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Affine-null metric formulation of Einstein's equations

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The details are presented of a new evolution algorithm for the characteristic initial-boundary value problem based upon an affine parameter rather than the areal radial coordinate used in the Bondi-Sachs formulation. The advantages over the Bondi-Sachs version are discussed, with particular emphasis on the application to the characteristic extraction of the gravitational waveform from Cauchy simulations of general relativistic astrophysical systems.

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I. INTRODUCTION

There has been important progress in computing accurate gravitational waveforms by means of Cauchy-characteristic extraction (CCE) [1], whereby data from a Cauchy simulation provides the inner boundary data for a characteristic evolution extending to future null infinity $\mathcal{I}^+$, where the waveform is defined unambiguously. CCE has become an important tool for gravitational wave data analysis [2]. It has been applied to compute waveforms from simulations of binary black hole inspiral and mergers [3–5], from rotating stellar core collapse [6,7], to explore the memory effect [8] and to study the effect of spin on gravitational waves from precessing binary black holes [9].

A CCE module [10] has been prepared for public use as part of the Einstein toolkit [11]. The module is based upon the PITT null code [12,13], which implements the world-tube–null-cone version [14] of the Bondi-Sachs [15,16] characteristic initial-boundary value problem. There are technical complications in applying the Bondi-Sachs formulation to CCE arising from the use of an areal radial coordinate to parametrize the outgoing null geodesics. This paper considers an alternative approach to the world-tube-null-cone problem, which replaces the areal coordinate by an affine parameter. The details of an evolution algorithm for the affine system of Einstein equations are presented. The comparative advantages with the Bondi-Sachs version for application to CCE are discussed.

Recent success in simulating general relativistic astrophysical systems has been achieved by Cauchy codes, which evolve the spacetime metric inside an artificially constructed outer boundary. In doing so, it is common practice to compute the gravitational waveform from data on an extraction world tube inside the outer boundary, using perturbative methods based upon introducing a Schwarzschild background in the exterior region. This has been carried out using the Regge-Wheeler-Zerilli [17,18] treatment of the perturbed metric, as reviewed in Ref. [19], and also by calculating the Newman-Penrose [20] Weyl curvature component Ψ_4, as first done for the binary black hole problem in Refs. [21–24]. In this approach, errors arise from the finite size of the extraction world tube, from nonlinearities and from gauge ambiguities involved in the arbitrary introduction of a background metric. The gauge ambiguities might seem less severe in the case of Ψ_4 (vs metric) extraction, but there are still delicate problems associated with the choices of a preferred null tetrad and preferred worldlines along which to measure the waveform (see Ref. [25] for an analysis).

In order to properly approximate the waveform at $\mathcal{I}^+$, the extraction world tube must be sufficiently large but at the same time causally and numerically isolated from errors propagating in from the outer boundary. Considerable improvement in the perturbative approach has resulted from techniques for dealing with large outer boundaries and extrapolating the extracted waveform to infinity. However, this is not an ideally efficient approach. It is especially impractical in simulations of stellar collapse, where it is most strategic to restrict the computational domain to just outside the stellar surface. CCE is a different approach, which is specifically tailored to study radiation at $\mathcal{I}^+$.

In problems with isolated sources, the radiation zone can be compactified inside a finite grid boundary with the metric rescaled by $1/r^2$ as an implementation of Penrose's [26] conformal boundary at $\mathcal{I}^+$. Because $\mathcal{I}^+$ is a null hypersurface, no extraneous outgoing radiation condition or other artificial boundary condition is required. In CCE, Cauchy data on the extraction world tube provides the inner boundary data for a characteristic evolution extending to a compactified $\mathcal{I}^+$, where the waveform is defined unambiguously by geometric methods. This eliminates waveform error due to asymptotic approximations and gauge ambiguities introduced by the choice of extraction world tube. In addition, the extraction world tube can be placed in the near zone surrounding the sources in order to enhance computational efficiency. See Ref. [27] for a review.

High accuracy waveforms from a binary inspiral and merger are important for the design of the detection templates that are critical for the success of gravitational wave astronomy. This has stimulated efforts to increase the accuracy of characteristic evolution for use in CCE.

Another global approach applicable to isolated systems is to base the Cauchy problem itself on the analogue of the hyperboloidal Cauchy hypersurfaces in Minkowski space, which asymptote to $\mathcal{I}^+$. This approach, first extensively developed by Friedrich [28], is potentially the basis for a very attractive numerical approach to simulate gravitational wave production. For reviews of progress on the numerical implementation, see Refs. [29–31]. In spite of the attractiveness of the hyperboloidal approach and its recent success with model problems [32–35], considerable work remains to make it applicable to systems of astrophysical interest.

The Cauchy evolution codes have incorporated increasingly sophisticated numerical techniques, such as mesh refinement, multidomain decomposition, pseudospectral collocation and high order (in some cases eighth order) finite difference approximations. Work has begun to incorporate such techniques in characteristic codes [36]. However, such high accuracy methods cannot by themselves cure some of the major complications and sources of error arising in CCE. The timelike extraction world tube $\mathcal{T}$ at the inner boundary of CCE is constructed from a coordinate sphere $x^2 + y^2 + z^2 = \mathcal{R}^2$, $\mathcal{R} = \text{const}$, cut out from the Cartesian Cauchy grid. However, the radial grid points of the Bondi-Sachs system are based upon an areal coordinate r, with the angular grid lying on the spheres $r = R = \text{const}$. As a result, the extraction world tube $\mathcal{T}$ does not lie on the grid points of the Bondi-Sachs system (except for special cases such as spherical symmetry). This necessitates the introduction of an auxiliary characteristic coordinate system in the neighborhood of $\mathcal{T}$ in which the radial coordinate is replaced by an affine parameter λ along the outgoing null rays. By taking advantage of the affine freedom, $\mathcal{T}$ can then be parametrized by $\lambda = 0$. The Cauchy data is first transformed into the affine characteristic system and expanded about $\lambda = 0$ to a sufficient power of λ to determine data for the inner r-grid points of the Bondi-Sachs system in the neighborhood of $\mathcal{T}$. This is a complicated procedure which introduces interpolation error and has even led to inconsistent inner boundary conditions in the initial implementation of CCE (see Ref. [10] for a discussion).

In view of this, the question naturally arises: Why not use the affine-null system in the first place for the characteristic evolution algorithm and grid? The history behind this choice goes far back. It has to do with the simple hierarchical structure that the Einstein equations take in the Bondi-Sachs system, but which is seemingly broken in the affine system. We explain this in Sec. II.

The difference in behavior between an areal coordinate r and an affine parameter arises from focusing effects on the null rays. The affine coordinate λ only becomes singular at caustics, whereas the areal coordinate r also becomes singular at points where the expansion of the null rays vanishes. We deal here with the vacuum Einstein equations, where such focusing effects do not arise in the spherically symmetric case, and the areal coordinate is also an affine parameter along the radial null geodesics. However, there is another important application of characteristic coordinates to cosmology where, due to the lensing effect of matter, even in spherical symmetry the areal coordinate is not affine. The natural role of the past null cone in astronomical observations has been incorporated into to a new approach to cosmology [37,38]. Bishop and his collaborators [39–41] have initiated a program to implement this null cone version of observational cosmology by means of a characteristic evolution code based upon the Bondi-Sachs formalism. In principle, data obtained from observations on the past null cone could be evolved backward in time to obtain the earlier history of the Universe. At present, their simulations have been confined to the spherically symmetric case, but they recognized that an areal coordinate would limit the approach to the region of the Universe prior to refocusing. They developed a stable, convergent characteristic evolution code in which the areal coordinate was replaced by an affine parameter. The techniques developed in the present paper could be easily generalized to include matter and applied to extend their treatment to anisotropic cosmologies.

In Sec. III, we discuss the details of the Einstein equations in the affine-null system and show how the hierarchical structure of the evolution system can be restored. This provides the basis for a new world-tube–null-cone evolution algorithm.

II. NULL SPHERICAL COORDINATE SYSTEMS

The coordinates of both the Bondi-Sachs system and null-affine system are based upon a family of outgoing null hypersurfaces emanating from the spherical cross sections of a timelike world tube $\mathcal{T}$, where the null coordinate u labels these hypersurfaces and the angular coordinates x^A ($A = 2, 3$) label the spherical set of null geodesic rays. In the Bondi-Sachs system, the surface area coordinate r labels the points along the outgoing null rays. In the resulting $x^\alpha = (u, r, x^A)$ coordinates, the metric takes the Bondi-Sachs form [15,16]:

$$ds^2 = -\left(e^{2\beta}\frac{V}{r} - r^2 h_{AB}U^A U^B\right)du^2 - 2e^{2\beta}dudr - 2r^2 h_{AB}U^B du dx^A + r^2 h_{AB}dx^A dx^B, \tag{2.1}$$

where $\det(h_{AB}) = \det(q_{AB}) = q(x^C)$, with $q_{AB}(x^C)$ some standard choice of unit round-sphere metric. The fields β, U^A, V and h_{AB} are functions of (u, r, x^A). Here h_{AB} is the metric of the topological two-spheres ($u = \text{const}$, $r = \text{const}$) after conformal rescaling by $1/r^2$ to surface area 4π. Its inverse is defined by $h^{AC}h_{CB} = \delta^A_B$.

The affine-null system is similarly based upon the outgoing null hypersurfaces $u = \text{const}$ emanating from $\mathcal{T}$ with coordinates x^A labeling the null rays, but now an affine parameter λ is used to coordinatize points along the rays. The affine freedom

$$\lambda \to A(u, x^A)\lambda + B(u, x^A)$$

is used to prescribe the normalization $(\nabla^a u)\nabla_a \lambda = -1$ and set $\lambda = 0$ on $\mathcal{T}$. In the resulting $x^\alpha = (u, \lambda, x^A)$ coordinates, the metric takes the form

$$ds^2 = -(\mathcal{V} - g_{AB}W^A W^B)du^2 - 2dud\lambda - 2g_{AB}W^A dudx^B + g_{AB}dx^A dx^B. \quad (2.2)$$

In addition, we again set $g_{AB} = r^2 h_{AB}$, where $\det h_{AB} = \det q_{AB}$ with $q_{AB}(x^C)$ a unit round-sphere metric. However, r is now a metric function of (u, λ, x^A) along with W^A, $\mathcal{V}$ and h_{AB}.

At the metric level, such affine-null coordinates were introduced by Sachs [42] in formulating a double-null initial value problem. They are also the natural coordinates adopted in the Newman-Penrose [20] formulation of the Einstein equations in terms of a null tetrad and the associated Weyl tensor components. The affine coordinate λ is singular only at caustics, whereas the areal coordinate r is also singular at points where the expansion of the null rays vanishes. In particular, this occurs at the points on a stationary event horizon. As a result, codes based upon an areal coordinate have poor accuracy in tracking the late time tail preceding black hole formation; cf. Ref. [43] for a discussion in the context of black hole perturbation theory. The affine-null metric and the Bondi-Sachs metric are related by the transformation $\lambda(u, r, x^A)$ determined by

$$\partial_r \lambda(u, r, x^A) = e^{2\beta}. \quad (2.3)$$

However, the simplicity of this transformation is misleading because the surfaces $r = \text{const}$ which determine the partial derivatives ∂_A and ∂_u in the Bondi-Sachs system differ from the $\lambda = \text{const}$ surfaces that determine the partial derivatives ∂_A and ∂_u in the affine-null system. It is important to keep this distinction in mind in the following comparison of the corresponding evolution systems.

The role of the different components of the Einstein equations in formulating a characteristic initial value problem can be best described in terms of an orthonormal null tetrad $(L^a, N^a, M^a, \bar{M}^a)$, corresponding to the metric decomposition:

$$g_{ab} = -L_{(a}N_{b)} + M_{(a}\bar{M}_{b)}, \qquad N^a L_a = -2, \qquad M^a \bar{M}_a = 2. \quad (2.4)$$

We choose $L_a = -u_a$ to be the future pointing normal to the null hypersurfaces (so that L^a is tangent to the outgoing rays) and choose M^a to be a complex spatial vector tangent to the null hypersurfaces. This uniquely determines N^a. Then the vacuum Einstein equations $G_{ab} = 0$ decompose into the main equations

$$L^b G_{ab} = 0 \quad (2.5)$$

$$M^a M^b G_{ab} = 0 \quad (2.6)$$

and the supplementary equations

$$N^b R_{ab} = 0, \qquad G_{ab} = R_{ab} - \frac{1}{2} g_{ab} R. \quad (2.7)$$

It is a consequence of the Bianchi identities that if the main equations are satisfied, then $N^b R_{ab}$ satisfies a first order ordinary differential equation along the null rays. As a result, if the main equations are satisfied and the supplementary equations $N^b R_{ab} = 0$ are satisfied on the world tube $\mathcal{T}$, then they will be satisfied everywhere. This result was first demonstrated for the Bondi-Sachs system in Refs. [15,16], but it also holds for the affine-null system. See Ref. [44] for a recent discussion of the supplementary equations as a system of world tube conservation laws that impose symmetric hyperbolic constraints on the world tube data. In CCE, the world tube data is supplied by solutions of the Einstein equations determined by the Cauchy evolution, and it is assumed this data is consistent with the supplementary equations. Thus, we concentrate here on the main equations.

First consider the Bondi-Sachs equations. In that case, following the formalism developed in Ref. [45], the main equations (2.5) take the schematic form of hypersurface equations:

$$\beta_{,r} = \mathcal{N}_\beta[h_{CD}] \quad (2.8)$$

$$(r^4 e^{-2\beta} h_{AB} U^B_{,r})_{,r} = \mathcal{N}_U[h_{CD}, \beta] \quad (2.9)$$

$$V_{,r} = \mathcal{N}_V[h_{CD}, \beta, U^C], \quad (2.10)$$

where a comma denotes partial derivatives, e.g. $\beta_{,r} = \partial_r \beta$, and the main equations (2.6) take the form of evolution equations:

$$M^A M^B (r h_{AB,u})_{,r} = \mathcal{N}_h[h_{CD}, \beta, U^C, V]. \quad (2.11)$$

Here the $\mathcal{N}$ terms on the right-hand sides of Eqs. (2.8)–(2.11) can be calculated from the values of their arguments on a given $u = \text{const}$ null hypersurface. Moreover, each $\mathcal{N}$ term only depends upon previous members in the sequential order $[h_{CD}, \beta, U^C, V]$. Because of this hierarchical structure of the system, given h_{AB} on an initial null hypersurface $u = 0$, the main equations can be integrated radially in sequential order to determine the initial values of β, U^A, V and $h_{AB,u}$ at $u = 0$ in terms of their integration constants on $\mathcal{T}$, i.e.

$$\beta|_{\mathcal{T}}, U^A|_{\mathcal{T}}, U^A_{,r}|_{\mathcal{T}}, V|_{\mathcal{T}}, h_{AB,u}|_{\mathcal{T}}. \quad (2.12)$$

In addition, the location of the world tube, specified by $r|_{\mathcal{T}} = R(u, x^A)$, is another essential part of the data. After determining $h_{AB,u}$ at $u = 0$, the hypersurface data h_{AB} can be advanced to $u = \Delta u$ by a finite difference procedure. Given the world tube data (2.12), this procedure can be iterated to form a world-tube-null-cone evolution algorithm. This evolution algorithm is extremely simple and economical compared to Cauchy evolution algorithms. It is the algorithm underlying the PITT null code.

Now consider the affine-null system, for which the main equations take the schematic form,

$$r^{-1}r_{,\lambda\lambda} = \mathcal{H}_r[h_{CD}] \tag{2.13}$$

$$(r^4 h_{AB} W^B_{,\lambda})_{,\lambda} = \mathcal{H}_W[h_{CD}, r] \tag{2.14}$$

$$(2(r^2)_{,u} - \mathcal{V}(r^2)_{,\lambda})_{,\lambda} = \mathcal{H}_\mathcal{V}[h_{CD}, r, W^C] \tag{2.15}$$

$$M^A M^B (r h_{AB})_{,u\lambda} = \mathcal{H}_h[h_{CD}, r, W^C, \mathcal{V}], \tag{2.16}$$

where the $\mathcal{H}$ terms on the right hand sides of Eqs. (2.13)–(2.16) can again be calculated from the values of their arguments on a given $u =$ const null hypersurface.

As in the Bondi-Sachs case, the $\mathcal{H}$ terms depend upon the metric functions in sequential order, in this case in the order $[h_{CD}, r, W^C, \mathcal{V}]$. However, the hierarchical structure of the radial integration scheme is broken by the appearance of the term $(r^2)_{,u}$ term on the left-hand side of Eq. (2.15). Thus, Eq. (2.15) is not a pure hypersurface equation, and the radial integration scheme does not produce an evolution algorithm in the same way as for the Bondi-Sachs system. This was the reason that the affine-null formulation was not chosen in building the PITT null code.

However, by reformulating the hypersurface equation (2.15) by the introduction of an auxiliary variable, the pure hypersurface form of the radial integration scheme can be restored. This new formulation is described in Sec. III after presenting the details of the main equations of the null-affine system.

III. THE NULL-AFFINE EVOLUTION SYSTEM

In presenting the details of the Einstein equations for the affine-null system, we begin with some useful formula for describing the metric (2.2) and its associated connection and curvature. We then proceed to describe the construction of a numerical evolution algorithm.

A. Calculation of the Einstein tensor

The contravariant components of the metric (2.2) are given by

$$\begin{aligned} &g^{u\lambda} = -1, \qquad g^{uu} = g^{uA} = 0, \qquad g^{\lambda A} = -W^A, \\ &g^{\lambda\lambda} = \mathcal{V}, \qquad g^{AB} = r^{-2}h^{AB}. \end{aligned} \tag{3.1}$$

It is convenient to introduce a dyad vector m_A to represent the two-metric by

$$\begin{aligned} &h_{AB} = m_{(A}\bar{m}_{B)}, \qquad h^{AB} = m^{(A}\bar{m}^{B)}, \\ &m^A = h^{AB}m_B \qquad h_{AB}m^A m^B = 2. \end{aligned} \tag{3.2}$$

For that purpose, we chose the vector M^a forming the null tetrad (2.4) to lie tangent to the surfaces ($u =$ const, $r =$ const), so that it has components $M^a = (0, 0, M^A)$. We then set $M^A = r^{-1}m^A$. Here we raise and lower indices of two-dimensional vector and tensor fields on the sphere with h_{AB} and h^{AB}, e.g. $W_A = h_{AB}W^B$. We recall that $\det h_{AB} = \det q_{AB}$, where $q_{AB}(x^C)$ is some standard choice of unit round-sphere metric. The determinant condition implies

$$\begin{aligned} h^{AB}h_{AB,u} &= m^A\bar{m}^B h_{AB,u} = 0, \\ h^{AB}h_{AB,\lambda} &= m^A\bar{m}^B h_{AB,\lambda} = 0. \end{aligned} \tag{3.3}$$

We fix the spin rotation freedom in the dyad $m^A \to e^{i\varphi(u,\lambda,x^C)}m^A$ by requiring

$$\bar{m}^A m_{A,\lambda} = 0 \tag{3.4}$$

and

$$\bar{m}^A m_{A,u}|_{\lambda=0} = 0. \tag{3.5}$$

Note that the determinant condition (3.3) also implies

$$\bar{m}^A m_{A,\lambda} + m^A\bar{m}_{A,\lambda} = 0. \tag{3.6}$$

The spin rotation freedom then reduces to a phase factor $\varphi(x^A)$, which is determined by the choice of conventions at $(u = 0, \lambda = 0)$. Given these conventions, h_{AB} and m_A are in one-to-one correspondence.

As an example, for stereographic coordinates $x^A = (\eta, \rho)$ on the unit round-sphere with metric

$$\begin{aligned} q_{AB}dx^A dx^B &= \sqrt{q}(d\eta^2 + d\rho^2), \\ \sqrt{q} &= \frac{4}{1 + \eta^2 + \rho^2}, \end{aligned} \tag{3.7}$$

the rescaled metric on the general curved topological sphere can be represented as

$$h_{AB} = \sqrt{q}\begin{pmatrix} e^{2\gamma}\cosh 2\alpha & \sinh 2\alpha \\ \sinh 2\alpha & e^{-2\gamma}\cosh 2\alpha \end{pmatrix}, \tag{3.8}$$

$$h^{AB} = \frac{1}{\sqrt{q}}\begin{pmatrix} e^{-2\gamma}\cosh 2\alpha & -\sinh 2\alpha \\ -\sinh 2\alpha & e^{2\gamma}\cosh 2\alpha \end{pmatrix}, \tag{3.9}$$

where γ and α represent the two degrees of freedom. A specific choice of polarization dyad associated with this representation is

$$\begin{aligned} m_A = q^{1/4}(&e^{\gamma}(\cosh\alpha + i\sinh\alpha), \\ &ie^{-\gamma}(\cosh\alpha - i\sinh\alpha)), \end{aligned} \tag{3.10}$$

$$\begin{aligned} m^A = q^{-1/4}(&e^{-\gamma}(\cosh\alpha - i\sinh\alpha), \\ &ie^{\gamma}(\cosh\alpha + i\sinh\alpha)). \end{aligned} \tag{3.11}$$

The components of the Einstein tensor can be calculated in terms of the metric functions h_{AB}, r, W^A, $\mathcal{V}$ from the components of the curvature tensor,

$$\begin{aligned} R^a{}_{bcd} &= \partial_c\Gamma^a{}_{bd} - \partial_d\Gamma^a{}_{bc} + \Gamma^a{}_{ce}\Gamma^e{}_{bd} - \Gamma^a{}_{de}\Gamma^e{}_{bc}, \\ R_{ab} &= R^c{}_{acb}, \end{aligned} \tag{3.12}$$

where

$$\Gamma^a{}_{bc} = \frac{1}{2} g^{ad}(\partial_b g_{cd} + \partial_c g_{bd} - \partial_d g_{bc}) \tag{3.13}$$

are the Christoffel symbols. The components of the Christoffel symbols in terms of the metric functions are given in the appendix. We denote two-dimensional covariant derivatives of tensor fields on the sphere with respect to h_{AB} by a colon, e.g.

$$W^A{}_{:B} = \partial_B W^A + {}^{(h)}\Gamma^A_{BC} W^C, \tag{3.14}$$

where

$${}^{(h)}\Gamma^A_{BC} = \frac{1}{2} h^{AD}(\partial_B h_{CD} + \partial_C h_{BD} - \partial_D h_{BC}) \tag{3.15}$$

is the Christoffel symbol associated with h_{AB}.

In terms of these conventions and notation, the main affine-null equations (2.13)–(2.16) have the specific form

$$0 = R^b_\lambda u_{,b} = \frac{2r_{,\lambda\lambda}}{r} - \frac{1}{4} h^{CD}_{,\lambda} h_{CD,\lambda} \tag{3.16}$$

$$\begin{aligned} 0 &= R^b_A u_{,b} \\ &= -\frac{1}{2r^2}(r^4 h_{AB} W^B_{,\lambda})_{,\lambda} + \left(\frac{r_{,\lambda}}{r}\right)_{,A} - \left(\frac{r_{,B}}{r}\right) h^{BC} h_{AC,\lambda} \\ &\quad - \frac{1}{2} h^{BC} h_{AC,\lambda:B} \end{aligned} \tag{3.17}$$

$$\begin{aligned} 0 &= h^{AB} R_{AB} \\ &= {}^{(2)}\mathcal{R} + (2(r^2)_{,u} - \mathcal{V}(r^2)_{,\lambda})_{,\lambda} + \left(\frac{1}{r^2}(r^4 W^A)_{,\lambda}\right)_{:A} \\ &\quad - (\ln r^2)^{:A}{}_{:A} - \frac{r^4}{2} h_{AB} W^A_{,\lambda} W^B_{,\lambda} \end{aligned} \tag{3.18}$$

$$\begin{aligned} 0 &= m^A m^B R_{AB} \\ &= m^A m^B \Big\{ r(rh_{AB})_{,u\lambda} - \frac{1}{2}(r^2 \mathcal{V} h_{AB,\lambda})_{,\lambda} \\ &\quad + r r_{:C} W^C h_{AB,\lambda} + (r^2 W_{A:B})_{,\lambda} \\ &\quad + \frac{r^2}{4} h_{AB,\lambda} \bar{m}_C m_D (W^{C:D} - W^{D:C}) \\ &\quad + \frac{r^2}{2} W^C h_{AB,\lambda:C} - \frac{r^4}{2} h_{AC} h_{BD} W^C_{,\lambda} W^D_{,\lambda} \Big\}. \end{aligned} \tag{3.19}$$

B. Restoration of the hypersurface equation hierarchy

The strategy now is to use the auxiliary variable

$$\mathcal{Y} = \mathcal{V} - \frac{2r_{,u}}{r_{,\lambda}} \tag{3.20}$$

to eliminate the explicit appearance of the $r_{,u}$ derivative in Eq. (3.25) and reexpress it as a hypersurface equation for $\mathcal{Y}$. In that process, the substitution of $\mathcal{Y}$ for $\mathcal{V}$ in the evolution equation (3.19) leads to the intermediate expression

$$\begin{aligned} &m^A m^B \Big\{ r(rh_{AB})_{,u\lambda} - \frac{1}{2}(r^2 \mathcal{V} h_{AB,\lambda})_{,\lambda} \Big\} \\ &\quad = r m^A m^B \Big\{ r h_{AB,u} - \frac{r r_{,u}}{r_{,\lambda}} h_{AB,\lambda} \Big\}_{,\lambda} \\ &\qquad - m^A m^B \frac{1}{2}(r^2 \mathcal{Y} h_{AB,\lambda})_{,\lambda} \end{aligned} \tag{3.21}$$

or, using Eq. (3.4),

$$\begin{aligned} &m^A m^B \Big\{ r(rh_{AB})_{,u\lambda} - \frac{1}{2}(r^2 \mathcal{V} h_{AB,\lambda})_{,\lambda} \Big\} \\ &\quad = r \Big\{ m^A m^B r \Big(h_{AB,u} - \frac{r_{,u}}{r_{,\lambda}} h_{AB,\lambda} \Big) \Big\}_{,\lambda} \\ &\qquad - m^A m^B \frac{1}{2}(r^2 \mathcal{Y} h_{AB,\lambda})_{,\lambda}. \end{aligned} \tag{3.22}$$

When Eq. (3.22) is inserted back into Eq. (3.19), it gives a hypersurface equation for a combination of the time derivatives $m^A m^B h_{AB,u}$ and $r_{,u}$. In order to complete an evolution system, we need an additional radial equation for $r_{,u}$. This is obtained from the u derivative of the Raychauduri equation (3.16), which determines the rate of change of the expansion of the outgoing rays. In order to formulate a hierarchical radial integration scheme, we introduce the auxiliary variables

$$\rho = r_{,u} \tag{3.23}$$

and

$$k_{AB} = h_{AB,u}. \tag{3.24}$$

Then the u derivative of Eq. (3.16) gives

$$\rho_{,\lambda\lambda} = \frac{\rho}{8} h^{CD}_{,\lambda} h_{CD,\lambda} + \frac{r}{4} h^{BC}_{,\lambda} k_{BC,\lambda}, \tag{3.25}$$

where the determinant condition implies that the undifferentiated k_{AB} terms vanish, i.e.

$$h^{BD} h^{CE}_{,\lambda} h_{ED,\lambda} k_{BC} = 0. \tag{3.26}$$

In assembling the foregoing results into a radial integration hierarchy, simplifications result from using the spin-weighted scalars,

$$\sigma = \frac{1}{4} m^A m^B h_{AB,\lambda} = \frac{1}{2} m^A m_{A,\lambda}, \tag{3.27}$$

$$\kappa = \frac{1}{4} m^A m^B k_{AB}, \qquad k_{AB} = \kappa \bar{m}_A \bar{m}_B + \bar{\kappa} m_A m_B \tag{3.28}$$

and

$$J = 4(r_{,\lambda}\kappa - \rho\sigma) = r_{,\lambda} m^A m^B h_{AB,u} - r_{,u} m^A m^B h_{AB,\lambda}. \tag{3.29}$$

Then Eq. (3.22) becomes

$$m^A m^B\left\{r(rh_{AB})_{,u\lambda}-\frac{1}{2}(r^2\eth h_{AB,\lambda})_{,\lambda}\right\}$$
$$=r\left(\frac{rJ}{r_{,\lambda}}\right)_{,\lambda}-m^A m^B\frac{1}{2}(r^2\eth\eth h_{AB,\lambda})_{,\lambda}, \quad (3.30)$$

and Eq. (3.25) becomes

$$\rho_{,\lambda\lambda}=-\rho\sigma\bar{\sigma}-r\bar{\sigma}\kappa_{,\lambda}-r\sigma\bar{\kappa}_{,\lambda}$$
$$=-\frac{\rho}{r_{,\lambda}}(r\sigma\bar{\sigma})_{,\lambda}-2r\sigma\bar{\sigma}\left(\frac{\rho}{r_{,\lambda}}\right)_{,\lambda}-\frac{r\bar{\sigma}}{4}\left(\frac{J}{r_{,\lambda}}\right)_{,\lambda}$$
$$-\frac{r\sigma}{4}\left(\frac{\bar{J}}{r_{,\lambda}}\right)_{,\lambda}, \quad (3.31)$$

where, from Eqs. (3.16) and (3.27),

$$r\sigma\bar{\sigma}=-r_{,\lambda\lambda}. \quad (3.32)$$

C. Evolution algorithm

By assembling the results in Sec. III B, the main equations (3.16)–(3.19), along with Eq. (3.31), now take the desired form:

$$r^{-1}r_{,\lambda\lambda}=-\sigma\bar{\sigma}=H_r[h_{CD}] \quad (3.33)$$

$$(r^4h_{AB}W^B_{,\lambda})_{,\lambda}=2r^2\left(\frac{r_{,\lambda}}{r}\right)_{,A}-2rr_{,B}h^{BC}h_{AC,\lambda}-r^2h^{BC}h_{AC,\lambda:B}$$
$$=H_W[h_{CD},r] \quad (3.34)$$

$$(\eth(r^2)_{,\lambda})_{,\lambda}={}^{(2)}\mathcal{R}+\left(\frac{1}{r^2}(r^4W^A)_{,\lambda}\right)_{:A}-(\ln r^2)^{:A}{}_{:A}$$
$$-\frac{r^4}{2}h_{AB}W^A_{,\lambda}W^B_{,\lambda}=H_\eth[h_{CD},r,W^C] \quad (3.35)$$

$$\left(\frac{rJ}{r_{,\lambda}}\right)_{,\lambda}=m^Am^B\left\{\frac{1}{2r}(r^2\eth h_{AB,\lambda})_{,\lambda}-r_{:C}W^Ch_{AB,\lambda}\right.$$
$$-\frac{1}{r}(r^2W_{A:B})_{,\lambda}-\frac{r}{4}h_{AB,\lambda}\bar{m}_Cm_D(W^{C:D}-W^{D:C})$$
$$\left.-\frac{r}{2}W^Ch_{AB,\lambda:C}+\frac{r^3}{2}h_{AC}h_{BD}W^C_{,\lambda}W^D_{,\lambda}\right\}$$
$$=H_J[h_{CD},r,W^C,\eth] \quad (3.36)$$

$$\left(\frac{\rho}{r_{,\lambda}}\right)_{,\lambda\lambda}=-\frac{r}{4r_{,\lambda}}\left(\bar{\sigma}\left(\frac{J}{r_{,\lambda}}\right)_{,\lambda}+\sigma\left(\frac{\bar{J}}{r_{,\lambda}}\right)_{,\lambda}\right)=H_\rho[h_{CD},r,J], \quad (3.37)$$

where the H terms can be calculated from the values of their arguments on a $u=$ const null hypersurface. Given $h_{AB}(u,\lambda,x^C)$, the system (3.33)–(3.37) forms a hierarchy of radial hypersurface equations which can be integrated to determine the remaining variables in the sequential order $[r, W^A, \eth, J, \rho]$.

This gives rise to the following initial-boundary value problem. Specify the initial hypersurface data

$$[h_{AB},r], \qquad u=0, \qquad \lambda\geq 0 \quad (3.38)$$

and the initial boundary data

$$[r_{,\lambda},W^A,W^A_{,\lambda},\eth,J,\rho,\rho_{,\lambda}], \qquad u=0, \qquad \lambda=0, \quad (3.39)$$

subject to the constraint (3.33), which constitutes an ordinary differential radial equation. On the boundary, specify

$$[W^A,W^A_{,\lambda},\eth,J,\rho,\rho_{,\lambda}], \qquad u>0, \qquad \lambda=0, \quad (3.40)$$

subject to the conditions (2.7). Using the initial data, integrate Eqs. (3.33)–(3.37) to determine the initial values of $[r, W^A, \eth, J, \rho]$.

Given this initialization and the boundary data, the evolution system can be integrated by a finite difference approximation. The initial values $\rho(0,\lambda,x^A)$ and $J(0,\lambda,x^A)$ determine the values of r and h_{AB} at $u=\Delta u$ through Eqs. (3.23) and (3.29). Using the boundary data, Eqs. (3.34)–(3.37) can then be integrated in sequential order to determine $[W^A,\eth,J,\rho]$ at $u=\Delta u$. Now $[h_{AB},r,W^A,\eth,J,\rho]$ are known at $u=\Delta u$, and this process can be repeated to provide a finite difference evolution algorithm. If the algorithm converges as $\Delta u\to 0$, then it produces a solution to the affine-null initial-boundary value problem for Einstein's equations.

IV. DISCUSSION

We have constructed an evolution algorithm based upon the affine-null system, which, like the Bondi-Sachs system, is based upon a hierarchy of radial equations along the outgoing characteristics. It has the additional advantage of the flexibility in describing an arbitrary inner world tube boundary as a coordinate surface. This is especially important for application to CCE, where the inner boundary, which is constructed in terms of the Cauchy coordinates, generically differs from the $r=$ const Bondi-Sachs world tubes. As a result, the affine-null algorithm offers the possibility of increased economy and accuracy.

The formal solution of the null-affine problem constructed in Sec. III C yields an exact solution provided the finite difference approximation converges as $\Delta u\to 0$. A necessary condition for this is the well-posedness of the underlying analytic initial-boundary value problem. Well-posedness, i.e. the existence of a unique solution which depends continuously on the data, is a necessary condition for a successful numerical treatment. Although characteristic evolution codes based upon the Bondi-Sachs formalism have been demonstrated to be stable in a large number of test cases [13,46], there remains some lingering doubt because well-posedness of the analytic problem has not yet been established. Rendall [47] has shown that the affine-null problem is well-posed in the double null case where the inner boundary is also a null hypersurface.

However, Rendall's approach cannot be applied to the corresponding problem where the inner boundary is a timelike world tube. The well-posedness of the world-tube–null-cone characteristic initial-boundary value problem for Einstein's equations remains an outstanding issue.

The only source of error in CCE which does not decrease with numerical resolution arises from the mismatch between the initial Cauchy data and initial characteristic data. This results because the radius $\mathcal{R}_2$ of the outer Cauchy boundary is larger than the radius $\mathcal{R}_1$ of the inner world tube boundary of the characteristic evolution. Whereas the Cauchy data in the region $\mathcal{R}_1 \le \mathcal{R} \le \mathcal{R}_2$ is chosen, say, by some constraint solver for binary black hole initial data, the characteristic initial data for $\mathcal{R} \ge \mathcal{R}_1$ is chosen to suppress the initial radiation content by requiring that the Newman-Penrose Weyl component $\Psi_0 = 0$. This mismatch between the initial Cauchy and characteristic data leads to an extraneous error in the extracted waveform which is related to the spurious radiation content in the Cauchy data. This error decreases as $\mathcal{R}_2 \to \mathcal{R}_1$, but present day Cauchy codes require that the outer boundary be in the far zone of a binary black hole to avoid incoming radiation generated by the outer boundary condition. As a result, the full potential of CCE is not realized.

This mismatch can be eliminated by Cauchy-characteristic matching (CCM) [48]. CCE is one of the pieces of CCM in which the characteristic world tube data is extracted from the Cauchy evolution. In CCM, data on the outer Cauchy boundary are in turn obtained from the characteristic evolution. In doing so, it is possible to place the radius $\mathcal{R}_2$ of the Cauchy boundary just outside the radius $\mathcal{R}_1$ of the characteristic extraction world tube. In fact, in a finite difference implementation of CCM for a model scalar wave problem [49], it has been possible to arrange that $\mathcal{R}_2 \to \mathcal{R}_1$ in the continuum limit. This resulted in a seamless interface between the Cauchy and characteristic evolutions with no mismatch in the initial data.

The success of CCM depends upon the proper mathematical and computational treatment of the initial-boundary value problem for the Cauchy evolution. At present, the only successful 3D application of CCM in general relativity has been to the linearized matching problem between a characteristic code and a Cauchy code based upon harmonic coordinates [50]. Considerable work remains to apply it to astrophysical systems. The linearized harmonic code satisfied a well-posed initial-boundary value problem, which seems to be a critical missing ingredient in earlier attempts at CCM in general relativity. More recently, a well-posed initial-boundary value problem has been established for fully nonlinear harmonic evolution [51,52], which should facilitate the extension of CCM to the nonlinear case.

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APPENDIX: CHRISTOFFEL SYMBOLS

The calculation of the Ricci tensor

$$R_{\alpha\beta} = \Gamma^{\nu}_{\alpha\beta,\nu} - \Gamma^{\nu}_{\nu\alpha,\beta} + \Gamma^{\rho}_{\alpha\beta}\Gamma^{\nu}_{\rho\nu} - \Gamma^{\rho}_{\nu\alpha}\Gamma^{\nu}_{\rho\beta} \quad \text{(A1)}$$

which enters the main equations can be carried out explicitly in terms of the Christoffel symbols for the affine-null metric (2.2). The components are listed according to the notation $(x^0, x^1, x^A) = (u, \lambda, x^A)$,

$$\Gamma^{\alpha}_{\alpha\nu} = \partial_\nu \ln (r^2\sqrt{q}) \quad \text{(A2)}$$

$$\Gamma^{0}_{1\alpha} = 0 \quad \text{(A3)}$$

$$\Gamma^{\alpha}_{11} = 0 \quad \text{(A4)}$$

$$\Gamma^{0}_{AB} = r r_{,\lambda} h_{AB} + \frac{r^2}{2} h_{AB,\lambda} \quad \text{(A5)}$$

$$\Gamma^{1}_{AB} = \frac{r^2}{2}(h_{AC}W^{C}_{\ :B} + h_{BC}W^{C}_{\ :A} + h_{AB,u} - \mathcal{V} h_{AB,\lambda}) + r h_{AB}(r_{,C}W^{C} + r_{,u} - \mathcal{V} r_{,\lambda}) \quad \text{(A6)}$$

$$\Gamma^{C}_{AB} = r r_{,\lambda} W^{C} h_{AB} + \frac{1}{2} r^2 W^{C} h_{AB,\lambda} + \frac{1}{r}(r_{,B}\delta^{C}_{A} + r_{,A}\delta^{C}_{B} - r_{,D}h^{CD}h_{AB}) + {}^{(h)}\Gamma^{C}_{AB} \quad \text{(A7)}$$

$$\Gamma^{0}_{0A} = -\frac{r^2}{2}(h_{AB,\lambda}W^{B} + h_{AB}W^{B}_{,\lambda}) - r r_{,\lambda} h_{AB} W^{B} \quad \text{(A8)}$$

$$\Gamma^{1}_{10} = \frac{1}{2}\mathcal{V}_{,\lambda} - \frac{r^2}{2} h_{AB} W^{A} W^{B}_{,\lambda} \quad \text{(A9)}$$

$$\Gamma^{1}_{1A} = \frac{r^2}{2} h_{AB} W^{B}_{,\lambda} \quad \text{(A10)}$$

$$\Gamma^{0}_{00} = -\frac{1}{2}\mathcal{V}_{,\lambda} + \frac{1}{2}(r^2 h_{AB} W^{A} W^{B})_{,\lambda} \quad \text{(A11)}$$

$$\Gamma^{C}_{A1} = \frac{1}{r} r_{,\lambda}\delta^{C}_{A} + \frac{1}{2} h^{CD} h_{AD,\lambda} \quad \text{(A12)}$$

$$\Gamma^{1}_{00} = \frac{1}{2} W_{,0} + \frac{1}{2}(r^2 h_{AB})_{,0} W^{A} W^{B} + \frac{1}{2}\mathcal{V}(\mathcal{V} - r^2 h_{AB} W^{A} W^{B})_{,\lambda} - \frac{1}{2} W^{A}(\mathcal{V} - r^2 h_{BC} W^{B} W^{C})_{,A} \quad \text{(A13)}$$

$$\Gamma^1_{0A} = \frac{1}{2}W_{,A} + \frac{1}{2}\mathcal{V}(r^2 h_{AB}W^B)_{,\lambda} - rr_{,B}W^B h_{AC}W^C - \frac{r^2}{2}W^B(h_{BC}W^C_{:A} + h_{AC}W^C_{:B}) - \frac{1}{2}W^B(r^2 h_{AB})_{,0} \tag{A14}$$

$$\Gamma^B_{0A} = -\frac{1}{2}W^B(r^2 h_{AC}W^C)_{,\lambda} + \frac{1}{2}g^{BC}(r^2 h_{AC})_{,0} - \frac{r_{,A}}{r}W^B - \frac{1}{2}W^B_{:A} + \frac{r_{,D}}{r}h^{BD}h_{AC}W^C + \frac{1}{2}h^{BC}h_{AD}W^D_{:C} \tag{A15}$$

$$\Gamma^A_{01} = -\frac{r_{,\lambda}}{r}W^A - \frac{1}{2}h^{AB}(h_{BC}W^C)_{,\lambda} \tag{A16}$$

$$\Gamma^A_{00} = -\frac{1}{2}W^A\mathcal{V}_{,\lambda} + \frac{1}{2}W^A(r^2 h_{BC}W^B W^C)_{,\lambda} - 2W^A\frac{r_{,u}}{r} - h^{AB}(h_{BC}W^C)_{,u} + \frac{1}{2r^2}h^{AB}\mathcal{V}_{,B} - \frac{r_{,B}}{r}h^{AB}h_{CD}W^C W^D - \frac{1}{2}h^{AB}(h_{CD}W^C W^D)_{,B}. \tag{A17}$$

[1] N. T. Bishop, R. Gómez, L. Lehner, and J. Winicour, Phys. Rev. D **54**, 6153 (1996).
[2] P. Ajith *et al.* (NINJA Project), Classical Quantum Gravity **29**, 124001 (2012).
[3] C. Reisswig, N. T. Bishop, D. Pollney, and B. Szilágyi, Phys. Rev. Lett. **103**, 221101 (2009).
[4] C. Reisswig, N. T. Bishop, D. Pollney, and B. Szilágyi, Classical Quantum Gravity **27**, 075014 (2010).
[5] M. C. Babiuc, J. Winicour, and Y. Zlochower, Classical Quantum Gravity **28**, 134006 (2011).
[6] C. Reisswig, C. D. Ott, U. Sperhake, and E. Schnetter, Phys. Rev. D **83**, 064008 (2011).
[7] C. D. Ott, C. Reisswig, E. Schnetter, E. O'Connor, U. Sperhake, F. Löffler, P. Diener, E. Abdikamalov, I. Hawke, and A. Burrows, Phys. Rev. Lett. **106**, 161103 (2011).
[8] C. Reisswig and D. Pollney, Astrophys. J. Lett. **732**, L13 (2011).
[9] C. Reisswig, S. Husa, L. Rezzolla, E. N. Dorband, D. Pollney, and J. Seiler, Phys. Rev. D **80**, 124026 (2009).
[10] M. C. Babiuc, B. Szilágyi, J. Winicour, and Y. Zlochower, Phys. Rev. D **84**, 044057 (2011).
[11] http://www.einsteintoolkit.org.
[12] R. A. Isaacson, J. S. Welling, and J. Winicour, J. Math. Phys. (N.Y.) **24**, 1824 (1983).
[13] N. T. Bishop, R. Gómez, L. Lehner, M. Maharaj, and J. Winicour, Phys. Rev. D **56**, 6298 (1997).
[14] L. A. Tamburino and J. Winicour, Phys. Rev. **150**, 1039 (1966).
[15] H. Bondi, M. van der Burg, and A. W. K. Metzner, Proc. R. Soc. A **269**, 21 (1962).
[16] R. K. Sachs, Proc. R. Soc. A **270**, 103 (1962).
[17] T. Regge and J. A. Wheeler, Phys. Rev. **108**, 1063 (1957).
[18] F. Zerilli, Phys. Rev. D **2**, 2141 (1970).
[19] A. Nagar and L. Rezzolla, Classical Quantum Gravity **22**, R167 (2005).
[20] E. T. Newman and R. Penrose, J. Math. Phys. (N.Y.) **3**, 566 (1962).
[21] J. Baker, M. Campanelli, C. O. Lousto, and R. Takahashi, Phys. Rev. D **65**, 124012 (2002).
[22] F. Pretorius, Phys. Rev. Lett. **95**, 121101 (2005).
[23] M. Campanelli, C. O. Lousto, P. Marronetti, and Y. Zlochower, Phys. Rev. Lett. **96**, 111101 (2006).
[24] J. G. Baker, J. Centrella, D-I. Choi, M. Koppitz, and J. van Meter, Phys. Rev. D **73**, 104002 (2006).
[25] L. Lehner and O. M. Moreschi, Phys. Rev. D **76**, 124040 (2007).
[26] R. Penrose, Phys. Rev. Lett. **10**, 66 (1963).
[27] J. Winicour, Living Rev. Relativity **15**, 2 (2012).
[28] H. Friedrich, Commun. Math. Phys. **91**, 445 (1983).
[29] J. Frauendiener, Living Rev. Relativity **7**, 1 (2004).
[30] S. Husa, Lect. Notes Phys. **604**, 239 (2002).
[31] S. Husa, Lect. Notes Phys. **617**, 159 (2003).
[32] A. Zenginoğlu, Classical Quantum Gravity **25**, 195025 (2008).
[33] V. Moncrief and O. Rinne, Classical Quantum Gravity **26**, 125010 (2009).
[34] O. Rinne, Classical Quantum Gravity **27**, 035014 (2010).
[35] J. M. Bardeen, O. Sarbach, and L. T. Buchman, Phys. Rev. D **83**, 104045 (2011).
[36] C. Reisswig, N. T. Bishop, and D. Pollney, Gen. Relativ. Gravit. **45**, 1069 (2013).
[37] J. Kristian and R. K. Sachs, Astrophys. J. **143**, 379 (1966).
[38] G. F. R. Ellis, S. D. Nel, R. Maartens, W. R. Stoeger, and A. P. Whitman, Phys. Rep. **124**, 315 (1985).
[39] N. T. Bishop and P. Haines, Quaestiones Mathematicae **19**, 259 (1996).
[40] P. J. van der Walt and N. T. Bishop, Phys. Rev. D **82**, 084001 (2010).
[41] P. J. van der Walt and N. T. Bishop, Phys. Rev. D **85**, 044016 (2012).
[42] R. K. Sachs, J. Math. Phys. (N.Y.) **3**, 908 (1962).
[43] S. Husa, Y. Zlochower, R. Gómez, and J. Winicour, Phys. Rev. D **65**, 084034 (2002).
[44] J. Winicour, Gen. Relativ. Gravit. **43**, 3269 (2011).

[45] J. Winicour, J. Math. Phys. (N.Y.) **24**, 1193 (1983).
[46] R. Gómez, L. Lehner, R. Marsa, and J. Winicour, Phys. Rev. D **57**, 4778 (1998).
[47] A. D. Rendall, Proc. R. Soc. A **427**, 221 (1990).
[48] N. T. Bishop, R. Gómez, L. Lehner, B. Szilágyi, J. Winicour, and R. A. Isaacson, *Black Holes, Gravitational Radiation and the Universe*, edited by B. Iyer and B. Bhawal (Academic, New York, 1998).
[49] N. T. Bishop, R. Gómez, P. R. Holvorcem, R. A. Matzner, P. Papadopoulos, and J. Winicour, J. Comput. Phys. **136**, 140 (1997).
[50] B. Szilágyi and J. Winicour, Phys. Rev. D **68**, 041501 (2003).
[51] H. O. Kreiss and J. Winicour, Classical Quantum Gravity **23**, S405 (2006).
[52] H-O. Kreiss, O. Reula, O. Sarbach, and J. Winicour, Commun. Math. Phys. **289**, 1099 (2009).