Emergent symmetry on black hole horizons

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For a stationary and axisymmetric black hole, there is a natural way to split the fields into a probe sector and a background sector. The equations of motion for the probe sector enjoy a significantly enhanced symmetry on the black hole horizon. The extended symmetry is conformal in four dimensions, while in higher dimensions it is much bigger. This puts conformal symmetry at the bottom of the ladder of symmetries that can arise on black hole horizons in generic dimensions.

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A black hole is characterized by the presence of a horizon. So symmetries attached to the horizon presumably hold the key to the mysteries of black holes. There has been good evidence for this in relation to the black hole entropy.

For extremal black holes, one can zoom in the near horizon region and find evidence that quantum gravity in this background is dual to a conformal field theory in a state having the same Bekenstein-Hawking entropy as the black hole (see, e.g., [1,2]). For nonextremal black holes, one can similarly calculate the black hole entropy through a putative dual conformal field theory by imposing certain boundary conditions on (near) the horizon and finding conformal symmetries there (see, e.g., [3–7]).

All these works rely on imposing extra conditions on a chosen boundary, but sometimes one can find more than one working boundary condition for a given background. Since the dynamics governing the properties of a black hole is unique, this raises the question of whether external information like an artificial boundary condition is necessary in calculating the black hole entropy.

In this respect, we have found previously that some conformal symmetries actually emerge on the black hole horizons by themselves [8]. (See also [5,9,10].) The construction only depends on a few very general properties needed for the metric to describe a black hole,[1] but requires no extra boundary conditions. Since the radial direction is interpreted as the renormalization scale, such symmetries can be viewed as emergent from the duality perspective.

The purpose of this paper is to show that, in higher dimensions ($D > 4$), the emergent symmetry group is in fact much larger than that of the conformal symmetries. As a proof of principle, we will only present the result for the simplest nontrivial case. Comparing to four dimensions, gravity in higher dimensions is known to display much richer features, such as more possibilities for the topology of a horizon [11]. What we find here adds another potentially useful piece to the category.

Our starting point is the metric for a stationary and axisymmetric black hole,

$$ds^2 = f\left[-\frac{\Delta}{v^2}dt^2 + \frac{dr^2}{\Delta}\right] + h_{ij}d\theta^i d\theta^j + g_{ab}(d\phi^a - w^a dt)(d\phi^b - w^b dt), \tag{1}$$

where the details are explained in [8]. Here we only note that ϕ^a and t are ignorable coordinates: f, v, h_{ij}, g_{ab} and w^a only depend on r and θ^i, while $\Delta = \Delta(r)$; the horizon is located at r_0 with $\Delta(r_0) = 0$, where all other functions are regular. One can write (1) as $ds^2 = \tilde{G}_{\mu\nu}dX^\mu dX^\nu = H_{IJ}dx^I dx^J + G_{AB}dy^A dy^B$, where $x^I \in \{r, \theta^i\}$ and $y^A \in \{\phi^a, t\}$. Reducing the Einstein-Hilbert action on the ignorable coordinates ϕ^a and t, we find

$$\begin{aligned}S &= \int d^{\tilde{k}}x d^k y\sqrt{-\tilde{G}}(\tilde{R} - 2\Lambda) = (2\pi)^{k-1}T\int d^{\tilde{k}}x\mathcal{L},\\
\mathcal{L} &= \sqrt{|H|}\sqrt{|G|}\Big\{R - 2\Lambda + \frac{1}{4}\partial G_{AB}\partial G^{AB} + \Big(\partial\ln\sqrt{|G|}\Big)^2\\
&\quad - \frac{2}{\sqrt{|G|}}\nabla^2\sqrt{|G|}\Big\}\\
&= \sqrt{Hg/\varrho}\Big\{R - 2\Lambda + \frac{1}{4}\partial g_{ab}\partial g^{ab} + \frac{\varrho}{2}g_{ab}\partial w^a\partial w^b\\
&\quad + (\partial\ln\sqrt{g/\varrho})^2 - (\partial\ln\sqrt{\varrho})^2 - \frac{2\nabla^2\sqrt{g/\varrho}}{\sqrt{g/\varrho}}\Big\},\end{aligned} \tag{2}$$

where $\tilde{k} = D - k$, $k = [\frac{D+1}{2}]$ is the total number of ignorable coordinates, T is the total age of the system, $\tilde{R}$ and R are the Ricci scalar for $\tilde{G}_{\mu\nu}$ and H_{IJ}, respectively, $H = \det|H_{IJ}| = \frac{f}{\Delta}\det|h_{ij}|$, $G = \det|G_{AB}| = -g/\varrho$, $g = \det|g_{ab}|$, and $\varrho = \frac{v^2}{f\Delta}$. The indices from H_{IJ} have been suppressed in (2).

The basic idea is to treat (2) as an action for the fields G_{AB} living in the fixed background of H_{IJ}. Without

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[1]The analysis only depends on the general metric (1), which covers other interesting objects such as black rings [11]. But we will refer to all of them as black holes just for simplicity.

considering the backreaction on H_{IJ}, there is more possibility for G_{AB} than in the full theory. The extended transformations found below will leave the equations of G_{AB}, i.e., $\delta S/\delta G_{AB} = 0$, invariant on the horizon, but will not for $\delta S/\delta H_{IJ} = 0$. Although this may appear *ad hoc*, it is not too much different from introducing an external field to probe the black hole background [9,10]. In the present case, H_{IJ} is playing the role of a background, while G_{AB} is playing the role of a probe. As it happens, one relies on the hope that the extended transformations could become a symmetry of the full system when the Einstein-Hilbert action is replaced by that of a complete theory of quantum gravity.

We will focus on the equations of G_{AB} from now on. Varying G_{AB}, we find

$$\delta\mathcal{L} = \sqrt{H}\Big[E_{ab}\delta g^{ab} + E_a\delta w^a + E_{(g/\varrho)}\delta\sqrt{g/\varrho} + \nabla_I J^I_\delta\Big], \quad (3)$$

$$J^I_\delta = \pi^{Iab}\delta g_{ab} + \pi^I_a\delta w^a + \pi^I_{(g/\varrho)}\delta\sqrt{g/\varrho} - 2\partial^I\delta\sqrt{g/\varrho},$$
$$\pi^{Iab} = \sqrt{g/\varrho}\left(\frac{1}{2}\partial^I g^{ab} - \partial^I \ln\sqrt{\varrho}g^{ab}\right),$$
$$\pi^I_a = \sqrt{g/\varrho}\varrho g_{ab}\partial^I w^b, \qquad \pi^I_{(g/\varrho)} = 2\partial^I \ln\sqrt{g}, \quad (4)$$

where the index I goes over r and θ^i. One can first read off the equations of motion $E_{ab} = E_a = E_{(g/\varrho)} = 0$, which are equivalent to (the indices A, B run over ϕ^a and t)

$$E_{AB} = -\frac{1}{2}\frac{\nabla\left(\sqrt{-G}\partial G_{AB}\right)}{\sqrt{-G}} + \frac{1}{2}\partial G_{AC}\partial G_{BD}G^{CD} - \frac{2\Lambda}{D-2}G_{AB} \propto \frac{\delta S}{\delta G^{AB}} = 0. \quad (5)$$

All the extended symmetries that we will find satisfy

$$\sqrt{H}\nabla_I J^I_\delta \sim \delta E_{AB} \sim \mathcal{O}(\Delta). \quad (6)$$

Our calculation will rely on the following universal property:

$$w^a(r,\theta^i) = w^a_0(r) + w^a_1(r,\theta^i)\Delta + \cdots, \quad (7)$$

where "…" stands for higher order terms in the near horizon limit. This property is necessary for any black hole in (1) to have an intrinsically regular horizon. Our construction will crucially depends on $w^a \neq 0$. In a coordinate system that is static at the spatial infinity (with ϕ^a's normalized to be of period 2π), $w^a_0(r_0) = \Omega^a$ is the constant angular velocity of the horizon along ϕ^a. For a Schwarzschild-like black hole without an intrinsic rotation, one can first change to a rotating frame and then the same construction can still be applied.

Because of the freedom in redefining the ignorable coordinates, the action (2) has a rigid $GL(k,R)$ symmetry, $G_{AB} \to (\mathcal{V}\cdot G\cdot\mathcal{V}^T)_{AB}$, where $\mathcal{V}$ is a $k\times k$ constant matrix. The scaling factor in $GL(k,R)$ is orthogonal to all other transformations and will not be needed here. We will only focus on the $SL(k,R)$ subsector in this work.[2]

In [8] it has been found that some $SL(2,R)$ subsector of this $SL(k,R)$ can be extended, on the black hole horizon, to a centerless Virasoro algebra (Witt algebra),

$$[\delta_m,\delta_n] = (m-n)\delta_{m+n}, \qquad m,n = 0,\pm1,\pm2,\cdots. \quad (8)$$

To do this, one of the rotations (say ϕ^1) is singled out, labeled simply as ϕ, while all other rotations are indexed with $\tilde{a}$, $\tilde{b}$, $\cdots$, i.e.,

$$(G_{AB}) = \begin{pmatrix} g_{\phi\phi} & g_{\phi\tilde{b}} & -w_\phi \\ g_{\tilde{a}\phi} & g_{\tilde{a}\tilde{b}} & -w_{\tilde{a}} \\ -w_\phi & -w_{\tilde{b}} & -\frac{1}{\varrho}+w^2 \end{pmatrix}. \quad (9)$$

Then one considers the $SL(2,R)$ generators, which are $k\times k$ matrices, with only the following nontrivial elements:

$$(L_1)_{1k} = -2(L_0)_{11} = 2(L_0)_{kk} = -(L_{-1})_{k1} = 1. \quad (10)$$

Using $\delta G = -(L\cdot G + G\cdot L^T)$, one can find how δ_0 and $\delta_{\pm1}$ act on the fields of G_{AB}. Then the extension to other generators is constructed. Technically, only $\delta_{\pm2}$ were given explicitly in [8], which is enough to reconstruct the whole algebra. Here we write down the general result,[3]

$$\delta_m g_{\phi\phi} = -(m+1)\left[(m-1)g_{\phi\phi} - m\frac{w_\phi}{w^\phi} + m(m-1) \times \frac{w'^\phi(w_\phi - g_{\phi\phi}w^\phi)}{(w^\phi)^2\Delta'/\Delta}\right](w^\phi)^m,$$
$$\delta_m g_{\tilde{a}\phi} = -\frac{m+1}{2}\left[(m-1)g_{\tilde{a}\phi} - m\frac{w_{\tilde{a}}}{w^\phi} + m(m-1) \times \frac{w'^\phi(w_{\tilde{a}} - g_{\tilde{a}\phi}w^\phi)}{(w^\phi)^2\Delta'/\Delta}\right](w^\phi)^m,$$
$$\delta_m g_{\tilde{a}\tilde{b}} = 0, \quad \Rightarrow \quad \delta_m\varrho = \varrho(m+1)(w^\phi)^m,$$
$$\delta_m w^\phi = -\left[w^\phi + \frac{m(m+1)}{2}\frac{g^{\phi\phi}}{\varrho w^\phi}\right](w^\phi)^m,$$
$$\delta_m w^{\tilde{a}} = -\frac{m+1}{2}\left[w^{\tilde{a}} + m\frac{g^{\tilde{a}\phi}}{\varrho w^\phi}\right](w^\phi)^m, \quad (11)$$

[2]Consequently, one has $\delta\sqrt{-G} = \delta\sqrt{g/\varrho} = 0$, which is then assumed to be true for all the extended symmetries. Since this assumption is made in the whole spacetime, it is not a boundary condition that we set out to avoid. When this assumption is dropped, one will be looking at an extension involving the constant scaling factor of $GL(k,R)$. This possibility has not been considered carefully.

[3]This result differs from that in [8] at the subleading order. The result here satisfies the Witt algebra (8) at the leading order and also obeys (6), while those in [8] satisfy the Witt algebra up to the subleading order, but has $\delta E_{AB} \sim \Delta'(r_0) + \mathcal{O}(\Delta)$.

where a "prime" means a derivative with respect to r. Note the transformation of ϱ is always determined by $\delta\sqrt{g/\varrho}=0 \Rightarrow \delta\varrho = \varrho g^{ab}\delta g_{ab}$.

The subleading terms (those containing a factor $1/\varrho\sim\Delta$) are important for the symmetry to work. Without them, one would have found

$$\sqrt{H}\nabla_I J^I_\delta \sim \delta E_{AB} \sim \Delta'(r_0)+\mathcal{O}(\Delta). \tag{12}$$

But for extremal black holes, which has $\Delta'(r_0)\sim T_H=0$, where T_H is the Hawking temperature of the black hole, the symmetry works without the need for the subleading terms. Technically this will bring a significant simplification over the calculation when one tries to make the extension of the full $SL(k,R)$ algebra.[4]

Once there is an extension to the $SL(2,R)$ subsector, the dressing up of the rest of the $SL(k,R)$ generators is straightforward. One can simply do this by taking commutators of the other $SL(k,R)$ generators with (11) repeatedly. In four dimensions, $k=2$ and the conformal symmetry is all that is there. The simplest nontrivial case is that of the five dimensional extremal black holes.

In five dimensions, $k=3$ and we will have the following fields to consider: g_{11}, g_{12}, g_{22}, w^1, and w^2. As said above, one can drop the subleading $(1/\varrho\sim\Delta)$-terms for extremal black holes. After some effort, we find that all the symmetry generators organize into the following two, $\delta^+_{m,s}$ and $\delta^-_{m,s}$,

$$\begin{aligned}
\delta^+_{m,p}g_{11} &= \frac{4}{3}(m+1)g_{11}(w^1)^m(w^2)^p,\\
\delta^+_{m,p}g_{12} &= \left(\frac{m+1}{3}g_{12}w^2+pg_{11}w^1\right)(w^1)^m(w^2)^{p-1},\\
\delta^+_{m,p}g_{22} &= -2\left(\frac{m+1}{3}g_{22}w^2-pg_{12}w^1\right)(w^1)^m(w^2)^{p-1},\\
\delta^+_{m,p}w^1 &= -(w^1)^{m+1}(w^2)^p, \delta^+_{m,p}w^2=0,\\
\delta^-_{m,p}g_{11} &= -2\left(\frac{p+1}{3}g_{11}w^1-mg_{12}w^2\right)(w^1)^{m-1}(w^2)^p,\\
\delta^-_{m,p}g_{12} &= \left(\frac{p+1}{3}g_{12}w^1+mg_{22}w^2\right)(w^1)^{m-1}(w^2)^p,\\
\delta^-_{m,p}g_{22} &= \frac{4}{3}(p+1)g_{22}(w^1)^m(w^2)^p,\\
\delta^-_{m,p}w^1 &= 0, \delta^-_{m,p}w^2=-(w^1)^m(w^2)^{p+1},
\end{aligned} \tag{13}$$

where $m\in\mathbb{Z}$, $p\geq 0$ for $\delta^+_{m,p}$ and $p\geq -1$ for $\delta^-_{m,p}$. The range on p is the minimal truncation of (13) which includes all the $SL(3,R)$ generators; but (6) is satisfied by arbitrary values of m and p for both operators $\delta^\pm_{m,p}$. However, since all the fields are real while w^1 and w^2 can be negative, the indices m and p only take integer values.

The algebra satisfied by (13) is

$$\begin{aligned}
[\delta^+_{m,p},\delta^+_{n,q}] &= (m-n)\delta^+_{m+n,p+q},\\
[\delta^-_{m,p},\delta^-_{n,q}] &= (p-q)\delta^-_{m+n,p+q},\\
[\delta^+_{m,p},\delta^-_{n,q}] &= p\delta^+_{m+n,p+q}-n\delta^-_{m+n,p+q}.
\end{aligned} \tag{14}$$

Let $\delta^1_{\mathbf{m}}=\delta^+_{\mathbf{m}}$ and $\delta^2_{\mathbf{m}}=\delta^-_{\mathbf{m}}$, with $\mathbf{m}=(m^1,m^2)$ standing for the vector of indices, one cast the above algebra into an even more compact form

$$[\delta^i_{\mathbf{m}},\delta^j_{\mathbf{n}}]=m^j\delta^i_{\mathbf{m}+\mathbf{n}}-n^i\delta^j_{\mathbf{m}+\mathbf{n}}, \tag{15}$$

where $i,j\in\{1,2\}$. This algebra is formally a type of generalization to the usual Witt algebra. It has the general structure of that for the map $\mathbb{R}'^n\mapsto\mathbb{R}'^n$, (where $\mathbb{R}'$ means $\mathbb{R}$ with the points 0 and $\pm\infty$ pinched off) for which $i,j\in\{1,\cdots,n\}$ and $\mathbf{m}=\{m^1,\cdots,m^n\}$. It will be interesting to see if (15) also appears in generic dimensions, for which we expect $n=k-1$. [Loosely, one can first consider diagonally embedding $SL(3,R)$ into $SL(4,R)$, then $SL(4,R)$ into $SL(5,R)$, and so on, and (15) is the most natural result to expect.]

There is no fundamental obstacle to apply the same construction to general situations. But for nonextremal black holes, the subleading terms will make the calculation significantly more involved. There is also the interesting question of what happens if one starts with different embeddings of $SL(2,R)$ into $SL(k,R)$. The choice (10) is a diagonal embedding of $SL(2,R)$ into $SL(k,R)$. We have also tried the principle embedding of $SL(2,R)$ into $SL(3,R)$, but without success. [Our preliminary result suggests that, for $SL(3,R)$, there is no extension of the principally embedded $SL(2,R)$ to the Witt algebra.] It should be interesting to find all possible algebras that can arise from this construction in generic dimensions. Also, the present construction is only based on the universal properties of a black hole metric as described below (1) and in (7), and this is independent of the topology of the black hole horizon, so it gives the same result for, e.g., a Emparan-Reall black ring and a five dimensional Myers-Perry black hole.[5]

Our result indicates that, in higher dimensions, the symmetry governing the near horizon physics of a black hole could be much larger than the previously known conformal symmetry. For black holes with multiple independent rotating planes, it is known that each nonvanishing rotation can give raise to a Virasoro algebra and each is as

[4]The possibility of an extension of the full $SL(k,R)$ was first suggested to me by Evgeny Skvortsov.

[5]It is possible that conspiracy among the metric elements in certain solutions could give rise to further enlargement of the near horizon symmetry, but this should be studied on a case by case basis and is beyond the scope of the present work.

good in reproducing the Bekenstein-Hawking entropy through Cardy's formula [12]. Although there is a huge difference between our construction and the calculations done previously, all the methods must be related somehow if they are going to describe the same physics. Our result then suggests that, through similarity transformations, the different Virasoros corresponding to different rotations are in fact equivalent to each other, and they are only part of a even larger symmetry such as in (15).

At the moment, an obvious task is to find a way to abstract physical information from the newly found symmetries, which we have not succeeded in doing yet. Technically, this is partially due to the fact that the symmetry found here is not the usual diffeomorphisms, but is some internal gauge symmetry. As a result, the successful techniques used in previous calculations (e.g., those mentioned at the beginning of this paper) are not directly applicable here. Finding a way out of this is a major goal of our next step. Apart from helping us understand the properties of black holes in better detail, it is also possible that the general features found in generic dimensions can help us answer some of the more direct questions in four dimensions.

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