Embedding tensor of Scherk-Schwarz flux compactifications from eleven dimensions

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We study the Scherk-Schwarz reduction of $D = 11$ supergravity with background fluxes in the context of a recently developed framework pertaining to $D = 11$ supergravity. We derive the embedding tensor of the associated four-dimensional maximal gauged theories *directly* from eleven dimensions by exploiting the generalized vielbein postulates, and by analyzing the couplings of the full set of 56 electric and magnetic gauge fields to the generalized vielbeine. The treatment presented here will apply more generally to other reductions of $D = 11$ supergravity to maximal gauged theories in four dimensions.

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I. INTRODUCTION

Recently, a reformulation [1] of $D = 11$ supergravity [2] that emphasizes the exceptional $E_{7(7)}$ duality symmetry [3] and is based on the SU(8) invariant reformulation of $D = 11$ supergravity [4] has been constructed. The central object in this reformulation is an $E_{7(7)}$ 56-bein *in eleven dimensions*, which can be thought of as the eleven-dimensional ancestor of the 56-bein in four dimensions containing the 70 scalars of the reduced maximal theory. The four generalized vielbeine [1,4,5] that comprise the 56-bein in eleven dimensions are derived by analyzing the supersymmetry transformations of the 56 vector fields in the SU(8) invariant reformulation, generalizing and completing the construction of [4] (similar new structures also appear in the SO(16) invariant formulation of $D = 11$ supergravity where the relevant vielbein belongs to $E_{8(8)}$ [6,7]). The emphasis on supersymmetry as the origin of the generalized exceptional geometry obtained in this way is the main distinctive feature in comparison with other approaches to generalized geometry.[1] The 56-bein satisfies certain differential identities called "generalised vielbein postulates" [1,4] due to their similarities with the usual vielbein postulate in differential geometry, and these relations will be at the center of our construction.

The very nature of the reformulation in that it emphasizes structures in eleven dimensions that become apparent upon reduction to four dimensions makes it a useful framework in which to study questions regarding four-dimensional maximal gauged theories from a higher dimensional perspective. This feature extends the attributes of the SU(8) invariant reformulation, which leads to a nonlinear metric ansatz [10] and a proof [11,12] of the consistency of the S^7 reduction [13] of $D = 11$ supergravity. In particular, the new structures found in [1,5] give rise to nonlinear *Ansätze* for the internal components of the three-form [5] (see also [14]) and six-form [15] potentials. In fact, *Ansätze* can be given for the full uplift to eleven dimensions for any solution (and, in particular, the stationary points of the potential) of the four-dimensional theory; the possibility to perform such nontrivial tests of all formulas is another distinctive feature of the present approach. Furthermore, the generalized vielbein postulates reduce to the consistency requirements of the four-dimensional maximal gauged theory. In particular, there is a direct relation [1,15] between the set of generalized vielbein postulates with derivatives along four dimensions and the $E_{7(7)}$ Cartan equation of the maximal gauged theory [16–18], in which the gauging is defined via the embedding tensor [16,19,20].

The formalism developed in [1] has already been applied to an extensive study of the S^7 reduction [15]. In particular, nonlinear *Ansätze* are given for the uplift of four-dimensional solutions of SO(8) gauged maximal supergravity [21] to eleven dimensions, including dual fields. In addition, the embedding tensor of SO(8) gauged maximal supergravity is recovered directly by reducing the generalized vielbein postulates with derivatives along four dimensions. While the S^7 reduction is highly nontrivial from the perspective of the nonlinearity of uplift *Ansätze* and the field content in four dimensions, the gauging, and therefore the embedding tensor, is relatively simple in that the gauging only involves electric vectors, and moreover is uniform.

In this paper, we study Scherk-Schwarz [22][2] reductions of $D = 11$ supergravity with background flux [25–35]

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[1]For a summary of recent developments and a complete bibliography see [8,9].

[2]In fact, the essential idea of reducing on a group manifold appears in [23]; for a useful historical account of Kaluza-Klein theory, see [24].

within the context of the formalism developed in [1]. The Scherk-Schwarz flux compactification has principally been studied from a four-dimensional gauge algebra perspective by associating background fields to particular representations in the GL(7) decomposition of the **912** representation of $E_{7(7)}$ in which the embedding tensor lives. Here, we concentrate on obtaining the embedding tensor of such theories *directly from eleven dimensions* by analyzing the couplings of the 56 vector fields (28 electric and 28 magnetic vectors) via the generalized vielbein postulates. Hence, our approach should be contrasted with recent work [36–39] aiming to construct the embedding tensor for nongeometric compactifications obtained by generalized Scherk-Schwarz reductions of extended generalized geometries.

While the Scherk-Schwarz reduction is much simpler than the S^7 reduction, the novelty of the Scherk-Schwarz reduction as far as we are interested in is the potential for gaugings involving a combination of electric and magnetic vectors leading to a more complicated embedding tensor [28,32]. We derive the embedding tensor of Scherk-Schwarz flux compactifications directly and explicitly from the $D = 11$ generalized vielbein postulates. This constitutes a further nontrivial demonstration of the utility of the formalism developed in Ref. [1] and gives further credence to the interpretation of the generalized vielbein postulates as the higher dimensional origin of the embedding tensor. More generally, the results of Ref. [1] can be applied to any compactification of $D = 11$ supergravity to maximal gauged theories in four dimensions yielding nonlinear uplift *Ansätze* and the embedding tensor.

The outline of the paper is as follows. In Sec. II, we present a self-contained review of Scherk-Schwarz reductions with background flux including a discussion of the background field equations (Sec. II A), which to the best of our knowledge does not appear in previous literature. The Jacobi-like constraints on the background fluxes as well as the background field equations form the complete set of equations that must be satisfied for a *bona fide* Scherk-Schwarz flux compactification. The nontriviality of these constraints, particularly the background field equations, illustrates the difficulty of providing a complete classification of such compactifications.

In Sec. III, we briefly review the embedding tensor formalism [16–20] and give a general solution of the linear constraint satisfied by the embedding tensor. The reduction *Ansätze* defined in Sec. II are applied to the generalized vielbein postulates in Sec. IV yielding the embedding tensor of Scherk-Schwarz flux compactifications. This embedding tensor can be cast in the form of the general solution of the linear constraint given in Sec. III. Furthermore, in Appendix B, we verify that the quadratic constraints are satisfied. Finally, in Sec. V, we demonstrate explicitly in the simple example of a flat group reduction that indeed less than or equal to 28 electric or magnetic vectors are gauged as is expected from general results of the embedding tensor formalism [35]. We make concluding remarks in Sec. VI.

Conventions.—In this paper, we reserve the use of ϵ for an alternating *tensor* with respect to some metric structure, while we use η to denote the *tensor density*, alias the alternating symbol. It is important to note that *all* objects denoted with a caret above them depend only on the external coordinates, that is, are only x dependent.

II. SCHERK-SCHWARZ REDUCTION

Consider a reduction of $D = 11$ supergravity such that the elfbein takes the form

$$E_M{}^A(z) = \begin{pmatrix} \hat{\Delta}^{-1/2}(x)\hat{e}_\mu{}^\alpha(x) & \hat{B}_\mu{}^m(x)\hat{e}_m{}^a(x) \\ 0 & U_m{}^n(y)\hat{e}_n{}^a(x) \end{pmatrix}, \quad (1)$$

where the eleven-dimensional coordinates have been split as $\{z^M\} \equiv \{x^\mu, y^m\}$, and where

$$\hat{e} = \det(\hat{e}_\mu{}^\alpha), \qquad \hat{\Delta} = \det(\hat{e}_m{}^a) \quad (2)$$

(recall that all hatted quantities depend only on the four-dimensional coordinates x^μ). The matrices $U_m{}^n(y)$ depend only on the internal coordinates and satisfy the property that

$$\partial_{[m}U_{n]}{}^p = -\frac{1}{2}f^p{}_{rs}U_m{}^rU_n{}^s. \quad (3)$$

The y-independent structure constants f importantly satisfy a unimodularity property, viz.

$$f^m{}_{mn} = 0, \quad (4)$$

which is equivalent to

$$\partial_n[U(U^{-1})_m{}^n] = 0, \quad (5)$$

where

$$U \equiv \det(U_m{}^n). \quad (6)$$

The condition of unimodularity, emphasized in [22], ensures that the measure is invariant under seven-dimensional diffeomorphisms.[3]

Furthermore, the following integrability condition is satisfied:

$$f^q{}_{[mn}f^r{}_{p]q} = 0. \quad (7)$$

[3]The importance of unimodularity was discussed in the context of Bianchi cosmology by Sneddon [40] slightly before Scherk and Schwarz, and shown to be required for consistency of the reduction to a homogeneous cosmology.

This is equivalent to the Jacobi identity for the associated Lie algebra.

Specifically, in terms of the following parametrization of the elfbein

$$E_M{}^A = \begin{pmatrix} \Delta^{-1/2} e'_\mu{}^\alpha & B_\mu{}^n e_n{}^a \\ 0 & e_m{}^a \end{pmatrix}, \tag{8}$$

where $\Delta = \det e_n{}^a = U\hat{\Delta}$, we assume the following reduction *Ansätze* for the elfbein components:

$$e'_\mu{}^\alpha(x,y) = U^{1/2}\hat{e}_\mu{}^\alpha(x), \tag{9}$$

$$B_\mu{}^m(x,y) = (U^{-1})_n{}^m \hat{B}_\mu{}^n(x), \tag{10}$$

$$e_n{}^a(x,y) = U_m{}^n \hat{e}_n{}^a(x). \tag{11}$$

In general, the reduction *Ansätze* for fields is such that all seven-dimensional covariant tensor indices are contracted with U, which contains all the y dependence, while seven-dimensional contravariant tensor indices are contracted with U^{-1}, as should be clear from the *Ansätze* for $B_\mu{}^m$ and $e_n{}^a$ given above.

The reduction *Ansatz* for the three-form potential is similarly defined, except that some components have background contributions as well:

$$A_{\mu\nu\rho}(x,y) = \hat{A}_{\mu\nu\rho}(x) + \hat{\zeta}_{\mu\nu\rho}(x), \tag{12}$$

$$A_{\mu\nu m}(x,y) = U_m{}^n \hat{A}_{\mu\nu n}(x), \tag{13}$$

$$A_{\mu mn}(x,y) = U_m{}^p U_n{}^q \hat{A}_{\mu pq}(x), \tag{14}$$

$$A_{mnp}(x,y) = A'_{mnp}(x,y) + a_{mnp}(y), \tag{15}$$

where

$$A'_{mnp}(x,y) = U_m{}^q U_n{}^r U_p{}^s \hat{A}_{qrs}(x), \tag{16}$$

and $\hat{\zeta}_{\mu\nu\rho}$ and a_{mnp} are defined such that

$$4!\partial_{[\mu}\hat{\zeta}_{\nu\rho\sigma]} = i\mathfrak{f}_{FR}\hat{\Delta}^{-3}\hat{\epsilon}_{\mu\nu\rho\sigma}, \tag{17}$$

$$4!\partial_{[m}a_{npq]} = g_{rstu}U_m{}^r U_n{}^s U_p{}^t U_q{}^u, \tag{18}$$

for some constant $\mathfrak{f}_{FR}$ and totally antisymmetric constant g_{mnpq}. The above equations give the background values of the field strength $F_{\mu\nu\rho\sigma}$ and F_{mnpq}, respectively. We will see later that the special y dependence with *constant* g_{mnpq} in (18) is required for the consistency of both the equations of motion and the generalized vielbein postulates.

The exterior derivative of Eq. (18), which corresponds to the closure of the background field strength, implies the following constraint [34]:

$$f^s{}_{[mn}g_{pqr]s} = 0. \tag{19}$$

We will find later that this constraint plays a crucial role in defining a consistent gauge algebra. In fact, this constraint was first found by considering the consistency of the gauge algebra, in particular, the Jacobi identity [25].

In order to determine the form of the dual six-form under this reduction, we consider its defining equation

$$\frac{i}{4!}\epsilon_{M_1\ldots M_{11}}F^{M_8\ldots M_{11}} = 7!\partial_{[M_1}A_{M_2\ldots M_7]} + 7!\frac{\sqrt{2}}{2}A_{[M_1\ldots M_3}\partial_{M_4}A_{M_5\ldots M_7]}, \tag{20}$$

where it is important to note that indices on F^{MNPQ} have been raised using the eleven-dimensional metric, and where we have ignored fermion bilinear contributions. Consider the $m_1\ldots m_7$ components of the above equation. Using the fact that

$$\epsilon_{m_1\ldots m_7\mu\nu\rho\sigma} = U\hat{\Delta}^{-1}\hat{\epsilon}_{\mu\nu\rho\sigma}\eta_{m_1\ldots m_7} \tag{21}$$

the left-hand side of Eq. (20) simplifies to

$$\frac{i}{4!}\epsilon_{m_1\ldots m_7\mu\nu\rho\sigma}F^{\mu\nu\rho\sigma} = -U\mathfrak{f}_{FR}\eta_{m_1\ldots m_7} + U(x\text{-dependent terms}), \tag{22}$$

where $\eta_{m_1\ldots m_7}$ is defined with respect to a flat seven-dimensional metric and the x-dependent terms in the remainder of the expression have contributions from $\hat{A}_{\mu\nu\rho}$, $\hat{A}_{\mu\nu m}$, $\hat{A}_{\mu mn}$, $\hat{A}_{mnp}$, and g_{mnpq} as well as $f^p{}_{mn}$. This is due to the fact that the inverse metric is not diagonal. We stress once more that the indices on the four-form F in Eq. (22) have been raised with the eleven-dimensional metric.

The right-hand side of Eq. (20) reduces to

$$7!\partial_{[m_1}\left(A_{m_2\ldots m_7]} + \frac{\sqrt{2}}{2}A'_{m_2m_3m_4}a_{m_5m_6m_7]}\right) + \frac{7!\sqrt{2}}{2}\Big(A'_{[m_1m_2m_3}\partial_{m_4}(A'_{m_5m_6m_7]} + 2a_{m_5m_6m_7]}) + a_{[m_1m_2m_3}\partial_{m_4}a_{m_5m_6m_7]}\Big). \tag{23}$$

Now, defining an *Ansatz* for $A_{m_1\ldots m_6}$ of the form

$$A_{m_1\ldots m_6} = A'_{m_1\ldots m_6}(x,y) + \frac{\sqrt{2}}{2} a_{[m_1m_2m_3}A'_{m_4m_5m_6]} + a_{m_1\ldots m_6}(y), \tag{24}$$

where

$$A'_{m_1\ldots m_6}(x,y) = U_{m_1}{}^{n_1}\ldots U_{m_6}{}^{n_6}\hat{A}_{n_1\ldots n_6}(x) \tag{25}$$

and $a_{m_1\ldots m_6}$ is such that

$$7!\partial_{[m_1}a_{m_2\ldots m_7]} = -U\mathfrak{f}_{FR}\eta_{m_1\ldots m_7} - \frac{7!\sqrt{2}}{2}a_{[m_1m_2m_3}\partial_{m_4}a_{m_5m_6m_7]}. \tag{26}$$

Equation (20) reduces to a purely x-dependent, rather complicated, relation between $\hat{A}_{m_1\ldots m_6}$ and components of the three-form potential $\hat{A}$. Note that duality relation (26) is the duality relation satisfied by the background solution.

A. Background solution

In the context of formulating a well-defined reduction, an important consideration is the background field equations and the constraints these imply on the background fields.

The background of the Scherk-Schwarz reduction is given by

$$E_M{}^A = \begin{pmatrix} \hat{e}_\mu{}^\alpha(x) & 0 \\ 0 & U_m{}^n(y)\delta_n^a \end{pmatrix}, \qquad A_{mnp} = a_{mnp},$$
$$F_{\mu\nu\rho\sigma} = i\mathfrak{f}_{FR}\hat{\epsilon}_{\mu\nu\rho\sigma}. \tag{27}$$

Thus, the internal metric is

$$g_{mn} = U_m{}^pU_n{}^q\delta_{pq}, \qquad g^{mn} = (U^{-1})_p{}^m(U^{-1})_q{}^n\delta^{pq}. \tag{28}$$

The field equations of eleven-dimensional supergravity are

$$R_{MN} = \frac{1}{72}g_{MN}F^2_{PQRS} - \frac{1}{6}F_{MPQR}F_N{}^{PQR}, \tag{29}$$

$$E^{-1}\partial_M(EF^{MNPQ}) = \frac{\sqrt{2}}{1152}i\epsilon^{NPQR_1\ldots R_4S_1\ldots S_4}F_{R_1\ldots R_4}F_{S_1\ldots S_4}. \tag{30}$$

For the background solution, the component of these equations along the internal directions are

$$\frac{1}{6}g_{mpqr}g_n{}^{qpr} = \frac{1}{4}(\delta_{mp}\delta_{nq}\delta^{rs}\delta^{tu}f^p{}_{rt}f^q{}_{su} - 2\delta_{pq}\delta^{rs}f^p{}_{mr}f^q{}_{ns} - 2f^p{}_{mq}f^q{}_{np}) - \frac{1}{3}\delta_{mn}\mathfrak{f}^2_{FR} + \frac{1}{72}\delta_{mn}g_{pqrs}g^{pqrs}, \tag{31}$$

$$f^{[m_1}{}_{pq}g^{m_2m_3]pq} = -\frac{\sqrt{2}}{72}\mathfrak{f}_{FR}\eta^{m_1\ldots m_7}g_{m_4\ldots m_7}, \tag{32}$$

where the indices on g_{mnpq} are raised with the Kronecker δ symbol. We note that by putting the theory on shell, this operation breaks the GL(7) symmetry to SO(7) or a subgroup thereof, in the same way as the rigid SU(8) symmetry of maximal supergravity is broken to (a subgroup of) SO(8) in any given vacuum.[4] The special dependence on $U(y)$ in (18) is now seen to be necessary for the "Maxwell equation" (30) to become y independent, and thus to reduce to an equation relating the *constant* tensors $f^m{}_{np}$ and g_{mnpq}, (32). We note that, while the background constraints for the case with no flux appear already in Ref. [22], the constraints implied on the background of a Scherk-Schwarz reduction with flux have never been fully spelled out in the literature to the best of our knowledge. In particular, Eq. (32) is a nontrivial restriction on the class of viable Scherk-Schwarz reductions. These constraints, which are imposed by the background field equations, are independent of the constraints imposed by the consistency of the gauge algebra [25] (see also [35]).

The components of the Einstein equation along the four-dimensional spacetime directions fixes the radius of the four-dimensional anti-de Sitter space

$$\hat{R}_\mu{}^\nu = \left(\frac{2}{3}\mathfrak{f}^2_{FR} + \frac{1}{72}g^{mnpq}g_{mnpq}\right)\delta^\nu_\mu. \tag{33}$$

All other equations of motion are trivially satisfied.

III. THE EMBEDDING TENSOR FORMALISM

The embedding tensor formalism,[5] which was initially developed in the context of three-dimensional maximal gauged supergravities [19,20] and later developed in the context of four-dimensional maximal gauged supergravities [16–18], is the most efficient framework in which to understand gaugings. The embedding tensor formalism uses the fact that the ungauged supergravity, of which the gauged theory is a deformation, is controlled by a global symmetry group that is larger than what one would naively expect—an observation first made in the context of the four-dimensional maximal theory [3].

[4] We thank Henning Samtleben for a discussion on this point.

[5] See Ref. [35] for a lucid account of the embedding tensor formalism.

In four dimensions, the scalars, which parametrize the $E_{7(7)}$ vielbein $\mathcal{V}$, satisfy the following equation:

$$\partial_\mu \mathcal{V}_{\mathcal{M}ij} + \mathcal{Q}_\mu{}^k{}_{[i}\mathcal{V}_{\mathcal{M}j]k} - \mathcal{P}_{\mu ijkl}\mathcal{V}_{\mathcal{M}}{}^{kl} - g\mathcal{A}_\mu{}^{\mathcal{P}} X_{\mathcal{P}\mathcal{M}}{}^{\mathcal{N}}\mathcal{V}_{\mathcal{N}ij} = 0. \quad (34)$$

Objects that are of particular interest in the above equation are $(X_\mathcal{M})_\mathcal{N}{}^\mathcal{P}$. These generate the gauge algebra and are related to the embedding tensor[6] $\Theta_\mathcal{M}{}^\alpha$ via the $E_{7(7)}$ generators t_α, viz.

$$X_\mathcal{M} = \Theta_\mathcal{M}{}^\alpha t_\alpha. \quad (35)$$

The embedding tensor satisfies two algebraic constraints. The first, linear constraint, comes from a consideration of the supersymmetric consistency of the gauged theory. In the case of maximal four-dimensional theories, this translates to the statement that the embedding tensor lives in the **912** representation of $E_{7(7)}$

$$\Theta_\mathcal{M}{}^\alpha + 2(t_\beta)_\mathcal{M}{}^\mathcal{N}(t^\alpha)_\mathcal{N}{}^\mathcal{P}\Theta_\mathcal{P}{}^\beta = 0, \quad (36)$$

where the $E_{7(7)}$ index α is raised with the inverse Killing-Cartan form κ^{-1}, which is given in Appendix A. More specifically, the above relation follows by requiring that the projectors $\mathbb{P}_{\mathbf{56}}$ and $\mathbb{P}_{\mathbf{6480}}$ annihilate $\Theta_\mathcal{M}{}^\alpha$ [16]. In terms of the gauge group generators, the linear constraint is

$$X_{\mathcal{M}\mathcal{N}}{}^\mathcal{P} + 2X_{\mathcal{R}\mathcal{M}}{}^\mathcal{Q}(\kappa^{-1})^{\alpha\beta}(t_\alpha)_\mathcal{Q}{}^\mathcal{R}(t_\beta)_\mathcal{N}{}^\mathcal{P} = 0. \quad (37)$$

The general solution of the linear constraint is given by

$$\begin{aligned}
&X_{\mathsf{MN}}{}^{\mathsf{PQ}}{}_{\mathsf{RS}} = \delta^{[\mathsf{P}}_{[\mathsf{R}}\mathsf{T}^{\mathsf{Q}]}{}_{\mathsf{S}]\mathsf{MN}}, && X_{\mathsf{MN}\,\mathsf{PQ}}{}^{\mathsf{RS}} = -\delta^{[\mathsf{R}}_{[\mathsf{P}}\mathsf{T}^{\mathsf{S}]}{}_{\mathsf{Q}]\mathsf{MN}},\\
&X^{\mathsf{MN}}{}_{\mathsf{PQ}\,\mathsf{RS}} = -2\delta^{[\mathsf{M}}_{[\mathsf{P}}\mathsf{T}^{\mathsf{N}]}{}_{\mathsf{QRS}}, && X^{\mathsf{MN}\,\mathsf{PQRS}} = -\frac{2}{4!}\eta^{\mathsf{PQRS}[\mathsf{M}|\mathsf{T}_1\mathsf{T}_2\mathsf{T}_3|}\mathsf{T}^{\mathsf{N}]}{}_{\mathsf{T}_1\mathsf{T}_2\mathsf{T}_3},\\
&X^{\mathsf{MN}}{}_{\mathsf{PQ}}{}^{\mathsf{RS}} = \delta^{[\mathsf{R}}_{[\mathsf{P}}\mathsf{T}_{\mathsf{Q}]}{}^{\mathsf{S}]\mathsf{MN}}, && X^{\mathsf{MN}\,\mathsf{PQ}}{}_{\mathsf{RS}} = -\delta^{[\mathsf{P}}_{[\mathsf{R}}\mathsf{T}_{S]}{}^{\mathsf{Q}]\mathsf{MN}},\\
&X_{\mathsf{MN}}{}^{\mathsf{PQRS}} = -2\delta^{[\mathsf{P}}_{[\mathsf{M}}\mathsf{T}_{\mathsf{N}]}{}^{\mathsf{QRS}]}, && X_{\mathsf{MN}\,\mathsf{PQ}\,\mathsf{RS}} = -\frac{2}{4!}\eta_{\mathsf{PQ}\,\mathsf{RS}[\mathsf{M}|\mathsf{T}_1\mathsf{T}_2\mathsf{T}_3|}\mathsf{T}_{\mathsf{N}]}{}^{\mathsf{T}_1\mathsf{T}_2\mathsf{T}_3},
\end{aligned} \quad (38)$$

where

$$\begin{aligned}
\mathsf{T}^{\mathsf{M}}{}_{\mathsf{NPQ}} &= -\frac{3}{4}\mathsf{A}_2{}^{\mathsf{M}}{}_{\mathsf{NPQ}} - \frac{3}{2}\delta^{\mathsf{M}}_{[\mathsf{P}}\mathsf{A}_{1\,\mathsf{Q}]\mathsf{N}},\\
\mathsf{T}_{\mathsf{M}}{}^{\mathsf{NPQ}} &= -\frac{3}{4}\mathsf{A}_{2\,\mathsf{M}}{}^{\mathsf{NPQ}} - \frac{3}{2}\delta^{[\mathsf{P}}_{\mathsf{M}}\mathsf{A}_1{}^{\mathsf{Q}]\mathsf{N}}.
\end{aligned} \quad (39)$$

Note that the solution above applies more generally to other compactifications. Structures $\mathsf{A}_1{}^{\mathsf{MN}}$, $\mathsf{A}_1{}^{\mathsf{MN}}\,\mathsf{A}_2{}^{\mathsf{M}}{}_{\mathsf{NPQ}}$, and $\mathsf{A}_{2\,\mathsf{M}}{}^{\mathsf{NPQ}}$ satisfy the following properties:

$$\begin{aligned}
\mathsf{A}_{1\,[\mathsf{MN}]} &= 0, & \mathsf{A}_1{}^{[\mathsf{MN}]} &= 0,\\
\mathsf{A}_2{}^{\mathsf{M}}{}_{[\mathsf{NPQ}]} &= \mathsf{A}_2{}^{\mathsf{M}}{}_{\mathsf{NPQ}}, & \mathsf{A}_2{}^{\mathsf{M}}{}_{\mathsf{MPQ}} &= 0,\\
\mathsf{A}_{2\,\mathsf{M}}{}^{[\mathsf{NPQ}]} &= \mathsf{A}_{2\,\mathsf{M}}{}^{\mathsf{NPQ}}, & \mathsf{A}_{2\,\mathsf{M}}{}^{\mathsf{MPQ}} &= 0.
\end{aligned} \quad (40)$$

Equivalently,

$$\begin{aligned}
(\Theta_{\mathsf{MN}})_{\mathsf{P}_1}{}^{\mathsf{P}_2} &= \frac{1}{2}\mathsf{T}^{\mathsf{P}_2}{}_{\mathsf{P}_1\mathsf{MN}},\\
(\Theta^{\mathsf{MN}})^{\mathsf{P}_1\ldots\mathsf{P}_4} &= -\frac{2}{4!}\eta^{\mathsf{P}_1\ldots\mathsf{P}_4[\mathsf{M}|\mathsf{Q}_1\mathsf{Q}_2\mathsf{Q}_3|}\mathsf{T}^{\mathsf{N}]}{}_{\mathsf{Q}_1\mathsf{Q}_2\mathsf{Q}_3},\\
(\Theta^{\mathsf{MN}})_{\mathsf{P}_1}{}^{\mathsf{P}_2} &= -\frac{1}{2}\mathsf{T}_{\mathsf{P}_1}{}^{\mathsf{P}_2\mathsf{MN}},\\
(\Theta_{\mathsf{MN}})^{\mathsf{P}_1\ldots\mathsf{P}_4} &= -2\delta^{[\mathsf{P}_1}_{[\mathsf{M}}\mathsf{T}_{\mathsf{N}]}{}^{\mathsf{P}_2\mathsf{P}_3\mathsf{P}_4]}.
\end{aligned} \quad (41)$$

The corresponding objects $(\Theta^{\mathsf{MN}})_{\mathsf{P}_1\ldots\mathsf{P}_4}$ and $(\Theta_{\mathsf{MN}})_{\mathsf{P}_1\ldots\mathsf{P}_4}$ are obtained by contracting $(\Theta^{\mathsf{MN}})^{\mathsf{P}_1\ldots\mathsf{P}_4}$ and $(\Theta^{\mathsf{MN}})^{\mathsf{P}_1\ldots\mathsf{P}_4}$ with the permutation symbol in accordance with the equations in Appendix A.

It is important to note at this point that $\mathsf{T}^{\mathsf{M}}{}_{\mathsf{NPQ}}$ and $\mathsf{T}_{\mathsf{M}}{}^{\mathsf{NPQ}}$ are real and completely independent. This is because they are written in terms of SL(8) indices and there is no relation between an upper SL(8) index and a lower one. This is in contrast to objects with SU(8) indices where upper and lower indices are related to one another via conjugation. The T tensor, which has SU(8) indices, can be derived by dressing the T-tensors above with the $E_{7(7)}$ vielbein $\mathcal{V}_{\mathcal{M}ij}$

$$T_{i_1i_2}{}^{j_1j_2}{}_{k_1k_2} = -\Omega^{\mathcal{M}\mathcal{Q}}\Omega^{\mathcal{N}\mathcal{R}}\mathcal{V}_{\mathcal{Q}i_1i_2}\mathcal{V}_\mathcal{R}{}^{j_1j_2}\mathcal{V}_{\mathcal{P}k_1k_2}X_{\mathcal{M}\mathcal{N}}{}^\mathcal{P}, \quad (42)$$

where

$$T_{i_1i_2}{}^{j_1j_2}{}_{k_1k_2} = \delta^{[j_1}_{[i_1}T^{j_2]}{}_{i_2]k_1k_2}, \quad (43)$$

and

[6] Indices $\alpha, \beta, \ldots = 1, \ldots, 133$ label the $E_{7(7)}$ generators and are not to be confused with the four-dimensional tangent space indices, which are also labeled by lower Greek letters from the beginning of the alphabet.

$$T^i{}_{jkl} = -\frac{3}{4} A_2{}^i{}_{jkl} - \frac{3}{2} \delta^i_{[k} A_{1\,l]j}. \tag{44}$$

Since the T-tensor has SU(8) indices, $T_i{}^{jkl}$ is simply the complex conjugate of $T^i{}_{jkl}$. Note that this is in contrast to the properties satisfied by the T-tensors which satisfy no such relation, as pointed out above.

Furthermore, the embedding tensor satisfies a quadratic constraint, which is necessary for the gauge algebra generated by $X_{\mathcal{M}}$ to close

$$X_{\mathcal{M}\mathcal{Q}}{}^{\mathcal{R}} X_{\mathcal{N}\mathcal{R}}{}^{\mathcal{P}} - X_{\mathcal{N}\mathcal{Q}}{}^{\mathcal{R}} X_{\mathcal{M}\mathcal{R}}{}^{\mathcal{P}} + X_{\mathcal{M}\mathcal{N}}{}^{\mathcal{R}} X_{\mathcal{R}\mathcal{Q}}{}^{\mathcal{P}} = 0. \tag{45}$$

However, notice that the above constraint is stronger than the closure of the algebra since $X_{(\mathcal{M}\mathcal{N})}{}^{\mathcal{P}}$ does not trivially vanish. In fact, the quadratic condition comes from the requirement that the embedding tensor be invariant under the action of the gauge group

$$\delta_{\mathcal{M}} \Theta_{\mathcal{N}}{}^{\alpha} = \Theta_{\mathcal{M}}{}^{\beta} \delta_{\beta} \Theta_{\mathcal{N}}{}^{\alpha} = 0. \tag{46}$$

Equivalently, given that the embedding tensor satisfies the linear constraint and lives in the **912** representation of $E_{7(7)}$, the quadratic constraint is [35]

$$\Omega^{\mathcal{M}\mathcal{N}} \Theta_{\mathcal{M}}{}^{\alpha} \Theta_{\mathcal{N}}{}^{\beta} = 0. \tag{47}$$

In this form, it is clear to see that viewed as a matrix, the row rank of the embedding tensor is at most half-maximal. Therefore, we are guaranteed that only at most 28 out of the possible 56 vectors will be gauged [35].

IV. GENERALIZED VIELBEIN POSTULATES AND THE EMBEDDING TENSOR

The generalized vielbein postulates provide an understanding of various aspects of the reduction. In particular, for the case of the S^7 compactification, they are a necessary ingredient in the proof of the consistency of the reduction. Specifically, the $d = 4$ generalized vielbein postulates reduce to the $E_{7(7)}$ Cartan equation of gauged maximal supergravity in that case [11,15].

The generalized vielbeine combine the would-be scalar degrees of freedom originating from the siebenbein, the three-form and the six-form into a single object, and are explicitly given by [1]

$$e^m_{AB} = i\Delta^{-1/2} \Gamma^m_{AB}, \tag{48}$$

$$e_{mn\,AB} = -\frac{\sqrt{2}}{12} i\Delta^{-1/2} (\Gamma_{mn\,AB} + 6\sqrt{2} A_{mnp} \Gamma^p_{AB}), \tag{49}$$

$$\begin{aligned} e_{m_1\ldots m_5\,AB} = \frac{1}{6!\sqrt{2}} i\Delta^{-1/2} \Big[&\Gamma_{m_1\ldots m_5\,AB} + 60\sqrt{2} A_{[m_1 m_2 m_3} \Gamma_{m_4 m_5]AB} \\ &- 6!\sqrt{2} \Big(A_{pm_1\ldots m_5} - \frac{\sqrt{2}}{4} A_{p[m_1 m_2} A_{m_3 m_4 m_5]} \Big) \Gamma^p_{AB} \Big], \end{aligned} \tag{50}$$

$$\begin{aligned} e_{m_1\ldots m_7,n\,AB} = -\frac{2}{9!} i\Delta^{-1/2} \Big[&(\Gamma_{m_1\ldots m_7} \Gamma_n)_{AB} + 126\sqrt{2} A_{n[m_1 m_2} \Gamma_{m_3\ldots m_7]\,AB} \\ &+ 3\sqrt{2} \times 7! \Big(A_{n[m_1\ldots m_5} + \frac{\sqrt{2}}{4} A_{n[m_1 m_2} A_{m_3 m_4 m_5} \Big) \Gamma_{m_6 m_7]AB} \\ &+ \frac{9!}{2} \Big(A_{n[m_1\ldots m_5} + \frac{\sqrt{2}}{12} A_{n[m_1 m_2} A_{m_3 m_4 m_5} \Big) A_{m_6 m_7]p} \Gamma^p{}_{AB} \Big]. \end{aligned} \tag{51}$$

We emphasize again that these objects depend on all eleven coordinates. By virtue of their definition, they satisfy certain differential constraints, the so-called generalized vielbein postulates. Along the external $d = 4$ directions these are of the form

$$\mathcal{D}_\mu e^m_{AB} + \frac{1}{2} \partial_n B_\mu{}^n e^m_{AB} + \partial_n B_\mu{}^m e^n_{AB} + \mathcal{Q}^C_{\mu\,[A} e^m_{B]C} + \mathcal{P}_{\mu ABCD} e^{mCD} = 0, \tag{52}$$

$$\mathcal{D}_\mu e_{mnAB} + \frac{1}{2}\partial_p B_\mu{}^p e_{mnAB} + 2\partial_{[m} B_{|\mu|}{}^p e_{n]pAB} + 3\partial_{[m} B_{|\mu|np]} e^p_{AB} + \mathcal{Q}^C_{\mu\,[A} e_{mnB]C} + \mathcal{P}_{\mu ABCD} e_{mn}{}^{CD} = 0, \tag{53}$$

$$\mathcal{D}_\mu e_{m_1\ldots m_5 AB} + \frac{1}{2}\partial_p B_\mu{}^p e_{m_1\ldots m_5 AB} - 5\partial_{[m_1} B_{|\mu|}{}^p e_{m_2\ldots m_5]pAB} + \frac{3}{\sqrt{2}}\partial_{[m_1} B_{|\mu|m_2 m_3} e_{m_4 m_5]AB} - 6\partial_{[m_1} B_{|\mu|m_2\ldots m_5 p]} e^p_{AB} + \mathcal{Q}^C_{\mu\,[A} e_{m_1\ldots m_5 B]C} + \mathcal{P}_{\mu ABCD} e_{m_1\ldots m_5}{}^{CD} = 0, \tag{54}$$

$$\mathcal{D}_\mu e_{m_1\ldots m_7,nAB} - \frac{1}{2}\partial_p B_\mu{}^p e_{m_1\ldots m_7,nAB} - \partial_n B_\mu{}^p e_{m_1\ldots m_7,pAB} + 5\partial_{[m_1} B_{|\mu|m_2 m_3} e_{m_4\ldots m_7]nAB} - 2\partial_{[m_1} B_{|\mu|m_2\ldots m_6} e_{m_7]nAB} + \mathcal{Q}^C_{\mu\,[A} e_{m_1\ldots m_7,nB]C} + \mathcal{P}_{\mu ABCD} e_{m_1\ldots m_7,n}{}^{CD} = 0, \tag{55}$$

where

$$\mathcal{D}_\mu \equiv \partial_\mu - B_\mu{}^m \partial_m \tag{56}$$

and the connection coefficients are of the form

$$\mathcal{Q}^A_{\mu\,B} = -\frac{1}{2}[e^m{}_a \partial_m B_\mu{}^n e_{nb} - (e^p{}_a \mathcal{D}_\mu e_{pb})]\Gamma^{ab}_{AB} - \frac{\sqrt{2}}{12} e_\mu{}^\alpha (F_{\alpha abc}\Gamma^{abc}_{AB} - \eta_{\alpha\beta\gamma\delta}F^{\beta\gamma\delta a}\Gamma_{aAB}), \tag{57}$$

$$\mathcal{P}_{\mu ABCD} = \frac{3}{4}[e^m{}_a \partial_m B_\mu{}^n e_{nb} - (e^p{}_a \mathcal{D}_\mu e_{pb})]\Gamma^a_{[AB}\Gamma^b_{CD]} - \frac{\sqrt{2}}{8} e_\mu{}^\alpha F_{abc\alpha}\Gamma^a_{[AB}\Gamma^{bc}_{CD]} - \frac{\sqrt{2}}{48} e_{\mu\,\alpha}\eta^{\alpha\beta\gamma\delta} F_{a\beta\gamma\delta}\Gamma_{b[AB}\Gamma^{ab}_{CD]}. \tag{58}$$

Below we will consider and analyze these equations in the context of Scherk-Schwarz reduction.

Note the general triangular feature of the equations, whereby certain generalized vielbeine and vectors appear more frequently than others. More specifically, as one moves through Eq. (52) to (55), as well as the generalized vielbeine and vectors that appeared before, a new generalized vielbein and vector contribute in turn. This pattern is broken in Eq. (55), where $B_{\mu m_1\ldots m_7,n}$, which is associated with dual gravity degrees of freedom and the supersymmetry transformation of which gives generalized vielbein $e_{m_1\ldots m_7,nAB}$ does not contribute. This is a completely general feature of the eleven-dimensional theory and, therefore, applies to any compactification. An important consequence of this seems to be that any four-dimensional gauged theory obtained as a consistent reduction of $D = 11$ supergravity cannot have gauge vectors associated with the gauging of these particular seven vectors. This implies an additional constraint on the embedding tensor of any theory that is obtained from a reduction of $D = 11$ supergravity. However, we know that one can take a full set of 28 magnetic vectors in four dimensions and gauge these to obtain an SO(8) gauged maximal supergravity [41]. While it is true [41] (see also [5]) that this theory is equivalent to the original SO(8) gauged maximal supergravity of [21], the very fact that a full set of magnetic vectors can be gauged in four dimensions and that this has no corresponding higher-dimensional original is significant in understanding the extent to which the deformed SO(8) gauged maximal supergravities of [41] can be realized as a reduction from $D = 11$ supergravity.[7]

Let us first consider the connection coefficients $\mathcal{Q}_\mu$ and $\mathcal{P}_\mu$. The y dependence in both connection coefficients come from the same three terms, viz.

[7] An interesting question is whether a deformation of the $D = 11$ 56-bein $\mathcal{V}$ [1] of the form

$$\mathcal{V} \to \begin{pmatrix} \cos\omega & \sin\omega \\ -\sin\omega & \cos\omega \end{pmatrix} \begin{pmatrix} \mathcal{V}^{\mathrm{MN}} \\ \mathcal{V}_{\mathrm{MN}} \end{pmatrix},$$

in analogy with the rotation introduced in Ref. [5], allows the possibility of further gauging of magnetic vectors. This would clearly point to the existence of a genuine deformation of $D = 11$ supergravity. Such a consistent deformation could then provide a higher-dimensional origin of the deformed maximal SO(8) gauged supergravities of Ref. [41].

$$[e^m{}_a\partial_m B_\mu{}^n e_{nb} - (e^p{}_a\mathcal{D}_\mu e_{pb})], \quad e_\mu{}^\alpha F_{\alpha abc}, \quad e_{\mu\alpha}\eta^{\alpha\beta\gamma\delta}F_{\beta\gamma\delta a}. \tag{59}$$

Using the *Ansatz* for $B_\mu{}^m$ and $e^m{}_a$, Eqs. (10) and (11) and property (3) satisfied by U, it is simple to show that

$$e^m{}_a\partial_m B_\mu{}^n e_{nb} - (e^p{}_a\mathcal{D}_\mu e_{pb}) = -\hat{e}^p{}_a(\partial_\mu \hat{e}_{pb} - f^n{}_{pq}\hat{B}^q_\mu \hat{e}_{nb}). \tag{60}$$

Hence, the y dependence drops out. Now, consider

$$e_\mu{}^\alpha F_{\alpha abc} = (F_{\mu npq} - B_\mu{}^r F_{rnpq})e^n{}_a e^p{}_b e^q{}_c. \tag{61}$$

Notice that curved 7d indices enter only as dummy indices. Furthermore, from Eq. (18) we note that the y dependence of the field strength in the second term is cancelled by the y dependence of $B_\mu{}^r$ and the inverse siebenbein. Therefore, the only potential obstacle to the dropping out of the y dependence in the expression above is when a 7d derivative acts on the potential. However, the 7d derivative always acts as an exterior derivative. Hence, using Eqs. (3) and (18), we will always obtain a y-independent piece along with the appropriate U contractions. However, these U factors will be cancelled for the same reason as stated above: that there is no free curved 7d index. The same argument can be used to show the y independence of the third term. Therefore, we conclude that the connection coefficients are y independent.

The eleven-dimensional fields enter the generalized vielbein postulates via the four generalized vielbeine and three of the vectors. The reduction *Ansätze* for the generalized vielbeine can be found using the *Ansätze* for the fields that define them, Eqs. (48)–(51). They are as follows:

$$e^m_{AB} = U^{-1/2}(U^{-1})_n{}^m \hat{e}^n_{AB}(x), \tag{62}$$

$$e_{mnAB} = U^{-1/2}U_m{}^p U_n{}^q \hat{e}_{pqAB}(x) - a_{mnp}e^p_{AB}, \tag{63}$$

$$\begin{aligned} e_{m_1\ldots m_5AB} &= U^{-1/2}U_{m_1}{}^{n_1}\ldots U_{m_5}{}^{n_5}\hat{e}_{n_1\ldots n_5AB}(x) \\ &\quad - \frac{\sqrt{2}}{2}a_{[m_1m_2m_3}e_{m_4m_5]AB} \\ &\quad - \left(a_{pm_1\ldots m_5} + \frac{\sqrt{2}}{4}a_{p[m_1m_2}a_{m_3m_4m_5]}\right)e^p_{AB}, \end{aligned} \tag{64}$$

$$\begin{aligned} e_{m_1\ldots m_7,nAB} &= U^{1/2}U_n{}^p\hat{e}_{m_1\ldots m_7,pAB}(x) - a_{n[m_1m_2}e_{m_3\ldots m_7]AB} \\ &\quad + \left(a_{n[m_1\ldots m_5} - \frac{\sqrt{2}}{4}a_{n[m_1m_2}a_{m_3m_4m_5}\right)e_{m_6m_7]AB} \\ &\quad + \left(a_{n[m_1\ldots m_5} - \frac{\sqrt{2}}{12}a_{n[m_1m_2}a_{m_3m_4m_5}\right)a_{m_6m_7]p}e^p_{AB}, \end{aligned} \tag{65}$$

where $\hat{e}^n_{AB}$, $\hat{e}_{pqAB}$, $\hat{e}_{n_1\ldots n_5AB}$, and $\hat{e}_{m_1\ldots m_7,pAB}$ are the generalized vielbeine that appear in the torus reduction and are therefore directly related to the four-dimensional scalars.

The reduction *Ansätze* for the vectors are found by using the fact that the supersymmetry transformation of the vectors [1],

$$\delta B_\mu{}^m = \frac{\sqrt{2}}{8}e^m_{AB}[2\sqrt{2}\bar{\varepsilon}^A\varphi^B_\mu + \bar{\varepsilon}_C\gamma'_\mu\chi^{ABC}] + \text{H.c.}, \tag{66}$$

$$\delta B_{\mu mn} = \frac{\sqrt{2}}{8}e_{mnAB}[2\sqrt{2}\bar{\varepsilon}^A\varphi^B_\mu + \bar{\varepsilon}_C\gamma'_\mu\chi^{ABC}] + \text{H.c.}, \tag{67}$$

$$\delta B_{\mu m_1\ldots m_5} = \frac{\sqrt{2}}{8}e_{m_1\ldots m_5AB}[2\sqrt{2}\bar{\varepsilon}^A\varphi^B_\mu + \bar{\varepsilon}_C\gamma'_\mu\chi^{ABC}] + \text{H.c.}, \tag{68}$$

$$\delta B_{\mu m_1\ldots m_7,n} = \frac{\sqrt{2}}{8}e_{m_1\ldots m_7,nAB}[2\sqrt{2}\bar{\varepsilon}^A\varphi^B_\mu + \bar{\varepsilon}_C\gamma'_\mu\chi^{ABC}] + \text{H.c.}, \tag{69}$$

should reproduce the respective generalized vielbeine.[8] The reduction ansatz for $B_\mu{}^m$ is give in Eq. (10), while the reduction *Ansätze* for $B_{\mu mn}$ and $B_{\mu m_1.....m_5}$ are listed below:

$$B_{\mu mn} = U_m{}^p U_n{}^q \hat{B}_{\mu pq}(x) - (U^{-1})_p{}^q \hat{B}_\mu{}^p a_{qmn}, \tag{70}$$

[8]The factor of $U^{-1/2}$ are absent in the *Ansätze* for the vectors because they are cancelled by a redefinition of the vierbein that contracts the fermions.

$$B_{\mu m_1\ldots m_5} = U_{m_1}{}^{n_1}\ldots U_{m_5}{}^{n_5}\hat{B}_{\mu n_1\ldots n_5}(x) - \frac{\sqrt{2}}{2}a_{[m_1m_2m_3}\hat{B}_{|\mu|m_4m_5]}(x) - \left(a_{pm_1\ldots m_5} - \frac{\sqrt{2}}{4}a_{p[m_1m_2}a_{m_3m_4m_5]}\right) \times (U^{-1})_q{}^p\hat{B}_\mu{}^q. \tag{71}$$

Substituting the above *Ansätze* into the generalized vielbein postulates (52)–(55), a straightforward yet tedious calculation shows that the y dependence in all the equations factorizes out. Importantly, we find that the two terms that vanished due to properties of Killing spinors on S^7 in the case of the S^7 compactification [15], i.e.

$$\partial_m B_\mu{}^m \quad \text{and} \quad \partial_{[m}B_{|\mu|np]},$$

do not vanish in this case. In particular,

$$\partial_m B_\mu{}^m = \partial_m(U^{-1})_n{}^m\hat{B}_\mu{}^n = (U^{-1})_n{}^m\partial_m \log U\hat{B}_\mu{}^n, \tag{72}$$

$$\partial_{[m}B_{|\mu|np]} = U_{[m}{}^qU_n{}^rU_{p]}{}^sf^t{}_{qr}\hat{B}_{\mu st} - \partial_{[m}(a_{np]q}(U^{-1})_r{}^q)\hat{B}_\mu{}^r, \tag{73}$$

where in the first line we have used Eq. (5).

The generalized vielbein postulates reduce to the following equations:

$$\partial_\mu\hat{e}^m_{AB} - f^m{}_{pq}\hat{B}_\mu{}^p\hat{e}^q_{AB} + \mathcal{Q}^C_{\mu\,[A}\hat{e}^m_{B]C} + \mathcal{P}_{\mu ABCD}\hat{e}^{mCD} = 0, \tag{74}$$

$$\partial_\mu\hat{e}_{mnAB} - 2f^p{}_{q[m}\hat{e}_{n]pAB}\hat{B}_\mu{}^q + 3f^q{}_{[mn}\hat{B}_{|\mu|p]q}\hat{e}^p_{AB} + \frac{1}{6}g_{mnpq}\hat{B}_\mu{}^p\hat{e}^q_{AB} + \mathcal{Q}^C_{\mu\,[A}\hat{e}_{mnB]C} + \mathcal{P}_{\mu ABCD}\hat{e}_{mn}{}^{CD} = 0, \tag{75}$$

$$\partial_\mu\hat{e}_{m_1\ldots m_5AB} + 5f^p{}_{q[m_1}\hat{e}_{m_2\ldots m_5]pAB}\hat{B}_\mu{}^q - \frac{3\sqrt{2}}{2}f^p{}_{[m_1m_2}\hat{B}_{|\mu p|m_3}\hat{e}_{m_4m_5]AB} + 15f^p{}_{[m_1m_2}\hat{B}_{|\mu p|m_3m_4m_5q]}\hat{e}^q_{AB} + \frac{\sqrt{2}}{12}\hat{B}_\mu{}^pg_{p[m_1m_2m_3}\hat{e}_{m_4m_5]AB} + \frac{\sqrt{2}}{8}\hat{B}_{\mu[m_1m_2}g_{m_3m_4m_5p]}\hat{e}^p_{AB} - \frac{1}{6!}\mathfrak{f}_{FR}\eta_{pqm_1\ldots m_5}\hat{B}_\mu{}^p\hat{e}^q_{AB} + \mathcal{Q}^C_{\mu\,[A}\hat{e}_{m_1\ldots m_5B]C} + \mathcal{P}_{\mu ABCD}\hat{e}_{m_1\ldots m_5}{}^{CD} = 0, \tag{76}$$

$$\partial_\mu\hat{e}_{m_1\ldots m_7,nAB} + f^p{}_{qn}\hat{B}_\mu{}^q\hat{e}_{m_1\ldots m_7,pAB} - 5f^p{}_{[m_1m_2}\hat{B}_{|\mu p|m_3}\hat{e}_{m_4\ldots m_7]nAB} + 5f^p{}_{[m_1m_2}\hat{B}_{|\mu p|m_3\ldots m_6}\hat{e}_{m_7]nAB} + \frac{5}{18}\hat{B}_\mu{}^pg_{p[m_1m_2m_3}\hat{e}_{m_4\ldots m_7]nAB} + \frac{\sqrt{2}}{24}\hat{B}_{\mu[m_1m_2}g_{m_3\ldots m_6}\hat{e}_{m_7]nAB} + \frac{1}{3\cdot 7!}\mathfrak{f}_{FR}\eta_{m_1\ldots m_7}\hat{B}_\mu{}^p\hat{e}_{pnAB} + \mathcal{Q}^C_{\mu\,[A}\hat{e}_{m_1\ldots m_7,nB]C} + \mathcal{P}_{\mu ABCD}\hat{e}_{m_1\ldots m_7,n}{}^{CD} = 0. \tag{77}$$

As emphasized before, the y-independent, hatted generalized vielbeine and vectors in the generalized vielbein postulates above are directly related to the respective four-dimensional quantities. In particular, since the reduction of these eleven-dimensional quantities is taken to be that of a simple toroidal nature, the conversion of "curved" SU(8) indices $A, B, C, \ldots$ to flat SU(8) indices $i, j, k, \ldots$ is trivial.

With this in mind, define an $E_{7(7)}$ vielbein[9]

$$\mathcal{V}_{\mathcal{M}ij} = (\mathcal{V}_{\mathsf{MN}\,ij}, \mathcal{V}^{\mathsf{MN}}{}_{ij}) \tag{78}$$

that is related to the hatted generalized vielbeine via the following relations:

$$\mathcal{V}^{m8}{}_{ij} = \frac{\sqrt{2}}{8}i\hat{e}^m_{ij},$$
$$\mathcal{V}_{mnij} = -\frac{3}{2}i\hat{e}_{mnij},$$
$$\mathcal{V}^{mn}{}_{ij} = \frac{3}{2}i\eta^{mnp_1\ldots p_5}\hat{e}_{p_1\ldots p_5ij},$$
$$\mathcal{V}_{m8ij} = -\frac{9\sqrt{2}}{2}i\eta^{n_1\ldots n_7}\hat{e}_{n_1\ldots n_7,mij}. \tag{79}$$

As expected $\mathcal{V}$ satisfies the $E_{7(7)}$ properties, as can be checked explicitly using Eqs. (48)–(51) and (79),

[9]Strictly speaking, $\mathcal{V}$ is not an $E_{7(7)}$ group element because it is acted upon by SU(8) transformations on the right, whereas the indices on the left are to be regarded as SL(8) indices. The true $E_{7(7)}$ group element is obtained by a complex rotation of this matrix (see, for example, Ref. [42] for more details).

$$\mathcal{V}_{\mathcal{M}ij}\mathcal{V}_{\mathcal{N}}{}^{ij} - \mathcal{V}_{\mathcal{M}}{}^{ij}\mathcal{V}_{\mathcal{N}ij} = i\Omega_{\mathcal{M}\mathcal{N}},$$
$$\Omega^{\mathcal{M}\mathcal{N}}\mathcal{V}_{\mathcal{M}}{}^{ij}\mathcal{V}_{\mathcal{N}kl} = i\delta^{ij}_{kl},$$
$$\Omega^{\mathcal{M}\mathcal{N}}\mathcal{V}_{\mathcal{M}}{}^{ij}\mathcal{V}_{\mathcal{N}}{}^{kl} = 0, \tag{80}$$

where the symplectic form Ω is such that

$$\Omega^{\mathsf{MN}}{}_{\mathsf{PQ}} = \delta^{\mathsf{MN}}_{\mathsf{PQ}}, \qquad \Omega_{\mathsf{MN}}{}^{\mathsf{PQ}} = -\delta^{\mathsf{PQ}}_{\mathsf{MN}},$$
$$\Omega_{\mathsf{MNPQ}} = 0, \qquad \Omega^{\mathsf{MNPQ}} = 0. \tag{81}$$

Similarly, we combine the vectors into a **56** of $E_{7(7)}$ defined by

$$\mathcal{A}^{\mathcal{M}}_{\mu} = (\mathcal{A}^{\mathsf{MN}}_{\mu}, \mathcal{A}_{\mu\,\mathsf{MN}}) \tag{82}$$

where

$$\mathcal{A}_{\mu}{}^{m8} = -\frac{1}{2}\hat{B}_{\mu}{}^{m}, \qquad \mathcal{A}_{\mu mn} = -3\sqrt{2}\hat{B}_{\mu mn},$$
$$\mathcal{A}_{\mu}{}^{mn} = -3\sqrt{2}\eta^{mnp_1\ldots p_5}\hat{B}_{\mu p_1\ldots p_5},$$
$$\mathcal{A}_{\mu m8} = -18\eta^{n_1\ldots n_7}\hat{B}_{\mu n_1\ldots n_7,m}. \tag{83}$$

In the notation introduced above, the supersymmetry transformations of the generalized vielbeine and vectors take a very compact form

$$\delta\mathcal{V}_{\mathcal{M}ij} = \sqrt{2}\Sigma_{ijkl}\mathcal{V}_{\mathcal{M}}{}^{kl}, \tag{84}$$

$$\delta\mathcal{A}_{\mu}{}^{\mathcal{M}} = i\Omega^{\mathcal{M}\mathcal{N}}\mathcal{V}_{\mathcal{N}ij}(2\sqrt{2}\bar{\varepsilon}^{i}\varphi^{j}_{\mu} + \bar{\varepsilon}_{k}\hat{\gamma}_{\mu}\chi^{ijk}) + \text{H.c.} \tag{85}$$

In order to relate our results for the Scherk-Schwarz reduction with the four-dimensional understanding of gaugings as embodied in the embedding tensor formalism, we need to rewrite the reduced generalized vielbein postulates (74)–(77) in terms of the notation introduced above, that is in terms of $E_{7(7)}$ objects $\mathcal{V}$ and $\mathcal{A}$. A straightforward calculation shows that upon substitution of $\mathcal{V}$ and $\mathcal{A}$ components, as defined by Eqs. (79) and (83), Eqs. (74)–(77) become

$$\partial_{\mu}\mathcal{V}^{m}_{ij} + \mathcal{Q}^{k}_{\mu[i}\mathcal{V}^{m}_{j]k} - \mathcal{P}_{\mu ijkl}\mathcal{V}^{mkl} + 2\mathcal{A}_{\mu}{}^{p}f^{m}{}_{pq}\mathcal{V}^{q}_{ij} = 0, \tag{86}$$

$$\partial_{\mu}\mathcal{V}_{mn\,ij} + \mathcal{Q}^{k}_{\mu[i}\mathcal{V}_{|mn|j]k} - \mathcal{P}_{\mu ijkl}\mathcal{V}_{mn}{}^{kl} + 4\mathcal{A}_{\mu}{}^{p}\delta^{[r}_{[m}f^{s]}{}_{n]p}\mathcal{V}_{rsij} + 6\mathcal{A}_{\mu pq}\delta^{[p}_{[r}f^{q]}{}_{mn]}\mathcal{V}^{r}_{ij} + 2\sqrt{2}\mathcal{A}_{\mu}{}^{p}g_{mnpq}\mathcal{V}^{q}_{ij} = 0, \tag{87}$$

$$\partial_{\mu}\mathcal{V}^{mn}_{ij} + \mathcal{Q}^{k}_{\mu[i}\mathcal{V}^{mn}_{j]k} - \mathcal{P}_{\mu ijkl}\mathcal{V}^{mnkl} - 4\mathcal{A}_{\mu}{}^{p}\delta^{[m}_{[r}f^{n]}{}_{s]p}\mathcal{V}^{rs}_{ij} + \frac{1}{2}\mathcal{A}_{\mu pq}\eta^{mnturs[p}f^{q]}{}_{tu}\mathcal{V}_{rsij} - 2\mathcal{A}_{\mu}{}^{pq}\delta^{[m}_{r}f^{n]}{}_{pq}\mathcal{V}^{r}_{ij}$$
$$+ \frac{\sqrt{2}}{6}\mathcal{A}_{\mu}{}^{p}\eta^{mnqrstu}g_{pqrs}\mathcal{V}_{tuij} - \frac{\sqrt{2}}{12}\mathcal{A}_{\mu pq}\delta^{[m}_{s}\eta^{n]pqr_1\ldots r_4}g_{r_1\ldots r_4}\mathcal{V}^{s}_{ij} + 4\sqrt{2}\mathfrak{f}_{FR}\mathcal{A}_{\mu}{}^{p}\delta^{mn}_{pq}\mathcal{V}^{q}_{ij} = 0, \tag{88}$$

$$\partial_{\mu}\mathcal{V}_{mij} + \mathcal{Q}^{k}_{\mu[i}\mathcal{V}_{mj]k} - \mathcal{P}_{\mu ijkl}\mathcal{V}_{m}{}^{kl} - 2\mathcal{A}_{\mu}{}^{p}f^{q}{}_{pm}\mathcal{V}_{qij} + 3\mathcal{A}_{\mu pq}\delta^{[p}_{[m}f^{q]}{}_{rs]}\mathcal{V}^{rs}_{ij} + \mathcal{A}_{\mu}{}^{pq}\delta^{[r}_{m}f^{s]}{}_{pq}\mathcal{V}_{rsij}$$
$$+ \sqrt{2}\mathcal{A}_{\mu}{}^{p}g_{pqrm}\mathcal{V}^{qr}_{ij} - \frac{\sqrt{2}}{24}\mathcal{A}_{\mu pq}\eta^{pqr_1\ldots r_4[s}\delta^{t]}_{m}g_{r_1\ldots r_4}\mathcal{V}_{stij} - 2\sqrt{2}\mathfrak{f}_{FR}\mathcal{A}_{\mu}{}^{p}\delta^{rs}_{pm}\mathcal{V}_{rsij} = 0. \tag{89}$$

Now, the components of $X_{\mathcal{M}}$ in terms of GL(7) indices can be simply read off by comparing Eq. (34) and Eqs. (86)–(89) listed above[10]

$$X_{m8}{}^{p8}{}_{r8} = -X_{m8\,r8}{}^{p8} = -\frac{1}{2}f^{p}{}_{mr}, \quad X_{m8}{}^{pq}{}_{r8} = -X_{m8\,r8}{}^{pq} = -\sqrt{2}\delta^{pq}_{mr}\mathfrak{f}_{FR}, \quad X_{m8}{}^{pq}{}_{rs} = -X_{m8\,rs}{}^{pq} = 2\delta^{[p}_{[r}f^{q]}{}_{s]m},$$
$$X_{mn}{}^{pq}{}_{r8} = -X_{mn\,r8}{}^{pq} = \delta^{[p}_{r}f^{q]}{}_{mn}, \quad X^{mn}{}_{p8\,rs} = X^{mn}{}_{rs\,p8} = -3\delta^{[m}_{[p}f^{n]}{}_{rs]}, \quad X^{mn\,pq\,rs} = -\frac{1}{2}\eta^{pqrstu[m}f^{n]}{}_{tu},$$
$$X^{mn}{}_{p8}{}^{rs} = -X^{mn\,rs}{}_{p8} = -\frac{\sqrt{2}}{24}\delta^{[r}_{p}\eta^{s]mntuvw}g_{tuvw}, \quad X_{m8}{}^{pq\,rs} = -\frac{\sqrt{2}}{12}\eta^{pqrstuv}g_{mtuv}, \quad X_{m8\,p8\,rs} = X_{m8\,rs\,p8} = -\frac{\sqrt{2}}{2}g_{mprs}. \tag{90}$$

[10]For brevity, we have left out a factor of the gauge coupling g in these expressions.

The components of $X_{\mathcal{M}}$ presented above agree in their general form with the components given already in the literature [32].[11] Written in terms of SL(8) indices, they take the form of the general solution given in Eq. (38) with

$$\mathsf{A}_{1\,88} = -\frac{8\sqrt{2}}{3}\mathfrak{f}_{FR}, \qquad \mathsf{A}_2{}^m{}_{np8} = -\frac{8}{3}f^m{}_{np},$$
$$\mathsf{A}_{2\,8}{}^{mnp} = \frac{\sqrt{2}}{9}\eta^{mnpr_1\ldots r_4}g_{r_1\ldots r_4}, \tag{91}$$

and all other components vanishing. The appearance of these structures can be understood from a group-theoretic point of view by considering the branching of the **912** representation of $E_{7(7)}$ in which the embedding tensor lives with respect to GL(7) [28,32,35]

$$\mathbf{912} \to \mathbf{1}_{+7} + \overline{\mathbf{35}}_{+5} + (\mathbf{7} + \mathbf{140})_{+3} + (\overline{\mathbf{21}} + \overline{\mathbf{28}} + \overline{\mathbf{224}})_{+1}$$
$$+ (\mathbf{21} + \mathbf{28} + \mathbf{224})_{-1} + (\bar{\mathbf{7}} + \overline{\mathbf{140}})_{-3}$$
$$+ \mathbf{35}_{-5} + \mathbf{1}_{-7}, \tag{92}$$

where the subscript represents the charge under $GL(1) \subset GL(7)$. Hence [28,32]

$$\mathfrak{f}_{FR} \leftrightarrow \mathbf{1}_{+7},$$
$$g_{mnpq} \leftrightarrow \overline{\mathbf{35}}_{+5},$$
$$f^p{}_{mn} \leftrightarrow \mathbf{140}_{+3},$$
$$f^p{}_{pm} \leftrightarrow \mathbf{7}_{+3}.$$

Of course, $f^p{}_{pm} = 0$, so $\mathbf{7}_{+3}$ does not contribute.

Note that we have used

$$\eta_{m_1\ldots m_7 8} = \eta_{m_1\ldots m_7}. \tag{93}$$

The quadratic constraint (45) is satisfied for the $X_{\mathcal{M}}$ derived from the generalized vielbein postulates. The constraints must be verified for each component and they are shown to be satisfied using Schouten identities, the unimodularity property (4), the Jacobi identity (7), and the background Bianchi identity (19). We refer the reader to Appendix B for details.

The calculations involved in the verification of the quadratic constraint are highly nontrivial. However, the fact that $X_{\mathcal{M}}$ as derived from the *eleven-dimensional* generalized vielbein postulates not only satisfy the linear constraint but also the more nontrivial quadratic constraint shows that there is indeed a *bona fide* gauge algebra for the gauging in the reduction. More generally, it points yet again to the deep relation between our eleven-dimensional formalism, developed in Refs. [1,15], and the embedding tensor formalism [16–20] that describes gauged supergravity.

Note that the verification of the linear and quadratic constraints did not require the use of the background consistency Eqs. (31) and (32). These are extra constraints that must be satisfied by the background solution if the reduction is to be consistent.

V. SCHERK-SCHWARZ REDUCTION WITH NO FLUX

An object $\Theta_{\mathcal{M}}{}^\alpha$, satisfying the embedding tensor constraints, is guaranteed to have at most half-maximal row rank [35] as was explained in Sec. III. However, even though we have shown that $\Theta_{\mathcal{M}}{}^\alpha$ as derived from the generalized vielbein postulates satisfies the embedding tensor constraints, it is not immediately obvious that always less than 28 vectors will be gauged, as is required by consistency. In fact a naive counting suggests that 49 vectors contribute, since this is the number of vectors that remain in the generalized vielbein postulates after the reduction *Ansätze* are substituted in. This is in contrast to the case of the S^7 reduction considered in [15]. There it is clear from the onset that $B_{\mu\,mn}$ drop out of the generalized vielbein postulates because of properties of Killing vectors. This leaves $B_\mu{}^m$ and $B_\mu{}^{mn}$, which are indeed the 28 vectors that are gauged in the S^7 reduction.

The fact that general results of the embedding tensor formalism guarantee that less than or equal to 28 vectors are gauged means that our naive counting of the contributing vectors is over-simplified and that constraints such as those placed on structure constants $f^p{}_{mn}$ for consistency of the reduction will conspire to reduce the number of gauged vectors to less than 28.

In this section, we explicitly demonstrate this for the simplifying case corresponding to the original reduction considered in [22], where there is no flux, i.e.

$$\mathfrak{f}_{FR} = 0, \qquad g_{mnpq} = 0. \tag{94}$$

The background Eq. (33) implies that the four-dimensional spacetime is Minkowski and that the group under consideration is "flat" [22], i.e.

$$2\delta_{pq}\delta^{rs}f^p{}_{mr}f^q{}_{ns} + 2f^p{}_{mq}f^q{}_{np} - \delta_{mp}\delta_{nq}\delta^{rs}\delta^{tu}f^p{}_{rt}f^q{}_{su}$$
$$= 0. \tag{95}$$

In this case the generalized vielbein postulates (74)–(77) take a simpler form

$$\partial_\mu \hat{e}^m_{AB} - f^m{}_{pq}\hat{B}_\mu{}^p\hat{e}^q_{AB} + \mathcal{Q}^C_{\mu[A}\hat{e}^m_{B]C} + \mathcal{P}_{\mu ABCD}\hat{e}^{mCD} = 0, \tag{96}$$

[11] There are some discrepancies in numerical factors (see Eq. (4.16) of Ref. [32]). In any case, here we verify that both the linear and quadratic constraints are satisfied for the components of $X_{\mathcal{M}}$ given in Eq. (90).

[10] For brevity, we have left out a factor of the gauge coupling g in these expressions.

$$\partial_\mu \hat{e}_{mnAB} - 2f^p{}_{q[m}\hat{e}_{n]pAB}\hat{B}_\mu{}^q + 3f^q{}_{[mn}\hat{B}_{|\mu|p]q}\hat{e}^p_{AB} + \mathcal{Q}^C_{\mu[A}\hat{e}_{mnB]C} + \mathcal{P}_{\mu ABCD}\hat{e}_{mn}{}^{CD} = 0, \tag{97}$$

$$\partial_\mu \hat{e}_{m_1\ldots m_5 AB} + 5\hat{B}_\mu{}^q f^p{}_{q[m_1}\hat{e}_{m_2\ldots m_5]pAB} - \frac{3\sqrt{2}}{2}f^p{}_{[m_1 m_2}\hat{B}_{|\mu p|m_3}\hat{e}_{m_4 m_5]AB} + 15 f^p{}_{[m_1 m_2}\hat{B}_{|\mu p|m_3 m_4 m_5 q]}\hat{e}^q_{AB} + \mathcal{Q}^C_{\mu[A}\hat{e}_{m_1\ldots m_5 B]C} + \mathcal{P}_{\mu ABCD}\hat{e}_{m_1\ldots m_5}{}^{CD} = 0, \tag{98}$$

$$\partial_\mu \hat{e}_{m_1\ldots m_7,nAB} + f^p{}_{qn}\hat{B}_\mu{}^q\hat{e}_{m_1\ldots m_7,pAB} - 5f^p{}_{[m_1 m_2}\hat{B}_{|\mu p|m_3}\hat{e}_{m_4\ldots m_7]nAB} + 5f^p{}_{[m_1 m_2}\hat{B}_{|\mu p|m_3\ldots m_6}\hat{e}_{m_7]nAB} + \mathcal{Q}^C_{\mu[A}\hat{e}_{m_1\ldots m_7,nB]C} + \mathcal{P}_{\mu ABCD}\hat{e}_{m_1\ldots m_7,n}{}^{CD} = 0. \tag{99}$$

A simple example of a flat group is given by [22]

$$U_m{}^n = (\exp My^1)_m{}^n, \tag{100}$$

where the seven-dimensional coordinates $y^m = (y^1, y^{\tilde{m}})$ with $\tilde{m} = 2, \ldots, 7$ and M is a constant traceless matrix with zeros in the first row and column, i.e.

$$M_m{}^n = \begin{pmatrix} 0 & \underline{0}^T \\ \underline{0} & \tilde{M}_{\tilde{m}}{}^{\tilde{n}} \end{pmatrix}. \tag{101}$$

Using the fact that

$$\partial_m U_n{}^p = \delta^1_m U_n{}^q M_q{}^p, \tag{102}$$

we find that

$$f^p{}_{mn} = 2M_{[m}{}^p\delta^1_{n]}. \tag{103}$$

In particular, we find that the only nonzero components of the structure constant are $f^{\tilde{p}}{}_{1\tilde{n}}$. Inspecting the generalized vielbein postulates (96)–(99) we find that $\hat{B}_{\mu mn}$ and $\hat{B}_{\mu m_1\ldots m_5}$ enter the equations in the form

$$f^q{}_{[mn}\hat{B}_{\mu p]q} \quad \text{and} \quad f^p{}_{[m_1 m_2}\hat{B}_{\mu m_3\ldots m_6]p}.$$

Hence, only

$$\hat{B}_{\mu\tilde{m}\tilde{n}} \quad \text{and} \quad \hat{B}_{\mu\tilde{m}_1\ldots\tilde{m}_5}$$

contribute. Along with $\hat{B}_\mu{}^1$ and $\hat{B}_\mu{}^{\tilde{m}}$ this gives a total of

$$28 = 1 + 6 + 6 + 15 = 13\,\text{electric} + 15\,\text{magnetic}$$

vectors appearing in the generalized vielbein postulates, which is kinematically consistent. Of course, one should here distinguish between the kinematics of the gauge couplings and the dynamics of the theory, which determines the vacuum and thus decides which vectors will remain as massless gauge bosons, and which will acquire a mass through spontaneous symmetry breaking. Indeed, for generic Scherk-Schwarz compactifications, the majority of the candidate 28 vectors fields will become massive in the reduction and can therefore not be gauged. In fact, $\hat{B}_\mu{}^1$ is the only vector that becomes gauged in the reduced theory. An analysis of all possible gaugings from a Scherk-Schwarz reduction with no background flux is given in Ref. [43]. It is shown that only electric vectors become gauged in this case.

In general, the Scherk-Schwarz reduction with background fluxes will have less than or equal to 28 gauge vectors contributing, kinematically, as is expected from general arguments. However, the distribution between electric and magnetic vectors can be varied—although as pointed out before, no more than 21 magnetic vectors can be gauged in this symplectic frame. In the context of Scherk-Schwarz flux compactifications this has already been observed in [28].

VI. CONCLUDING REMARKS

In this paper, we have investigated the Scherk-Schwarz reduction of $D = 11$ supergravity with background flux. In this case, the reduction *Ansatz* immediately gives a relation between the 56-bein in eleven dimensions and the 56-bein that parametrizes the scalars in four dimensions, Eqs. (62)–(65). In this form, the reduction *Ansatz* is applied to the generalized vielbein postulates yielding the embedding tensor of the respective gauged maximal theories in four dimensions. Furthermore, the reduction *Ansatz* written in the form (62)–(65) is suggestive of the fact that Scherk-Schwarz flux reductions can be thought of as an $E_{7(7)}$ generalized Scherk-Schwarz reduction of the form

$$\mathcal{V}_{\mathcal{M}AB}(x,y) = \mathcal{U}_{\mathcal{M}}{}^{\mathcal{N}}(y)\hat{\mathcal{V}}_{\mathcal{N}AB}(x), \tag{104}$$

$$\mathcal{B}_{\mu\mathcal{M}}(x,y) = U^{1/2}\mathcal{U}_{\mathcal{M}}{}^{\mathcal{N}}(y)\mathcal{A}_{\mu\mathcal{N}}(x), \tag{105}$$

where

$$\mathcal{V}_{\mathcal{M}AB} = \begin{pmatrix} \mathcal{V}_{m8\,AB} \\ \mathcal{V}^{mn}{}_{AB} \\ \mathcal{V}_{mnAB} \\ \mathcal{V}^{m8}{}_{AB} \end{pmatrix}, \qquad \mathcal{B}_{\mu\mathcal{M}} = \begin{pmatrix} \mathcal{B}_{\mu\,m8} \\ \mathcal{B}_{\mu}{}^{mn} \\ \mathcal{B}_{\mu\,mn} \\ \mathcal{B}_{\mu}{}^{m8} \end{pmatrix},$$

and $\hat{\mathcal{V}}_{\mathcal{M}AB}$ and $\mathcal{A}_{\mu\mathcal{N}}$ (similarly defined) are the 56-bein and the set of 56 vectors appropriate for the torus reduction, respectively. Moreover, $\mathcal{U}(y)$ is an $E_{7(7)}$ matrix of the form

$$\begin{pmatrix} U^{1/2}U_m{}^p & 3\sqrt{2}U^{1/2}a_{mrs}(U^{-1})_p{}^r(U^{-1})_q{}^s & U^{-1/2}S^{rs}_{+\,m}U_r{}^pU_s{}^q & U^{-1/2}S_{ms}(U^{-1})_p{}^s \\ 0 & U^{1/2}(U^{-1})_{[p}{}^m(U^{-1})_{q]}{}^n & U^{-1/2}S^{mnrs}U_r{}^pU_s{}^q & -2U^{-1/2}S^{mn}_{-\;s}(U^{-1})_p{}^s \\ 0 & 0 & U^{-1/2}U_{[m}{}^pU_{n]}{}^q & 6\sqrt{2}U^{-1/2}a_{mnr}(U^{-1})_p{}^r \\ 0 & 0 & 0 & U^{-1/2}(U^{-1})_p{}^m \end{pmatrix}, \tag{107}$$

where

$$S^{mn}_{\pm\;s} = 3\sqrt{2}\eta^{mnr_1\dots r_5}\left(a_{sr_1\dots r_5} \pm \frac{\sqrt{2}}{4}a_{sr_1r_2}a_{r_3r_4r_5}\right), \tag{108}$$

$$S_{mn} = -36\eta^{r_1\dots r_7}a_{mr_1r_2}\left(a_{nr_3\dots r_7} - \frac{\sqrt{2}}{12}a_{nr_3r_4}a_{r_5r_6r_7}\right), \tag{109}$$

$$S^{mnpq} = \frac{\sqrt{2}}{2}\eta^{mnpqr_1r_2r_3}a_{r_1r_2r_3}. \tag{110}$$

Equation (104) is to be compared with Eq. (64) of Ref. [1]:

$$\mathcal{V}_{\mathcal{M}AB}(x,y) = \mathcal{V}_{\mathcal{M}}{}^{\mathcal{A}}(x,y)\Gamma_{\mathcal{A}AB}, \tag{111}$$

where

$$\Gamma_{\mathcal{A}AB} = \begin{pmatrix} \Gamma_{aAB} \\ \Gamma^{ab}_{AB} \\ i\Gamma_{abAB} \\ i\Gamma^{a}_{AB} \end{pmatrix}. \tag{112}$$

In this case, one finds that the form of matrix $\mathcal{U}(y)$ is exactly the same as the form of $\mathcal{V}_{\mathcal{M}}{}^{\mathcal{A}}$ with the following identifications:

$$U_m{}^n \leftrightarrow e_m{}^a, \quad a_{mnp} \leftrightarrow A_{mnp}, \quad a_{m_1\dots m_6} \leftrightarrow A_{m_1\dots m_6}. \tag{113}$$

In particular, in Ref. [1], $\mathcal{V}_{\mathcal{M}}{}^{\mathcal{A}}$ is identified with the $E_{7(7)}$ coset element constructed in Ref. [44].

An interesting question is whether new reductions can be found by considering an *Ansatz* of the form (104), (105). A direction related to this is pursued in [37–39] in the context of extended generalized geometry, where $\mathcal{U}_{\mathcal{M}}{}^{\mathcal{N}}$ is assumed to depend on all extended coordinates. One should, however, keep in mind that (107) is already the most general $E_{7(7)}$ matrix (albeit in a triangular gauge), which does not leave much room for more exotic possibilities.

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APPENDIX A: $E_{7(7)}$ ALGEBRA AND IDENTITIES

In this appendix we review the SL(8) decomposition of the $E_{7(7)}$ algebra. In such a decomposition, the generators in the adjoint representation can be written

$$\begin{aligned} (t^{\mathsf{M}}{}_{\mathsf{N}})^{\mathsf{PQ}}{}_{\mathsf{RS}} &= 2\left(\delta^{\mathsf{PQ}}_{\mathsf{N[S}}\delta^{\mathsf{M}}_{\mathsf{R]}} - \frac{1}{8}\delta^{\mathsf{M}}_{\mathsf{N}}\delta^{\mathsf{PQ}}_{\mathsf{RS}}\right), \\ (t^{\mathsf{M}}{}_{\mathsf{N}})_{\mathsf{PQ}}{}^{\mathsf{RS}} &= -2\left(\delta^{\mathsf{PQ}}_{\mathsf{N[S}}\delta^{\mathsf{M}}_{\mathsf{R]}} - \frac{1}{8}\delta^{\mathsf{M}}_{\mathsf{N}}\delta^{\mathsf{PQ}}_{\mathsf{RS}}\right), \\ (t_{\mathsf{PQRS}})^{\mathsf{T}_1\dots\mathsf{T}_4} &= \delta^{\mathsf{T}_1\dots\mathsf{T}_4}_{\mathsf{PQRS}}, \\ (t_{\mathsf{PQRS}})_{\mathsf{T}_1\dots\mathsf{T}_4} &= \frac{1}{4!}\eta_{\mathsf{PQRST}_1\dots\mathsf{T}_4}. \end{aligned} \tag{A1}$$

It can be explicitly checked that the generators satisfy the following familiar commutation relations:

$$[t^{\mathsf{M}}{}_{\mathsf{N}}, t^{\mathsf{P}}{}_{\mathsf{Q}}] = \delta^{\mathsf{M}}_{\mathsf{Q}} t^{\mathsf{P}}{}_{\mathsf{N}} - \delta^{\mathsf{P}}_{\mathsf{N}} t^{\mathsf{M}}{}_{\mathsf{Q}},$$
$$[t^{\mathsf{M}}{}_{\mathsf{N}}, t_{\mathsf{PQRS}}] = -4\left(\delta^{\mathsf{M}}_{[\mathsf{P}} t_{\mathsf{QRS}]\mathsf{N}} + \frac{1}{8}\delta^{\mathsf{M}}_{\mathsf{N}} t_{\mathsf{PQRS}}\right), \tag{A2}$$

$$[t_{\mathsf{MNPQ}}, t_{\mathsf{RSTU}}] = \frac{1}{72}(\eta_{\mathsf{VMNPQ[RST}} t^{\mathsf{V}}{}_{\mathsf{U}]} - \eta_{\mathsf{VRSTU[MNP}} t^{\mathsf{V}}{}_{\mathsf{Q}]}). \tag{A3}$$

It is sometimes convenient to also define coset generators with upper indices

$$t^{\mathsf{MNPQ}} = \frac{1}{4!}\eta^{\mathsf{MNPQRSTU}} t_{\mathsf{RSTU}} \tag{A4}$$

keeping in mind that these are not independent generators. Furthermore, the components of the Killing metric are

$$\kappa^{\mathsf{M}}{}_{\mathsf{N}},{}^{\mathsf{P}}{}_{\mathsf{Q}} = 12\left(\delta^{\mathsf{M}}_{\mathsf{Q}}\delta^{\mathsf{P}}_{\mathsf{N}} - \frac{1}{8}\delta^{\mathsf{M}}_{\mathsf{N}}\delta^{\mathsf{P}}_{\mathsf{Q}}\right),$$
$$\kappa_{\mathsf{MNPQ,RSTU}} = \frac{2}{4!}\eta_{\mathsf{MNPQRSTU}},$$
$$(\kappa^{-1})_{\mathsf{N}}{}^{\mathsf{M}},{}_{\mathsf{Q}}{}^{\mathsf{P}} = \frac{1}{12}\left(\delta^{\mathsf{M}}_{\mathsf{Q}}\delta^{\mathsf{P}}_{\mathsf{N}} - \frac{1}{8}\delta^{\mathsf{M}}_{\mathsf{N}}\delta^{\mathsf{P}}_{\mathsf{Q}}\right),$$
$$(\kappa^{-1})^{\mathsf{MNPQ,RSTU}} = \frac{1}{2\cdot 4!}\eta^{\mathsf{MNPQRSTU}}. \tag{A5}$$

APPENDIX B: THE QUADRATIC CONSTRAINT

The quadratic constraint on the embedding tensor is required in order for the algebra of the gauge group to close

$$[X_{\mathcal{M}}, X_{\mathcal{N}}] = -X_{\mathcal{M}\mathcal{N}}{}^{\mathcal{P}} X_{\mathcal{P}}, \tag{B1}$$

or equivalently,

$$X_{\mathcal{M}\mathcal{Q}}{}^{\mathcal{R}} X_{\mathcal{N}\mathcal{R}}{}^{\mathcal{P}} - X_{\mathcal{N}\mathcal{Q}}{}^{\mathcal{R}} X_{\mathcal{M}\mathcal{R}}{}^{\mathcal{P}} = -X_{\mathcal{M}\mathcal{N}}{}^{\mathcal{R}} X_{\mathcal{R}\mathcal{Q}}{}^{\mathcal{P}}. \tag{B2}$$

Note that this constraint is highly nontrivial even to the extent that the left-hand side of the above equations is manifestly antisymmetric under the interchange of indices $\mathcal{M}$ and $\mathcal{N}$, whereas

$$X_{\mathcal{M}\mathcal{N}}{}^{\mathcal{P}}$$

is not in general antisymmetric under such an operation. We can therefore split this object into two tensors, viz.

$$X_{\mathcal{M}\mathcal{N}}{}^{\mathcal{P}} = X_{[\mathcal{M}\mathcal{N}]}{}^{\mathcal{P}} + Z_{\mathcal{M}\mathcal{N}}{}^{\mathcal{P}}, \tag{B3}$$

where the components of $X_{\mathcal{M}\mathcal{N}}{}^{\mathcal{P}}$ in a GL(7) decomposition is given in (90) and

$$Z_{\mathcal{M}\mathcal{N}}{}^{\mathcal{P}} \equiv X_{(\mathcal{M}\mathcal{N})}{}^{\mathcal{P}}.$$

In (90) we had already derived all the components of $X_{\mathcal{M}\mathcal{N}}{}^{\mathcal{P}}$ from the generalized vielbein postulates, so we can now explicitly exhibit the nonzero components of the symmetric tensor $Z_{\mathcal{M}\mathcal{N}}{}^{\mathcal{P}}$ as

$$Z_{m8}{}^{p8}{}_{r8} = Z^{p8}{}_{m8\,r8} = -\frac{1}{4}f^{p}{}_{mr}, \qquad Z_{m8}{}^{pq}{}_{r8} = Z^{pq}{}_{m8\,r8} = \frac{\sqrt{2}}{2}\delta^{pq}_{mr}\mathfrak{f}_{FR}, \qquad Z_{m8}{}^{pq}{}_{rs} = Z^{pq}{}_{m8\,rs} = -\frac{1}{2}\delta^{[p}_{m}f^{q]}{}_{rs},$$
$$Z_{m8\,rs}{}^{pq} = Z_{rs\,m8}{}^{pq} = -\frac{3}{2}\delta^{[p}_{[r}f^{q]}{}_{mn]}, \qquad Z_{mn}{}^{pq}{}_{r8} = Z^{pq}{}_{mn\,r8} = -\delta^{[p}_{[m}, f^{q]}{}_{n]r}, \qquad Z_{m8\,rs\,p8} = Z_{rs\,m8\,p8} = \frac{\sqrt{2}}{4}g_{mprs},$$
$$Z^{mn}{}_{p8}{}^{rs} = Z_{p8}{}^{mn\,rs} = -\frac{\sqrt{2}}{16}\delta^{[r}_{[p}\eta^{s]mntuvw}g_{tu]vw}, \qquad Z^{mn\,rs}{}_{p8} = Z^{rs\,mn}{}_{p8} = \frac{\sqrt{2}}{48}(\delta^{[m}_{p}\eta^{n]rstuvw} + \delta^{[r}_{p}\eta^{s]mntuvw})g_{mtuv},$$
$$Z^{mn\,pq\,rs} = Z^{pq\,mn\,rs} = \frac{1}{4}(\eta^{pqrstu[m}f^{n]}{}_{tu} + \eta^{mnrstu[p}f^{q]}{}_{tu}). \tag{B4}$$

The contraction given on the right-hand side of Eq. (B2) is indeed symmetric under the interchange of $\mathcal{M}$ and $\mathcal{N}$ [18].

The components of $X_{\mathcal{M}\mathcal{N}}{}^{\mathcal{P}}$ as derived from the generalized vielbein postulates, (90), satisfy the linear constraint since they can be put into a form compatible with the general solution of the linear constraint (38) (see Sec. III). However, the quadratic constraint is not necessarily satisfied by the general solution (38) and Eq. (B2) must be considered for the particular solution given by Eq. (90).

The components of X, given in (90), satisfy

$$X_{\mathcal{M}}{}^{\mathsf{PQ}}{}_{\mathsf{RS}} = -X_{\mathcal{M}\mathsf{RS}}{}^{\mathsf{PQ}}, \qquad X_{\mathcal{M}}{}^{\mathsf{PQRS}} = X_{\mathcal{M}}{}^{\mathsf{RSPQ}},$$
$$X_{\mathcal{M}\mathsf{PQRS}} = X_{\mathcal{M}\mathsf{RSPQ}}. \tag{B5}$$

We will verify Eq. (B2) for each component in turn:

(1)

$$X_{\mathsf{MNPQ}}{}^{\mathcal{R}}X_{\mathsf{TU}\mathcal{R}\mathsf{VW}} - X_{\mathsf{TUPQ}}{}^{\mathcal{R}}X_{\mathsf{MN}\mathcal{R}\mathsf{VW}} = -X_{\mathsf{MNTU}}{}^{\mathcal{R}}X_{\mathcal{R}\mathsf{PQVW}}. \tag{B6}$$

The only components for which both sides of the above equation are nontrivial are

$$(\mathsf{MN},\mathsf{PQ},\mathsf{TU},\mathsf{VW}) = (m8,p8,t8,vw) \text{ or } (m8,pq,t8,v8).$$

The latter case above is equivalent to the former, since from Eq. (B5) both sides of Eq. (B6) are symmetric under the interchange of PQ and VW. Therefore, we only need to consider

$$\begin{aligned}
&X_{m8\,p8}{}^{\mathcal{R}}X_{t8\,\mathcal{R}\,vw} - X_{t8p8}{}^{\mathcal{R}}X_{m8\,\mathcal{R}\,vw} + X_{m8\,t8}{}^{\mathcal{R}}X_{\mathcal{R}\,p8\,\,vw}\\
&= [2X_{m8\,p8}{}^{r8}X_{t8\,\,r8\,vw} + X_{m8\,p8\,rs}X_{t8}{}^{rs}{}_{vw} - (m\leftrightarrow t)] + X_{m8\,t8}{}^{r8}X_{r8\,p8\,vw} + X_{m8\,t8\,rs}X^{rs}{}_{p8\,vw},\\
&= -\left[\frac{\sqrt{2}}{2}f^{r}{}_{mp}g_{trvw} + \sqrt{2}g_{mp[v|s}f^{s}{}_{|w]t} - (m\leftrightarrow t)\right] - \frac{\sqrt{2}}{2}f^{r}{}_{tm}g_{pvwr} - \frac{3\sqrt{2}}{2}f^{r}{}_{[vw}g_{p]tmr},\\
&= -\frac{3\sqrt{2}}{2}f^{r}{}_{m[p}g_{vw]tr} - \frac{\sqrt{2}}{4}f^{r}{}_{tm}g_{pvwr} - \frac{3\sqrt{2}}{4}f^{r}{}_{[vw}g_{p]tmr} - (m\leftrightarrow t),\\
&= -5\sqrt{2}f^{r}{}_{[tm}g_{pvw]r},
\end{aligned}$$

which vanishes by Eq. (19).

(2)

$$X_{\mathsf{MNPQ}}{}^{\mathcal{R}}X_{\mathsf{TU}\mathcal{R}}{}^{\mathsf{VW}} - X_{\mathsf{TUPQ}}{}^{\mathcal{R}}X_{\mathsf{MN}\mathcal{R}}{}^{\mathsf{VW}} = -X_{\mathsf{MNTU}}{}^{\mathcal{R}}X_{\mathcal{R}\mathsf{PQ}}{}^{\mathsf{VW}}. \tag{B7}$$

The components of the above equation where both sides of the equation are nontrivial are given by

$$(\mathsf{MN},\mathsf{PQ},\mathsf{TU},\mathsf{VW}) = \begin{cases}(m8,\,p8,\,t8,\,v8)\\(m8,\,p8,\,tu,\,v8)\\(m8,\,p8,\,t8,\,vw)\\(mn,\,p8,\,t8,\,vw)\end{cases}. \tag{B8}$$

In the first case, we have

$$\begin{aligned}
&X_{m8\,p8}{}^{\mathcal{R}}X_{t8\,\mathcal{R}}{}^{v8} - X_{t8\,p8}{}^{\mathcal{R}}X_{m8\,\mathcal{R}}{}^{v8} + X_{m8\,p8}{}^{\mathcal{R}}X_{\mathcal{R}\,p8}{}^{v8}\\
&= -f^{s}{}_{p[t|}f^{v}{}_{s|m]} + \frac{1}{2}f^{s}{}_{tm}f^{v}{}_{ps},\\
&= \frac{3}{2}f^{s}{}_{[tm}f^{v}{}_{p]s},
\end{aligned}$$

which vanishes by Eq. (7). Similarly, the second case also vanishes by Eq. (7).
Consider the third case in (B12),

$$\begin{aligned}
&X_{m8\,p8}{}^{\mathcal{R}}X_{t8\,\mathcal{R}}{}^{vw} - X_{t8\,p8}{}^{\mathcal{R}}X_{m8\,\mathcal{R}}{}^{vw} + X_{m8\,t8}{}^{\mathcal{R}}X_{\mathcal{R}\,p8}{}^{vw}\\
&= -\frac{1}{6}\eta^{vwr_1\ldots r_5}g_{[m|r_1r_2r_3}g_{|t]r_4r_5p} + \frac{1}{24}\delta_p^{[v}\eta^{w]r_1\ldots r_6}g_{mtr_1r_2}g_{r_3\ldots r_6},\\
&= -\frac{1}{6}\eta^{vwr_1\ldots r_5}g_{[m|r_1r_2r_3}g_{|t]r_4r_5p} + \frac{1}{6}\delta_p^{[v}\eta^{wr_1\ldots r_6]}g_{mtr_1r_2}g_{r_3\ldots r_6} - \frac{1}{8}\eta^{vwr_1\ldots r_5}g_{mt[pr_1}g_{r_2\ldots r_5]},\\
&= \frac{1}{6}\delta_p^{[v}\eta^{wr_1\ldots r_6]}g_{mtr_1r_2}g_{r_3\ldots r_6} - \frac{7}{24}\eta^{vwr_1\ldots r_5}g_{[mtpr_1}g_{r_2\ldots r_5]}.
\end{aligned}$$

Both of the terms above vanish because they contain antisymmetrizations over eight indices. Moreover, it is simple to show that Eq. (B11) is satisfied for the fourth case, as in this case both sides of Eq. (B2) are equal to

$$\delta_p^{[v}f^{w]}{}_{ts}f^{s}{}_{mn}.$$

(3)

$$X_{\mathsf{MN}}{}^{\mathsf{PQ}\mathcal{R}}X_{\mathsf{TU}\mathcal{R}\mathsf{VW}} - X_{\mathsf{TU}}{}^{\mathsf{PQ}\mathcal{R}}X_{\mathsf{MN}\mathcal{R}\mathsf{VW}} = -X_{\mathsf{MNTU}}{}^{\mathcal{R}}X_{\mathcal{R}}{}^{\mathsf{PQ}}{}_{\mathsf{VW}}. \tag{B9}$$

Using the identities given in (B7), the above equation reduces to

$$X_{\mathsf{MN}}{}^{\mathcal{R}\mathsf{PQ}}X_{\mathsf{TUVW}\mathcal{R}} - X_{\mathsf{TU}}{}^{\mathcal{R}\mathsf{PQ}}X_{\mathsf{MNVW}\mathcal{R}} = X_{\mathsf{MNTU}}{}^{\mathcal{R}}X_{\mathcal{R}\mathsf{VW}}{}^{\mathsf{PQ}}, \tag{B10}$$

which is equivalent to Eq. (B11).

(4)

$$X_{\mathsf{MN}}{}^{\mathsf{PQ}\mathcal{R}}X_{\mathsf{TU}\mathcal{R}}{}^{\mathsf{VW}} - X_{\mathsf{TU}}{}^{\mathsf{PQ}\mathcal{R}}X_{\mathsf{MN}\mathcal{R}}{}^{\mathsf{VW}} = -X_{\mathsf{MNTU}}{}^{\mathcal{R}}X_{\mathcal{R}}{}^{\mathsf{PQVW}}. \tag{B11}$$

There is only one component of Eq. (B11) for which both sides of the above equation are nonvanishing:

$$
\begin{aligned}
&X_{m8}{}^{pq\mathcal{R}}X_{t8\mathcal{R}}{}^{vw}-X_{t8}{}^{pq\mathcal{R}}X_{m8\mathcal{R}}{}^{vw}+X_{m8t8}{}^{\mathcal{R}}X_{\mathcal{R}}{}^{pq\,vw}\\
&=-\frac{\sqrt{2}}{3}\eta^{pqr_1\ldots r_4[v}g_{[m|r_1r_2r_3}f^{w]}{}_{r_4|t]}-\frac{\sqrt{2}}{3}\eta^{vwr_1\ldots r_4[p}g_{[m|r_1r_2r_3}f^{q]}{}_{r_4|t]}+\frac{\sqrt{2}}{4}\eta^{pqvwu_1u_2u_3}f^{s}{}_{u_1u_2}g_{mtu_3s}+\frac{\sqrt{2}}{12}\eta^{pqvwu_1u_2u_3}f^{s}{}_{mt}g_{u_1u_2u_3s},\\
&=-\frac{2\sqrt{2}}{3}\eta^{r_1\ldots r_4[pqv}g_{[m|r_1r_2r_3}f^{w]}{}_{r_4|t]}+\frac{5\sqrt{2}}{6}\eta^{pqvwu_1u_2u_3}f^{s}{}_{[mt}g_{u_1u_2u_3]s}+\frac{\sqrt{2}}{2}\eta^{pqvwu_1u_2u_3}f^{s}{}_{u_1[t}g_{m]u_2u_3s},\\
&=-\frac{4\sqrt{2}}{3}\eta^{[r_1\ldots r_4pqv}g_{[m|r_1r_2r_3}f^{w]}{}_{r_4|t]}+\frac{\sqrt{2}}{6}\eta^{pqvwr_1\ldots r_3}g_{[m|r_1r_2r_3}f^{s}{}_{s|t]}+\frac{5\sqrt{2}}{6}\eta^{pqvwu_1u_2u_3}f^{s}{}_{[mt}g_{u_1u_2u_3]s},
\end{aligned}
$$

which vanishes by unimodularity, (4), and Eq. (19).

(5)

$$
X_{\mathsf{MNPQ}}{}^{\mathcal{R}}X^{\mathsf{TU}}{}_{\mathcal{R}\mathsf{VW}}-X^{\mathsf{TU}}{}_{\mathsf{PQ}}{}^{\mathcal{R}}X_{\mathsf{MN}\mathcal{R}\mathsf{VW}}=-X_{\mathsf{MN}}{}^{\mathsf{TU}\mathcal{R}}X_{\mathcal{R}\mathsf{PQVW}}. \tag{B12}
$$

The only nontrivial components to consider in this case are

$$
(\mathsf{MN},\mathsf{PQ},\mathsf{TU},\mathsf{VW})=(m8,p8,tu,vw)\quad\text{or}\quad (m8,pq,tu,v8). \tag{B13}
$$

Both cases reduce to the same equation; hence, we only consider the first case:

$$
\begin{aligned}
X_{m8\,p8}{}^{\mathcal{R}}X^{tu}{}_{\mathcal{R}}{}^{vw}-X^{tu}{}_{p8}{}^{\mathcal{R}}X_{m8\mathcal{R}}{}^{vw}+X_{m8}{}^{tu\mathcal{R}}X_{\mathcal{R}\,p8}{}^{vw}=&-\frac{\sqrt{2}}{24}f^{[v}{}_{mp}\eta^{w]tus_1\ldots s_4}g_{s_1\ldots s_4}-\frac{\sqrt{2}}{4}f^{[t}{}_{s_1s_2}\eta^{u]vws_1\ldots s_4}g_{mps_3s_4}\\
&-\frac{\sqrt{2}}{12}\delta^{[v}_{[p}f^{w]}{}_{s]m}\eta^{stuq_1\ldots q_4}g_{q_1\ldots q_4}-\frac{\sqrt{2}}{4}\delta^{[t}_{[p}f^{u]}{}_{rs]}\eta^{rsvwq_1\ldots q_3}g_{mq_1\ldots q_3}\\
&-\frac{\sqrt{2}}{12}\delta^{[t}_{r}f^{u]}{}_{sm}\delta^{[v}_{p}\eta^{w]rsq_1\ldots q_4}g_{q_1\ldots q_4}+\frac{\sqrt{2}}{12}\delta^{[v}_{p}f^{w]}{}_{rs}\eta^{tursq_1\ldots q_3}g_{mq_1\ldots q_3}.
\end{aligned}
$$

Using Schouten identities, the first, third, and fifth terms in the expression on the right-hand side reduce to

$$
\frac{\sqrt{2}}{6}\delta^{[v}_{p}\eta^{w]tur_1\ldots r_4}f^{s}{}_{mr1}g_{r_2\ldots r_4s} \tag{B16}
$$

and similarly the second and fourth terms simplify to

$$
-\frac{\sqrt{2}}{6}f^{[t}{}_{r1r2}\eta^{u][v|r_1\ldots r_5}\delta^{|w]}_{p}g_{mr_3\ldots r_5}. \tag{B17}
$$

Therefore,

$$
\begin{aligned}
X_{m8\,p8}{}^{\mathcal{R}}X^{tu}{}_{\mathcal{R}}{}^{vw}-X^{tu}{}_{p8}{}^{\mathcal{R}}X_{m8\mathcal{R}}{}^{vw}+X_{m8}{}^{tu\mathcal{R}}X_{\mathcal{R}\,p8}{}^{vw}=&\frac{\sqrt{2}}{6}\delta^{[v}_{p}\eta^{w]tur_1\ldots r_4}f^{s}{}_{mr_1}g_{r_2\ldots r_4s}-\frac{\sqrt{2}}{6}f^{[t}{}_{r_1r_2}\eta^{u][v|r_1\ldots r_5}\delta^{|w]}_{p}g_{mr_3\ldots r_5}\\
&+\frac{\sqrt{2}}{12}\delta^{[v}_{p}f^{w]}{}_{rs}\eta^{tursq_1\ldots q_3}g_{mq_1\ldots q_3},\\
=&\frac{5\sqrt{2}}{24}\delta^{[v}_{p}\eta^{w]tur_1\ldots r_4}f^{s}{}_{[mr_1}g_{r_2\ldots r_4]s}+\frac{\sqrt{2}}{6}\delta^{[v}_{p}\eta^{w]tur_1\ldots r_4}f^{s}{}_{sr_1}g_{mr_2\ldots r_4},
\end{aligned}
$$

$$
\begin{aligned}
&X_{m8\,p8}{}^{\mathcal{R}}X^{tu}{}_{\mathcal{R}\,vw}-X^{tu}{}_{p8}{}^{\mathcal{R}}X_{m8\,\mathcal{R}\,vw}+X_{m8}{}^{tu\mathcal{R}}X_{\mathcal{R}\,p8\,vw}\\
&=6\delta^{r}_{[v}f^{s}{}_{w]m}\delta^{[t}_{[p}f^{u]}{}_{rs]}+3f^{r}{}_{pm}\delta^{[t}_{[r}f^{v]}{}_{vw]}-6\delta^{[r}_{[p}f^{s}{}_{vw]}\delta^{[t}_{r}f^{u]}{}_{sm},\\
&=3\delta^{[t|}_{v}f^{s}{}_{[pm}f^{|u]}{}_{w]s}-3\delta^{[t|}_{w}f^{s}{}_{[pm}f^{|u]}{}_{v]s}+3\delta^{[t|}_{p}f^{s}{}_{[vw}f^{|u]}{}_{m]s},
\end{aligned}
$$

which vanishes by Eq. (7).

(6)

$$
X_{\mathsf{MNPQ}}{}^{\mathcal{R}}X^{\mathsf{TU}}{}_{\mathcal{R}}{}^{\mathsf{VW}}-X^{\mathsf{TU}}{}_{\mathsf{PQ}}{}^{\mathcal{R}}X_{\mathsf{MN}\mathcal{R}}{}^{\mathsf{VW}}=-X_{\mathsf{MN}}{}^{\mathsf{TU}\mathcal{R}}X_{\mathcal{R}\mathsf{PQ}}{}^{\mathsf{VW}}. \tag{B14}
$$

It is straightforward to see that all terms in the above equation vanish trivially unless

$$
(\mathsf{MN},\mathsf{PQ},\mathsf{TU},\mathsf{VW})=(m8,p8,tu,vw). \tag{B15}
$$

In this case,

where we have again used Schouten identities. It is now clear that Eq. (B23) holds as a result of Eqs. (4) and (19).

(7)

$$X_{\mathsf{MN}}{}^{\mathsf{PQ}\mathcal{R}}X^{\mathsf{TU}}{}_{\mathcal{R}\mathsf{VW}}-X^{\mathsf{TUPQ}\mathcal{R}}X_{\mathsf{MN}\mathcal{R}\mathsf{VW}} = -X_{\mathsf{MN}}{}^{\mathsf{TU}\mathcal{R}}X_{\mathcal{R}}{}^{\mathsf{PQ}}{}_{\mathsf{VW}}. \tag{B18}$$

Using the relations in (B7), this equation is equivalent to Eq. (B23), which we have already verified.

(8)

$$X_{\mathsf{MN}}{}^{\mathsf{PQ}\mathcal{R}}X^{\mathsf{TU}}{}_{\mathcal{R}}{}^{\mathsf{VW}}-X^{\mathsf{TUPQ}\mathcal{R}}X_{\mathsf{MN}\mathcal{R}}{}^{\mathsf{VW}} = -X_{\mathsf{MN}}{}^{\mathsf{TU}\mathcal{R}}X_{\mathcal{R}}{}^{\mathsf{PQVW}}. \tag{B19}$$

The only nontrivial equation to consider in this case is

$$X_{m8}{}^{pq\mathcal{R}}X^{tu}{}_{\mathcal{R}}{}^{vw}-X^{tu\,pq\mathcal{R}}X_{m8\,\mathcal{R}}{}^{vw}+X_{m8}{}^{tu\mathcal{R}}X_{\mathcal{R}}{}^{pq\,vw} = \frac{3}{2}\eta^{pqvws_1s_2[t}f^{u]}{}_{r[m}f^{r}{}_{s_1s_2]},$$

where we have used Schouten identities. Therefore, Eq. (B19) is satisfied.

(9)

$$X^{\mathsf{MN}}{}_{\mathcal{Q}}{}^{\mathcal{R}}X_{\mathsf{TU}\mathcal{R}}{}^{\mathcal{P}}-X_{\mathsf{TU}\mathcal{Q}}{}^{\mathcal{R}}X^{\mathsf{MN}}{}_{\mathcal{R}}{}^{\mathcal{P}} = -X^{\mathsf{MN}}{}_{\mathsf{TU}}{}^{\mathcal{R}}X_{\mathcal{R}\mathcal{Q}}{}^{\mathcal{P}}. \tag{B20}$$

Note that the left-hand side of this equation is of the same form as the left-hand side of cases 5–8. Therefore, it remains to show that

$$-X^{\mathsf{MN}}{}_{\mathsf{TU}}{}^{\mathcal{R}}X_{\mathcal{R}\mathcal{Q}}{}^{\mathcal{P}} = X_{\mathsf{TU}}{}^{\mathsf{MN}\mathcal{R}}X_{\mathcal{R}\mathcal{Q}}{}^{\mathcal{P}}. \tag{B21}$$

This can be simply verified using Schouten identities and Eqs. (4), (7), and (19) for all components.

(10)

$$X^{\mathsf{MN}}{}_{\mathsf{PQ}}{}^{\mathcal{R}}X^{\mathsf{TU}}{}_{\mathcal{R}\mathsf{VW}}-X^{\mathsf{TU}}{}_{\mathsf{PQ}}{}^{\mathcal{R}}X^{\mathsf{MN}}{}_{\mathcal{R}\mathsf{VW}} = -X^{\mathsf{MNTU}\mathcal{R}}X_{\mathcal{R}\mathsf{PQVW}}. \tag{B22}$$

This equation is trivially satisfied.

(11)

$$X^{\mathsf{MN}}{}_{\mathsf{PQ}}{}^{\mathcal{R}}X^{\mathsf{TU}}{}_{\mathcal{R}}{}^{\mathsf{VW}}-X^{\mathsf{TU}}{}_{\mathsf{PQ}}{}^{\mathcal{R}}X^{\mathsf{MN}}{}_{\mathcal{R}}{}^{\mathsf{VW}} = -X^{\mathsf{MNTU}\mathcal{R}}X_{\mathcal{R}\mathsf{PQ}}{}^{\mathsf{VW}}. \tag{B23}$$

The only nontrivial components to consider is

$$\begin{aligned}
X^{mn}{}_{p8}{}^{\mathcal{R}}X^{tu}{}_{\mathcal{R}}{}^{vw}-X^{tu}{}_{p8}{}^{\mathcal{R}}X^{mn}{}_{\mathcal{R}}{}^{vw}+X^{mn\,tu\mathcal{R}}X_{\mathcal{R}\,p8}{}^{vw} &= \frac{3}{2}\eta^{vwrsq_1q_2[m}f^{n]}{}_{q_1q_2}\delta^{[t}_{[r}f^{u]}{}_{sp]}-\frac{3}{2}\eta^{vwrsq_1q_2[t}f^{u]}{}_{q_1q_2}\delta^{[m}_{[r}f^{n]}{}_{sp]}\\
&\quad-\frac{1}{2}\eta^{tursq_1q_2[m}f^{n]}{}_{q_1q_2}\delta^{[v}_{p}f^{w]}{}_{rs},\\
&= \frac{1}{2}\delta^{[v}_{p}\eta^{w]m[t|r_1\dots r_4}f^{n}{}_{r_1r_2}f^{|u]}{}_{r_3r_4}-\frac{1}{2}\delta^{[v}_{p}\eta^{w]n[t|r_1\dots r_4}f^{m}{}_{r_1r_2}f^{|u]}{}_{r_3r_4}\\
&\quad-\frac{1}{2}\eta^{tursq_1q_2[m}f^{n]}{}_{q_1q_2}\delta^{[v}_{p}f^{w]}{}_{rs},
\end{aligned}$$

where in the second equality we have used Schouten identities to simplify the first two terms on the second line. Further use of Schouten identities gives

$$X^{mn}{}_{p8}{}^{\mathcal{R}}X^{tu}{}_{\mathcal{R}}{}^{vw}-X^{tu}{}_{p8}{}^{\mathcal{R}}X^{mn}{}_{\mathcal{R}}{}^{vw}+X^{mn\,tu\mathcal{R}}X_{\mathcal{R}\,p8}{}^{vw} = \frac{1}{2}\delta^{[v}_{p}\eta^{w]tur_1\dots r_4}f^{[m}{}_{r_1r_2}f^{n]}{}_{r_3r_4} + 2\delta^{[v}_{p}\eta^{w]tur_1r_2r_3[m}f^{n]}{}_{[r_1r_2}f^{r_4}{}_{r_3r_4]}.$$

The first term vanishes as a consequence of the fact that

$$f^{[m}{}_{[r1r2}f^{n]}{}_{r3r4]}$$

is antisymmetric under the interchange of m and n, but symmetric under the interchange of pairs $[r_1r_2]$ and $[r_3r_4]$. Furthermore, the second term vanishes either by the unimodularity property (4) or the Jacobi identity (7). Hence Eq. (B22) is satisfied.

(12)

$$X^{\mathsf{MNPQ}\mathcal{R}}X^{\mathsf{TU}}{}_{\mathcal{R}\mathsf{VW}}-X^{\mathsf{TUPQ}\mathcal{R}}X^{\mathsf{MN}}{}_{\mathcal{R}\mathsf{VW}} = -X^{\mathsf{MNTU}\mathcal{R}}X_{\mathcal{R}}{}^{\mathsf{PQ}}{}_{\mathsf{VW}}. \tag{B24}$$

Using Eq. (B7), this case is equivalent to case 11, which we have already verified.

(13)

$$X^{\mathsf{MNPQ}\mathcal{R}}X^{\mathsf{TU}}{}_{\mathcal{R}}{}^{\mathsf{VW}}-X^{\mathsf{TUPQ}\mathcal{R}}X^{\mathsf{MN}}{}_{\mathcal{R}}{}^{\mathsf{VW}} = -X^{\mathsf{MNTU}\mathcal{R}}X_{\mathcal{R}}{}^{\mathsf{PQVW}}. \tag{B25}$$

The above equation is trivially satisfied.

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