Quantum field theory on timelike hypersurfaces in Rindler space

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The general boundary formulation of quantum field theory is applied to a massive scalar field in two-dimensional Rindler space. The field is quantized according to both the Schrödinger-Feynman quantization prescription and the holomorphic one in two different spacetime regions: a region bounded by two Cauchy surfaces and a region bounded by one timelike curve. An isomorphism is constructed between the Hilbert spaces associated with these two boundaries. This isomorphism preserves the probabilities that can be extracted from the free and the interacting quantum field theories, proving the equivalence of the *S*-matrices defined in the two settings, when both apply.

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I. INTRODUCTION

The general boundary formulation (GBF) provides a new axiomatic approach to describe the dynamics of quantum fields [1–17]. The set of axioms, inspired by topological quantum field theory [18,19], assigns algebraic structures to geometrical ones and ensures the consistency of these assignments. In particular, amplitude maps are associated with general spacetime regions, and state spaces are associated with their corresponding boundaries. A generalization of Born's rule [20] guaranties a consistent physical interpretation of such structures.

The main motivation for the development of the GBF has been represented by conceptual difficulties inherent in the attempt to formulate a quantum theory of gravity [19,21] like the so-called problem of time [22], the problem of providing a fully local description of the quantum dynamics in a quantum gravitational context, and the measurement problem. From this perspective, a remarkable aspect of the GBF is the following: no background metric is required for the implementation of the GBF.

On one hand, it is very useful to consider quantum field theories of matter fields on fixed Lorentzian spacetimes to test the GBF and to gain insight into its structure. On the other hand, in the standard formulation of these field theories, only regions with spacelike initial and final data hypersurfaces are usually considered. Within the GBF, a much wider class of setups can be implemented. Indeed, the GBF offers the possibility to construct quantum field theories in general spacetime regions, in particular, compact spacetime regions with just one connected boundary with spacelike and timelike parts. This means that the GBF enables us to have a completely new perspective on the well-established quantum theory of matter fields.

In recent years, the GBF was applied to many different physical setups [3,10–16,23], which led to many interesting results, like the crossing symmetry of the *S*-matrix of perturbative quantum field theory (QFT) which is a derived property within the GBF [15,16] or the rigorous construction in anti–de Sitter space [10] of an asymptotic amplitude that can be interpreted as an *S*-matrix for *spatial asymptotic states*.

In this article, we apply the GBF to study the quantum theory of a massive scalar field in two-dimensional Rindler space in two different spacetime regions: a region bounded by two Cauchy surfaces given by hyperplanes of constant Rindler time and a region bounded by one timelike hypersurface of a constant Rindler spatial coordinate. The first region is usually considered in the standard formulation of QFT and represents an important test for the ability of the GBF to reproduce known results. In contrast, the timelike boundary of the second region makes the applicability of the standard techniques of quantization difficult and represents a significant departure from the traditional description of dynamics in QFT. We will show that the GBF can deal with this second setting with no difficulty, and, moreover, we shall prove that a one-to-one relation can be established between the state spaces in the two settings. This result extends previous results obtained in Minkowski space [15,16][1] and de Sitter spaces [24,25].

The article is structured as follows: In Sec. II, we introduce the GBF and its main structures. In Sec. III, the two spacetime regions of interest here are introduced, and the solutions of the

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[1]The one-to-one correspondence established for the standard spacelike bounded regions in Minkowski space and a particular family of regions with timelike boundaries was used, in particular, to show explicitly that the crossing symmetry of QFT is generic in the GBF.

classical equations of motion are specified. In Sec. IV, we present the quantization of the scalar field in both regions, and in Sec. V, we establish an isomorphism between the two quantum theories and show that it preserves amplitudes and probabilities in the free quantum field theory. In Sec. VI C, we show that this is also true for the interacting theory. Our conclusions and outlooks are summarized in Sec. VII.

II. THE GENERAL BOUNDARY FORMULATION OF QUANTUM FIELD THEORY

In this section, we give a short review on the Schrödinger-Feynman representation [1] and the holomorphic representation [4] in which the GBF axioms presented in Ref. [1] have been so far implemented. We introduce the main structures that will be used in the rest of the paper such as state spaces and amplitude maps for both representations.

Let $S_M(\phi) = \int_M \mathrm{d}^N x \mathcal{L}(\phi, \partial\phi, x)$ be the action of a linear real scalar field theory in a spacetime region M of an N-dimensional Lorentzian manifold $(\mathcal{M}, g)$. Denoting the boundary[2] of the region M with Σ, we associate with this hypersurface the space L_Σ of solutions of the Euler-Lagrange equations defined in a neighborhood of Σ.[3] The symplectic potential on Σ results to be

$$(\theta_\Sigma)_\phi(X) := \int_\Sigma \mathrm{d}^{N-1}\sigma X(x(\sigma))\left(n^\mu \frac{\delta\mathcal{L}}{\delta\partial_\mu\phi}\right)(x(\sigma)), \qquad (1)$$

where n^μ is the unit normal vector to Σ. We define the bilinear map $[\cdot,\cdot]_\Sigma\colon L_\Sigma \times L_\Sigma \to \mathbb{R}$ as $[\xi,\eta]_\Sigma := (\theta_\Sigma)_\xi(\eta)$ for each $\xi, \eta \in L_\Sigma$. Moreover, the space L_Σ is equipped with the symplectic structure defined as the antisymmetric bilinear map $\omega_\Sigma\colon L_\Sigma \times L_\Sigma \to \mathbb{R}$ given by $\omega_\Sigma(\xi,\eta) := \frac{1}{2}[\xi,\eta]_\Sigma - \frac{1}{2}[\eta,\xi]_\Sigma$. The last ingredient for the quantum theory we need to specify is a compatible complex structure J_Σ represented by the linear map $J_\Sigma\colon L_\Sigma \to L_\Sigma$ such that $J_\Sigma^2 = -\mathrm{id}$ and $\omega_\Sigma(J_\Sigma\cdot, J_\Sigma\cdot) = \omega_\Sigma(\cdot,\cdot)$ and $\omega_\Sigma(\cdot, J_\Sigma\cdot)$ is a positive definite bilinear map. Note that all ingredients but the complex structure J_Σ are classical data uniquely defined by specifying the action.

These basic ingredients can now be used in different ways to specify the Hilbert spaces, which, according to the axioms of the GBF, are associated with the boundary hypersurface Σ.[4] In the following sections, we introduce the two representations developed so far within the GBF, namely, the Schrödinger representation, usually associated with the Feynman path integral quantization prescription, and the holomorphic representation.

A. The holomorphic representation

From the complex structure J_Σ, we define the symmetric bilinear form $g_\Sigma\colon L_\Sigma \times L_\Sigma \to \mathbb{R}$ as

$$g_\Sigma(\xi,\eta) := 2\omega_\Sigma(\xi, J_\Sigma\eta) \quad \forall\, \xi,\eta \in L_\Sigma \qquad (2)$$

and assume that this form is positive definite. Next, we introduce the sesquilinear form

$$\{\xi,\eta\}_\Sigma := g_\Sigma(\xi,\eta) + 2\mathrm{i}\omega_\Sigma(\xi,\eta) \quad \forall\, \xi,\eta \in L_\Sigma. \qquad (3)$$

The completion of L_Σ with the inner product $\{\cdot,\cdot\}_\Sigma$ turns it into a complex Hilbert space. The Hilbert space $\mathcal{H}^h_\Sigma = H^2(L_\Sigma, \mathrm{d}\nu_\Sigma)$,[5] namely, the set of square integrable holomorphic functions on L_Σ, is the closure of the set of all coherent states[6] [4],

$$K^h_{\Sigma,\xi}(\phi) := e^{\frac{1}{2}\{\xi,\phi\}_\Sigma}, \qquad (4)$$

where $\xi \in L_\Sigma$ and the closure is taken with respect to the inner product

$$\langle K^h_{\Sigma,\xi}, K^h_{\Sigma,\xi'}\rangle := \int_{L_\Sigma} \mathrm{d}\nu_\Sigma(\phi)\overline{K^h_{\Sigma,\xi}(\phi)}K^h_{\Sigma,\xi'}(\phi), \qquad (5)$$

where $\mathrm{d}\nu_\Sigma$ is a Gaussian probability measure constructed from the metric g_Σ [4]. It can be represented formally as $\mathrm{d}\nu_\Sigma(\phi) = \mathrm{d}\mu_\Sigma(\phi)e^{\frac{1}{4}g_\Sigma(\phi,\phi)}$ with a certain translation-invariant measure $\mathrm{d}\mu_\Sigma$.

Associated to each spacetime region M, there is an amplitude ϱ_M defined for states belonging to the Hilbert space associated with the boundary Σ of this region,

$$\varrho_M(\psi^h) := \int_{L_{\tilde{M}}} \mathrm{d}\nu_{\tilde{M}}(\phi)\psi^h(\phi), \qquad (6)$$

where $L_{\tilde{M}} \subseteq L_\Sigma$ is the set of all global solutions on M mapped to L_Σ by just considering the solutions in a neighborhood of Σ.[7] The measure $\mathrm{d}\nu_{\tilde{M}}$ is again a Gaussian probability measure constructed from the metric g_Σ [4].[8] This amplitude for coherent states turns out to be[9]

[2]Notice that whether the boundary hypersurface Σ is a Cauchy surface (or a disjoint union of Cauchy surfaces) has no bearing on the following treatment.

[3]More precisely, L_Σ is the space of germs of solutions at Σ, which is the set of all equivalence classes of solutions where two solutions are equivalent if there exists a neighborhood of Σ such that the two solutions coincide in this whole neighborhood.

[4]If the boundary of the region considered is given by the disjoint union of two hypersurfaces, say $\Sigma = \Sigma_1 \cup \Sigma_2$, the associated Hilbert space is a tensor product of the Hilbert spaces defined on each hypersurface, $\mathcal{H}_\Sigma = \mathcal{H}_{\Sigma_1} \otimes \mathcal{H}^*_{\Sigma_2}$, where the different orientation of the hypersurface Σ_2 with respect to Σ_1 is responsible for the dualization of the corresponding Hilbert space.

[5]To make this mathematically precise, one actually has to construct $\mathcal{H}^h_\Sigma = H^2(\hat{L}_\Sigma, \mathrm{d}\nu_\Sigma)$, where $\hat{L}_\Sigma$ is a certain extension of L_Σ. For more details about the construction of $\hat{L}_\Sigma$ and $\mathrm{d}\nu_\Sigma$, we refer the reader to Ref. [4].

[6]States in the holomorphic representation are denoted with a superscript h.

[7]More precisely, global solutions are mapped to the corresponding germs at Σ.

[8]Again, we refer the reader to Ref. [4], where the constructions are given that make all the objects used here well defined. Additionally, in Ref. [5], it was shown that the one-to-one correspondence between maps Ω_Σ (which is an important ingredient of the Schrödinger-Feynman representation and will be defined in the next section) and complex structures J_Σ leads also to mathematically well-defined constructions for all the expressions in Sec. II B.

[9]See Eq. (31) of Ref. [6] for normalized coherent states and Eq. (43) in Ref. [4] as well as Ref. [5].

$$\varrho_M(K^h_\xi) = \exp\left(\frac{1}{4}g_\Sigma(\xi^R,\xi^R) - \frac{1}{4}g_\Sigma(\xi^I,\xi^I) - \frac{\mathrm{i}}{2}g_\Sigma(\xi^R,\xi^I)\right), \tag{7}$$

where ξ^R, $\xi^I \in L_{\tilde{M}}$, and $\xi = \xi^R + J_\Sigma \xi^I$. A consistent probability interpretation can be given to this amplitude using the generalized Born rule [1,20], defined in the GBF.

B. The Schödinger-Feynman representation

In this section, we introduce the Schrödinger-Feynman representation of the GBF. However, we will not start from the symplectic form and complex structure we established in the beginning but directly from the action $S_M(\phi)$. This is the way the Schrödinger-Feynman representation was established originally. The construction of the Schrödinger-Feynman representation from the symplectic form and complex structure will be the content of the next section, which will illuminate the relation between the two representations.

In the Schrödinger-Feynman representation, quantum states in the Hilbert space associated with the boundary Σ are represented as wave functionals of the space of field configurations.[10] The amplitude associated with the region M is given by the linear map ϱ_M: $\mathcal{H}_\Sigma \to \mathbb{C}$,

$$\varrho_M(\psi^S) = \int \mathcal{D}\varphi\, \psi^S(\varphi) Z_M(\varphi), \tag{8}$$

where the integral is extended over all the configurations φ on the boundary of the region M, and $Z_M(\varphi)$ is the field propagator, formally defined as

$$Z_M(\varphi) = \int_{\phi|_\Sigma=\varphi} \mathcal{D}\phi\, e^{\mathrm{i}S_M(\phi)}, \tag{9}$$

where $S_M(\phi)$ is the action of the field in M and the integral is extended to the spacetime field configurations ϕ that reduce to the configuration φ on the boundary hypersurface Σ.

As in the holomorphic representation, coherent states can be defined in the Schrödinger representation, too. They are given as

$$K^S_{\Sigma,\xi}(\varphi) = \kappa_{\Sigma,\xi} \exp\left(\int \mathrm{d}^3 s\, \xi(s)\varphi(s) - \frac{1}{2}\Omega_\Sigma(\varphi,\varphi)\right), \tag{10}$$

where $\kappa_{\Sigma,\xi}$ is a normalization constant and Ω_Σ is a bilinear map from two copies of the space of field configurations on the boundary hypersurface Σ to the complex numbers. The vacuum state is obtained from Eq. (10) by setting $\xi = 0$.

With the coherent states above, we can again define the Hilbert space associated with the boundary Σ as the closure of the space of coherent states with respect to an inner product. In the Schrödinger representation, this is the expression

$$\langle \psi_\Sigma | \psi'_\Sigma \rangle := \int \mathcal{D}\varphi\, \overline{\psi^S_\Sigma(\varphi)}\, \psi^{S\prime}_\Sigma(\varphi). \tag{11}$$

[10] We denote states in the Schrödinger-Feynman representation with a superscript S.

C. Relation between the two representations

In this section, we show how to develop the Schrödinger-Feynman representation starting from the symplectic form and the complex structure. We also clarify the relation between the two representations.

We start by defining what plays the role of the "space of momentum" in the Schrödinger-Feynman representation:

$$M_\Sigma := \{\eta \in L_\Sigma : [\xi, \eta] = 0\ \forall \xi \in L_\Sigma\}. \tag{12}$$

It can be shown that M_Σ is a Lagrangian subspace of L_Σ.[11] Next, we consider the quotient space $Q_\Sigma := L_\Sigma / M_\Sigma$, which corresponds to the space of all field configurations on Σ. We denote the quotient map $L_\Sigma \to Q_\Sigma$ by q_Σ. The last ingredient needed for the Schrödinger representation is the bilinear map defining the vacuum state,

$$\begin{aligned} &\Omega_\Sigma : Q_\Sigma \times Q_\Sigma \to \mathbb{C}, \\ &(\varphi, \varphi') \mapsto 2\omega_\Sigma(j_\Sigma(\varphi), J_\Sigma j_\Sigma(\varphi')) - \mathrm{i}[j_\Sigma(\varphi), \varphi']_\Sigma, \end{aligned} \tag{13}$$

where j_Σ is the unique linear map $Q_\Sigma \to L_\Sigma$ such that $q_\Sigma \circ j_\Sigma = \mathrm{id}_{Q_\Sigma}$ and $j_\Sigma(Q_\Sigma) \subseteq J_\Sigma M$. Coherent states are given in terms of Ω_Σ by the expressions

$$K^S_{\Sigma,\xi}(\varphi) = \exp\Big(\Omega_\Sigma(q_\Sigma(\xi), \varphi) + \mathrm{i}[\xi, \varphi]_\Sigma - \frac{1}{2}\Omega_\Sigma(q_\Sigma(\xi), q_\Sigma(\xi)) - \frac{\mathrm{i}}{2}[\xi, \xi]_\Sigma - \frac{1}{2}\Omega_\Sigma(\varphi, \varphi)\Big). \tag{14}$$

It was shown in Ref. [5] that there is a one-to-one correspondence between bilinear maps Ω_Σ appropriate for the Schrödinger representation and complex structures J_Σ. This means that given a complex structure, we uniquely fix all the algebraic structures of the two representations.[12] In particular, an isomorphism exists between the Hilbert spaces in the holomorphic representation and the Schrödinger-Feynman representation that preserves the amplitude map. Hence, the two representations can be used equivalently.

III. CLASSICAL THEORY

Rindler space $\mathcal{R}$ is given by the metric $ds^2 = \rho^2 d\eta^2 - d\rho^2$, where $\rho \in \mathbb{R}^+$ and $\eta \in \mathbb{R}$. The free action of the Klein-Gordon field in a spacetime region M is

$$S_{M,0}(\phi) = \frac{1}{2}\int_M \mathrm{d}\eta \mathrm{d}\rho\, \rho\left(-(\partial_\rho\phi)^2 + \frac{1}{\rho^2}(\partial_\eta\phi)^2 - m^2\phi^2\right), \tag{15}$$

where m is the mass of the field and ∂_ρ and ∂_η denote the partial derivatives with respect to ρ and η, respectively. From the action, we can deduce the equation of motion:

[11] It is this subspace M_Σ that defines the Schrödinger polarization of the prequantum Hilbert space constructed from L_Σ; see Ref. [5] for details.

[12] This one-to-one correspondence sends Eq. (14) into Eq. (10).

$$(-\rho\partial_\rho\rho\partial_\rho + \partial_\eta^2 + m^2\rho^2)\phi = 0. \qquad (16)$$

Solutions of the field equation (16) can be expressed in terms of the modes

$$\chi_p(x) = \frac{\mathrm{i}}{2}(\sinh(p\pi))^{-1/2} I_{\mathrm{i}p}(m\rho)e^{-\mathrm{i}p\eta},$$
$$\phi_p(x) = \frac{(\sinh(p\pi))^{1/2}}{\pi} K_{\mathrm{i}p}(m\rho)e^{-\mathrm{i}p\eta}, \qquad p \geq 0, \qquad (17)$$

where $I_{\mathrm{i}p}$ and $K_{\mathrm{i}p}$ are the modified Bessel functions of the first and second kind, respectively; see the Appendix.

In the following, we will study the field in two different spacetime regions. The first is a region M_1 bounded by two semilines of constant Rindler time η_1 and η_2, respectively, with $\eta_1 < \eta_2$; namely, $M_1 = [\eta_1, \eta_2] \times \mathbb{R}^+$ and all the relevant quantities referring to this region will be indicated with the subscript $[\eta_1, \eta_2]$. Additionally, we will consider the region M_2 bounded by one hyperbola of constant $\rho = \rho_1$, namely, $M_2 = \mathbb{R} \times [\rho_1, \infty)$. Because of the asymptotic behavior (A4), in both regions, the field will be expanded in the basis of the modes $\phi_p(x)$.

A. Region with spacelike boundary: M_1

Consider the region bounded by the two semilines of constant η, namely, the region M_1. We denote by φ_1 and φ_2 the configurations of the field on the boundaries Σ_1 at $\eta = \eta_1$ and Σ_2 at $\eta = \eta_2$, respectively: $\phi|_{\Sigma_1} = \varphi_1$ and $\phi|_{\Sigma_2} = \varphi_2$. It will be useful to express the solution of the Klein-Gordon equation (16) in terms of these boundary field configurations. In particular, the general solution to Eq. (16) can be written as

$$\phi(\eta, \rho) = (X_a(\eta)Y_a)(\rho) + (X_b(\eta)Y_b)(\rho), \qquad (18)$$

where each $X_i(\eta)$ is understood as an operator acting on a mode decomposition of Y_i. In particular, we can choose $X_a(\eta) = \cos(p\eta)$ and $X_b(\eta) = \sin(p\eta)$. Expressing each Y_i in terms of the boundary field configurations φ_i leads to

$$\phi(\eta,\rho) = \left(\frac{\sin p(\eta_2-\eta)}{\sin p(\eta_2-\eta_1)}\varphi_1\right)(\rho) + \left(\frac{\sin p(\eta-\eta_1)}{\sin p(\eta_2-\eta_1)}\varphi_2\right)(\rho), \qquad (19)$$

where p is to be understood as the operator $p := \sqrt{(\rho\partial_\rho)^2 - m^2}$ acting on a mode decomposition of the boundary field configurations. As mentioned above, the divergent character of $I_{\mathrm{i}p}$ at infinity forces us to retain in this mode expansion only the modified Bessel function of the second kind, $K_{\mathrm{i}p}$, also known as Macdonald function; see the Appendix. The free action (15) in terms of the boundary field configurations reads

$$S_{[\eta_1,\eta_2],0}(\varphi_1,\varphi_2) = \frac{1}{2}\int_0^\infty \frac{\mathrm{d}\rho}{\rho}(\varphi_1 \ \ \varphi_2)W_{[\eta_1,\eta_2]}\begin{pmatrix}\varphi_1\\ \varphi_2\end{pmatrix}, \qquad (20)$$

where the $W_{[\eta_1,\eta_2]}$ is a 2×2 matrix given by

$$W_{[\eta_1,\eta_2]} = \frac{p}{\sin p(\eta_2-\eta_1)}\begin{pmatrix}\cos p(\eta_2-\eta_1) & -1\\ -1 & \cos p(\eta_2-\eta_1)\end{pmatrix}. \qquad (21)$$

B. Region with timelike boundary: M_2

In contrast to the spacetime region considered before, the region M_2 presents only one boundary Σ_{ρ_1} defined by the hyperbola $\rho = \rho_1$, i.e., $M_2 = \mathbb{R} \times [\rho_1, \infty)$. The subscript ρ_1 will be used for the quantities referring to this region. The field configurations will then contain only the modified Bessel function of the first kind, and a solution of the Klein-Gordon equation in this region, reducing to the boundary configuration φ at ρ_1, can be written as

$$\phi(\eta,\rho) = \left(\frac{K_{\mathrm{i}p}(m\rho)}{K_{\mathrm{i}p}(m\rho_1)}\varphi\right)(\eta), \qquad (22)$$

where $\frac{K_{\mathrm{i}p}(m\rho)}{K_{\mathrm{i}p}(m\rho_1)}$ has to be understood as an operator acting on the field configuration $\varphi(\eta)$ as

$$\frac{K_{\mathrm{i}p}(m\rho)}{K_{\mathrm{i}p}(m\rho_1)}e^{ip'\eta} = \frac{K_{\mathrm{i}p'}(m\rho)}{K_{\mathrm{i}p'}(m\rho_1)}e^{ip'\eta}. \qquad (23)$$

The action of the field (22) in the region M_2 is expressed in terms of φ as

$$S_{\rho_1,0}(\varphi) = \frac{1}{2}\int_{-\infty}^{\infty}\mathrm{d}\eta\varphi(\eta)\rho\left(\frac{\mathrm{d}}{\mathrm{d}\rho}\frac{K_{\mathrm{i}p}(m\rho)}{K_{\mathrm{i}p}(m\rho_1)}\varphi\right)(\eta)|_{\rho=\rho_1}. \qquad (24)$$

IV. QUANTUM THEORY

In this section, the quantum theory of the free field in the different regions considered above will be presented. In Ref. [11], a general treatment of the GBF description of the quantum dynamics of a scalar field in a certain class of spacetimes and spacetime regions has been presented. The scalar field in the two spacetime regions in Rindler spacetime considered here satisfies the conditions of Ref. [11], and the results obtained there can then be used in the present work.

A. Quantization in M_1

1. Holomorphic representation

To constitute valid initial data on the hypersurfaces Σ_1 and Σ_2, the field ϕ must vanish at spacelike infinity, which excludes the modes containing the Bessel functions of the first kind and leaves us with the decomposition

$$\phi(x) = \int_0^\infty \mathrm{d}p(\phi(p)\phi_p(x) + \mathrm{c.c.}). \qquad (25)$$

From the second variation of the action in Eq. (15), we obtain the symplectic form as

$$\omega_{\Sigma_i}(\phi,\phi') = \frac{1}{2}\int_0^\infty \frac{\mathrm{d}\rho}{\rho}(\phi\partial_\eta\phi' - \phi'\partial_\eta\phi)(\rho). \qquad (26)$$

Now, we obtain for two modes ϕ_p and $\phi_{p'}$ at Σ_i with $i = 1, 2$ the following expressions:

$$\omega_{\Sigma_i}(\overline{\phi_p}, \phi_{p'}) = \delta(p - p'),$$
$$\omega_{\Sigma_i}(\phi_p, \phi_{p'}) = \omega_{\Sigma_i}(\overline{\phi_p}, \overline{\phi_{p'}}) = 0. \qquad (27)$$

With the complex structure

$$J_{\Sigma_i} = \frac{\partial_\eta}{\sqrt{-\partial_\eta^2}}, \qquad (28)$$

which corresponds to the timelike Killing vector field ∂_η, we obtain, for two general solutions ϕ and ψ,

$$\omega_{\Sigma_i}(\phi, \psi) = \frac{i}{2}\int_0^\infty \mathrm{d}p(\overline{\phi(p)}\psi(p) - \phi(p)\overline{\psi(p)}), \qquad (29)$$

$$g_{\Sigma_i}(\phi, \psi) = \int_0^\infty \mathrm{d}p(\overline{\phi(p)}\psi(p) + \phi(p)\overline{\psi(p)}), \qquad (30)$$

$$\{\phi, \psi\}_{\Sigma_i} = g_{\Sigma_i}(\phi, \psi) + 2i\omega_{\Sigma_i}(\phi, \psi) = 2\int_0^\infty \mathrm{d}p\phi(p)\overline{\psi(p)}. \qquad (31)$$

These are all the algebraic objects necessary for the holomorphic quantization of the Klein-Gordon field in the region M_1.

2. Schrödinger-Feynman representation

Substituting in Eq. (9) the free action (20) of the classical solution (19) in the spacetime region M_1, we can express the field propagator in terms of the boundary field configurations φ_1 and φ_2,

$$Z_{[\eta_1,\eta_2],0}(\varphi_1, \varphi_2) = \left(\det\frac{-\mathrm{i}p}{2\pi\sin p(\eta_2 - \eta_1)}\right)^{-1/2} e^{\mathrm{i}S_{[\eta_1,\eta_2],0}(\varphi_1,\varphi_2)}, \qquad (32)$$

where again $p = \sqrt{(\rho\partial_\rho)^2 - m^2}$ has to be understood as an operator. This field propagator satisfies the composition property

$$Z_{[\eta_1,\eta_3],0}(\varphi_1, \varphi_3) = \int \mathcal{D}\varphi_2 Z_{[\eta_1,\eta_2],0}(\varphi_1, \varphi_2)Z_{[\eta_2,\eta_3],0}(\varphi_2, \varphi_3). \qquad (33)$$

Following Refs. [11,16,25], we define by Eq. (10) the coherent states in the Hilbert space $\mathcal{H}_\eta$ associated to the semiline of constant Rindler time η. These states have the property to remain coherent under the evolution implemented by the field propagator (32). In the interaction picture, they take the form

$$K^S_{\eta,\xi}(\varphi) = \exp\left(-\frac{1}{2}\int_0^\infty \mathrm{d}p\frac{1}{2p}(e^{-2\mathrm{i}p\eta}\xi^2(p) + |\xi(p)|^2)\right) \times \exp\left(\int_0^\infty \mathrm{d}pe^{-\mathrm{i}p\eta}\xi(p)\varphi(p)\right)\psi_{\eta,0}(\varphi), \qquad (34)$$

where $\psi_{\eta,0}$ is the vacuum state[13] in $\mathcal{H}_\eta$,

$$\psi_{\eta,0}(\varphi) = \det\left(\frac{p}{\pi e^{\mathrm{i}p\eta}}\right)^{1/4}\exp\left(-\frac{1}{2}\int \mathrm{d}p'\varphi(p')p'\varphi(p')\right). \qquad (35)$$

We have now at our disposal all the ingredients to compute explicitly the free amplitude for a coherent state in the spacetime region M_1. In particular, we consider the coherent state defined by two complex functions ξ_1 and ξ_2 as $K^S_{\eta_1,\xi_1}\otimes\overline{K^S_{\eta_2,\xi_2}}$ in the Hilbert space $\mathcal{H}_{\eta_1}\otimes\mathcal{H}^*_{\eta_2}$ associated with the boundary of M_1. The free amplitude results to be

$$\varrho_{[\eta_1,\eta_2]}(K^S_{\eta_1,\xi_1}\otimes\overline{K^S_{\eta_2,\xi_2}}) = \int \mathcal{D}\varphi_1\mathcal{D}\varphi_2\overline{K^S_{\eta_2,\xi_2}(\varphi_2)}K^S_{\eta_1,\xi_1}(\varphi_1)Z_{[\eta_1,\eta_2],0}(\varphi_1, \varphi_2) = \exp\left(-\frac{1}{2}\int_0^\infty\frac{\mathrm{d}p}{2p}(|\xi_1(p)|^2 + |\xi_2(p)|^2 - 2\overline{\xi_2(p)}\xi_1(p))\right), \qquad (36)$$

where we used again the expansion of the functions $\xi_{1,2}(\rho)$ in the basis of the modes $u_p(\rho)$. Notice that this amplitude does not depend on the Rindler times η_1 and η_2.

B. Quantization in M_2

In this section, we will give all elements of the two representations of the GBF in the region M_2.

1. Holomorphic quantization

For the holomorphic representation, we start with the symplectic form:

$$\omega_{\Sigma_{\rho_1}}(\phi, \phi') = \frac{1}{2}\int_{-\infty}^\infty \mathrm{d}\eta\rho(\phi\partial_\rho\phi' - \phi'\partial_\rho\phi)(\eta). \qquad (37)$$

The solutions to the Klein-Gordon equation in Rindler space at Σ_{ρ_1} only exist locally around Σ_{ρ_1} and thus are not expected to vanish at spacelike infinity. They can be parametrized using the modified Bessel functions of the first kind as[14]

$$\phi_{\Sigma_{\rho_1}}(x) = \int_{-\infty}^\infty \mathrm{d}p(\phi_{\Sigma_{\rho_1}}(p)\chi_p(x) + \text{c.c.}), \qquad (38)$$

[13]In the notation of Ref. [11], the coefficients c_a and c_b have be chosen to be 1 and i, respectively.

[14]The only solutions we have to consider at the boundary are the Bessel functions of the first kind since the Bessel functions of the second kind are not independent solutions [see Eq. (A1)].

In contrast, solutions in the interior of M_2 must vanish for $\rho \to \infty$ and thus can be parameterized using just the modified Bessel functions of the second kind as

$$\phi_{M_2}(x) = \int_0^\infty \mathrm{d}p(\phi_{M_2}(p)\phi_p(x) + \text{c.c.}). \quad (39)$$

For the parameterization in Eq. (38) and with the Wronskian of two Bessel functions of the first kind [see Eq. (A2)], we find for the symplectic form the expression

$$\omega_{\Sigma_{\rho_1}}(\phi_{\Sigma_{\rho_1}}, \phi'_{\Sigma_{\rho_1}}) = -\frac{\mathrm{i}}{2}\int_{-\infty}^\infty \mathrm{d}p(\phi_{\Sigma_{\rho_1}}(p)\overline{\phi'_{\Sigma_{\rho_1}}(p)} - \text{c.c.}). \quad (40)$$

To obtain the metric and the inner product on the space of solutions $L_{\Sigma_{\rho_1}}$ at Σ_{ρ_1} we define the action of the complex structure $J_{\Sigma_{\rho_1}}$ as $J_{\Sigma_{\rho_1}}\chi_p(x) = -\mathrm{i}\chi_p(x)$. Hence, we obtain

$$J_{\Sigma_{\rho_1}}\phi_{\Sigma_{\rho_1}}(x) = -\mathrm{i}\int_{-\infty}^\infty \mathrm{d}p(\phi_{\Sigma_{\rho_1}}(p)\chi_p(x) - \text{c.c.}) \quad (41)$$

and the metric and the inner product result,

$$g_{\Sigma_{\rho_1}}(\phi_{\Sigma_{\rho_1}}, \phi'_{\Sigma_{\rho_1}}) = 2\omega_{\Sigma_M}(\phi_{\Sigma_{\rho_1}}, J_{\Sigma_{\rho_1}}\phi'_{\Sigma_{\rho_1}}) = \int_{-\infty}^\infty \mathrm{d}p(\phi_{\Sigma_{\rho_1}}(p)\overline{\phi'_{\Sigma_{\rho_1}}(p)} + \text{c.c.}), \quad (42)$$

$$\{\phi_{\Sigma_{\rho_1}}, \phi'_{\Sigma_{\rho_1}}\}_{\Sigma_{\rho_1}} = g_{\Sigma_{\rho_1}}(\phi_{\Sigma_{\rho_1}}, \phi'_{\Sigma_{\rho_1}}) + 2\mathrm{i}\omega_{\Sigma_M}(\phi_{\Sigma_{\rho_1}}, \phi'_{\Sigma_{\rho_1}}) = 2\int_{-\infty}^\infty \mathrm{d}p\,\phi_{\Sigma_{\rho_1}}(p)\overline{\phi'_{\Sigma_{\rho_1}}(p)}. \quad (43)$$

By defining coherent states and their amplitudes, we obtain the free quantum theory for the Klein-Gordon field in the region M_2. In the next section, we will establish the identification between states on the boundary of M_2 and M_1.

2. Schrödinger-Feynman quantization

The field propagator is expressed in terms of the action (24) as

$$Z_{\rho_1,0}(\varphi) = \det\left(\frac{4\pi^2 K^2_{\mathrm{i}|p|}(m\rho_1)}{m\sinh(|p|\pi)}\right)^{-1/4} e^{\mathrm{i}S_{\rho_1,0}(\varphi)}, \quad (44)$$

where the expression in the determinant is to understood as an operator acting as

$$\frac{4\pi^2 K^2_{\mathrm{i}|p|}(m\rho_1)}{m\sinh(|p|\pi)} e^{ip'\eta} = \frac{4\pi^2 K^2_{\mathrm{i}|p'|}(m\rho_1)}{m\sinh(|p'|\pi)} e^{ip'\eta} \quad (45)$$

on the Fourier expansion of field configurations. We will consider the vacuum state

$$\psi_{\rho_1,0}(\varphi) = C_{\rho_1}\exp\left(-\frac{1}{2}\int \mathrm{d}\eta\,\varphi(\eta) \times \left(\mathrm{i}\rho\frac{\mathrm{d}}{\mathrm{d}\rho}\ln(\overline{I_{\mathrm{i}|p|}(m\rho)})\varphi\right)(\eta)|_{\rho=\rho_1}\right), \quad (46)$$

giving rise to the Hilbert space $\mathcal{H}_{\Sigma_{\rho_1}}$. C_{ρ_1} in the above equation is the normalization factor of the vacuum state. A coherent state in $\mathcal{H}_{\Sigma_{\rho_1}}$, in the interaction picture, reads

$$K^S_{\rho_1,\xi}(\varphi) = \kappa_{\rho_1,\xi}\exp\left(\int_0^\infty \frac{\mathrm{d}p}{\overline{I_{\mathrm{i}|p|}(m\rho_1)}}[\xi(p)\varphi(-p) + \xi(-p)\varphi(p)]\right)\psi_{\rho_1,0}(\varphi), \quad (47)$$

where $\xi(p)$ and $\varphi(p)$ are the coefficients of the expansion of $\xi(\eta)$ and $\varphi(\eta)$, respectively, in the basis of the plane waves $e^{ip\eta}/\sqrt{2\pi}$. $\kappa_{\rho_1,\xi}$ is the normalization factor given by

$$\kappa_{\rho_1,\xi} = \exp\left(-\int_0^\infty \mathrm{d}p\frac{\pi}{4\sinh(p\pi)}\left(\frac{I_{\mathrm{i}p}(m\rho_1)}{\overline{I_{\mathrm{i}|p|}(m\rho_1)}}2\xi(p)\xi(-p) + |\xi(p)|^2 + |\xi(-p)|^2\right)\right). \quad (48)$$

The free amplitude for a coherent state results to be

$$\varrho_{\rho_1}(K^S_{\rho_1,\xi}) = \int \mathcal{D}\varphi\,\psi_{\rho_1,\xi}Z_{\rho_1,0}(\varphi) = \exp\left(-\frac{1}{2}\int_0^\infty \mathrm{d}p\frac{\pi}{2\sinh(p\pi)}(|\xi(p)|^2 + |\xi(-p)|^2 + 2\xi(p)\xi(-p))\right), \quad (49)$$

which is independent of ρ_1, as it should be.

V. IDENTIFICATION OF STATES

In the last section, we derived all the objects necessary for the GBF on M_2. We will now establish an isomorphism between the states on the boundary Σ_{ρ_1} and ∂M_1 using the coherent states. Since the coherent states form a dense subset in the respective Hilbert spaces, it suffices if we can identify them.

A. Holomorphic representation

We have for the amplitude for a generic region M and a coherent state K^h_τ the following expression:

$$\varrho_M(K^h_\tau) = \exp\left(\frac{1}{4}g_{\partial M}(\hat{\tau}, \hat{\tau})\right), \quad (50)$$

with $\hat{\tau} = \tau^R - i\tau^I$ and $\tau^R, \tau^I \in L_{\tilde{M}}$ such that $\tau = \tau^R + J_{\partial M}\tau^I$. The reader can easily verify that Eq. (50) coincides with Eq. (7).

For region M_1, we obtain for solutions ϕ, ϕ' in $L_{\tilde{M}_1}$ that

$$g_{\partial M_1}(\phi, \phi') = 2\int_0^\infty \mathrm{d}p(\phi(p)\overline{\phi'(p)} + \text{c.c.}). \quad (51)$$

For the solution $\phi_{\Sigma_{\rho_1}}$ in $L_{\tilde{M}_2} \subset L_{\Sigma_{\rho_1}}$, we obtain by projecting the solution ϕ to a neighborhood of Σ_{ρ_1} with the decomposition (38) and using relation (A1) the identities

$$\phi_{\Sigma_{\rho_1}}(p) = \phi(p) \quad \text{and} \quad \phi_{\Sigma_{\rho_1}}(-p) = \overline{\phi(p)}.$$

For the metric, we find then

$$g_{\Sigma_{\rho_1}}(\phi_{\Sigma_{\rho_1}}, \phi'_{\Sigma_{\rho_1}}) = \int_{-\infty}^{\infty} \mathrm{d}p(\phi_{\Sigma_{\rho_1}}(p)\overline{\phi'_{\Sigma_{\rho_1}}(p)} + \text{c.c.}) \tag{52}$$

$$= 2\int_0^{\infty} \mathrm{d}p(\phi(p)\overline{\phi'(p)} + \text{c.c.}). \tag{53}$$

Hence, identifying the expression in Eq. (50) for the amplitude in M_2 and M_1 is equivalent to the identification

$$\hat{\tau}_{\Sigma_{\rho_1}} = \hat{\tau}_{\partial M_1} \tag{54}$$

for the two different regions. For region M_2, let us define $\tau^R_{\tilde{M}_2}(p)$ and $\tau^I_{\tilde{M}_2}(p)$ such that

$$\begin{aligned} \tau^R_{\Sigma_{\rho_1}} &= \int_0^{\infty} \mathrm{d}p(\tau^R_{\tilde{M}_2}(p)\phi_p(x) + \text{c.c.}), \\ \tau^I_{\Sigma_{\rho_1}} &= \int_0^{\infty} \mathrm{d}p(\tau^I_{\tilde{M}_2}(p)\phi_p(x) + \text{c.c.}). \end{aligned} \tag{55}$$

Then, we obtain with the action of the complex structure corresponding to Σ_{ρ_1} the identity

$$\begin{aligned} \tau_{\Sigma_{\rho_1}} &= \tau^R_{\Sigma_{\rho_1}} + J_{\Sigma_{\rho_1}}\tau^I_{\Sigma_{\rho_1}} \\ &= \int_0^{\infty} \mathrm{d}p[(\tau^R_{\tilde{M}_2}(p) - \mathrm{i}\tau^I_{\tilde{M}_2}(p))\chi_p(x) + (\tau^R_{\tilde{M}_2}(p) \\ &\quad + \mathrm{i}\tau^I_{\tilde{M}_2}(p))\overline{\chi_{-p}(x)} + \text{c.c.}]. \end{aligned} \tag{56}$$

By comparing this with Eq. (38) (replacing $\phi_{\Sigma_{\rho_1}}$ by τ), we obtain

$$\begin{aligned} \tau_{\Sigma_{\rho_1}}(p) &= \tau^R_{\tilde{M}_2}(p) - \mathrm{i}\tau^I_{\tilde{M}_2}(p), \\ \overline{\tau_{\Sigma_{\rho_1}}(-p)} &= \tau^R_{\tilde{M}_2}(p) + \mathrm{i}\tau^I_{\tilde{M}_2}(p), \end{aligned} \tag{57}$$

for $p > 0$, which can be inverted as

$$\begin{aligned} \tau^R_{\tilde{M}_2}(p) &= \frac{1}{2}(\tau_{\Sigma_{\rho_1}}(p) + \overline{\tau_{\Sigma_{\rho_1}}(-p)}), \\ \tau^I_{\tilde{M}_2}(p) &= \frac{\mathrm{i}}{2}(\tau_{\Sigma_{\rho_1}}(p) - \overline{\tau_{\Sigma_{\rho_1}}(-p)}). \end{aligned} \tag{58}$$

Then, we find the expression

$$\hat{\tau}_{\Sigma_{\rho_1}} = \int_0^{\infty} \mathrm{d}p[\tau_{\Sigma_{\rho_1}}(p)\phi_p(x) + \tau_{\Sigma_{\rho_1}}(-p)\overline{\phi_p(x)}]. \tag{59}$$

For region M_1, we have for a solution $(\tau_1, \tau_2) \in L_{\Sigma_1} \oplus L_{\overline{\Sigma_2}} = L_{\partial M_1}$ that $\tau^R = 1/2(\tau_1 + \tau_2, \tau_1 + \tau_2)$ and $J_{\partial M_1}\tau^I = 1/2(\tau_1 - \tau_2, \tau_2 - \tau_1)$, and hence $\hat{\tau} = 1/2(1 + \mathrm{i}J_{\Sigma_1})\tau_1 + 1/2(1 - \mathrm{i}J_{\Sigma_1})\tau_2$, and we obtain

$$\hat{\tau}_{\partial M_1}(x) = \int_0^{\infty} \mathrm{d}p(\phi_p(x)\tau_1(p) + \overline{\phi_p(x)\tau_2(p)}), \tag{60}$$

which leads to the identification

$$\tau_1(p) = \tau_{\Sigma_{\rho_1}}(p), \qquad \overline{\tau_2(p)} = \tau_{\Sigma_{\rho_1}}(-p), \tag{61}$$

with $p > 0$. These expressions give an isomorphism between the Hilbert spaces on the boundary of M_2 and M_1. In particular, this isomorphism preserves the amplitude by construction and, thus, preserves the probability for the quantum field theory. It also preserves the vacuum state since $\psi_{0;\Sigma_h} = K_{0;\Sigma_h}$ is mapped to $\psi_{0;\partial M_\eta} = K_{0;\partial M_\eta}$.

In Sec. VI, we will find that also the observable amplitudes for certain Weyl observables of the form $W = \exp(\mathrm{i}D)$ with $D(\phi) = \int \mathrm{d}^2x\sqrt{-\det g(x)}\mu(x)\phi(x)$ and $\mu(x)$ a general test function are preserved. Since the corresponding amplitude can be used as a generating functional for the perturbative quantization of interacting scalar field theories, this means that the amplitudes for interacting scalar field theories in the two regions are equivalent.

B. Schrödinger-Feynman representation

In Schrödinger-Feynman representation, we proceed in a way analogous to what we did in the holomorphic representation. Based on previous results [14–16], and, in particular, according to formula (75) of Ref. [11], in the region M_1, we have

$$\hat{\xi}(\rho, \eta) = -\frac{\mathrm{i}}{2p}(e^{-\mathrm{i}p\eta}\xi_1(\rho) + e^{\mathrm{i}p\eta}\overline{\xi_2(\rho)}), \tag{62}$$

where $\frac{e^{\pm \mathrm{i}p\eta}}{2p}$ is to be understood as an operator; expanding the function $\xi_{1,2}(\rho)$ according to Eq. (A6), we get

$$\begin{aligned} \hat{\xi}(\rho, \eta) = -\mathrm{i}\int_0^{\infty} \frac{\mathrm{d}p}{2p} \frac{\sqrt{2p\sinh(\pi p)}}{\pi} K_{\mathrm{i}p}(m\rho)(e^{-\mathrm{i}p\eta}\xi_1(p) \\ + e^{\mathrm{i}p\eta}\overline{\xi_2(p)}). \end{aligned} \tag{63}$$

On the other hand, in the region M_2, according to formula (91) of Ref. [11], we have

$$\hat{\xi}(\rho, \eta) = -K_{\mathrm{i}|p|}(m\rho)\xi(\eta), \tag{64}$$

where $K_{\mathrm{i}p}(m\rho)$ is to be understood as an operator; the substitution of $\xi(\eta)$ with its expansion $\xi(\eta) = \int \frac{\mathrm{d}p}{\sqrt{2\pi}} e^{\mathrm{i}p\eta}\xi(p)$ leads to

$$\begin{aligned} \hat{\xi}(\rho, \eta) &= -\int_{-\infty}^{\infty} \frac{\mathrm{d}p}{\sqrt{2\pi}} K_{\mathrm{i}|p|}(m\rho)e^{\mathrm{i}p\eta}\xi(p) \\ &= -\int_0^{\infty} \frac{\mathrm{d}p}{\sqrt{2\pi}} K_{\mathrm{i}p}(m\rho)(e^{\mathrm{i}p\eta}\xi(p) + e^{-\mathrm{i}p\eta}\xi(-p)). \end{aligned} \tag{65}$$

Identifying Eq. (63) with Eq. (65) leads to the following relations, valid for $p > 0$:

$$\xi(p) = \mathrm{i}\sqrt{\frac{\sinh(\pi p)}{\pi p}}\overline{\xi_2(p)}, \quad \text{and} \qquad \xi(-p) = \mathrm{i}\sqrt{\frac{\sinh(\pi p)}{\pi p}}\xi_1(p). \tag{66}$$

Then, the substitution of these expressions for $\xi(\pm p)$ in the free amplitude (49) in region M_2 reduces to the free amplitude (36) in region M_1. It must be noted that the isomorphism implemented by Eq. (66) results to be an isometric isomorphism.

1. Equivalence of states on the boundary of Rindler space

Consider the vacuum state (46) defined on the hyperbola. We notice that the surface of constant ρ in the limit where ρ tends to zero approaches the union of the surfaces defined by $\eta \to -\infty$ and $\eta \to +\infty$.[15] It is then to be expected that the vacuum state (46) at $\rho = 0$ reduces to the tensor product of two vacuum states (35) for $\eta \to -\infty$ and $\eta \to +\infty$, which implies that the operator appearing in the exponential of Eq. (46) tends to the one in the exponential of Eq. (35). This can be easily checked by the asymptotic property (A3) of the modified Bessel function $I_{\mathrm{i}|p|}$,

$$\lim_{\rho\to 0} \mathrm{i}\rho \frac{d}{d\rho} \ln(\overline{I_{\mathrm{i}|p|}(m\rho)}) = |p|, \tag{67}$$

which is indeed the operator characterizing the vacuum state (35). The normalization factor C_{ρ_1} appearing in Eq. (46) satisfies

$$|C_{\rho_1}|^2 = \det\left(-\frac{\mathrm{i}}{2\pi}\rho_1 \frac{\mathrm{d}}{\mathrm{d}\rho_1}\ln(I_{\mathrm{i}|p|}(m\rho_1)) + \frac{\mathrm{i}}{2\pi}\rho_1\frac{\mathrm{d}}{\mathrm{d}\rho_1}\ln(\overline{I_{\mathrm{i}|p|}(m\rho_1)})\right)^{1/2} = \det\left(\frac{1}{\pi^2}\frac{\sinh(|p|\pi)}{|I_{\mathrm{i}|p|}(m\rho_1)|^2}\right)^{1/2}. \tag{68}$$

In the limit $\rho_1 \to 0$, using Eq. (A3) we have that

$$|I_{\mathrm{i}|p|}(m\rho_1)|^2 \sim |\Gamma(\mathrm{i}|p|+1)|^{-2} = |\mathrm{i}|p|\Gamma(\mathrm{i}|p|)|^{-2} = \frac{\sinh(\pi|p|)}{\pi|p|}. \tag{69}$$

The modulus square of the normalization factor C_{ρ_1}, in the limit $\rho \to 0$ can then be written as

$$|C_{\rho_1}|^2 = \det\left(\frac{|p|}{\pi}\right)^{1/2}, \tag{70}$$

and the vacuum state reads in this limit

$$\psi_{\rho_1\to 0,0}(\varphi_0) = \det\left(\frac{|p|}{\pi}\right)^{1/4} e^{\mathrm{i}\arg(C_{\rho_1})} \times \exp\left(-\int_0^\infty \mathrm{d}p\,\varphi_0(p)p\varphi_0(-p)\right). \tag{71}$$

In order for this state to correspond to the state $\psi_{\eta\to-\infty,0} \otimes \overline{\psi_{\eta\to\infty,0}} \in \mathcal{H}_{-\infty} \otimes \mathcal{H}^*_{\infty}$,

$$\psi_{\eta\to-\infty,0}(\varphi_{-\infty}) \otimes \overline{\psi_{\eta\to\infty,0}(\varphi_\infty)} = \det\left(\frac{p}{\pi}\right)^{1/4}\exp\left(-\frac{1}{2}\int_0^\infty \mathrm{d}p[\varphi_\infty(p)p\varphi_\infty(p) + \varphi_{-\infty}(p)p\varphi_{-\infty}(p)]\right), \tag{72}$$

the following equality must be satisfied:

$$\frac{1}{2}[\varphi_\infty(p)\varphi_\infty(p) + \varphi_{-\infty}(p)\varphi_{-\infty}(p)] = \varphi_0(p)\varphi_0(-p). \tag{73}$$

With this equality, which relates the coefficient of the modes expansion of the field in the asymptotic hypersurfaces $\eta \to \pm\infty$ and $\rho_1 \to 0$, it can be shown that also asymptotic coherent states coincide, namely,

$$\psi_{\rho_1\to 0,\xi}(\varphi_0) = \psi_{\eta\to-\infty,\xi_1}(\varphi_{-\infty}) \otimes \overline{\psi_{\eta\to\infty,\xi_2}(\varphi_\infty)}, \tag{74}$$

where $\psi_{\rho_1\to 0,\xi} \in \mathcal{H}_{\rho_1\to 0}$ and $\psi_{\eta\to-\infty,\xi_1} \otimes \overline{\psi_{\eta\to\infty,\xi_2}} \in \mathcal{H}_{-\infty} \otimes \mathcal{H}^*_{\infty}$.

2. Equivalence of probability

In this section, we show how the probability computed in the two regions M_1 and M_2 are related. In the GBF, probabilities can be computed from the amplitude maps and are encoded in the formula

$$P(\mathcal{A}/\mathcal{S}) = \frac{\langle \varrho_M \diamond P_{\mathcal{S}}, \varrho_M \diamond P_{\mathcal{A}}\rangle}{\langle \varrho_M \diamond P_{\mathcal{S}}, \varrho_M \diamond P_{\mathcal{S}}\rangle}, \tag{75}$$

where $\mathcal{A}$ and $\mathcal{S}$ are subspaces of the Hilbert space $\mathcal{H}_{\partial M}$ associated to the boundary ∂M of the region M and $P_{\mathcal{A}}$ and $P_{\mathcal{S}}$ the orthogonal projectors onto these subspaces. The symbol $\diamond$ denotes the composition of maps. Consequently, $\varrho_M \diamond P_{\mathcal{S}}$ and $\varrho_M \diamond P_{\mathcal{A}}$ are linear maps from $\mathcal{H}_{\partial M}$ to the complex numbers. Two conditions must be required for this composition: (i) the maps $\varrho_M \diamond P_{\mathcal{S}}$ and $\varrho_M \diamond P_{\mathcal{A}}$ are continuous, and (ii) the map $\varrho_M \diamond P_{\mathcal{S}}$ does not vanish. Then, these maps can be viewed as elements in the dual Hilbert space $\mathcal{H}^*_{\partial M}$, and the inner product $\langle\cdot,\cdot\rangle$ appearing in Eq. (75) is the inner product of this dual Hilbert space. $P(\mathcal{A}/\mathcal{S})$ represents the conditional probability for observing $\mathcal{A}$ given that $\mathcal{S}$ has been prepared.

We consider first the region M_1. In this case, there exists a natural decomposition of the boundary Hilbert space $\mathcal{H}_{\partial M_1}$, namely, $\mathcal{H}_{\partial M_1} = \mathcal{H}_1 \otimes \mathcal{H}^*_2$. We can then choose the subspaces $\mathcal{S}_{M_1}$ and $\mathcal{A}_{M_1}$ as

[15] The Hilbert spaces associated to these hypersurfaces will be denoted as $\mathcal{H}_{-\infty}$ and $\mathcal{H}_{\infty}$, respectively.

$$\mathcal{S}_{M_1} = \{\psi \otimes \xi : \xi \in \mathcal{H}_2^*\} \quad \text{and} \qquad (76)$$
$$\mathcal{A}_{M_1} = \{\psi \otimes \xi : \psi \in \mathcal{H}_1\}.$$

In order to evaluate the numerator and denominator of Eq. (75), it is convenient to introduce an orthonormal basis of the boundary Hilbert space $\mathcal{H}_{\partial M_1}$. In particular, since $\mathcal{H}_{\partial M_1}$ decomposes as the tensor product $\mathcal{H}_1 \otimes \mathcal{H}_2^*$, we introduce two orthonormal bases $\{\nu_k^1\}$ and $\{\nu_k^2\}$ for the spaces $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively. Then, we have

$$\langle \varrho_M \diamond P_{\mathcal{S}_{M_1}}, \varrho_M \diamond P_{\mathcal{A}_{M_1}} \rangle = \sum_{k,l} \overline{\varrho_{M_1} \diamond P_{\mathcal{S}_{M_1}}(\nu_k^1 \otimes \nu_l^2)} \times \varrho_{M_1} \diamond P_{\mathcal{A}_{M_1}}(\nu_k^1 \otimes \nu_l^2)$$
$$\langle \varrho_M \diamond P_{\mathcal{S}_{M_1}}, \varrho_M \diamond P_{\mathcal{S}_{M_1}} \rangle = \sum_{k,l} |\varrho_{M_1} \diamond P_{\mathcal{S}_{M_1}}(\nu_k^1 \otimes \nu_l^2)|^2. \qquad (77)$$

Without loss of generality, we can choose $\nu_1^1 = \psi$ and $\nu_1^2 = \xi$, and the probability (75) takes the form

$$P(\mathcal{A}_{M_1}/\mathcal{S}_{M_1}) = \frac{|\varrho_{M_1}(\psi \otimes \xi)|^2}{\sum_l |\varrho_{M_1}(\psi \otimes \nu_l^2)|^2}. \qquad (78)$$

Also without loss of generality, we can choose the states ψ and ξ to be coherent states that we denote as K_{ξ_1} and $\overline{K_{\xi_2}}$, respectively:

$$P(\mathcal{A}_{M_1}/\mathcal{S}_{M_1}) = \frac{|\varrho_{M_1}(K_{\xi_1} \otimes \overline{K_{\xi_2}})|^2}{\sum_l |\varrho_{M_1}(K_{\xi_1} \otimes \nu_l^2)|^2}. \qquad (79)$$

In order to give a more useful expression of the denominator, we use the resolution to the identity provided by the coherent states to obtain

$$\sum_l |\varrho_{M_1}(K_{\xi_1} \otimes \nu_l^2)|^2 = \sum_l \left| D^{-1} \int d\zeta d\bar{\zeta}\, \overline{C_{\nu_k^2,\zeta}}\, \varrho_{M_1}(K_{\xi_1} \otimes \overline{K_\zeta}) \right|^2, \qquad (80)$$

where $\overline{C_{\nu_k^2,\zeta}} = \langle \nu_k^2, K_\zeta \rangle_{\mathcal{H}_2}$ and D is the coefficient appearing in the resolution of the identity satisfied by the coherent states [11]. The isomorphism expressed by the relations (66) can be used to map the subspaces $\mathcal{A}_{M_1}$ and $\mathcal{S}_{M_1}$ of the Hilbert space associated to the boundary of the region M_2 to the corresponding subspaces $\mathcal{A}_{M_2}$ and $\mathcal{S}_{M_2}$ defined for the theory in the region M_2. In particular, as we have seen, the relations in equation Eq. (66) transform the free amplitude $\varrho_{M_1}(K_{\xi_1} \otimes \overline{K_{\xi_2}})$ into the free amplitude $\varrho_{M_2}(K_{\hat{\xi}})$; moreover, the number $\overline{C_{\nu_k^2,\zeta}}$ is invariant under the action of the isometric isomorphism (66). We can consequently conclude that the probabilities computed in the region M_1 for the free theory are the same as the one computed in the region M_2,

$$P(\mathcal{A}_{M_1}/\mathcal{S}_{M_1})|_{(\xi_1,\hat{\xi}_2)=\hat{\xi}} = P(\mathcal{A}_{M_2}/\mathcal{S}_{M_2}). \qquad (81)$$

VI. PRESERVATION OF AMPLITUDES IN THE INTERACTING THEORY

In this section, we will first compare the observable amplitude for Weyl observables $W(\phi) = \exp(iD(\phi))$ with $D(\phi) = \int d^2x \sqrt{-\det g(x)}\mu(x)\phi(x)$, where $\mu(x)$ is a general test function in the regions M_1 and M_2.

A. Holomorphic representation

For a general region M, we have from proposition 4.3 of Ref. [6] the following expression for the observable amplitude:

$$\varrho_M^W(K_\tau) = \varrho_M(K_\tau) \exp\Big(i \int_M d^2x \sqrt{-\det g(x)}\mu(x)\hat{\tau}(x) + \frac{i}{2}\int_M d^2x d^2x' \sqrt{\det g(x) \det g(x')}\mu(x) \times G_F^M(x,x')\mu(x') \Big), \qquad (82)$$

where $G_F^M(x,x')$ is the Feynman propagator constructed such that

$$(\eta_D - iJ_{\partial M}\eta_D)(x) = \int_M d^2x \sqrt{-\det g(x)} G_F^J(x,x')\mu(x'), \qquad (83)$$

where η_D is the unique element of $J_{\partial M} L_{\tilde{M}}$ fulfilling the condition $D(\xi) = 2\omega_{\partial M}(\xi, \eta_D)$ for all $\xi \in L_{\tilde{M}}$.

Since we constructed the isomorphism between $\mathcal{H}_{\Sigma_{\rho_1}}$ and $\mathcal{H}_{\partial M_1}$ such that the expressions for $\hat{\tau}$ for the two regions coincide, we have that the observable maps coincide if the Feynman propagators coincide. In region M_1, we obtain for the Feynman propagator the following expression [23]:

$$G_F^{M_1}(x,x') = i \int dp (\theta(\eta' - \eta)\overline{\phi_p^R(x)}\phi_p^R(x') + \theta(\eta - \eta')\phi_p^R(x)\overline{\phi_p^R(x')}),$$
$$= i \int_0^\infty \frac{dp}{2p} (\theta(\eta' - \eta) e^{ip(\eta - \eta')} + \theta(\eta - \eta') e^{ip(\eta' - \eta)}) K_{ip}(m\rho') K_{ip}(m\rho) \times \frac{p \sinh(p\pi)}{\pi^2} 2. \qquad (84)$$

For region M_2, we derive the Feynman propagator in the following. Let us assume that we are given a function $\phi_{\tilde{M}_2} \in L_{\tilde{M}_2} \subset L_{\Sigma_{\rho_1}}$ such that $J_{\Sigma_{\rho_1}}\phi_{\tilde{M}_2} = \eta_D$. Let us decompose $\phi_{\tilde{M}_2}$ as in Eq. (39). Then, we find that

$$\eta_D(x) = \int_{-\infty}^{\infty} \mathrm{d}p(\eta_D(p)\chi_p(x) + \mathrm{c.c.}),$$
$$= -i\int_0^{\infty} \mathrm{d}p(\phi_{\tilde{M}_2}(p)(\chi_p(x) - \overline{\chi_{-p}(x)})$$
$$- \overline{\phi_{\tilde{M}_2}(p)}(\overline{\chi_p(x)} - \chi_{-p}(x))), \quad (85)$$

from which we obtain that $\eta_D(p) = -\mathrm{i}\phi_{\tilde{M}_2}(p)$ for $p > 0$ and $\eta_D(p) = -\mathrm{i}\overline{\phi_{\tilde{M}_2}(-p)}$ for $p < 0$. Hence, we have for $\xi \in L_{\tilde{M}_2}$ using the identities in Eq. (52) that

$$\omega_{\Sigma_{\rho_1}}(\xi, \eta_D) = -\frac{\mathrm{i}}{2}\int_0^{\infty} \mathrm{d}p(\xi_{\Sigma_{\rho_1}}(p)\mathrm{i}\overline{\phi_{\tilde{M}_2}(p)}$$
$$+ \xi_{\Sigma_{\rho_1}}(-p)\mathrm{i}\phi_{\tilde{M}_2}(p) - \mathrm{c.c.}),$$
$$= \int_0^{\infty} \mathrm{d}p(\xi(p)\overline{\phi_{\tilde{M}_2}(p)} + \mathrm{c.c.}). \quad (86)$$

From the condition $D(\xi) = \int \mathrm{d}\eta \mathrm{d}\rho\rho\mu(x)\xi(x) = \omega_{\Sigma_{\rho_1}}(\xi, \eta_D)$, we obtain

$$\phi_{\tilde{M}_2}(p) = \int \mathrm{d}\eta' \mathrm{d}\rho'\rho'\mu(x')\overline{\phi_p(x')}, \quad (87)$$

and with Eq. (85), we find an expression for η_D. Now, we are interested in the projection of the Feynman propagator to the boundary $\eta_D - \mathrm{i}J_{\Sigma_{\rho_1}}\eta_D$. We obtain

$$\eta_D - \mathrm{i}J_{\Sigma_{\rho_1}}\eta_D = (\mathrm{i} + J_{\Sigma_{\rho_1}})\phi_{\tilde{M}_2}$$
$$= 2\mathrm{i}\int_0^{\infty} \mathrm{d}p(\phi_{\tilde{M}_2}(p)\overline{\chi_{-p}(x)}$$
$$+ \overline{\phi_{\tilde{M}_2}(p)}\,\overline{\chi_p(x)}). \quad (88)$$

Using that $\phi_p(x) = \overline{\phi_{-p}(x)}$, we find for the Feynman propagator the symmetrized expression

$$G_F^{M_2}(x, x') = \mathrm{i}\int_{-\infty}^{\infty} \mathrm{d}p[\theta(\rho' - \rho)\overline{\chi_{-p}(\eta, \rho)}\,\overline{\phi_p(\eta', \rho')}$$
$$+ \theta(\rho - \rho')\overline{\chi_{-p}(\eta', \rho')}\,\overline{\phi_p(\eta, \rho)}],$$
$$= \int_{-\infty}^{\infty} \frac{\mathrm{d}p}{2\pi}[\theta(\rho' - \rho)K_{\mathrm{i}|p|}(m\rho')\overline{I_{\mathrm{i}|p|}(m\rho)}$$
$$+ \theta(\rho - \rho')K_{\mathrm{i}|p|}(m\rho)\overline{I_{\mathrm{i}|p|}(m\rho')}]e^{\mathrm{i}p(\eta-\eta')}. \quad (89)$$

B. Schrödinger-Feynman quantization

A way to compute the expectation value of the Weyl observable W is to modify the action as

$$S_{M,\mu}(\phi) = S_{M,0}(\phi) + \int_M \mathrm{d}^2x\sqrt{-\det g(x)}\phi(x)\mu(x). \quad (90)$$

The form of the corresponding field propagator (9) can be obtained by shifting the integration variable by a classical solution ϕ_{cl} that matches the boundary configuration φ on the boundary ∂M,

$$Z_{M,\mu}(\varphi) = \int_{\phi|_{\partial M}=\varphi} \mathcal{D}\phi e^{\mathrm{i}S_{M,\mu}(\phi)}$$
$$= \int_{\phi|_{\partial M}=0} \mathcal{D}\phi e^{\mathrm{i}S_{M,\mu}(\phi_{\mathrm{cl}}+\phi)} = N_{M,\mu}e^{\mathrm{i}S_{M,\mu}(\phi_{\mathrm{cl}})}, \quad (91)$$

where $N_{M,\mu} = \int_{\phi|_{\partial M}=0} \mathcal{D}\phi e^{\mathrm{i}S_{M,\mu}(\phi)}$. The propagator can be expressed in terms of the propagator $Z_{M,0}(\varphi)$ of the free theory as

$$Z_{M,\nu}(\varphi) = Z_{M,0}(\varphi)\exp\Big(\mathrm{i}\int_M \mathrm{d}^2x\sqrt{-\det g(x)}\phi_{\mathrm{cl}}\mu(x)$$
$$+ \frac{\mathrm{i}}{2}\int_M \mathrm{d}^2x\sqrt{-\det g(x)}\alpha(x)\mu(x)\Big), \quad (92)$$

where the quantity α is the solution of the inhomogeneous equation $(-\rho\partial_\rho\rho\partial_\rho + \partial_\eta^2 + m^2\rho^2)\alpha(\eta, \rho) = \mu(\eta, \rho)$, with the vanishing boundary condition $\alpha|_{\partial M} = 0$. In the region M_1, a classical solution with boundary configurations φ_1 and φ_2 is given by Eq. (19), and the function α results to be

$$\alpha(\eta,\rho) = \int_{\eta_1}^{\eta_2} \mathrm{d}\eta'\rho\Big(\theta(\eta' - \eta)\frac{\sin p(\eta - \eta_1)\sin p(\eta_2 - \eta')}{p\sin p(\eta_2 - \eta_1)}$$
$$+ \theta(\eta - \eta')\frac{\sin p(\eta' - \eta_1)\sin p(\eta_2 - \eta)}{p\sin p(\eta_2 - \eta_1)}\Big). \quad (93)$$

Notice that $\alpha(\eta_1, \rho) = \alpha(\eta_2, \rho) = 0$. Substituting these quantities in the expression of the propagator (92) and performing the integration in Eq. (8) leads to the amplitude for a coherent state $K^S_{\eta_1,\xi_1} \otimes \overline{K^S_{\eta_2,\xi_2}}$:

$$\varrho^W_{[\eta_1,\eta_2]}(K^S_{\eta_1,\xi_1} \otimes \overline{K^S_{\eta_2,\xi_2}})$$
$$= \varrho_{[\eta_1,\eta_2]}(K^S_{\eta_1,\xi_1} \otimes \overline{K^S_{\eta_2,\xi_2}})\exp\Big(\int_{M_1} d^2x\sqrt{-g(x)}\hat{\xi}(x)\mu(x)\Big)$$
$$\times\exp\Big(\frac{i}{2}\int_{M_1} d^2xd^2x'\sqrt{g(x)g(x')}\mu(x)G_F^{M_1}(x, x')\mu(x')\Big), \quad (94)$$

where $\varrho_{[\eta_1,\eta_2]}(K^S_{\eta_1,\xi_1} \otimes \overline{K^S_{\eta_2,\xi_2}})$ is the free amplitude (36), $\hat{\xi}$ is the complex solution given by Eq. (62), and $G_F^{M_1}(x, x')$ is the Feynman propagator in region M_1 given by Eq. (84). Taking the limit $\eta_1 \to -\infty$ and $\eta_2 \to +\infty$ in the amplitude (94) reduces to substitute the subindex M_1 with the whole Rindler space.

In the region M_2, a classical solution with boundary configuration φ is given by Eq. (22), and α can be expressed in integral form as $\alpha(\eta, \rho) = \int_{\rho_1}^{\infty} \mathrm{d}\rho'\rho'\tilde{g}(\rho, \rho')\nu(\eta, \rho')$, where

$$\tilde{g}(\rho,\rho') = -\theta(\rho'-\rho)(L_{\mathrm{i}p}(m\rho')K_{\mathrm{i}p}(m\rho) - L_{\mathrm{i}p}(m\rho)K_{\mathrm{i}p}(m\rho')) + L_{\mathrm{i}p}(m\rho')K_{\mathrm{i}p}(m\rho) - K_{\mathrm{i}p}(m\rho)\frac{L_{\mathrm{i}p}(m\rho_1)}{K_{\mathrm{i}p}(m\rho_1)}K_{\mathrm{i}p}(m\rho'), \quad (95)$$

where $L_{\mathrm{i}p}$ is the real part of $I_{\mathrm{i}p}$. Notice that α satisfied the vanishing boundary condition $\alpha(\eta,\rho_1)=0$. The expression for the amplitude of a coherent state in the interacting theory results to be

$$\varrho^W_{M_2}(K^S_{\rho_1,\xi}) = \varrho_{M_2}(K^S_{\rho_1,\xi})\exp\left(\int_{M_2} d^2x\sqrt{-g(x)}\hat{\xi}(x)\mu(x) + \frac{i}{2}\int_{M_2} d^2x d^2x'\sqrt{g(x)g(x')}\mu(x)G_F^{M_2}(x,x')\mu(x')\right), \quad (96)$$

where x is a global notation for the coordinates η, ρ and $G_F^{M_2}(x,x')$ is given by Eq. (89). Taking the limit $\rho_1 \to 0$ in the amplitude (96) reduces to substitute the subindex M_2 with the whole Rindler space.

C. Equality of the Feynman propagators in region M_1 and M_2

In this section, we show in two different ways the equality of the propagators in the region M_1 and M_2, i.e., we show the identity $G_F^{M_2}(x,x') = G_F^{M_1}(x,x')$. This result means that the observable amplitudes $\varrho^W_{M_1}(\Psi)$ and $\varrho^W_{M_2}(\Psi')$ coincide for all Weyl observables of the form $W(\phi)=e^{iD(\phi)}$ with $D(\phi)=\int d^2x\sqrt{-\det g(x)}\mu(x)\phi(x)$ when the state Ψ is mapped to Ψ' with the isomorphism we identified in Sec. V and $\mu(x)$ has support in the interior of both regions. These amplitudes can be used as generating functionals to derive all the n-point functions of the field ϕ, which, thus, also coincide for the two regions. For a quantum field theory of two interacting scalar fields ϕ_1 and ϕ_2, the corresponding amplitude can also be generated using the amplitude in Eq. (82) as a generating functional [15]. Hence, the coincidence of the vacuum state, amplitudes, and probabilities is also valid for the interacting theory.

1. First method

We start from expression (84) of the Feynman propagator in region M_1. The integral can be extended to negative values of p by substituting p with $|p|$; then, using the relation

$$\frac{\mathrm{i}}{2|p|}(\theta(\eta'-\eta)e^{\mathrm{i}|p|(\eta-\eta')} + \theta(\eta-\eta')e^{\mathrm{i}|p|(\eta'-\eta)}) = -\lim_{\epsilon\to 0}\int_{-\infty}^{\infty}\frac{\mathrm{d}q}{2\pi}\frac{e^{-\mathrm{i}q(\eta-\eta')}}{q^2-p^2+\mathrm{i}\epsilon} \quad (97)$$

and expressing the Macdonald function in terms of the modified Bessel functions of the first kind, Eq. (A1), we obtain

$$G_F^{M_1}(x,x') = \frac{1}{4}\int_{-\infty}^{\infty}\frac{\mathrm{d}q}{2\pi}\int_{-\infty}^{\infty}\mathrm{d}p\frac{e^{-\mathrm{i}q(\eta-\eta')}}{q^2-p^2+\mathrm{i}\epsilon}(I_{-\mathrm{i}p}(m\rho') - I_{\mathrm{i}p}(m\rho'))(I_{-\mathrm{i}p}(m\rho)-I_{\mathrm{i}p}(m\rho))\frac{p}{\sinh(p\pi)}, = \int_{-\infty}^{\infty}\frac{\mathrm{d}q}{2\pi}e^{-\mathrm{i}q(\eta-\eta')}(\mathcal{I}_{++}+\mathcal{I}_{--}-\mathcal{I}_{-+}-\mathcal{I}_{+-}), \quad (98)$$

where we introduced the notation

$$\mathcal{I}_{lm} = \frac{1}{4}\int_{-\infty}^{\infty}\mathrm{d}p\frac{1}{q^2-p^2+\mathrm{i}\epsilon}\frac{p}{\sinh(p\pi)}I_{l\mathrm{i}p}(m\rho')I_{m\mathrm{i}p}(m\rho), \quad (l=+,-),(m=+,-). \quad (99)$$

In the following, we will perform the integration over p for every term $\mathcal{I}_{lm}$ with l, $m=\pm 1$ separately. First of all, we notice that each term $\mathcal{I}_{lm}$ apparently contains an infinite number of poles for $p=\mathrm{i}n$, where n is an integer. However, it can be shown that only the two poles $p_\pm = \pm(|q|+\mathrm{i}\epsilon)$ contribute to the sum in Eq. (98). We apply the complex contour integration to evaluate their contribution. We start with the integral $\mathcal{I}_{++}$, which is equal to

$$\mathcal{I}_{++} = -\frac{1}{4}\int_{-\infty}^{\infty}\mathrm{d}p\frac{1}{p^2-q^2-\mathrm{i}\epsilon}\frac{p}{\sinh p\pi}I_{\mathrm{i}p}(m\rho)I_{\mathrm{i}p}(m\rho'). \quad (100)$$

We rewrite this integral using formula (5.7.1) of Ref. [26],

$$I_\nu(z) = \sum_{k=0}^{\infty}\frac{(z/2)^{\nu+2k}}{\Gamma(k+1)\Gamma(k+\nu+1)}, \quad (101)$$

which is valid for $|z|<\infty$, $|\arg z|<\pi$. Substituting the above expression in $\mathcal{I}_{++}$, we get

$$\mathcal{I}_{++} = -\frac{1}{4}\int_{-\infty}^{\infty}\mathrm{d}p\frac{1}{p^2-q^2-\mathrm{i}\epsilon}\frac{p}{\sinh p\pi} \times \sum_{k,k'=0}^{\infty}\frac{(m\rho/2)^{2k}(m\rho'/2)^{2k'}}{\Gamma(k+1)\Gamma(k'+1)} \times \frac{(m\rho/2)^{\mathrm{i}p}(m\rho'/2)^{\mathrm{i}p}}{\Gamma(k+1+\mathrm{i}p)\Gamma(k'+1+\mathrm{i}p)}. \quad (102)$$

We compute this integral by closing the contour of integration in the complex p plane. To do this, we look at the behavior of the gamma functions for large values of the argument. We use the asymptotic expansion (1.4.23) of Ref. [26],

$$\Gamma(z) = e^{(z-1/2)\log z - z + 1/2\log 2\pi}(1+O(|z|^{-1})), \quad (103)$$

which is valid for $|\arg z|<\pi$. Substituting in $\mathcal{I}_{++}$, we get

$$I_{++} \approx -\frac{1}{4}\int_{-\infty}^{\infty} \mathrm{d}p \frac{1}{p^2 - q^2 - \mathrm{i}\epsilon} \frac{p}{\sinh p\pi} \sum_{k,k'=0}^{\infty} \frac{(m\rho/2)^{2k}(m\rho'/2)^{2k'}}{\Gamma(k+1)\Gamma(k'+1)} \times \exp(\mathrm{i}p(\log(m^2\rho\rho'/4) - \log(k+1+\mathrm{i}p) - \log(k'+1+\mathrm{i}p) + 2)) \times \exp(-(k+1/2)\log(k+1+\mathrm{i}p) - (k'+1/2)\log(k'+1+\mathrm{i}p) - \log(2\pi) + k + k' + 2). \quad (104)$$

We write $p = re^{\mathrm{i}\theta}$; consequently,

$$\begin{aligned}\log(k+1+\mathrm{i}p) &= \log(k+1+\mathrm{i}re^{\mathrm{i}\theta}) \\ &= \log(k+1+\mathrm{i}r\cos\theta - r\sin\theta) \\ &= \log\sqrt{(k+1-r\sin\theta)^2 + r^2\cos^2\theta} + \mathrm{i}\arctan\frac{r\cos\theta}{k+1-r\sin\theta} \\ &= \log\sqrt{(k+1)^2 - 2(k+1)r\sin\theta + r^2} + \mathrm{i}\arctan\frac{r\cos\theta}{k+1-r\sin\theta},\end{aligned} \quad (105)$$

which for $r \gg (k+1)$ reduces to $\log(k+1+\mathrm{i}p) \approx \log r + \mathrm{i}\arctan(-\cot\theta)$. Then, we have that the argument of the first exponential in Eq. (104) can be rewritten as

$$\begin{aligned}&\mathrm{i}p(\log(m^2\rho\rho'/4) - \log(k+1+\mathrm{i}p) - \log(k'+1+\mathrm{i}p) + 2) \\ &= \mathrm{i}re^{\mathrm{i}\theta}\Big(\underbrace{\log(m^2\rho\rho'/4) - 2\log r + 2}_{\tilde{r}} - 2\mathrm{i}\arctan(-\cot\theta)\Big) \\ &= \mathrm{i}(r\tilde{r}\cos\theta + 2r\sin\theta\arctan(-\cot\theta)) \\ &\quad - r(\tilde{r}\sin\theta - 2\cos\theta\arctan(-\cot\theta)).\end{aligned} \quad (106)$$

Let us have a close look at the factor in the last term:

$$\begin{aligned}&\tilde{r}\sin\theta - 2\cos\theta\arctan(-\cot\theta) \\ &= (\log(m^2\rho\rho'/4) - 2\log r + 2)\sin\theta \\ &\quad - 2\cos\theta\arctan(-\cot\theta).\end{aligned} \quad (107)$$

For finite ρ, ρ' and $\theta \in [-\pi, 0]$, we can always choose r large enough to get this factor positive. We find that we can close the contour of integration in the lower half plane, namely, $\theta \in [-\pi, 0]$, send $r \to \infty$, and apply the residue theorem. The pole in the lower half plane is located in $-|q| - \mathrm{i}\epsilon$, and the result of the integration is

$$I_{++} = -\frac{1}{4}\mathrm{i}\frac{\pi}{\sinh|q|\pi} I_{-\mathrm{i}|q|}(m\rho) I_{-\mathrm{i}|q|}(m\rho'). \quad (108)$$

We obtain the same expression for I_{--}, namely, $I_{++} = I_{--}$. For the integral I_{+-} and I_{-+}, applying similar techniques, we obtain

$$\begin{aligned}I_{+-} &= I_{-+} \\ &= -\frac{1}{4}\mathrm{i}\frac{\pi}{\sinh(|q|\pi)}(\theta(\rho-\rho')I_{\mathrm{i}|q|}(m\rho)I_{-\mathrm{i}|q|}(m\rho') \\ &\quad + \theta(\rho'-\rho)I_{-\mathrm{i}|q|}(m\rho)I_{\mathrm{i}|q|}(m\rho')).\end{aligned} \quad (109)$$

Finally, the Feynman propagator in the region M_1 results to be

$$\begin{aligned}&G_F^{M_1}(x,x') \\ &= \int_{-\infty}^{\infty}\frac{\mathrm{d}q}{2\pi} e^{-\mathrm{i}q(\eta-\eta')}[\theta(\rho-\rho')K_{\mathrm{i}|q|}(m\rho)I_{-\mathrm{i}|q|}(m\rho') \\ &\quad + \theta(\rho'-\rho)I_{-\mathrm{i}|q|}(m\rho)K_{\mathrm{i}|q|}(m\rho')],\end{aligned} \quad (110)$$

where relation (A1) has been used. This propagator coincides with the propagator (89) in the region M_2, namely, $G_F^{M_1}(x,x') = G_F^{M_2}(x,x')$.

2. Second method

We consider formula 7.213 of Ref. [27],

$$\int_0^\infty \frac{x\tanh(\pi x)}{\alpha^2 + x^2} P_{-\frac{1}{2}+\mathrm{i}x}(\cosh\beta)\mathrm{d}x = Q_{\alpha-\frac{1}{2}}(\cosh\beta), \quad \Re(a) > 0, \quad (111)$$

where P_n and Q_n are the associated Legendre functions of the first and second kind, respectively. We set $\alpha = \mathrm{i}\sqrt{p^2 - \mathrm{i}\epsilon} \simeq \mathrm{i}|p| + \epsilon$, with $\epsilon > 0$ and $\epsilon \ll 1$. Therefore,

$$\begin{aligned}&\int_0^\infty \frac{x\tanh(\pi x)}{-|p|^2 + \mathrm{i}\epsilon' + x^2} P_{-\frac{1}{2}+\mathrm{i}x}\left(\frac{a^2+b^2+c^2}{2ab}\right)\mathrm{d}x \\ &\simeq Q_{\mathrm{i}|p|+\epsilon-\frac{1}{2}}\left(\frac{a^2+b^2+c^2}{2ab}\right),\end{aligned} \quad (112)$$

where we also have replaced $\cosh\beta$ with $\frac{a^2+b^2+c^2}{2ab}$, $\epsilon' = |p|\epsilon$ is still very small, and equality holds for $\epsilon' \to 0$. Consequently, the above equation is valid for $\frac{a^2+b^2+c^2}{2ab} \geq 1$. We now consider formula 6.672.3 of Ref. [27],

$$\begin{aligned}\int_0^\infty K_\nu(ax)K_\nu(bx)\cos(cx)\mathrm{d}x &= \frac{\pi^2}{4\sqrt{ab}}\sec(\pi\nu) \\ &\quad \times P_{\nu-\frac{1}{2}}\left(\frac{a^2+b^2+c^2}{2ab}\right), \\ \Re(a+b) > 0,\ c > 0,\ &|\Re(\nu)| < \frac{1}{2}.\end{aligned} \quad (113)$$

We multiply by $\cos(cy)$, $(y > 0)$ both sides and then integrate with respect to c. It is easy to show that the integrals in the lhs of Eq. (113) result to be equal to $\frac{\pi}{2}K_\nu(ay)K_\nu(by)$. Then,

$$K_\nu(ay)K_\nu(by) = \frac{\pi}{2\sqrt{ab}}\sec(\pi\nu)\int_0^\infty \mathrm{d}c\cos(cy) \times P_{\nu-\frac{1}{2}}\left(\frac{a^2+b^2+c^2}{2ab}\right), \tag{114}$$

which is valid for $y > 0$, $\Re(a+b) > 0$, $|\Re(\nu)| < \frac{1}{2}$.

We now consider formula 6.672.4 of Ref. [27]:

$$\int_0^\infty K_\nu(ax)I_\nu(bx)\cos(cx)\mathrm{d}x = \frac{1}{2\sqrt{ab}}Q_{\nu-\frac{1}{2}}\left(\frac{a^2+b^2+c^2}{2ab}\right), \quad \Re(a) > |\Re(b)|, c > 0, \Re(\nu) > -\frac{1}{2}. \tag{115}$$

By applying the same technique, namely, by multiplying by $\cos(cy)$, $(y > 0)$, to both sides and then integrating with respect to c, we obtain

$$K_\nu(ay)I_\nu(by) = \frac{1}{\pi\sqrt{ab}}\int_0^\infty \mathrm{d}c\cos(cy)Q_{\nu-\frac{1}{2}}\left(\frac{a^2+b^2+c^2}{2ab}\right), \tag{116}$$

which is valid for $y > 0$, $\Re(a) > |\Re(b)|$, $\Re(\nu) > -\frac{1}{2}$.

We multiply by $\cos(cy)$, $(y > 0)$, both sides of Eq. (112) and then integrate with respect to c,

$$\int_0^\infty \mathrm{d}c\cos(cy)\int_0^\infty \frac{x\tanh(\pi x)}{-|p|^2 + \mathrm{i}\epsilon + x^2}P_{-\frac{1}{2}+\mathrm{i}x}\left(\frac{a^2+b^2+c^2}{2ab}\right)\mathrm{d}x \simeq \int_0^\infty \mathrm{d}c\cos(cy)Q_{\mathrm{i}|p|+\epsilon-\frac{1}{2}}\left(\frac{a^2+b^2+c^2}{2ab}\right), \tag{117}$$

and invert the integral on the lhs, which leads to, using Eqs. (114) and (116),

$$\int_0^\infty \frac{x\tanh(\pi x)}{-|p|^2 + \mathrm{i}\epsilon + x^2}K_{\mathrm{i}x}(ay)K_{\mathrm{i}x}(by)\frac{2\sqrt{ab}}{\pi}\cos(\pi \mathrm{i}x)\mathrm{d}x = \pi\sqrt{ab}K_{\mathrm{i}|p|+\epsilon}(ay)I_{\mathrm{i}|p|+\epsilon}(by) \tag{118}$$

or, equivalently,

$$\int_0^\infty \frac{x\sinh(\pi x)}{-|p|^2 + \mathrm{i}\epsilon + x^2}K_{\mathrm{i}x}(ay)K_{\mathrm{i}x}(by)\mathrm{d}x = \frac{\pi^2}{2}K_{\mathrm{i}|p|+\epsilon}(ay)I_{\mathrm{i}|p|+\epsilon}(by), \tag{119}$$

which is valid for $y > 0$, $\Re(a+b) > 0$, $\Re(a) > |\Re(b)|$, $\epsilon > 0$, $\epsilon \ll 1$.

We now rewrite the Feynman propagator in the region M_2 (89) as

$$G_F^{M_2}(x,x') = \lim_{\epsilon\to 0}\int_{-\infty}^\infty \frac{\mathrm{d}p}{2\pi}[\theta(\rho'-\rho)K_{\mathrm{i}|p|+\epsilon}(m\rho')\overline{I_{\mathrm{i}|p|+\epsilon}(m\rho)} + \theta(\rho-\rho')K_{\mathrm{i}|p|+\epsilon}(m\rho)\overline{I_{\mathrm{i}|p|+\epsilon}(m\rho')}]e^{\mathrm{i}p(\eta-\eta')}. \tag{120}$$

We use relation (119) with the following identifications [which satisfy the conditions for the validity of (119)]:

$$y = m > 0, \tag{121}$$

$$a = \rho, \qquad b = \rho', \quad \text{for } \rho > \rho', \tag{122}$$

$$a = \rho', \qquad b = \rho, \quad \text{for } \rho' > \rho; \tag{123}$$

we obtain

$$G_F^{M_2}(x,x') = \lim_{\epsilon\to 0}\int_{-\infty}^\infty \frac{\mathrm{d}p}{2\pi}\frac{2}{\pi^2}e^{\mathrm{i}p(\eta-\eta')}\int_0^\infty \frac{x\sinh(\pi x)}{-|p|^2 - \mathrm{i}\epsilon + x^2} \times K_{\mathrm{i}x}(m\rho)K_{\mathrm{i}x}(m\rho')\mathrm{d}x. \tag{124}$$

We invert the order of integration and perform first the integral over $\mathrm{d}p$. For $\eta > \eta'$, we close the contour of integration in the upper half plane, and we do the same for $\eta < \eta'$ in the lower half plane; the poles are $p_\pm = \pm(|x| - \mathrm{i}\epsilon)$. We obtain

$$\int_{-\infty}^\infty \frac{\mathrm{d}p}{2\pi}\frac{e^{\mathrm{i}p(\eta-\eta')}}{-p^2 - \mathrm{i}\epsilon + x^2} = \frac{\mathrm{i}}{2(|x| - \mathrm{i}\epsilon)}[\theta(\eta-\eta')e^{-\mathrm{i}(|x|-\mathrm{i}\epsilon)(\eta-\eta')} + \theta(\eta'-\eta)e^{\mathrm{i}(|x|-\mathrm{i}\epsilon)(\eta-\eta')}]. \tag{125}$$

The Feynman propagator takes the form

$$\begin{aligned} G_F^{M_2}(x,x') &= \lim_{\epsilon\to 0}\int_0^\infty \frac{\mathrm{i}}{2(|x| - \mathrm{i}\epsilon)}[\theta(\eta-\eta')e^{-\mathrm{i}(|x|-\mathrm{i}\epsilon)(\eta-\eta')} + \theta(\eta'-\eta)e^{\mathrm{i}(|x|-\mathrm{i}\epsilon)(\eta-\eta')}]K_{\mathrm{i}x}(m\rho) \times K_{\mathrm{i}x}(m\rho')\frac{2x\sinh(\pi x)}{\pi^2}\mathrm{d}x, \\ &= \int_0^\infty \frac{\mathrm{i}}{2x}[\theta(\eta-\eta')e^{-\mathrm{i}x(\eta-\eta')} + \theta(\eta'-\eta)e^{\mathrm{i}x(\eta-\eta')}]K_{\mathrm{i}x}(m\rho)K_{\mathrm{i}x}(m\rho') \times \frac{2x\sinh(\pi x)}{\pi^2}\mathrm{d}x, \end{aligned} \tag{126}$$

which coincides with the expression (84) of the Feynman propagator in region M_1, $G_F^{M_1}(x,x') = G_F^{M_2}(x,x')$.

VII. SUMMARY AND OUTLOOK

We constructed the general boundary quantum field theory for a scalar field in two-dimensional Rindler space in two different regions: a region M_1 with spacelike boundaries and a region M_2 with a purely timelike boundary. More specifically, the boundary of region M_1 was given by the disjoint union of two equal Rindler time hypersurfaces, and the boundary of region M_2 was given as a timelike curve of a constant Rindler spatial coordinate. We showed the existence of an isomorphism between the Hilbert spaces associated with these boundaries.

The isomorphism we identified preserves the amplitude map, and, thus, the probabilities that can be extracted from the free quantum field theories are also preserved. We showed that the amplitude is also preserved when an interaction of the quantum field with a classical source is considered. That was done by showing that the isomorphism preserves the generating functional for perturbative quantum field theory. To obtain this result, we showed that the Feynman propagators for the quantum field theories in the two regions are equivalent. Consequently, we have obtained two equivalent representations of the Feynman propagator in Rindler space. This generalizes previous results obtained for QFT in Rindler space [28].

In particular, the generating functional for a given source term is equivalent with the expectation value (operator amplitude [9]) of a particular local Weyl observable associated with that source term. We concluded that the expectation values for these observables are also preserved under the action of the isomorphism we identified.

Let us emphasize again that regions with timelike boundaries like M_2 cannot be considered in the standard formulation of quantum field theory. The case investigated in this article shows that pairs of regions exist in Rindler space where one of these regions has timelike boundaries and the other region has spacelike boundaries such that both regions can be used equivalently to describe the same physical situation. Analogous results have been obtained within the GBF in Minkowski space [15,16], a Euclidean space [14], and de Sitter space [24,25]. In Minkowski space, this result was used to show explicitly that the crossing symmetry is generic in the GBF.

The result presented here will find an immediate application in the context of the so-called Unruh effect, which is often derived from a comparison between the QFT in Minkowski and Rindler spaces. From such a perspective, it is of particular interest that the region M_2 does not extend to the spacelike infinity of Rindler space at $\rho = 0$. If Rindler space is embedded in Minkowski space as the right Rindler wedge, this point is mapped to the origin of Minkowski space. The mathematical problems arising from the singular behavior of the mode expansions used for the derivation of the Unruh effect at the origin of Minkowski space led to a critique of the mathematical basis of the Unruh effect by Narozhnyi *et al.* in Refs. [29–33].[16] By investigating the Unruh effect using region M_2, such problems would be completely avoided. Moreover, the hypercylinder region and isomorphism constructed between the Hilbert spaces used in the different regions can provide a new representation of the mixed state involved in the Unruh effect. This will offer the possibility to study the properties of such a state from a novel perspective. We shall elaborate on that elsewhere.

[16]See also the answer by Fulling and Unruh in Ref. [34] and a reply by Narozhnyi *et al.* in Ref. [35].

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APPENDIX: MODIFIED BESSEL FUNCTIONS

The modified Bessel function of the first kind $I_{\mathrm{i}p}$, with imaginary order, and the modified Bessel function of the second kind $K_{\mathrm{i}p}$, also known as the Macdonald function, are related by [27]

$$K_{ip} = \frac{\mathrm{i}\pi}{2\sinh(\pi p)}(I_{\mathrm{i}p} - \overline{I_{\mathrm{i}p}}). \tag{A1}$$

The Wronskian between the modified Bessel function of the first kind and its complex conjugate results to be

$$W_z(I_{i|p|}(z), \overline{I_{i|p|}(z)}) = \frac{2\sinh(\pi p)}{i\pi z}, \tag{A2}$$

Both these Bessel functions have an oscillatory behavior in a neighborhood of the origin ($\rho = 0$) [36],

$$\begin{aligned} I_{\mathrm{i}p}(m\rho) &\approx \left(\frac{m\rho}{2}\right)^{\mathrm{i}p} / \Gamma(\mathrm{i}p + 1), \\ K_{\mathrm{i}p}(m\rho) &\approx \sqrt{\frac{\pi}{p\sinh(\pi p)}} \cos\left(-p\ln\frac{m\rho}{2} + \arg\Gamma(\mathrm{i}p)\right). \end{aligned} \tag{A3}$$

The behavior of the Bessel function $K_{\mathrm{i}p}$ for a small value of the argument has been derived in Ref. [37]. For asymptotic values of their argument, the modified Bessel functions behave very differently,

$$\begin{aligned} I_{\mathrm{i}p}(m\rho) &\approx \frac{e^{m\rho}}{\sqrt{2\pi m\rho}}, \\ K_{\mathrm{i}p}(m\rho) &\approx \sqrt{\frac{\pi}{2m\rho}} e^{-m\rho}, \quad \text{for } \rho \gg 1. \end{aligned} \tag{A4}$$

The MacDonald function satisfies the identity

$$\int_0^\infty \frac{\mathrm{d}\rho}{\rho} K_{\mathrm{i}\mu}(\rho) K_{\mathrm{i}\mu'}(\rho) \frac{2\mu\sinh(\mu\pi)}{\pi^2} = \delta(\mu - \mu'), \tag{A5}$$

which allows us to expand the field configuration $\varphi(\rho)$ on the hypersurface of constant Rindler time as

$$\varphi(\rho) = \int \mathrm{d}p\, \varphi(p) \frac{\sqrt{2p\sinh(p\pi)}}{\pi} K_{\mathrm{i}p}(m\rho), \qquad p \geq 0. \tag{A6}$$

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