difference equation a family of polynomials $S_\ell(u_0, \mathbf{0}_\ell)$, $\ell \geqslant 0$, has been defined, and it has been shown that if $\mathcal{L} := \{\ell \mid S_\ell(u_0, \mathbf{0}_\ell)$ is not everywhere zero$\} \neq \emptyset$ and $L := \min(\mathcal{L}) \leqslant 5$ then $d \leqslant \max\{L, \deg(G_0)/(D-1)\}$ or d must be among the positive integer roots of $S_L(u_0, \mathbf{0}_L)$ (Theorem 4). Also, it has been shown that in this way one can bound d for all quadratic difference equations with $\tau_i = a + i - 1$, where $i = 1, 2, 3$, and all cubic difference equations with $\tau_i = a + i - 1$ where $i = 1, 2$. In general, with the presented approach it is impossible to bound the degree of solutions of difference equations for which $S_\ell(u_0, \mathbf{0}_\ell)$ are everywhere zero for all $0 \leqslant \ell \leqslant 5$. However, it has been proven that $d \leqslant \max\{D, \deg(G_0)/(D-1)\}$ for equations with $s = 2$, $\tau_1 = 0$ and $\tau_2 = \tau$, see Theorem 5.

An obvious direction of future research is applying the presented technique to polynomial difference equations with polynomial non-constant coefficients. A more challenging problem is to check if there are connections between the obtained results and Galois theory.

From the application point of view the obtained result improves polynomial resource analysis of computer programs developed in article Shkaravska et al. (2009). There the authors consider the size of an output as a polynomial function on the sizes of inputs. In the Charter project the authors developed the ResAna tool (Kersten et al., 2012) that applies polynomial interpolation to generate an upper bound on Java loop iterations. The tool requires the user to input the degree of the solution. The results of this article will help to automatically obtain the degree of the polynomial in many cases.

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Appendix A

This appendix assists the proofs of Lemmata 7 and 8, and Corollary 1. In Appendix A.1 the coefficients $B_{\mathbf{i}_\ell,\mu}(v_0, \mathbf{v}_\ell)$ and $H_{\mathbf{i}_L,\ell,\mu}(v_0, u_0)$ are listed. They are referred to in the proofs of Lemmata 7 and 8. In Appendix A.2 one finds the matrix of the linear system for $\bar{\alpha}$ which is used in the proof of Corollary 1.

A.1. *The coefficients $B_{\mathbf{i}_\ell,\mu}(v_0, \mathbf{v}_\ell)$ and $H_{\mathbf{i}_L,\ell,\mu}(v_0, u_0)$*

In this section we consider the polynomials $B_{\mathbf{i}_\ell,\mu}(v_0, \mathbf{v}_\ell)$ which are the coefficients of the monomials u_0^μ of $A_{\mathbf{i}_\ell}(v_0, \mathbf{v}_\ell, u_0)$. Moreover, we give the coefficients $H_{\mathbf{i}_L,\ell,\mu}(v_0, u_0)$ for the representations of $A_{\mathbf{i}_L}(v_0, \mathbf{v}_\ell, u_0)$ as the linear combinations of $B_{\mathbf{0}_\ell,\mu}(v_0, \mathbf{v}_\ell)$, for $\mathbf{i}_L \neq \mathbf{0}_L$. Calculations are performed in `lemma-7.mw` and `lemma-8.mw`.

- ($L = 0$.) In this case $E_0(v_0, (), u_0, ()) = 1$ and one gets $B_{(),0} = A_{()} = 1$ immediately by the definitions.
- ($L = 1$.) In this case $E_1(v_0, \mathbf{v}_1, u_0, \mathbf{u}_1) = -v_0 u_1 - v_1 u_0$, and therefore $A_{(0)}(v_0, \mathbf{v}_1, u_0) = -u_0 v_1$ and $A_{(1)}(v_0, \mathbf{v}_1, u_0) = -v_0$. From this it follows that $B_{(0),1}(v_0, v_1) = -v_1$ and $B_{(0),0}(v_0, v_1) = 0$. At the end of the day one obtains $H_{(1),0,0} = -v_0$ because of the identity $A_{(1)}(v_0, v_1, u_0) = -v_0 B_{(),0}$.

- ($L=2$.) Then $E_2(v_0,\mathbf{v}_2,u_0,\mathbf{u}_2)=(1/2)u_0^2v_1^2+(1/2)v_0^2u_1^2+v_1u_0v_0u_1-(1/2)v_2u_0-v_1u_1-(1/2)v_0u_2$ and $A_{(0,0)}(v_0,\mathbf{v}_2,u_0)=(1/2)v_1^2u_0^2-(1/2)v_2u_0$. From this it follows that

$B_{(0,0),2}(v_0,\mathbf{v}_2)=(1/2)v_1^2$	$B_{(0,0),1}(v_0,\mathbf{v}_2)=-(1/2)v_2$	$B_{(0,0),0}(v_0,\mathbf{v}_2)=0$

and

$H_{(1,0),1,1}=(-u_0v_0+1)$	$H_{(2,0),0,0}=(1/2)v_0^2$	$H_{(0,1),0,0}=-(1/2)v_0$

- ($L=3$.) Then for $E_3(v_0,\mathbf{v}_3,u_0,\mathbf{u}_3)$ one obtains $A_{(0,0,0)}=-(1/6)v_1^3u_0^3-(1/3)v_3u_0+(1/2)v_2u_0^2v_1$. From this it follows that

$B_{(0,0,0),3}(v_0,\mathbf{v}_3)=-(1/6)v_1^3$	$B_{(0,0,0),2}(v_0,\mathbf{v}_3)=(1/2)v_1v_2$	$B_{(0,0,0),1}(v_0,\mathbf{v}_3)=-(1/3)v_3$
$B_{(0,0,0),0}(v_0,\mathbf{v}_3)=0$		

Eventually,

$H_{(1,0,0),2,2}=-u_0^2v_0+2u_0$	$H_{(1,0,0),2,1}=-u_0v_0+2$	$H_{(2,0,0),1,1}=(1/2)v_0^2u_0-v_0$
$H_{(3,0,0),0,0}=-(1/6)v_0^3$	$H_{(0,1,0),1,1}=-(1/2)u_0v_0+1$	$H_{(1,1,0),0,0}=(1/2)v_0^2$
$H_{(0,0,1),0,0}=-(1/3)v_0$		

and the other $H_{\mathbf{i}_3,\ell,\mu}$ vanish.

- ($L=4$.) For $E_4(v_0,\mathbf{v}_4,u_0,\mathbf{u}_4)$ symbolic computation yields that

$B_{(0,0,0,0),4}=(1/24)v_1^4$	$B_{(0,0,0,0),3}=-(1/4)v_2v_1^2$	$B_{(0,0,0,0),2}=(1/3)v_3v_1+(1/8)v_2^2$
$B_{(0,0,0,0),1}=-(1/4)v_4$	$B_{(0,0,0,0),0}=0$	

and

$H_{(1,0,0,0),3,3}=-v_0u_0^3+3u_0^2$	$H_{(1,0,0,0),3,2}=-v_0u_0^2+3u_0$	$H_{(1,0,0,0),3,1}=-v_0u_0+3$
$H_{(2,0,0,0),2,2}=(1/2)v_0^2u_0^2-2v_0u_0+1$	$H_{(2,0,0,0),2,1}=(1/2)v_0^2u_0-2v_0$	$H_{(3,0,0,0),1,1}=-(1/6)v_0^3u_0+(1/2)v_0^2$
$H_{(4,0,0,0),0,0}=(1/24)v_0^4$	$H_{(0,1,0,0),2,2}=-(1/2)v_0u_0^2+2u_0$	$H_{(0,1,0,0),2,1}=-(1/2)v_0u_0+3$
$H_{(0,2,0,0),0,0}=(1/8)v_0^2$	$H_{(0,0,1,0),1,1}=-(1/3)v_0u_0+1$	$H_{(0,0,0,1),0,0}=-(1/4)v_0$
$H_{(1,1,0,0),1,1}=(1/2)v_0^2u_0-(3/2)v_0$	$H_{(2,1,0,0),0,0}=-(1/4)v_0^3$	$H_{(1,0,1,0),0,0}=(1/3)v_0^2$

The other $H_{\mathbf{i}_3,\ell,\mu}$ vanish.

- ($L=5$.) Symbolic computation of $E_5(v_0,\mathbf{v}_5,u_0,\mathbf{u}_5)$ gives

$B_{\mathbf{0}_5,5}(v_0,\mathbf{v}_5)=-(1/120)v_1^5$	$B_{\mathbf{0}_5,4}=(1/12)v_2v_1^3$	$B_{\mathbf{0}_5,3}=-(1/6)v_1^2v_3-(1/8)v_2^2v_1$
$B_{\mathbf{0}_5,2}=(1/4)v_4v_1+(1/6)v_3v_2$	$B_{\mathbf{0}_5,1}=-(1/5)v_5$	$B_{\mathbf{0}_5,0}=0$

Now we list the coefficients of $A_{\mathbf{i}_5}$ considered as linear combinations of $B_{\mathbf{0}_\ell,\mu}(v_0, \mathbf{v}_\ell)$ where $\ell < 5$:

$H_{(1,0,0,0,0),4,4} = -v_0u_0^4 + 4u_0^3$	$H_{(1,0,0,0,0),4,3} = 4u_0^2 - v_0u_0^3$
$H_{(1,0,0,0,0),4,2} = -v_0u_0^2 + 4u_0$	$H_{(1,0,0,0,0),4,1} = -u_0v_0 + 4$
$H_{(2,0,0,0,0),3,3} = (1/2)v_0^2u_0^3 - 3v_0u_0^2 + 3u_0$	$H_{(2,0,0,0,0),3,2} = (1/2)v_0^2u_0^2 - 3u_0v_0 + 2$
$H_{(2,0,0,0,0),3,1} = -3v_0 + (1/2)v_0^2u_0$	$H_{(3,0,0,0,0),2,2} = -(1/6)v_0^3u_0^2 + v_0^2u_0 - v_0$
$H_{(3,0,0,0,0),2,1} = -(1/6)v_0^3u_0 + v_0^2$	$H_{(4,0,0,0,0),1,1} = (1/24)v_0^4u_0 - (1/6)v_0^3$
$H_{(5,0,0,0,0),0,0} = -(1/120)v_0^5$	$H_{(0,1,0,0,0),3,3} = -(1/2)v_0u_0^3 + 3u_0^2$
$H_{(0,1,0,0,0),3,2} = -(1/2)v_0u_0^2 + 4u_0$	$H_{(0,1,0,0,0),3,1} = -(1/2)u_0v_0 + 6$
$H_{(0,2,0,0,0),1,1} = (1/8)v_0^2u_0 - (1/2)v_0$	$H_{(1,1,0,0,0),2,2} = (1/2)v_0^2u_0^2 - 3v_0u_0 + 2$
$H_{(1,1,0,0,0),2,1} = (1/2)v_0^2u_0 - 4v_0$	$H_{(1,2,0,0,0),1,1} = -(1/8)v_0^3$
$H_{(0,0,1,0,0),2,2} = -(1/3)v_0u_0^2 + 2u_0$	$H_{(0,0,1,0,0),2,1} = -(1/3)u_0v_0 + 4$
$H_{(1,0,1,0,0),1,1} = (1/3)v_0^2u_0 - (4/3)v_0$	$H_{(2,1,0,0,0),1,1} = -(1/4)v_0^3u_0 + v_0^2$
$H_{(2,0,1,0,0),0,0} = -(1/6)v_0^3$	$H_{(3,1,0,0,0),0,0} = (1/12)v_0^4$
$H_{(0,1,1,0,0),0,0} = (1/6)v_0^2$	$H_{(0,0,0,1,0),1,1} = -(1/4)u_0v_0 + 1$
$H_{(1,0,0,1,0),0,0} = (1/4)v_0^2$	$H_{(0,0,0,0,1),0,0} = -(1/5)v_0$

The coefficients that are not in the table vanish.

- ($L = 6$.) The representation of $A_{\mathbf{i}_6}(v_0, \mathbf{v}_6, u_0)$ is considered in detail in the technical report Shkaravska and van Eekelen (2010).

Now we give the coefficients for the representation of $A_{(0,1,0,0,0,0)}(v_0, \mathbf{v}_6, u_0)$ and $A_{(2,0,0,0,0,0)}(v_0, \mathbf{v}_6, u_0)$ via the alternative list of polynomials $B_{\mathbf{0}_\ell,\mu}(v_0, \mathbf{v}_\ell)$ where $\mu \leqslant \ell \leqslant 5$, with $B_{\mathbf{0}_2,1}(v_0, \mathbf{v}_2) = -v_2/2$ replaced by v_2^2:

$H'_{(0,1,0,0,0,0)v_2^2} = -(1/8)u_0$	$H'_{(0,1,0,0,0,0),4,1} = 10 - (1/2)u_0v_0$
$H'_{(0,1,0,0,0,0),4,2} = -(1/2)u_0^2v_0 + 7u_0$	$H'_{(0,1,0,0,0,0),4,3} = -(1/2)u_0^3v_0 + 5u_0^2$
$H'_{(0,1,0,0,0,0),4,4} = -(1/2)u_0^4v_0 + 4u_0^3$	
$H'_{(2,0,0,0,0,0),v_2^2} = -(1/8)$	$H'_{(2,0,0,0,0,0),4,1} = -4v_0 + (1/2)u_0v_0^2$
$H'_{(2,0,0,0,0,0),4,2} = (1/2)u_0^2v_0^2 + 3 - 4u_0v_0$	$H'_{(2,0,0,0,0,0),4,3} = 5u_0 + (1/2)u_0^3v_0^2 - 4u_0^2v_0$
$H'_{(2,0,0,0,0,0),4,4} = (1/2)u_0^4v_0^2 + 6u_0^2 - 4u_0^3v_0$	

In this table $H'_{(0,1,0,0,0,0)v_2^2}$ (resp. $H'_{(2,0,0,0,0,0),v_2^2}$) denotes the coefficient of v_2^2 in the representation of $A_{(0,1,0,0,0,0)}(v_0, \mathbf{v}_6, u_0)$ (resp. $A_{(2,0,0,0,0,0)}(v_0, \mathbf{v}_6, u_0)$) via the alternative list of polynomials.

A.2. Difference equations $G(P(x), P(x-1), P(x-2)) + G_0(x) = 0$, $D = 2$

In this section we give the matrix of the linear system for $\bar{\alpha}$, encountered in solving quadratic difference equations of the form $G(P(x), P(x-1), P(x-2)) + G_0(x) = 0$, see Corollary 1. The matrix is computed and the system is solved in `corollaries_Ds.mw`, which is available at http://resourceanalysis.cs.ru.nl/, in the archive mentioned in the Introduction.

$$\begin{pmatrix}
1 & 1 & 1 & 1 & 1 & 1 \\
0 & -1 & -2 & -2 & -3 & -4 \\
0 & -1/2 & -2 & -1 & -5/2 & -4 \\
0 & 1/2 & 2 & 2 & 9/2 & 8 \\
0 & -1/3 & -8/3 & -2/3 & -3 & -16/3 \\
0 & 1/2 & 4 & 2 & 15/2 & 16 \\
0 & -1/6 & -4/3 & -4/3 & -9/2 & -32/3 \\
0 & -1/4 & -4 & -1/2 & -17/4 & -8 \\
0 & 11/24 & 22/3 & 11/6 & 97/8 & 88/3 \\
0 & -1/4 & -4 & -2 & -45/4 & -32 \\
0 & 1/24 & 2/3 & 2/3 & 27/8 & 32/3 \\
0 & -1/5 & -32/5 & -2/5 & -33/5 & -64/5 \\
0 & 5/12 & 40/3 & 5/3 & 81/4 & 160/3 \\
0 & -7/24 & -28/3 & -7/3 & -183/8 & -224/3 \\
0 & 1/12 & 8/3 & 4/3 & 45/4 & 128/3 \\
0 & -1/120 & -4/15 & -4/15 & -81/40 & -128/15
\end{pmatrix}$$

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