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Wojciech Kamiski and Sebastian Steinhaus

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Coherent states, 6*j* symbols and properties of the next to leading order asymptotic expansions

Wojciech Kamiński[1,2,3,a)] and Sebastian Steinhaus[2,3,b)]
[1]*Wydział Fizyki, Uniwersytet Warszawski, Hoża 69, 00-681, Warsaw, Poland*
[2]*Perimeter Institute for Theoretical Physics, 31 Caroline Street North, Waterloo, Ontario N2L 2Y5, Canada*
[3]*Max Planck Institute for Gravitational Physics, Am Mühlenberg 1, D-14476 Potsdam, Germany*

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We present the first complete derivation of the well-known asymptotic expansion of the $SU(2)$ $6j$ symbol using a coherent state approach, in particular we succeed in computing the determinant of the Hessian matrix. To do so, we smear the coherent states and perform a partial stationary point analysis with respect to the smearing parameters. This allows us to transform the variables from group elements to dihedral angles of a tetrahedron resulting in an effective action, which coincides with the action of first order Regge calculus associated to a tetrahedron. To perform the remaining stationary point analysis, we compute its Hessian matrix and obtain the correct measure factor. Furthermore, we expand the discussion of the asymptotic formula to next to leading order terms, prove some of their properties and derive a recursion relation for the full $6j$ symbol. © *2013 AIP Publishing LLC*. [http://dx.doi.org/10.1063/1.4849515]

I. INTRODUCTION

Spin foam models[1–4] are candidate models for quantum gravity invented as a generalization of Feynman diagrams to higher dimensional objects. Their popularity is rooted in the fact that they were well adapted to describe 3D quantum gravity theories such as the Ponzano-Regge[5,6] or the Turaev-Viro model.[7] To examine whether these models are a quantum theory of 4D General Relativity, in particular whether one obtains gravity in a semi-classical limit is an active area of topical research. One of the strongest positive implications comes from the asymptotic analysis of single simplices in spin foam models: A first attempt to compute the asymptotic expansion of the amplitude associated to a 4-simplex in the Barrett-Crane model[8] can be found in Refs. 9 and 10. This was continued for the square of (the Euclidean and Lorentzian) $6j$ and $10j$ symbols in Ref. 11, whereas the most recent asymptotic results for modern spin foam models, i.e., the EPRL-model[12] or the FK-model,[13] were obtained using a coherent state approach.[14–19]

The basic amplitudes of the spin foam model are their vertex amplitudes ($SU(2)$ $6j$ symbols in the 3D Ponzano Regge model). They are defined in a representation theoretic way and can be constructed from coherent states of the underlying Lie group[20] as a multidimensional integral to which the stationary point approximation is applicable.[21] This method has proven to be very efficient in determining the dominating phase in the asymptotic formula as well as the geometric interpretation of the contributions to the asymptotic expansion in spin foam models.[14–19] In 3D, on the points of stationary phase, $6j$ symbols are geometrically interpreted as tetrahedra, their dominating phase given by the Regge action,[22,23] a discrete version of General Relativity on a triangulation. Similar results were proven by this method for the 4-simplex[9,14–18] in spin foam models. Until today, this is still one of the most promising evidences that spin foam models are viable quantum gravity theories.

[a)]wkaminsk@fuw.edu.pl
[b)]steinhaus.sebastian@gmail.com

Despite this success, the coherent state approach fails to produce the full amplitude. It has not yet been possible to compute the so-called measure factor, a proportionality constant (depending on the representation labels) in the asymptotic expansion, which is given by the determinant of the matrix of second derivatives, i.e., the Hessian matrix, evaluated on the stationary point. This failure even applies to the simplest spin foam model in 3D, the Ponzano-Regge model,[6] whose vertex amplitude is the $SU(2)$ $6j$ symbol. To the authors' best knowledge the Ponzano and Regge formula[5] has not yet been obtained this way; we can only refer to numerical results in Ref. 19. This is particularly troubling for the coherent state approach, since the full asymptotic formula for $SU(2)$ $6j$ symbols introduced in Ref. 5 has been proven in many different ways, for example, by geometric quantization,[24] Bohr-Sommerfeld approach,[25] Euler-MacLaurin approximation[26] or the character integration method.[11]

The source of the problem is the size of the Hessian matrix and the lack of immediate geometric formulas for its determinant. For the $6j$ symbol, for example, this matrix is 9 dimensional and its entries are basis dependent. This is a major drawback of the coherent state approach, in particular, since the full expansion is necessary to discuss and examine the properties of spin foams models of quantum gravity. To obtain this measure factor and compare it to other approaches,[27,28] the complete asymptotic expansion is indispensable. This is an important open issue for 4D spin foam models.

Our approach to overcome this problem can be seen (as we will show in Appendix H) as a combination of the coherent state approach[14–19] and the propagator kernel method.[29] It inherits nice geometric properties from the coherent state analysis with a similar geometric interpretation of the points of stationary phase. Moreover, the Hessian matrix is always described in terms of geometrical quantities and, most importantly, its determinant can be computed for the $6j$ symbol.

In addition to the computation of the asymptotic formula of the $6j$ symbol,[5] our approach allows us to propose a new way to compute higher order corrections to the asymptotic expansion. These corrections have already been discussed in Refs. 30 and 31: it was conjectured that the asymptotic expansion has an alternating form

$$\{6j\} = A_0 \cos\left(\sum \left(j_i + \frac{1}{2}\right)\theta_i + \frac{\pi}{4}\right) + A_1 \sin\left(\sum \left(j_i + \frac{1}{2}\right)\theta_i + \frac{\pi}{4}\right) + \ldots, \tag{1.1}$$

where A_n are consecutive higher order corrections and homogeneous functions in $j + \frac{1}{2}$. Our method allows us to prove this conjecture to any order in the asymptotic expansion.

A. Coherent states and integration kernels

The coherent state approach is based on the following principle: Invariants (under the action of the group) can be constructed by integration of a tensor product of vectors (living in the tensor product of vector spaces of irreducible representations) over the group, i.e., group averaging. Since the invariant subspace of the tensor product of three representations of $SU(2)$ is one-dimensional, the invariant is uniquely defined up to normalization. However, in order to apply the stationary point analysis the vectors in the construction above cannot be chosen arbitrarily. The choice, from which the method takes its name, is the coherent states class, which consists of eigenvectors of the generators of rotations with highest eigenvalues.[20] Although these states are very effective in obtaining the dominating phase of the amplitude, the associated Hessian matrix turns out to be very complicated. This problem occurs since the action is not purely imaginary, which is also related to the problem of choice of phase for the coherent states which has not yet been fully understood.

Both latter problems disappear if, instead of eigenstates with maximal eigenvalues, we take null eigenvectors for a generator of rotations L. Since this vector is trivially invariant with respect to rotations generated by L, the phase problem disappears. Similarly the contraction of invariants can be expanded in terms of an action that actually is purely imaginary. There is a trade-off, though: The quantity of stationary points increases and their geometric interpretation becomes more complicated. Moreover, frequently there exist no such eigenvectors for certain representations, (half-integer spins for $SU(2)$) and their tensor product gives thus vanishing invariants.

The solution to these issues comes from the simple observation that null eigenvectors can be obtained by the integration of a coherent state, pointing in direction perpendicular to the axis of L, over the rotations generated by L. Like that the geometric interpretation usually obtained when using coherent states is restored. Furthermore, if we first perform the partial stationary phase approximation with respect to the additional circle variables, we obtain a purely imaginary action. In the special case of the $6j$ symbol, our construction allows us to write the invariant purely in terms of edge lengths and dihedral angles of a tetrahedron, in particular we perform a variable transformation from group elements to dihedral angles of the tetrahedron. The resulting phase of the integral is given be the first order Regge action.[32]

B. Relation to discrete gravity

Regge calculus[22,23] is a discrete version of General Relativity defined upon a triangulation of the manifold. Influenced by Palatini's formulation, a first order Regge calculus was derived in Ref. 32, in which both edge lengths and dihedral angles are considered as independent variables and their respective equations of motion are first order differential equations. Additional constraints on the angles have to be imposed in order to reobtain their geometric interpretation. These consist of the vanishing of the angle Gram matrix that implies the existence of the flat n-simplex with the given angles. Our derivation of the Ponzano-Regge formula shows astonishing similarity to this procedure. Moreover, from our calculation one can deduce a suitable measure for first order (linearized) Quantum Regge calculus, such that the expected Ponzano-Regge factor $\frac{1}{\sqrt{V}}$ appears, which naturally leads to a triangulation invariant measure.[27]

Another version of 4D Regge calculus was explored in Ref. 33 with areas of triangles and (a class of) dihedral angles as fundamental and independent variables. Several local constraints guarantee that the geometry of a 4-simplex is uniquely determined. These variables were chosen in the pursuit to better understand the relation between discrete gravity and 4D spin foam models. The latter are based on a similar paradigm as the Ponzano-Regge or the Turaev-Viro models[5,7] in 3D, yet enhanced by the implementation of the simplicity constraints from the Plebanski formulation of General Relativity.[34] Area-angle variables as a discretization of Plebanski rather than Einstein-Hilbert formulation were conjectured to be more suitable to describe the semi-classical limit of those models.

Although it is known that the asymptotic limit of the amplitude of a 4-simplex for 4D gravity models is proportional to the cosine of the Regge action,[15–17] the proportionality factor still remains unknown. We hope that the method presented in this work can help in filling the gap.

C. Problem of the next to leading order (NLO) and complete asymptotic expansion

The asymptotic expansion for the $SU(2)$ $6j$ symbol, in particular for the next to leading order (NLO), is still a scarcely examined issue, since it is very non-trivial to write the (NLO) contributions in a compact form. Steps forward in this direction can be found in Refs. 30, 31, and 35, where the latter gives the complete expansion in the isosceles case of the $6j$ symbol.

The stationary point analysis applied in this work allows for a natural extension in a Feynman diagrammatic approach. From this approach the full expansion can be computed in principle, however in a very lengthy way. We derive a recursion relations of the Ward-Takesaki type, which is surprisingly similar to the one invented in Refs. 36 and 37 however in very different context, that, basically can be used in the asymptotic expansion to derive the NLO in a more concise way. Moreover, we can show explicitly that the consecutive terms in the expansion (1.1) are of the conjectured "sin/cos" form.

D. Organization of the paper

This paper is organized as follows: In Sec. II we will present our modified coherent states, how to use them to construct invariants and how to contract these invariants to compute spin network amplitudes. The contracted invariants will be used to define an action for the stationary point analysis, which will be examined whether it allows for the same geometric interpretation on its stationary points as other coherent state approaches.[14–19] Its symmetries as well as the group generated by the symmetry transformations will be discussed. Section III deals with the partial stationary point analysis with respect to the introduced circle variables. This will allow us to write the amplitude, after a variable transformation, purely in terms of angle variables, which will be identified as exterior dihedral angles of a polyhedron. In Sec. IV we focus on the example of the $6j$ symbol. After another variable transformation, we obtain the action of first order Regge calculus and perform the remaining stationary point analysis. Eventually, we obtain the asymptotic formula from Ref. 5. In Sec. V we prove the conjecture from Refs. 30 and 31 that the full asymptotic expansion is of alternating form (1.1) and derive the recursion relations for the full $6j$ symbol. We conclude with a discussion of the results and an outlook in Sec. VI.

We would like to point out that several results of this paper have been obtained by tedious calculations which we did not include in its main part to improve readability. Interested readers are welcome to look them up in Appendices A–H.

II. MODIFIED COHERENT STATES, SPIN-NETWORK EVALUATIONS AND SYMMETRIES

In this section we are going to present the modified coherent states, how to construct the spin-network evaluation from them and that they allow for the same geometric interpretation in the stationary point analysis as similar coherent state approaches. Furthermore, the symmetries of the action will be investigated.

Consider a three-valent spin network, i.e., a graph with three-valent nodes carrying $SU(2)$ intertwiners and edges carrying irreducible representations of $SU(2)$. For each edge of the spin network we introduce a (fiducial) orientation such that each node of the network can be denoted as the "source" $s(e)$ or the "target" $t(e)$ of the edge e. Later in this work we intend to give a geometrical meaning to the spin network, in terms of polyhedra, triangles, etc. so we denote the set of nodes by F and the set of edges by E, which will become the set of triangles/faces and set of edges of the triangulation, respectively. This dual identification is not always possible but we restrict our attention to the case of planar (spherical) graphs, where such notions are natural.

A. Intertwiners from modified coherent states

Intertwiners are invariant vectors (with respect to the action of the group) in the tensor product of vector spaces associated to irreducible representations of that group. In the case of 3 irreducible representations of $SU(2)$ the space of invariants is one dimensional and, moreover, there is a unique choice for the invariant for a given cyclic order of representations.[38,39]

Suppose $\xi \in V_{j_1} \otimes \cdots \otimes V_{j_n}$ is a vector in the tensor product of vector spaces of representations, then

$$\int_{SU(2)} dU\ U\xi \tag{2.1}$$

is invariant under the action of $SU(2)$. If ξ is chosen in a clever way, such an invariant is non-trivial. In the case of three representations it must be proportional to the unique invariant defined in Refs. 38 and 39. In the following we present a choice which has the advantage that the method of stationary phase can be directly applied.

For every face f, which is bounded by three edges, we choose a cyclic order of these edges $(j_{fe_1}, j_{fe_2}, j_{fe_3})$, labelled by the carried representations. These choices influence the orientation of the spin network[38,39] and are used to define and determine the sign of its amplitude, see also Appendix A. We introduce the following intertwiners for every face f:

$$C_f = \int_{SU(2)} dU_f\ U_f \int \prod_j \frac{d\phi_{ji}}{2\pi} f_f\left(\{\phi_{fe}\}_{e\in F}\right) \prod_{e\in F} \left(O_{\phi_{fe}}|1/2\rangle\right)^{2j_e}, \tag{2.2}$$

where f_f is a function of the three angles ϕ_{fe}, $e \subset f$, $|1/2\rangle$ is the basic state of the fundamental representation and O_ϕ is a rotation matrix on $\mathbb{R}^2$:

$$O_\phi = \begin{pmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{pmatrix}. \tag{2.3}$$

As mentioned above, (2.2) is invariant under the action of $SU(2)$.

Before moving on, we would like to outline the key differences between the approach described above and the usual coherent state approach.[15–17,19]

- Coherent states of $SU(2)$ are labelled by vectors in $\mathbb{R}^3$. On the stationary point with satisfied reality conditions, one obtains the geometric interpretation that for every face these three vectors form the edge vectors of a triangle. Later on we will prove the same geometric interpretation for the invariant C_f.
- Furthermore, we smear the coherent state by a rotation, which is the key ingredient of our approach. In addition to the stationary point analysis with respect to the $\{U_f\}$, we will also perform a stationary point analysis for the smearing angles $\{\phi_{fe}\}$. Clearly, this will result in more stationary points contributing to the final amplitude. To suppress their contributions, we introduce modifiers f_f which will be described in Sec. II A 1.

1. Prescription of the modifiers

In order to make (2.2) complete, we have to describe the function f_f.

For every face f we choose three vectors v_e ($e \subset f$) on $\mathbb{R}^2$ with norms j_e such that $\sum_{e\subset f} v_e = 0$, i.e., they form a triangle with edge lengths j_e. The vectors are ordered anti-clockwise, their choice is unique up to Euclidean transformations, i.e., rotations and translations (Fig. 1).

Let us denote the edges (in cyclic order) by 1, 2, 3. The angles (counted clockwise) between the vectors v_k and v_j are denoted by $2(\psi_{kj} - \pi)$, where $2(\psi_{kj} - \pi)$ is the $SO(3)$ angle taking values in $(0, \pi)$ for $(k, j) \in \{(2, 1), (3, 2), (1, 3)\}$. Due to the ordering, the $SU(2)$ angles ψ_{21}, ψ_{32}, and ψ_{13} are positive and smaller than 2π, in fact, one can also check that ψ_{kj} is in $(\pi, 2\pi)$. This choice contributes an overall sign to the invariant, to be more precise, there are two different choices of cyclic order giving two invariants that may differ by a sign factor. This will be discussed in more detail in Appendix A 2 a. In particular we compare them to the intertwiner introduced in Refs. 38 and 39. The angles ψ_{kj} satisfy the relation

$$\psi_{21} + \psi_{32} + \psi_{13} = 4\pi. \tag{2.4}$$

We introduce a function $f(x \bmod 2\pi, y \bmod 2\pi)$ such that

- it is equal to 1 in the neighbourhood of $x = \psi_{21}$, $y = \psi_{32}$,
- it is equal to zero in the neighbourhood of points

$$(x, y) = \pm(\psi_{21} + \pi, \psi_{32}), \pm(\psi_{21}, \psi_{32} + \pi), \pm(\psi_{21} + \pi, \psi_{32} + \pi), (-\psi_{21}, -\psi_{32}). \tag{2.5}$$

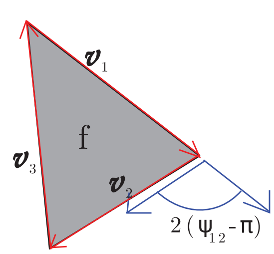

FIG. 1. The choice of vectors v_i.

Hence, we define

$$f_f(\phi_{f1}, \phi_{f2}, \phi_{f3}) = f(\phi_{f2} - \phi_{f1}, \phi_{f3} - \phi_{f2}). \tag{2.6}$$

2. The spin network evaluation

Given the definition of invariants in (2.2) it is straightforward to define the evaluation of a given spin network: The intertwiners are contracted with each other according to the combinatorics of the network. The resulting amplitude has to be normalized, i.e., divided by the product of norms of our intertwiners, see Sec. III D. It is, however, not sufficient in order to agree with the canonical definition.[38,39] The remaining sign ambiguity will be resolved in Appendix A.

As in the standard coherent state approach the amplitude (contraction of intertwiners) then reads

$$(-1)^s \int \prod_{f\in F} \mathrm{d}\, U_f \prod_{e\subset f} \frac{\mathrm{d}\,\phi_{fe}}{2\pi} \prod_f f_f\left(\{\phi_{fe}\}_{e\subset f}\right) \underbrace{\prod_{e\in E} \epsilon\Big(U_{s(e)} O_{\phi_{s(e)e}} |1/2\rangle\ ,\ U_{t(e)} O_{\phi_{t(e)e}} |1/2\rangle\Big)^{2j_e}}_{e^S}, \tag{2.7}$$

where s is the sign factor as prescribed in Refs. 38 and 39 (see Appendix A), and $\epsilon(\cdot,\cdot)$ is an invariant bilinear form defined by

$$\epsilon(|1/2\rangle, |1/2\rangle) = \epsilon(|-1/2\rangle, |-1/2\rangle) = 0, \quad \epsilon(|1/2\rangle, |-1/2\rangle) = -\epsilon(|-1/2\rangle, |1/2\rangle) = 1. \tag{2.8}$$

The choice of the orientation of edges, faces and the sign factor prescription will be described in the Appendix A 1. To perform the stationary point analysis we rewrite (part of) the integral kernel as an exponential function and define the 'action' S. From (2.7) one can deduce that

$$S = \sum_e S_e, \tag{2.9}$$

where the action S_e (labelled by the edge e) is given by

$$S_e = 2j_e \ln \epsilon\left(U_{s(e)} O_{\phi_{s(e)e}} |1/2\rangle\ ,\ U_{t(e)} O_{\phi_{t(e)e}} |1/2\rangle\right). \tag{2.10}$$

B. The action

In order to examine the geometric meaning of the action on its points of stationary phase, let us introduce the following geometric quantities. For each face $f \in F$ we introduce vectors n_f (as traceless Hermitian matrices, which can be naturally identified with vectors in $\mathbb{R}^3$) defined by

$$n_f = U_f H U_f^{-1}, \tag{2.11}$$

where

$$H = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}. \tag{2.12}$$

For each pair $\{f, e\}$ with $e \subset f$, we define vectors B_{fe} (also as traceless matrices):

$$\begin{aligned} B_{fe} &= j_e(2U_f O_{\phi_{fe}} |1/2\rangle\langle 1/2| O_{\phi_{fe}}^{-1} U_f^{-1} - \mathbb{I}) \\ &= U_f O_{\phi_{fe}} \left[j_e \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right] O_{\phi_{fe}}^{-1} U_f^{-1}. \end{aligned} \tag{2.13}$$

Note that the length of B_{fe} is equal to j_e.

We can already deduce that $\Re S \leq 0$. The stationary point analysis contains the conditions $\partial S = 0$ and $\Re S = 0$. These are as follows:

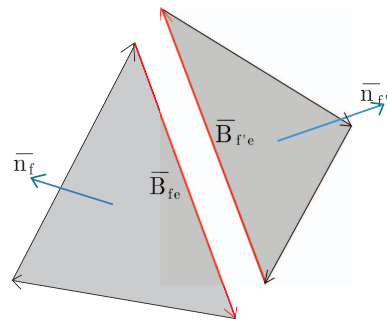

FIG. 2. Stationary point condition.

- The reality condition is satisfied if and only if

$$U_{s(e)}O_{\phi_{s(e)e}}|1/2\rangle \perp U_{t(e)}O_{\phi_{t(e)e}}|1/2\rangle, \tag{2.14}$$

  where $\perp$ means perpendicular in the $SU(2)$ invariant scalar product. This is equivalent to $B_{s(e)e} = -B_{t(e)e}$.
- Using both the reality condition and the definition of B_{fe} we obtain from the variation of S with respect to U_f:

$$X\frac{\partial S_e}{\partial U_f} = \begin{cases} \mathrm{Tr}\, XB_{fe} & e \subset f \\ 0 & e \not\subset f \end{cases}, \tag{2.15}$$

  where X is a generator of the Lie algebra. Hence the action is stationary with respect to U_f if

$$\sum_{e\subset f} B_{fe} = 0. \tag{2.16}$$

- Similarly we obtain for the variation of S with respect to ϕ_{fe} (again using the reality condition):

$$\frac{\partial S_e}{\partial \phi_{fe'}} = \begin{cases} \mathrm{Tr}\, n_f B_{fe} & e = e' \subset f \\ 0 & \text{otherwise} \end{cases}. \tag{2.17}$$

  So the condition from variation with respect to ϕ is

$$\forall_{e\subset f}\; n_f \perp B_{fe}. \tag{2.18}$$

Before we discuss the geometric meaning of the just derived conditions (see Fig. 2), we first have to examine the symmetries of the action to determine the amount of stationary points and their relations.

C. Symmetry transformations of the action

There exist several variable transformations that only change e^S by a sign such that a stationary point is transformed into another stationary point. Some of the transformations mentioned below are continuous so the stationary points form submanifolds of orbits under the action of these symmetries. We will explain the geometric interpretation of these orbits in Sec. II E, and show that these orbits are isolated for many spin networks, e.g., the $6j$ symbol.

The above mentioned transformations are as follows:

- u-symmetry: $\forall u \in SU(2)$,
$$\forall_{f\in F},\ U_f \to uU_f \tag{2.19}$$
applied to all U_f simultaneously preserves e^S. This is the only symmetry which has to be applied to all group elements simultaneously showing that one of the $SU(2)$ integrations in (2.7) is redundant (gauge).
- o_f-rotation: For a chosen face f and ϕ,
$$U_f \to U_f O_\phi, \quad \forall_{e\subset f}\phi_{fe} \to \phi_{fe} - \phi \tag{2.20}$$
preserves e^S, in fact, each e^{S_e} is preserved.
- $-u_f$-symmetry: For any chosen face $f \in F$,
$$U_f \to (-1)U_f \tag{2.21}$$
preserves e^S because for every face $\sum_{e\subset f} j_e$ is an integer.
- r_f-reversal transformation: For any chosen face f,
$$U_f \to U_f \underbrace{\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}}_{D}, \quad \forall_{e\subset f}\ \phi_{fe} \to -\phi_{fe}. \tag{2.22}$$
Because
$$D^{-1}O_\phi D = O_{-\phi}, \quad D|1/2\rangle = i|1/2\rangle\ , \tag{2.23}$$
e^S is multiplied by
$$i^{2\sum_{e\subset f} j_e} = (-1)^{\sum_{e\subset f} j_e}. \tag{2.24}$$
Let us notice that $2j_e \in \mathbb{Z}$ and $\sum_{e\subset f} j_e$ is an integer,
- $-o_{fe}$ transformation: For any chosen pair $e \subset f$
$$\phi_{fe} \to \phi_{fe} + \pi. \tag{2.25}$$
This multiplies the integrated term by $(-1)^{2j_e}$.

Note that the transformations o_f, $-u_f$ $-o_{fe}$, r_f are restricted to variables associated to one face. They transform the functions f_f as follows:

- o_f shifts all angles ϕ_{fe} on f by an angle ϕ:
$$f'_f(\{\phi_{fe}\}) = f_f(\{\phi_{fe} + \phi\}) = f_f(\{\phi_{fe}\}), \tag{2.26}$$
since f_f only depends on differences of angles.
- $-o_{fe}$
$$f'_f(\{\phi_{fe'}\}) = f_f(\{\phi_{fe'} + \delta_{ee'}\pi\}). \tag{2.27}$$
- r_f
$$f'_f(\{\phi_{fe}\}) = f_f(\{-\phi_{fe}\}). \tag{2.28}$$

To sum up, the functions f_f are preserved by u-, $-u_f$-, and o_f-transformations, since the first two do not affect the angles ϕ and the last one translates all angles by a constant.

In addition to that, let us also define an additional transformation c, which we call parity transformation:

$$\forall f: \quad U_f \to U_f \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{2.29}$$

It transforms the integral into its complex conjugate due to the fact that

$$\bar{U} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}^{-1} U \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \tag{2.30}$$

and the f_f, the matrix O_ϕ and the vectors $|\pm 1/2\rangle$ are real.

In Sec. II D we will examine which group is generated by the transformations, i.e., the symmetry group of the action.

D. Groups generated by symmetry transformations

The transformations described in Sec. II C generate a group $\tilde{G}$ with the following relations:

$$\begin{aligned} &u_{(-1)} = \prod_f (-u_f), \\ &\forall_f, \quad r_f^2 = (-u_f), \quad (-u_f)^2 = 1, \quad o_f(2\pi) = 1, \\ &\forall_{e\subset f}, \quad (-o_{fe})^2 = 1, \\ &\forall_f, \quad o_f(\pi) \prod_{e\subset f} (-o_{fe}) = 1\ , \end{aligned} \tag{2.31}$$

and all its elements commute besides u (that form $SU(2)$) and

$$\forall f,\ r_f o_f(\alpha) r_f^{-1} = o_f(-\alpha). \tag{2.32}$$

The group generated by all transformations except u is denoted by G.

In $\tilde{G}$ (resp. G), there is a normal subgroup generated by the transformations u, o_f, $-u_f$ (resp. o_f, $-u_f$), which preserves the modifiers f_f. We denote these subgroups by $\tilde{H}$ (and H respectively); their quotient groups are given by

$$K = \tilde{G}/\tilde{H} = G/H. \tag{2.33}$$

This is an Abelian group generated by

$$\forall_{e\subset f} \quad [r_f],\ [-o_{fe}] \tag{2.34}$$

with relations

$$\forall_f \prod_{e\subset f} [-o_{fe}] = 1,\ [r_f]^2 = [-o_{fe}]^2 = 1, \tag{2.35}$$

which show that K is isomorphic to $\mathbb{Z}_2^{3|F|}$.

In Secs. II E and II F, we will discuss the geometric interpretation of the points of stationary phase.

E. Geometric lemma

Our goal in this section is to describe the geometric interpretation of the stationary point orbits introduced in Sec. II B. In particular, we will show how these points are related to the standard stationary point interpretation in the coherent state method.

Lemma 1. For every set of vectors B_{fe} of length j_e satisfying

$$\begin{aligned} &\forall_e B_{s(e)e} = -B_{t(e)e}, \\ &\forall_f \sum_{e\subset f} B_{fe} = 0, \end{aligned} \tag{2.36}$$

there exist ϕ_{fe} and U_f being a point of stationary phase with vectors B_{fe}. Moreover, all these points are related via G transformations.

Proof. For every f we can choose the unit vector n_f perpendicular to all B_{fe} (for all $e \subset f$). Such a normal is only determined up to a sign. Let us choose U_f such that

$$n_f = U_f H U_f^{-1}. \tag{2.37}$$

Such a choice always exists, but it is not unique. U_f is only determined up to the transformation

$$U_f \to U_f D O_\phi \tag{2.38}$$

since D, defined in (2.22), stabilizes H up to a sign

$$DHD^{-1} = -H. \tag{2.39}$$

This is called the D_∞ group.

The vectors $U_f^{-1} B_{fe} U_f$ are orthogonal to H. The operators $U_f^{-1} B_{fe} U_f$ are thus real and we can choose their eigenvectors with positive eigenvalues as

$$\begin{pmatrix} \cos\phi_{fe} \\ \sin\phi_{fe} \end{pmatrix}. \tag{2.40}$$

Hence ϕ_{fe} is fixed (up to π).

It is straightforward to check that this construction gives a stationary point, in fact, each stationary point with vectors B_{fe} must be constructed in this way. The ambiguities in the choices above are all related by o_f-, $-u_f$-, $-o_f$-, and r_f-transformations, i.e., G-transformations. □

For every face f on the stationary point $\sum_{e\subset f} B_{fe} = 0$ and given the definition of f_f in Sec. II A 1, there is a unique choice (up to o_f transformations) of the stationary point angles ϕ_{fe} such that f_f is nonzero. In the neighbourhood of those stationary point $f_f = 1$, whereas around all remaining ones at least one of the functions f_f is zero:

Lemma 2. For given vectors B_{fe} satisfying (2.36), *there exists only one orbit (orbit of the action of the group $\tilde{H}$) of stationary points of the action, such that*

$$\prod f_f(\{\phi_{fe}\}) \neq 0, \tag{2.41}$$

and in the neighbourhood of this orbit

$$\prod f_f(\{\phi_{fe}\}) = 1. \tag{2.42}$$

Note that the normals to the faces change sign under r_f transformations:

$$n_f \to -n_f. \tag{2.43}$$

Under the c-transformation, the B_{fe} are inverted, but the normals to the faces are not affected, i.e., they behave as pseudovectors.

$$\begin{aligned} B_{fe} &\to -B_{fe}, \\ n_f &\to n_f. \end{aligned} \tag{2.44}$$

In Sec. II F we will specify the definition of the normals n_f.

F. Normal vectors to the faces

We will now give a precise geometric definition of n_f (normal to the face). To simplify notation we will omit the subscript ϕ in $O_{\phi_{fe}}$. Note that

$$n_f = U_f O_{fe} H O_{fe}^{-1} U_f^{-1} \tag{2.45}$$

for any edge $e \subset f$, since O_{fe} and H commute.

Take two consecutive edges $e_1, e_2 \subset f$ and their respective edge vectors B_{fe_i}:

$$B_{fe_1} = U_f O_{fe_1} \left[j_{e_1} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right] O_{fe_1}^{-1} U_f^{-1}, \tag{2.46}$$

$$B_{fe_2} = U_f O_{fe_1} (O_{fe_1}^{-1} O_{fe_2}) \left[j_{e_2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right] (O_{fe_1}^{-1} O_{fe_2})^{-1} O_{fe_1}^{-1} U_f^{-1}. \tag{2.47}$$

Rotating all three vectors by $U_f O_{fe_1}$ one obtains (the rotated vectors are denoted by B'_{fe_i}, n'_f):

$$n'_f = H, \quad B'_{fe_1} = j_{e_1} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad B_{fe_2} = O_{fe_1}^{-1} O_{fe_2} \left[j_{e_2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right] (O_{fe_1}^{-1} O_{fe_2})^{-1}. \tag{2.48}$$

For a stationary point with non-vanishing modifier f_f, $O_{fe_1}^{-1} O_{fe_2}$ describes the rotation by the $SO(3)$ angle $0 < 2(\psi_{12} - \pi) < \pi$. We thus conclude

$$n_f \cdot (B_{fe_1} \times B_{fe_2}) = n'_f \cdot (B'_{fe_1} \times B'_{fe_2}) > 0, \tag{2.49}$$

where we regard n_f and B_{fe_i} as vectors using the natural identification of hermitian matrices with $\mathbb{R}^3$ (tracial scalar product). Condition (2.49) fixes the sign of n_f and also completes the geometric interpretation of the points of stationary phase.

G. Interpretation of planar (spherical) spin-networks as polyhedra

In the last section we obtained an interpretation of the stationary points in terms of a set of vectors B_{fe} satisfying closure conditions for every face f

$$\sum_e B_{fe} = 0. \tag{2.50}$$

However, these conditions do not specify a unique reconstruction of the according surface dual to the spin network. In fact, already each triangle allows for two different configurations of B_{fe} vectors. Therefore, we will here describe a method to reconstruct the surface from B_{fe} vectors for the spherical case:

Let us draw the graph on the sphere (on the plane) as described in the Appendix A 1. From the possible ways of drawing it, which in the case of 2-edge irreducible spin networks is in one-to-one correspondence with the orientation of the spin network, we have to choose one. In the case of 2-edge irreducible graphs the polyhedra obtained from different choices only differ by orientation. In addition to nodes and edges, there is also a natural notion of two-cells. The latter are defined as areas bounded by loops of edges. We are mainly interested in the dual picture that in this case is a triangulation of the sphere. Thus there is a unique identification of the vertices in the dual picture. A cyclic ordering of the edges for each f is inherited from the orientation of the sphere.

In the following, we will construct an immersion (not an embedding) of this triangulation of the sphere into $\mathbb{R}^3$, such that every edge e is given by $B_{t(e)e}$ (with the right orientation).

Let us choose one vertex v_0. Every other vertex v' can be connected to v_0 by a path

$$v_0, e_0, v_1, e_1, \ldots, v'. \tag{2.51}$$

Every edge e_i in the sequence belongs to two faces. Exactly one of these faces is such that v_i, v_{i+1} are the consecutive vertices with respect to the cyclic order of the face. We denote this face by f_i (see Figure 3). We introduce the vector

$$\tilde{v}' = \sum_i B_{f_i e_i}. \tag{2.52}$$

One can prove that this vector does not depend on the chosen path. To see this, let us consider a basic move that consists of replacing v_i, e_i, v_{i+1} by v_i, e, v, e', v_{i+1} where all three vertices belong

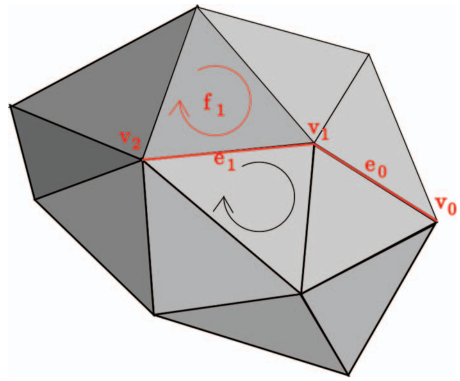

FIG. 3. Reconstruction of the surface.

to the same face f. Using the property

$$\sum_{e\subset f} B_{fe} = 0 \tag{2.53}$$

and the proper orientation, one can show that the vector $\tilde{v}'$ is invariant with respect to this move. In fact any two paths can be transformed into one another by a sequence of these basic moves (or their inverses) due to the fact that the graph is spherical. A different choice of v_0 gives a translated surface. It is straightforward to check that

$$\tilde{v}_b - \tilde{v}_a = B_{fe}, \tag{2.54}$$

where v_a and v_b are vertices joint by the edge e and f is the face such that (v_a, v_b) is the pair of consecutive vertices in the cyclic order of f.

Let us notice that from three vectors B_{fe} satisfying the closure condition one can form a triangle in two ways (see Figure 4), but only that one depicted on the left appears in the reconstruction discussed here. Moreover, the direction of the normal to the face coincides with the orientation inherited by the face from the cyclic order of its edges.

For non-planar graphs, in general, we can only reconstruct the universal cover of the surface.

Before we continue with the stationary point analysis for the angles ϕ_{fe} in Sec. III, let us briefly summarize the results of Sec. II: We have introduced a class of modified coherent states for irreducible representations of $SU(2)$, which contain an additional smearing parameter, and presented how to construct invariants from them. From the contraction of these invariants (according to the spin network) an effective action has been derived, whose points of stationary phase allow for the same geometrical interpretation as the standard coherent states.[19,20] The amount of stationary points is significantly increased by the smearing parameters, yet they are all related by symmetry transformations of the action; a certain set of them can be suppressed by the prescribed modifiers. Eventually, we have depicted a way to reconstruct a triangulation from planar spin networks.

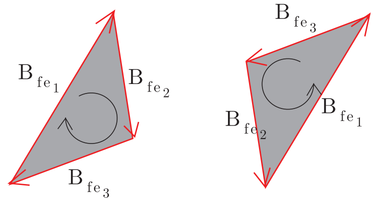

FIG. 4. Two possible triangles formed by B_{fe} satisfying closure condition.

III. VARIABLE TRANSFORMATION AND FINAL FORM OF THE INTEGRAL

In this section we focus on the stationary point analysis with respect to the angles ϕ_{fe}, which is the key modification in comparison to previously used coherent state approaches, see also Sec. II A. This analysis allows us to obtain an effective action for S_e associated to the edge e in terms of a phase, which we will identify as the angle between the normals of the faces sharing the edge e. Furthermore, we are able to expand the effective action for S_e in orders of $\frac{1}{j}$ and initiate the discussion of next-to-leading order contributions.

A. Partial integration over ϕ and the new action

Suppose that we have a non-degenerate configuration, i.e.,

$$\forall_e n_{s(e)} \cdot n_{t(e)} \neq \pm 1. \tag{3.1}$$

Then the partial stationary point analysis with respect to all ϕ_{fe} can be performed. Its result will be the sum over the contribution from all stationary points with respect to ϕ_{fe} for a given configuration of B_{fe} vectors, but for fixed U_f (so also fixed n_f).

1. Stationary points for S_e

In this section we will explain the contribution to the integral from the stationary point of the action S_e with respect to $\phi_{s(e)e}$, $\phi_{t(e)e}$. The $f_{s(e)}, f_{t(e)}$ terms can be ignored, since they are equal to 1 around the stationary point.

We can separately consider terms corresponding to each edge

$$\frac{1}{4\pi^2}\int d\phi_{s(e)e} d\phi_{t(e)e}\, \epsilon\left(U_{s(e)} O_{s(e)e}|1/2\rangle\ ,\ U_{t(e)} O_{t(e)e}|1/2\rangle\right)^{2j_e}, \tag{3.2}$$

and perform the stationary point analysis that gives the asymptotic result of the integration over $\phi_{s(e)e}$, $\phi_{t(e)e}$. The stationary point with respect to $\phi_{t(e)e}$ and $\phi_{s(e)e}$ is given by the conditions

$$U_{s(e)} O_{s(e)e}|1/2\rangle \perp U_{t(e)} O_{t(e)e}|1/2\rangle, \tag{3.3}$$

which is equivalent to

$$U = O^{-1}_{s(e)e} U^{-1}_{s(e)} U_{t(e)} O_{t(e)e} = (-1)^{\tilde{s}} e^{-i\tilde{\theta}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}} \begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}, \tag{3.4}$$

where $\tilde{\theta} \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ and $\tilde{s} \in \{0, 1\}$ are uniquely determined by this equation. In Sec. III A 2 we will show that $2\tilde{\theta}$ can be interpreted as the angle enclosed by the normal vectors $n_{s(e)}$ and $n_{t(e)}$ (with respect to the axis $B_{t(e)e}$). Hence, S_e on the stationary point is of the following form:

$$S_e = 2j_e \ln \epsilon(\cdots) = 2j_e\tilde{\theta} + i\ 2j_e\pi\tilde{s}. \tag{3.5}$$

As already discussed in Sec. II E, each stationary point is characterized by the existence of $B_{t(e)e} = -B_{s(e)e}$ orthogonal to both $n_{s(e)}$ and $n_{t(e)}$ (see also stationary point conditions in Sec. II B). There exist two such configurations that differ by a sign of $B_{s(e)e}$.

For every configuration one has 4 stationary points that can be obtained from one another by $-o_{s(e)}$- and $-o_{t(e)}$-transformations. In case j_e is an integer the contributions from the two stationary points are equal, see also Sec. II C.

Contributions from B_{fe} configurations with opposite signs are related by complex conjugation.

2. Geometric interpretation of the angle $\tilde{\theta}$

The missing piece of the description above is the exact value of the angle $\tilde{\theta}$. Here we will provide a geometric interpretation of this angle and its relation to the angle between faces. Let us recall

$$B_{s(e)e} = j_e U_{s(e)} O_{s(e)e} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} O^{-1}_{s(e)e} U^{-1}_{s(e)}, \tag{3.6}$$

$$n_{s(e)} = U_{s(e)} O_{s(e)e} H\, O^{-1}_{s(e)e} U^{-1}_{s(e)}, \tag{3.7}$$

$$n_{t(e)} = U_{t(e)} O_{t(e)e} H\, O^{-1}_{t(e)e} U^{-1}_{t(e)} = e^{i\tilde{\theta}\frac{B_{t(e)e}}{|B_{t(e)e}|}} n_{s(e)} e^{-i\tilde{\theta}\frac{B_{t(e)e}}{|B_{t(e)e}|}}. \tag{3.8}$$

The angle $2\tilde{\theta}$ is the angle by which one needs to rotate $n_{s(e)}$ around the axis $B_{t(e)e}$ to obtain $n_{t(e)}$. We will denote this $SO(3)$ angle by

$$\theta = 2\tilde{\theta}, \quad \theta \in (-\pi, \pi). \tag{3.9}$$

This remaining ambiguity of the sign factor $\tilde{s}$ will be resolved in Appendix A.

B. Partial integration over ϕ

We introduce new variables

$$\phi_1 = \phi_{s(e)} - \phi^0_{s(e)}, \quad \phi_2 = \phi_{t(e)} - \phi^0_{t(e)}\,, \tag{3.10}$$

where $\phi^0_{s(e)}$ and $\phi^0_{t(e)}$ denote the stationary points. Then using (3.4), we can write the action as

$$\frac{1}{4\pi^2}\int \mathrm{d}\,\phi_1\,\mathrm{d}\,\phi_2 (-1)^{2j_e\tilde{s}} \left(e^{i\tilde{\theta}}\cos\phi_1\cos\phi_2 + e^{-i\tilde{\theta}}\sin\phi_1\sin\phi_2\right)^{2j}, \tag{3.11}$$

where we integrate over ϕ_i. By splitting the terms in the bracket in real and imaginary part, we obtain

$$\cos\tilde{\theta}\,\underbrace{(\cos\phi_1\cos\phi_2 + \sin\phi_1\sin\phi_2)}_{\cos(\phi_1-\phi_2)} + i\sin\tilde{\theta}\,\underbrace{(\cos\phi_1\cos\phi_2 - \sin\phi_1\sin\phi_2)}_{\cos(\phi_1+\phi_2)}. \tag{3.12}$$

We define new variables

$$\alpha := \phi_1 - \phi_2, \quad \beta := \phi_1 + \phi_2, \tag{3.13}$$

and the Jacobian for this transformation is given by

$$\left|\frac{\partial\alpha\partial\beta}{\partial\phi_1\partial\phi_2}\right| = \begin{vmatrix} 1 & -1 \\ 1 & 1 \end{vmatrix} = 2. \tag{3.14}$$

Hence equation (3.11) becomes

$$\frac{1}{8\pi^2}\int d\alpha d\beta\, (-1)^{2j_e\tilde{s}} \left(\cos\tilde{\theta}\cos\alpha + i\sin\tilde{\theta}\cos\beta\right)^{2j_e} =$$

$$= \frac{1}{8\pi^2}\int d\alpha d\beta\, (-1)^{2j_e\tilde{s}} \exp\left\{\underbrace{2j_e\ln\left(\cos\tilde{\theta}\cos\alpha + i\sin\tilde{\theta}\cos\beta\right)}_{=:S'_e}\right\}, \tag{3.15}$$

where $S_e = S'_e + i\,2j_e\pi\tilde{s}$.

1. Expansion around stationary points

Given the definitions from Sec. III B, we compute the expansion of the following expression:

$$\frac{1}{8\pi^2}\int \mathrm{d}\,\alpha\,\mathrm{d}\,\beta\,(-1)^{2j\tilde{s}} e^{S'_e}. \tag{3.16}$$

The stationary point is given by $\alpha = \beta = 0$, which corresponds to $\phi_i = 0$, i.e., $\phi_{s(e)} = \phi^0_{s(e)}$, $\phi_{t(e)} = \phi^0_{t(e)}$. In this point the action associated to the edge e becomes

$$S'_e = 2j_e \ln\left(e^{i\tilde{\theta}_e}\right) = i2j_e\tilde{\theta}_e. \tag{3.17}$$

In order to compute the first order contribution, one has to consider the matrix of second derivatives (evaluated on the point of stationary phase):

$$\frac{\partial^2 S'_e}{\partial\alpha^2} = -2j_e\frac{\cos\tilde{\theta}\cos\alpha}{\cos\tilde{\theta}\cos\alpha + i\sin\tilde{\theta}\cos\beta}, \tag{3.18}$$

$$\frac{\partial^2 S'_e}{\partial\alpha\partial\beta} = 0 = \frac{\partial^2 S'_e}{\partial\beta\partial\alpha}, \tag{3.19}$$

$$\frac{\partial^2 S'_e}{\partial\beta^2} = -2ij_e\frac{\sin\tilde{\theta}\cos\beta}{\cos\tilde{\theta}\cos\alpha + i\sin\tilde{\theta}\cos\beta}. \tag{3.20}$$

Around the stationary point the action can be expanded (up to second order in the variables α, β):

$$S'_e = i2j_e\tilde{\theta} + \frac{1}{2}\begin{pmatrix}\alpha & \beta\end{pmatrix}\begin{pmatrix}-2j_e\cos\tilde{\theta}e^{-i\tilde{\theta}} & 0\\ 0 & -2ij_e\sin\tilde{\theta}e^{-i\tilde{\theta}}\end{pmatrix}\begin{pmatrix}\alpha\\ \beta\end{pmatrix} + \cdots. \tag{3.21}$$

In order to correctly perform the stationary phase approximation, it is indispensable to state the right branch of the square root, here for $\tilde{\theta} \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$:

$$\begin{aligned}\sqrt{\cos\tilde{\theta}e^{-i\tilde{\theta}}} &= \sqrt{|\cos\tilde{\theta}|}e^{-i\frac{1}{2}\tilde{\theta}}\\ \sqrt{i\sin\tilde{\theta}e^{-i\tilde{\theta}}} &= \sqrt{|\sin\tilde{\theta}|}e^{-i\frac{1}{2}\tilde{\theta}}\begin{cases} e^{-i\frac{\pi}{4}} & \tilde{\theta} \in \left(-\frac{\pi}{2}, 0\right)\\ e^{i\frac{\pi}{4}} & \tilde{\theta} \in \left(0, \frac{\pi}{2}\right)\end{cases}.\end{aligned} \tag{3.22}$$

Let us notice that

$$\operatorname{sgn}\sin\theta = \operatorname{sgn}\sin\tilde{\theta} \quad \text{for} \quad \tilde{\theta} \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right). \tag{3.23}$$

Hence, the leading order contribution from the stationary point is

$$\frac{1}{8\pi^2}\frac{2\pi(-1)^{2j_e\tilde{s}}}{\sqrt{2j^2\left|\sin 2\tilde{\theta}\right|}}e^{i2\tilde{\theta}\left(j+\frac{1}{2}\right)-i\frac{\pi}{4}\operatorname{sign}(\sin 2\tilde{\theta})}\left(1 + O\left(\frac{1}{j}\right)\right). \tag{3.24}$$

In Sec. III B 2 we will show an improvement of this result.

2. The total expansion of the edge integral

Let us introduce a number (see Appendix H for a motivation of its origin)

$$C_j = \frac{1}{4^j}\frac{\Gamma(2j+1)}{\Gamma(j+1)^2}. \tag{3.25}$$

We can multiply (3.24) by $\frac{C_j}{C_j} = 1$ and use the expansion $\frac{1}{C_j} = \sqrt{\pi j}\left(1 + O(\frac{1}{j})\right)$ derived in the Appendix D 2 a to write the result as

$$C_{j_e}\frac{(-1)^{s_e}}{4\sqrt{2\pi j_e|\sin\theta|}}e^{i\left(\theta\left(j_e+\frac{1}{2}\right)-\frac{\pi}{4}\operatorname{sign}(\sin\theta)\right)}\left(1 + O\left(\frac{1}{j_e}\right)\right). \tag{3.26}$$

By s_e we denoted the sign factor

$$s_e = \begin{cases} 0 & j_e \text{ integer},\\ 0 & j_e \text{ half-integer and } \tilde{s} = 0\\ 1 & j_e \text{ half-integer and } \tilde{s} = 1.\end{cases} \tag{3.27}$$

We will determine the sign s_e in Appendix A.

We introduce new "length" parameters

$$l_e := j_e + \frac{1}{2}, \tag{3.28}$$

and using the fact that $\frac{(-1)^{s_e}}{4\sqrt{2\pi l_e|\sin\theta|}} = \frac{(-1)^{s_e}}{4\sqrt{2\pi j_e|\sin\theta|}}\left(1+O(j^{-1})\right)$ we can express (3.26) in terms of l_e. Before we move on, we would like to present a first glimpse at the next-to-leading order contribution: As it will be shown in Sec. V B 4 by application of the stationary point analysis (3.26) and the recursion relation (5.29), the contribution (including NLO) from the integral of e^{S_e} over $\phi_{s(e)e}, \phi_{t(e)e}$ is given by

$$C_{j_e}\frac{(-1)^{s_e}}{4\sqrt{2\pi l_e\,|\sin\theta|}}e^{i(l_e\theta-\frac{\pi}{4}\operatorname{sign}(\sin\theta)-\frac{1}{8l_e}\cot\theta)}\left(1+O\left(\frac{1}{l_e^2}\right)\right), \tag{3.29}$$

where $\theta \in (-\pi, \pi)$ is the angle by which one has to rotate $n_{s(e)}$ around $B_{t(e)e}$ to obtain $n_{t(e)}$.

C. New form of the action

In Secs. III A and III B we have computed the contribution of one point of stationary phase with respect to the angles ϕ_{fe}. From Sec. III B we can also conclude that having one stationary point all others are obtained by application of transformations from $\tilde{G}$ that keep U_f fixed. These are given by compositions of

$$(-u_f)o_f(\pi),\ -o_{fe}\ \forall f. \tag{3.30}$$

However, only the orbit generated by the group of $(-u_f)o_f(\pi)$ from a non-trivial stationary point contributes, since all other stationary points are suppressed by the modifiers f_f. Therefore it is sufficient to compute the number of these stationary points. The group generated by $(-u_f)o_f(\pi)$ is equal to $Z_2^{|F|}$ and acts freely on the stationary points; the countability of the orbit is thus $2^{|F|}$.

Around the stationary orbit, the integral is hence of the form

$$(-1)^s 2^{|F|}\int\prod dU_f\prod_e(-1)^{s_e}C_{j_e}\frac{1}{4\sqrt{2\pi\left(j_e+\frac{1}{2}\right)|\sin\theta_e|}}e^{-i\frac{\pi}{4}\operatorname{sgn}\sin\theta_e}\,e^{i\left(j_e+\frac{1}{2}\right)\theta_e-\frac{1}{8(j_e+\frac{1}{2})}\cot\theta_e}, \tag{3.31}$$

where θ_e is the angle between $n_{s(e)}$ and $n_{t(e)}$ with the sign determined by left hand rule with respect to $B_{t(e)e}$. In the neighbourhood of the stationary point this definition is meaningful. The value of the product $\prod_e(-1)^{s_e}$ is discussed in the Appendix A 2 b. We use new "length" parameters introduced in Sec. III B 2

$$l_e := j_e + \frac{1}{2}, \tag{3.32}$$

and perform a change of variables

$$U_f \to n_f, \tag{3.33}$$

which is worked out in the Appendix B 1. The correct integral measure is given by

$$\mu = \frac{1}{2\pi}\delta(|n|^2-1)\,dn_1\,dn_2\,dn_3. \tag{3.34}$$

Thus, we can write the integral (integrating out o_f and $-u_f$ gauges) as

$$\frac{(-1)^s(-1)^{\sum_e s_e}e^{-i\frac{\pi}{4}\sum_e \operatorname{sgn}_e}\prod_e C_{j_e}}{2^{\frac{5}{2}|E|}\pi^{|F|+\frac{1}{2}|E|}} \int\prod_{f\in F}\delta(|n_f|^2-1)\,d^3n_f\frac{1}{\sqrt{\prod_e l_e|\sin\theta_e|}}e^{i\sum_e\left(l_e\theta_e+S_1^{l_e}(\theta_e)\right)}. \tag{3.35}$$

where from Sec. III B we know

$$S_1^{l_e}(\theta_e) = -\frac{1}{8l_e}\cot\theta_e + \dots. \tag{3.36}$$

The only present symmetry that has to be discussed is a u-symmetry, which is implemented by $SO(3)$ rotations

$$n_f \to u n_f u^{-1}. \tag{3.37}$$

If the configurations of the vectors B_{fe} is rigid, i.e., the only deformations of the configuration of the edges with given lengths are rotations, then the stationary u-orbit is isolated, i.e., there exists a neighbourhood of the orbit that does not intersect any other orbit.

1. c transformation as parity transformation

Furthermore, we would like to point out that given one orbit of stationary phase, we can always construct a different one via parity transformation of the B_{fe} vectors (see also Sec. II E about c transformations). After integrating out gauges these two points are related by

$$\begin{aligned} n'_f &= n_f, \\ B'_{fe} &= -B_{fe}, \end{aligned} \tag{3.38}$$

so also the angles are related by $\theta'_e = -\theta_e$ (n_f are preserved as pseudovectors). Finally, we see that the asymptotic contribution from the parity related stationary orbits is just the complex conjugate of the original one, such that the complete expansion is real.

In order to provide the correct expression of the action before performing the remaining stationary point analysis, it is necessary to compute the normalization of the intertwiners, the so-called "Theta" graph.

D. Normalization - "Theta" graph

We need to compute the self-contraction of the invariants C_f using the (in this case) symmetric bilinear form ϵ (as a generalization of the anti-symmetric form ϵ of spin $\frac{1}{2}$ to arbitrary representations). Its special properties allow us to relate the ϵ product $((\cdot,\cdot))$ to the scalar product on $SU(2)$:

$$(C_f, C_f) = \langle \overline{C_f}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} C_f \rangle = \langle C_f, C_f \rangle, \tag{3.39}$$

since C_f is real and $SU(2)$ invariant. The integral of the contraction of the intertwiner with itself is given by

$$\begin{aligned} &\int_{SU(2)^2 \times S_1^6} dU_1 dU_2 \prod_i \frac{\mathrm{d}\phi_{i1}}{2\pi} \prod_i \frac{\mathrm{d}\phi_{i2}}{2\pi} \\ &\qquad f_1(\{\phi_{i1}\}) f_2(\{\phi_{i2}\}) \prod_i \left(1/2|,\, O_{\phi_{i1}}^{-1} U_1^{-1} U_2 O_{\phi_{i2}} |1/2\right)^{2j_i}. \end{aligned} \tag{3.40}$$

Its stationary point conditions are

- $B_{i1} = -B_{i2}$.
- $\sum_i B_{i1} = \sum_i B_{i2} = 0$.

As the "Theta" graph itself is an evaluation of a spin network its effective action has the same transformations on the action as described in Sec. II C.

The u symmetry can be ruled out just by dropping the integration over U_1. Then one is left with the group G generated by the transformations

$$r_{f1},\ r_{f2},\ -u_2,\ o_{f1},\ o_{f2},\ -o_{f1,i},\ -o_{f2,i}. \tag{3.41}$$

On the stationary H orbits, i.e., the normal subgroup of G generated by $\{o_f,\ -u_f\}$, these transformations act as the group $K = G/H$, which gives $\mathbb{Z}_2^3 \times \mathbb{Z}_2^3$.

This group acts freely on the stationary H orbits and as before the modifiers suppress all but one of the H orbits. If we take $f_1 = f_2 = 1$ and restrict ourselves to the case where $\sum_i j_i$ is even (all j_i integer) then the action is invariant with respect to all transformations, thus every stationary orbit contribute the same $\frac{1}{2^6}$ of the overall result. In the case when $\sum j_e$ is not even, or some j_e are not integer, this choice leads to a vanishing invariant.

The computation of the full expansion of the theta graph in the even case also gives an expansion on the stationary orbit in the presence of f_i. This is briefly discussed in Sec. III D 1.

1. Theta graph for integer spins and $\sum j$ even

We will derive the complete expansion for $\sum j$ even. We need to compute

$$\prod_i C_{j_i} (C^{j_1 j_2 j_3}_{000})^2, \tag{3.42}$$

where C_{j_i} (see also the Appendix D 2 a) is the normalization of the $|0\rangle$ vector.

In the Appendix D 3 we show (following Ref. 40) that the theta graph $(C^{j_1 j_2 j_3}_{000})^2$ is equal to

$$\frac{1}{2\pi S}\left(1 + O\left(\frac{1}{l^2}\right)\right), \tag{3.43}$$

where S is the area of the triangle with edges $j_i + \frac{1}{2}$.

E. Final formula

Let us state the final formula normalized by the square roots of the "Theta" diagrams. Those are equal to

$$(-1)^{s_f} 2^{-7/2} \sqrt{\frac{\prod_{e \subset f} C_{j_e}}{\pi S_f}} \left(1 + O\left(\frac{1}{l^2}\right)\right), \tag{3.44}$$

where s_f is a sign factor necessary to be consistent with[38,39] that will be derived in the Appendix A 2 a.

To summarize the various calculations of this chapter, the contraction of normalized intertwiners has the following asymptotic expansion after the stationary phase approximation for the angles ϕ_{fe} has been performed and the asymptotic expansion from (3.44) has been inserted

$$\begin{gathered}\frac{(-1)^{s+\sum_f s_f + \sum_e s_e} e^{-i\frac{\pi}{4}\sum_e \mathrm{sgn}_e}}{2^{\frac{5}{2}|E| - \frac{7}{2}|F|}\pi^{\frac{1}{2}|F| + \frac{1}{2}|E|}} \frac{\prod_{f\in F} S_f^{\frac{1}{2}}}{\prod_{e\in E} l_e^{\frac{1}{2}}} \\ \int \prod_{f\in F} \delta(|n_f|^2 - 1)\, d^3 n_f \frac{1}{\sqrt{\prod_e |\sin\theta_e|}} e^{i\sum_e\left(l_e\theta_e - \frac{1}{8l_e}\cot\theta_e\right)}.\end{gathered} \tag{3.45}$$

As it will be shown in Appendix A, $s + \sum_f s_f + \sum_e s_e = 0 \bmod 2$ and thus the term

$$(-1)^{s+\sum_f s_f + \sum_e s_e} \tag{3.46}$$

in the integral can be omitted.

This is the contribution up to next-to-leading order. It is straightforward to generalize it to higher order due to the complete expansion of the edge amplitude (Sec. V B) and the expansion of "Theta" diagrams (Appendix D 3).

In Sec. IV we will focus our attention on the specific example of the 6j symbol. After another variable transformation to the set of exterior dihedral angles of the tetrahedron has been performed, we obtain the action of flat first order Regge Calculus, i.e., Regge Calculus in which both edge lengths and dihedral angles are considered as independent variables. The stationary point conditions (with respect to the dihedral angles) will reduce the action to ordinary Regge calculus, such that the geometry is entirely described by the set of edge lengths, where angles on the stationary point agree with the angles given for a tetrahedron built from the lengths. We will perform the stationary point analysis, in particular compute the determinant of the Hessian matrix, and obtain the correct asymptotic expression for the $SU(2)$ $6j$ symbol.[5]

IV. ANALYSIS OF 6J SYMBOL AND FIRST ORDER REGGE CALCULUS

In this section, we will perform the remaining integrations via stationary phase approximation starting from (3.45) in the case of the $6j$ symbol. As we are restricting the discussion to a specific spin network, we introduce the following notations:

This spin network consists of 4 faces f, which we will simply count by $i \in \{1, \ldots, 4\}$, and 6 edges e, which we will denote by ij, $i < j$, i.e., the faces sharing it. On the stationary point with respect to $\{\phi_{fe}\}$, we have two configurations of B_{fe}, which we will label accordingly as B_{ij} and similarly θ_{ij} using the convention that θ_{ij} is the angle at the edge l_{ij}.

In Ref. 19 it has been shown that the $6j$ symbol can be interpreted as a tetrahedron on the points of stationary phase (for non-degenerate configurations). In Sec. II B we have shown that our approach gives the same interpretation. Hence, we can assume that for one stationary point, the normals to the faces n_i of the tetrahedron are outward pointing and the B_{ij} vectors are oriented such that $\theta_{ij} \in (0, \pi)$. For the second stationary orbit, described by $B'_{ij} = -B_{ij}$, the angles are negative, hence this contributes the complex conjugate.

In order to perform the remaining stationary point analysis, it is necessary to perform another variable transformations from normals of faces n_i to angles between these normals θ_{ij} followed by integrating out gauge degrees of freedom corresponding to u transformations:

$$n_i \to \theta_{ij}. \tag{4.1}$$

This transformation is performed in the Appendix B 2 in great detail, and we obtain the following relation:

$$\prod_i d^3 n_i \delta(|n_i|^2 - 1) \to \prod_{ij} d\theta_{ij} \prod_{ij} |\sin\theta_{ij}| \delta(\det \tilde{G}), \tag{4.2}$$

where $\tilde{G}$ denotes the angle Gram matrix (for exterior dihedral angles) of a tetrahedron with components $G_{ij} = \cos(\theta_{ij})$, with $\theta_{ii} = 0$. Using (4.2) and simplifying (3.45) for the case of the $6j$ symbol, we obtain in the neighbourhood of the stationary point:

$$\frac{e^{-i\frac{6}{4}\pi}\prod_i S_i^{\frac{1}{2}}}{2\pi^3 \prod_{i<j} l_{ij}^{\frac{1}{2}}} \int \prod_{i<j} \mathrm{d}\,\theta_{ij} \underbrace{\prod_{i<j} |\sin\theta_{ij}| \delta(\det \tilde{G})}_{\text{Jacobian}} \frac{1}{\prod_{i<j}\sqrt{|\sin\theta_{ij}|}} e^{i\sum_{i<j}\left(l_{ij}\theta_{ij} - \frac{1}{8l_{ij}}\cot\theta_{ij}\right)}. \tag{4.3}$$

Let us consider one of the stationary points for which $\sin\theta_{ij} > 0$. The second one contributes the complex conjugate of the first because two points (orbits) are related by c (parity) transformations

$$\frac{i}{4\pi^4} \frac{|l| \prod_i S_i^{\frac{1}{2}}}{\prod_{i<j} l_{ij}^{\frac{1}{2}}} \int \mathrm{d}\,\rho \prod_{i<j} \mathrm{d}\,\theta_{ij} \prod_{i<j} \sqrt{\sin\theta_{ij}}\, e^{i\left(\sum_{i<j}\left(l_{ij}\theta_{ij} - \frac{1}{8l_{ij}}\cot\theta_{ij}\right) - |l|\rho \det\tilde{G}\right)}, \tag{4.4}$$

where $|l|^2 := \sum_{i<j} l_{ij}^2$ and ρ is a Lagrange multiplier.

It is worth to examine the action in (4.4) in more detail: This function of edge lengths l_{ij} and angles θ_{ij} is known as the action for "first order" Regge Calculus.[32] We will comment on this further in Sec. IV D.

In Sec. IV A we will perform a stationary phase approximation for the integrations over the angles θ_{ij}. We will use the improved action $\sum_{i<j} l_{ij}\theta_{ij}$, where we regard higher order corrections as the vertices of a Feynman diagram expansion, and the resulting points of stationary phase will correspond to perturbed stationary points obtained previously from the stationary point analysis with respect to the $SU(2)$ group elements U_f in Sec. II.

A. Stationary point analysis

The stationary point conditions for the action (4.4) are:

- Derivative with respect to θ_{ij}:

$$l_{ij} - |l|\rho \frac{\partial \det \tilde{G}}{\partial \theta_{ij}} = 0. \tag{4.5}$$

- Derivative with respect to ρ:

$$-|l| \det \tilde{G} = 0. \tag{4.6}$$

Equations (4.5) and (4.6) are exactly those equations stating that θ_{ij} are the exterior dihedral angles of a tetrahedron formed by edges of length l_{ij} (see Appendix C and Ref. 41). From the stationary point analysis with respect to group elements U_f we know that all normals n_i to the faces are outward directed. The point of stationary phase with respect to the angles θ_{ij} is only a small perturbation in comparison to the stationary point with respect to group elements. The areas of the respective face are denoted by S_i. For a flat tetrahedron, the following relation holds (see, for example, Refs. 41 and 42):

$$l_{ij} = \frac{2}{3}\frac{1}{V} S_i S_j \sin\theta_{ij}. \tag{4.7}$$

On the other hand, $\det \tilde{G} = 0$ holds, where a (single) null eigenvector of $\tilde{G}$ is given by the vector of areas of the triangles $(S_1, \ldots, S_4)$ (of the tetrahedron) as $\det \tilde{G} = 0$ imposes the closure of the flat tetrahedron. Thus follows

$$\frac{\partial \det \tilde{G}}{\partial \theta_{ij}} = -2\frac{\det' \tilde{G}}{\sum_k S_k^2} S_i S_j \sin\theta_{ij} \overset{(4.7)}{=} -3\frac{V \det' \tilde{G}}{\sum_k S_k^2}\, l_{ij}, \tag{4.8}$$

where $\det' \tilde{G} = \sum_i \tilde{G}^*_{ii}$ and $\tilde{G}^*_{ii}$ is the (i, i)th minor of $\tilde{G}$. $\det' \tilde{G}$ is computed in the Appendix C 1 b:

$$\det' \tilde{G} = \frac{3^4}{2^2}\Big(\sum_i S_i^2\Big)\frac{V^4}{\prod S_i^2}. \tag{4.9}$$

Using (4.8) and (4.9), we solve (4.5) for the Lagrange multiplier ρ:

$$\rho = -\frac{2^2 \prod S_i^2}{3^5 V^5 |l|}. \tag{4.10}$$

The quadratic order in the expansion around the stationary point, which we also call the kinetic term, i.e., the Hessian matrix of the action, is given by

$$\mathcal{H} := -i|l| \begin{pmatrix} 0 & \frac{\partial \det \tilde{G}}{\partial \theta_{ij}} \\ \frac{\partial \det \tilde{G}}{\partial \theta_{km}} & \rho \frac{\partial \det \tilde{G}}{\partial \theta_{ij} \partial \theta_{km}} \end{pmatrix}. \tag{4.11}$$

To complete the stationary point analysis, we have to compute the determinant of its inverse evaluated on the stationary point.

B. Propagator and Hessian

Let us introduce a function of lengths l:

$$\lambda = |l|\rho = -\frac{2^2 \prod S_i^2}{3^5 V^5}. \tag{4.12}$$

It is of scaling dimension 1 with respect to l.

1. Propagator

We will prove that the inverse of the kinetic term is equal to

$$\mathcal{H}^{-1} = i\begin{pmatrix} \frac{c}{|l|^2} & \frac{1}{|l|}\frac{\partial\lambda}{\partial l_{ij}} \\ \frac{1}{|l|}\frac{\partial\lambda}{\partial l_{kl}} & \frac{\partial\theta_{ij}}{\partial l_{kl}} \end{pmatrix}, \tag{4.13}$$

where c is a constant (defined in Lemma 7 in Appendix C). Let us compute

$$i\begin{pmatrix} \frac{c}{|l|^2} & \frac{1}{|l|}\frac{\partial\lambda}{\partial l_{ij}} \\ \frac{1}{|l|}\frac{\partial\lambda}{\partial l_{mn}} & \frac{\partial\theta_{ij}}{\partial l_{mn}} \end{pmatrix}(-i)|l|\begin{pmatrix} 0 & \frac{\partial \det\tilde{G}}{\partial\theta_{kl}} \\ \frac{\partial\det\tilde{G}}{\partial\theta_{mn}} & \rho\frac{\partial\det\tilde{G}}{\partial\theta_{kl}\partial\theta_{mn}} \end{pmatrix}. \tag{4.14}$$

This gives

$$|l|\begin{pmatrix} \frac{1}{|l|}\frac{\partial\lambda}{\partial l_{mn}}\frac{\partial\det\tilde{G}}{\partial\theta_{mn}} & \frac{\partial\det\tilde{G}}{\partial\theta_{ij}}\frac{c}{|l|^2} + \frac{1}{|l|}\frac{\partial\lambda}{\partial l_{mn}}\rho\frac{\partial\det\tilde{G}}{\partial\theta_{mn}\partial\theta_{ij}} \\ \frac{\partial\det\tilde{G}}{\partial\theta_{mn}}\frac{\partial\theta_{mn}}{\partial l_{ij}} & \frac{1}{|l|}\frac{\partial\lambda}{\partial l_{ij}}\frac{\partial\det\tilde{G}}{\partial\theta_{kl}} + \frac{\partial\theta_{ij}}{\partial l_{mn}}\rho\frac{\partial^2\det\tilde{G}}{\partial\theta_{mn}\partial\theta_{kl}} \end{pmatrix}, \tag{4.15}$$

using the results of the Appendix C 2, we see that (4.15) is equal to the identity.

2. Hessian

Similar to the angle Gram matrix discussed in Sec. IV A, $\det\frac{\partial\theta_{ij}}{\partial l_{kl}} = 0$ in the case of a flat tetrahedron. This is due to the fact that given a set of dihedral angles of a flat tetrahedron, the tetrahedron is only defined up to rotations and uniform scaling of its edge lengths. Hence, the null eigenvector of the matrix $\frac{\partial\theta_{ij}}{\partial l_{kl}}$ is given by the edge vector $\vec{l} := (l_{12}, \dots, l_{34})$. This is equivalent to the Schläfli identity in 3D: $\sum_{ij} l_{ij} d\theta_{ij} = 0$. We rewrite the matrix $\mathcal{H}^{-1}$ in the basis in which its second row is parallel to $\vec{l}$ and the next ones are perpendicular to $\vec{l}$:

$$i\begin{pmatrix} \frac{c}{|l|^2} & \frac{1}{|l|}\frac{\partial\lambda}{\partial l} & \cdots & \cdots \\ \frac{1}{|l|}\frac{\partial\lambda}{\partial l} & 0 & 0 & 0 \\ \vdots & 0 & \frac{\partial\theta_{ij}}{\partial l_{kl}} & \vdots \\ \vdots & 0 & \vdots & \vdots \end{pmatrix}. \tag{4.16}$$

The determinant of $(-\mathcal{H}^{-1})$ is thus equal to

$$\det(-\mathcal{H}^{-1}) = -(-i)^7\Big(\underbrace{\frac{1}{|l|}\frac{\partial\lambda}{\partial l}}_{=\frac{\lambda}{|l|^2}}\Big)^2 \det{}'\frac{\partial\theta_{ij}}{\partial l_{kl}}. \tag{4.17}$$

Since λ is of scaling dimension 1 (with respect to edge lengths), $l_{ij}\frac{\partial\lambda}{\partial l_{ij}} = \lambda$. More details and the tedious calculation of $\det'\frac{\partial\theta_{ij}}{\partial l_{kl}}$ can be found in Appendix C:

$$\det{}'\frac{\partial\theta_{ij}}{\partial l_{kl}} = \frac{3^3}{2^5}\frac{|l|^2}{\prod S_i^2}V^3. \tag{4.18}$$

Combining all these results, we obtain

$$\det(-\mathcal{H}^{-1}) = -i\frac{1}{|l|^4}\left(-\frac{2^2\prod S_i^2}{3^5 V^5}\right)^2\frac{3^3}{2^5}\frac{|l|^2}{\prod S_i^2}V^3 = -i\frac{1}{2\,3^7}\frac{\prod_i S_i^2}{|l|^2 V^7}, \tag{4.19}$$

and hence

$$\sqrt{|\det\mathcal{H}^{-1}|} = \frac{1}{\sqrt{2}\,3^{\frac{7}{2}}}\frac{\prod_i S_i}{|l|V^{\frac{7}{2}}}. \tag{4.20}$$

Since $\mathcal{H}^{-1}$ is antihermitian, it has only imaginary (and nonzero) eigenvalues. Therefore, it is important to count the number of $+i\mathbb{R}$ and $-i\mathbb{R}$ eigenvalues in order to pick the right branch of $\sqrt{\det(-\mathcal{H}^{-1})}$. The number of positive and negative imaginary eigenvalues is constant on the connected components of parameter spaces. For oriented tetrahedra (one of the two components) it can be computed in the equilateral case, i.e., all l_{ij} are equal. This was done in the Appendix D 4, then $\mathcal{H}^{-1}$ has 4 $i\mathbb{R}$ eigenvalues and 3 $-i\mathbb{R}$. Finally, we conclude:

$$\frac{1}{\sqrt{\det(-\mathcal{H})}} = e^{-4i\frac{\pi}{4}} e^{3i\frac{\pi}{4}} \sqrt{|\det \mathcal{H}^{-1}|} = \frac{1}{\sqrt{2}\, 3^{\frac{7}{2}}} e^{-i\frac{\pi}{4}} \frac{\prod S_i}{|l|\, V^{\frac{7}{2}}}. \tag{4.21}$$

The last step is to combine all the previous results to obtain the final formula for the asymptotics of the $6j$ symbol.

C. Final result

In this section, we will combine the results of the previous calculations step by step. First we perform the stationary point analysis for (4.4):

$$\begin{aligned} &\frac{i}{4\pi^4} \frac{|l| \prod_i S_i^{\frac{1}{2}}}{\prod_{i<j} l_{ij}^{\frac{1}{2}}} \prod_{i<j} \sqrt{\sin\theta_{ij}} \frac{(2\pi)^{\frac{7}{2}}}{\sqrt{\det(-\mathcal{H})}} e^{i\left(\sum_{ij}\left(l_{ij}\theta_{ij} - \frac{1}{8 l_{ij}} \cot\theta_{ij}\right) + \tilde{S}_1\right)} \\ &\qquad = i\, \frac{2^{\frac{3}{2}}}{\pi^{\frac{1}{2}}} \frac{|l| \prod_i S_i^{\frac{1}{2}} \prod_{i<j} \sqrt{\sin\theta_{ij}}}{\sqrt{\det(-\mathcal{H})} \prod_{i<j} l_{ij}^{\frac{1}{2}}} e^{i\left(\sum_{ij} l_{ij}\theta_{ij} + S_1\right)}, \end{aligned} \tag{4.22}$$

where S_1 is the NLO contribution. As a next step, we substitute $\sin\theta_{ij} = \frac{3}{2}\frac{l_{ij} V}{S_i S_j}$ (for $\sin\theta_{ij} > 0$) and (4.21) in (4.22):

$$\begin{aligned} &\frac{2^{\frac{3}{2}}}{\pi^{\frac{1}{2}}} \frac{|l| \prod S_i^{\frac{1}{2}}}{\prod l_{ij}^{\frac{1}{2}}} \left(\frac{3}{2}\right)^3 \frac{\prod l_{ij}^{\frac{1}{2}} V^3}{\prod S_i^{\frac{3}{2}}} e^{i\frac{\pi}{4}} \frac{\prod S_i}{\sqrt{2}\, 3^{\frac{7}{2}} |l|\, V^{\frac{7}{2}}} e^{i\left(\sum_{ij} l_{ij}\theta_{ij} + S_1\right)} \\ &\quad = \frac{1}{2} \frac{1}{\sqrt{12\pi\, V}} e^{i\frac{\pi}{4}} e^{i\left(\sum_{ij} l_{ij}\theta_{ij} + S_1\right)}. \end{aligned} \tag{4.23}$$

As previously discussed, the full contribution comes from two stationary points, which are related by parity transformations. Eventually, we obtain

$$\frac{1}{\sqrt{12\pi V}} \left(\cos\left(\sum_{ij} l_{ij}\theta_{ij} + \frac{\pi}{4} + S_1 \right) + O\left(|l|^{-2}\right) \right), \tag{4.24}$$

as in Ref. 5. In the formula above, we implicitly assumed that S_1 is real. This property will be proven in Sec. V.

D. First order Regge calculus

A first order formulation of Regge Calculus[32,43] is a discretization of General Relativity defined on the triangulation of the manifold in which both edge lengths and dihedral angles are considered as independent variables. Its introduction was motivated by Palatini's formulation of Relativity where equations of motion are first order differential equations. Its action in 3D is given by

$$S_R[l_e] = \sum_e l_e \epsilon_e, \quad \epsilon_e = 2\pi - \sum_{\tau \supset e} \theta_e^{(\tau)}, \tag{4.25}$$

where l_e denotes the length of the edge e, $\theta_e^{(\tau)}$ denotes the dihedral angle at edge e in the tetrahedron τ. By ϵ_e we denote the deficit angle at edge e. For every tetrahedron an additional constraint is imposed, namely,

$$\det \tilde{G} = 0 \tag{4.26}$$

that enter the action via a Lagrange multiplier.[32] $\tilde{G}$ is the angle Gram matrix of the tetrahedron. One can eliminate the θ_e^τ variables by partially solving the equations of motion (given by variations with respect to θ_e^τ), then

$$\theta_e^\tau = \theta_e^\tau(l) \tag{4.27}$$

turns out to be the dihedral angle at the edge e for a discrete geometry determined by the edge lengths $\{l_e\}$.

Our derivation of the 6j symbol asymptotics follows the same idea. It also suggests a suitable measure in the path integral quantization for (linearized) first order Regge calculus in order to reobtain the factor $\frac{1}{\sqrt{V}}$ from Ponzano-Regge asymptotics. We also hope that our methods might be applied in the 4D case, where a similar action, motivated by the construction of modern spin foam models, was proposed in Ref. 33. Furthermore, the present results could naturally provide and motivate a triangulation independent measure for first order Regge calculus following the approach in Ref. 27. Examining first order and area-angle (quantum) Regge calculus in 4D might also give new insights into possible measures for 4D spin foam models.

V. PROPERTIES OF THE NEXT TO LEADING ORDER AND COMPLETE ASYMPTOTIC EXPANSION

So far, we dealt with the asymptotic expansion of a spherical spin network evaluation in the leading order approximation and managed to work out the example of the 6j symbol. However, our method allows us to derive, in principle, the full asymptotic expansion of the evaluation by the higher order stationary point analysis, e.g., we have already mentioned the NLO corrections to the contribution from edges of the spin network (on the stationary points) in Sec. III B. Such corrections improve the asymptotic behaviour in particular for small spins. Therefore, we will apply our formalism in this section to derive new insights on the NLO corrections (to the 6j symbol).

NLO order corrections to the asymptotic formula of the $SU(2)$ 6j symbol have been thoroughly discussed in Refs. 30 and 31. In particular, the authors found evidence that the leading contributions in the expansion in $\frac{1}{l}$ are purely real and oscillating as $\cos(S_R + \frac{\pi}{4})$, whereas the next order term (also purely real) behaves like $\sin(S_R + \frac{\pi}{4})$, where S_R denotes the Regge action for the tetrahedron. Furthermore, this behaviour is conjectured to be alternating for consecutive orders.

We will refer to this behaviour introduced in Refs. 30 and 31 as "Dupuis-Livine" (DL) property and we will show that it holds for the full expansion of the asymptotics of any evaluation of spin networks, satisfying certain generic conditions, for example, the 6j symbol in the non-degenerate case. Furthermore we will derive a new recursion relation for the 6j symbol which can be applied to obtain a simpler form of the next to leading order correction to the Ponzano-Regge formula.

A. Properties of the Dupuis-Livine form

In this section we will give a definition to the Dupuis-Livine form and also discuss some of its basic properties.

Consider an asymptotic expansion in the variables $\{j\}$ of the following form:

$$\sum_i A_k(\{j\}) e^{i\sum j_i \theta_i}, \tag{5.1}$$

where A_k is a homogeneous function in all variables j of degree $k + \beta$. It can be rewritten in terms of the variables $\{l\}$ (with $l = j + \frac{1}{2}$):

$$\sum_i \tilde{A}_k(\{l\}) e^{i\sum l_i \theta_i}, \tag{5.2}$$

where $\tilde{A}_0 = e^{-\frac{i}{2}\sum \theta_i} A_0$.

We will say that it has the Dupuis-Livine (DL) property, if it can be written as

$$\tilde{A}_0(\{l\})e^{i\sum l_i\theta_i}\sum_k B_k, \tag{5.3}$$

where $i^k B_k$ is a real and homogeneous function of degree k. Note that if we write this expansion in the form

$$\tilde{A}_0(l)e^{i\sum l_i\theta_i+S} \tag{5.4}$$

then S also has DL form (and starts with degree 1). Furthermore, suppose that two asymptotic series f_1 and f_2 have the DL property then also

$$f_1 f_2, \quad \frac{1}{f_i} \tag{5.5}$$

have this property. In particular the last two relations are very useful for our discussion, since they allow us to examine the full expansion of the evaluation of the spin network in steps: first we examine the contributions from the edges, i.e., the partial integrations over the ϕ_{fe}, then the normalization factors until we eventually discuss the full expansion.

B. Partial integration over ϕ

In this subsection, we will examine whether the contributions from the partial integration over ϕ have the DL property. We will prove it by using a recurrence relation similar to Bonnet's formula for Legendre polynomials. Therefore, it will be necessary to introduce some technical definitions, from which we are able to derive recursion relations.

1. Weak equivalence

Let $\psi_i = f_i e^S$, where $S = kS_{-1} + \ldots$, $\Re S_{-1} \leq 0$ and f_i grows at most polynomially in k and admits a power series expansion in k.

Definition 1. ψ_1 is weakly equivalent to ψ_2 around the point x_0,

$$\psi_1 \equiv \psi_2, \tag{5.6}$$

if the expansion in k of the integral of both around x_0 is the same.

If $\psi = fe^S$ then

$$L^*\psi \equiv 0 \tag{5.7}$$

where

$$L^*\psi = L\psi + (\operatorname{div} L)\psi \tag{5.8}$$

and L is a vector field.

2. Equivalences and recursion relations

Let us introduce

$$L_\pm = \cos\tilde{\theta}\sin\alpha\frac{\partial}{\partial\alpha} \pm i\sin\tilde{\theta}\sin\beta\frac{\partial}{\partial\beta}, \tag{5.9}$$

$$A_\pm = \cos\tilde{\theta}\cos\alpha \pm i\sin\tilde{\theta}\cos\beta \tag{5.10}$$

that we regard as vector fields and functions of the variables α, β. It is straightforward to calculate

$$\operatorname{div} L_\pm = A_\pm, \tag{5.11}$$

and

$$L_+A_+ = L_-A_- = \frac{1}{2}A_+^2 + \frac{1}{2}A_-^2 - \cos 2\tilde{\theta}$$
$$L_-A_+ = L_+A_- = A_+A_- - 1. \tag{5.12}$$

Starting from $L_\pm^* A_+^k \equiv 0$ and using the above identities, we derive the following relation (see the Appendix D 1 for more details):

$$-\frac{(k+2)^2}{k+1}(A_+)^{k+2} + 2(k+1)\cos 2\tilde{\theta}\; A_+^k - (k-1)A_+^{k-2} \equiv 0. \tag{5.13}$$

Therefore, we introduce the following quantity:

$$\tilde{P}_l = \frac{1}{C_j}A_+^{2j}, \tag{5.14}$$

where $l = j + \frac{1}{2}$ and C_j is given by (3.25):

$$C_j = \frac{1}{4^j}\frac{\Gamma(2j+1)}{\Gamma(j+1)^2}. \tag{5.15}$$

Furthermore C_j admits a complete expansion in j, see also the Appendix D 2 a:

$$C_j = \frac{1}{\sqrt{\pi j}}\left(1 + O\left(\frac{1}{j}\right)\right). \tag{5.16}$$

Moreover, one can show that

$$C_{j+1} = \frac{2j+1}{2j+2}C_j, \quad C_{j-1} = \frac{2j}{2j-1}C_j. \tag{5.17}$$

Combining (5.14), (5.15), and (5.17) with (5.13) and substituting $k = 2j$ in (5.13) we obtain:

$$\begin{aligned} 0 \equiv & 2C_j\left[-\frac{(2j+2)(j+1)}{2j+1}\frac{2j+1}{2j+2}\;\tilde{P}_{l+1} + (2j+1)\cos 2\tilde{\theta}\,\tilde{P}_l - \frac{2j-1}{2}\frac{2j}{2j-1}\tilde{P}_{l-1}\right] = \\ = & 2C_j\left[-\left(l+\frac{1}{2}\right)\tilde{P}_{l+1} + 2l\cos 2\tilde{\theta}\,\tilde{P}_l - \left(l-\frac{1}{2}\right)\tilde{P}_{l-1}\right]. \end{aligned} \tag{5.18}$$

But C_j admits a nonzero asymptotic expansion, thus

$$-\left(l+\frac{1}{2}\right)\tilde{P}_{l+1} + 2l\cos 2\tilde{\theta}\,\tilde{P}_l - \left(l-\frac{1}{2}\right)\tilde{P}_{l-1} \equiv 0 \tag{5.19}$$

around any stationary point. With the definitions given here, (3.15), i.e., the amplitude associated to one edge, becomes

$$\frac{1}{8\pi^2}\int d\alpha d\beta A_+^{2j} = \frac{C_j}{8\pi^2}\int d\alpha d\beta\,\tilde{P}_l, \tag{5.20}$$

which establishes the connection to our previous calculations.

Let us notice that (5.19) is exactly Bonnet's recursion formula for Legendre polynomials.

3. Total expansion and DL property

Over any stationary point we have shown that the integral of $\tilde{P}_l$ can be expanded as

$$(-1)^s\sum_{k\geq 0}\frac{e^{il\theta}}{l^{k+\frac{1}{2}}}A_k(\theta) + O(l^{-\infty}), \tag{5.21}$$

where $\theta = 2\tilde{\theta}$ is now the $SO(3)$ angle and s is a sign factor that comes from the $SU(2)$ angle. Values of the integral for $\tilde{\theta}$ and $\tilde{\theta} + \pi$ differ by the factor $(-1)^s$. This restricts A_k to be of the form described above. Moreover, we know from Sec. V B 2 that

$$-\left(l+\frac{1}{2}\right)\tilde{P}_{l+1}+2l\cos\theta\,\tilde{P}_l-\left(l-\frac{1}{2}\right)\tilde{P}_{l-1}\equiv 0. \tag{5.22}$$

Applying the asymptotic form to the recursion relations, we obtain

Lemma 3. For every $m \geq 0$

$$\sum_{k\leq m}(2\beta^k_{m+1-k}+\beta^k_{m-k})A_k i^{m+1-k}\sin\left(\theta-\frac{\pi}{2}(m-k)\right)=0, \tag{5.23}$$

where

$$\beta^k_m=\frac{(-k-\frac{1}{2})_m}{m!}\in\mathbb{R}, \tag{5.24}$$

and

$$(a)_m=a\cdot(a-1)\cdot\ldots\cdot(a-m+1),\quad (a)_0=1. \tag{5.25}$$

We will prove the lemma in the Appendix D 2.

Consider the case where $k = m$ in (5.23). For any $m \geq 0$ one obtains that

$$2\beta^m_1+\beta^m_0=-2\left(m+\frac{1}{2}\right)+1=-2m, \tag{5.26}$$

such that (5.23) can be rewritten in the following way:

$$2mA_m i\,\sin(\theta)=\sum_{k<m}(2\beta^k_{m+1-k}+\beta^k_{m-k})A_k i^{m+1-k}\sin\left(\theta-\frac{\pi}{2}(m-k)\right). \tag{5.27}$$

Let us introduce

$$B_m := A_m i^{-m} e^{i\frac{\pi}{4}\operatorname{sign}\sin\theta}. \tag{5.28}$$

From the asymptotics of the integrations over $\tilde{P}_l$ follows that $B_0 \in \mathbb{R}$ and (5.27) can be rewritten as

$$2m\underbrace{A_m i^{-m}e^{i\frac{\pi}{4}\operatorname{sign}\sin(\theta)}}_{=B_m}\sin(\theta)=\sum_{k<m}\left(2\beta^k_{m+1-k}+\beta^k_{m-k}\right)\underbrace{A_k i^{-k}e^{i\frac{\pi}{4}\operatorname{sign}\sin(\theta)}}_{=B_k}\sin\left(\theta-\frac{\pi}{2}(m-k)\right)$$

$$\Longleftrightarrow\ 2mB_m\sin(\theta)=\sum_{k<m}\left(2\beta^k_{m+1-k}+\beta^k_{m-k}\right)B_k\sin\left(\theta-\frac{\pi}{2}(m-k)\right). \tag{5.29}$$

This implies that all $B_k \in \mathbb{R}$ and it proves that the asymptotic terms (in the connected component expansion - e^S) are of the form

$$\tilde{A}_k\in i^k\mathbb{R}\text{ for }k>0. \tag{5.30}$$

This proves that the contributions from the integration over ϕ evaluated on the points of stationary phase are of DL form.

4. The total expansion of the original integral

We know that the total expansion of the original integral around the stationary point is of the form given in (3.26). Using the recurrence relation (5.29) we can compute its next-to-leading order:

$$C_j\frac{(-1)^s}{4\sqrt{2\pi l\,|\sin\theta|}}e^{i(l\theta-\frac{\pi}{4}\operatorname{sign}(\sin\theta)-\frac{1}{8l}\cot\theta)}\left(1+O\left(\frac{1}{l^2}\right)\right). \tag{5.31}$$

As a next step, we will examine whether the normalization factors computed from the self-contraction of intertwiners is of DL form as well.

C. Different forms of intertwiners and DL property

To examine whether the normalization factors satisfy the DL property, we will construct different forms of invariants. Since the (three-valent) intertwiner is unique, all new constructions are proportional to the original one.

Let U_i be distinct group elements from a sufficiently small neighbourhood of the identity. Let

$$\begin{aligned} C_{U_i,f} = \int dU \int \prod \frac{d\phi_i}{2\pi} f(\phi_1, \phi_2, \phi_3) \\ UU_1 O_{\phi_1} |1/2\rangle^{2j_{e_1}} \otimes UU_2 O_{\phi_2} |1/2\rangle^{2j_{e_2}} \otimes UU_3 O_{\phi_3} |1/2\rangle^{2j_{e_3}} \end{aligned} \tag{5.32}$$

be the new invariant, where f is such a function that it is constant in the neighbourhood of the angles, which satisfy the stationary phase conditions, i.e., where all

$$B_i = j_{e_i} UU_1 O_{\phi_1} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} (UU_1 O_{\phi_1})^{-1} \tag{5.33}$$

sum to zero. We will choose U_i in such a way (described below) that such points are separated. In such a case we can choose f to be nonzero around only one of them.

Let us now describe U_i. For given three vectors B_i in the plane perpendicular to $H = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}$ (see also Sec. II for more details) such that

$$\sum B_i = 0, \tag{5.34}$$

we choose U_i in the neighborhood of identity such that $U_i H U_i^{-1} \perp B_i$. There are many such choices which will be used in the sequel.

Let us take contraction of such a $C_{U_i,f}$ with the intertwiner $C_{f'}$ obtained with the help of modifiers,

$$\left(C_{U_i,f}, C_{f'}\right). \tag{5.35}$$

Due to the definition of $C_{U_i,f}$, there is only one $-u$ and o_f orbit of stationary points on which f and f' are nonzero. These are given by the conditions

$$\begin{aligned} &B_{fe_i} = -B_i, \\ &n_f \perp B_{fe_i}, \\ &\underbrace{UU_i H U_i^{-1} U^{-1}}_{n_{f'}} \perp B_{fe_i}. \end{aligned} \tag{5.36}$$

Hence, on the stationary point U is of the form

$$U = \begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix}. \tag{5.37}$$

If we choose U_i in such a way that

$$U_i H U_i^{-1} \cdot H \neq \pm 1, \tag{5.38}$$

i.e., the two normal vectors are not (anti)parallel, then also

$$UU_i H U_i^{-1} U^{-1} \cdot H \neq \pm 1. \tag{5.39}$$

This guarantees that this configuration is non-degenerate, such that the partial integration over ϕ (see Sec. V B) and the fixing of the o_f and u_f symmetry can be performed. Following the same method as presented in Sec. V B we prove that the asymptotic expansion of $\left(C_{U_i,f}, C_{f'}\right)$ has the DL property.

Similar considerations apply to

$$\left(C_{U_i^1,f^1}, C_{U_i^2,f^2}\right) \tag{5.40}$$

if $U_i^1 H(U_i^1)^{-1} \cdot U_i^2 H(U_i^2)^{-1} \neq \pm 1$.

Finally, using the uniqueness of the intertwiner, we obtain

$$\left(C_f, C_f\right) = \pm \frac{\left(C_f, C_{U_i^2,f^2}\right)\left(C_{U_i^1,f^1}, C_f\right)}{\left(C_{U_i^1,f^1}, C_{U_i^2,f^2}\right)}. \tag{5.41}$$

As a product of functions whose asymptotic expansion is of DL form, it follows directly that (5.41) is of DL form, too.

D. Leading order expansion and a recursion relation for the 6*j* symbol

In Secs. V B and V C we have shown that both the contributions from partial integrations over ϕ and the normalization factors satisfy the DL property. Hence using properties explained in Appendix E we have proven the conjecture from Refs. 30 and 31.

In this section we will discuss the next-to-leading order expansion for the 6*j*-symbol. Therefore, we do a brief recap of the results of Sec. IV.

From the stationary point (with outward pointing normals) we have contributions from the Hessian, i.e., the kinetic term, and higher order terms, which are computed using a Feynman diagrammatic approach

$$\propto \frac{1}{\sqrt{\det(-\mathcal{H})}} e^{i\sum l_{ij}\theta_{ij} + S_1}, \tag{5.42}$$

where S_1 are the evaluations of the connected Feynman diagrams of the expansion in $\{\theta,\ \rho\}$ evaluated on the stationary point of the action $i\sum l_{ij}\theta_{ij}$, using $-\mathcal{H}^{-1}$ as the propagator of this theory. We are interested only in $|l|^{-1}$ contributions, the respective Feynman rules are briefly discussed in Appendix G.

The expansion up to the next to leading order is of the form (see also Sec. IV C):

$$\frac{1}{2}\frac{1}{\sqrt{12\pi V}} e^{i\left(\sum_{ij}\left(l_{ij}\theta_{ij} - \frac{1}{8 l_{ij}}\cot\theta_{ij}\right) + \tilde{S}_1\right)} = \frac{1}{2}\frac{1}{\sqrt{12\pi V}} e^{i\left(\sum_{ij} l_{ij}\theta_{ij} + S_1\right)}, \tag{5.43}$$

where S_1 is of order $|l|^{-1}$. The full contribution comes from two stationary point that are related via parity transformation, see also Sec. III C 1; their contributions are related by complex conjugation. Hence, we obtain up to $|l|^{-1}$:

$$\frac{1}{\sqrt{12\pi V}}\left(\cos\left(\sum_{ij} l_{ij}\theta_{ij} + \frac{\pi}{4} + S_1\right) + O\left(|l|^{-2}\right)\right). \tag{5.44}$$

The next to leading order expansion is briefly described in Appendix G. Although, this method is algorithmically more involved than the method proposed in Refs. 30 and 31, the final expression is also more geometric. We will now derive a recursion relation for the full 6*j* symbol using a similar idea as in Refs. 36 and 37 that, we hope, can serve to compute the NLO expansion in more concise way.

1. Recursion relation for 6j symbols

In this section we derive a recursion relation for the whole 6*j* symbol. First, let us introduce a multiplication operator

$$N(l) = \sqrt{\prod_i \Theta_i(l)}, \tag{5.45}$$

where Θ_i is normalization (of a three-valent intertwiner) computed from the theta graph. Furthermore, we define the operator T^v_{ij} via its action on a function of edge lengths $\{l\}$:

$$T^v_{ij}\, C(l) = \left(1 + v\frac{1}{2l_{ij}}\right) C(\{l_{km} + v\delta_{(ij)(km)}\}). \tag{5.46}$$

We assume that $T^v_{ii} = 1$.

As a next step, recall the definition of $\tilde{P}_l$ (5.14) and its recursion relation (5.19). The latter can be written as follows:

$$\cos\theta\, \tilde{P}_l \equiv \left(\frac{1}{2} + \frac{1}{4l}\right)\tilde{P}_{l+1} + \left(\frac{1}{2} - \frac{1}{4l}\right)\tilde{P}_{l-1}. \tag{5.47}$$

and we can write the non-normalized $6j$ amplitude as

$$Z''(l) = \int \prod \mathrm{d}\,\theta_{ij} \prod_{(ij)} \sin\theta_{ij} \prod_{(ij)} \tilde{P}_{l_{ij}}(\theta_{ij})\delta(\det\tilde{G}). \tag{5.48}$$

In order to derive the recursion relation, we insert an additional $\det\tilde{G}$ into (5.48):

$$\int \prod \mathrm{d}\,\theta_{ij} \det\tilde{G} \prod_{(ij)} \sin\theta_{ij} \prod_{(ij)} \tilde{P}_{l_{ij}}(\theta_{ij})\delta(\det\tilde{G}) = 0, \tag{5.49}$$

since $\det\tilde{G}$ is constrained to vanish. Similar to Refs. 36 and 37, $\det\tilde{G}$ can be expanded as a sum over perturbations:

$$\det\tilde{G} = \sum_{\sigma\in S_4} \mathrm{sgn}\sigma \frac{1}{16} \sum_{\vec{v}\in\{-1,1\}^4} e^{i v_i \theta_{i\sigma_i}}, \tag{5.50}$$

with the convention that $\theta_{ij} = \theta_{ji}$ and $\theta_{ii} = 0$. Using (5.50), Eq. (5.49) can be rewritten as

$$\sum_{\sigma\in S_4} \mathrm{sgn}\,\sigma \frac{1}{16} \sum_{\vec{v}\in\{-1,1\}^4} \prod_i T^{v_i}_{i\sigma_i} Z''(l) = 0. \tag{5.51}$$

On the other hand, we know from previous calculations that

$$\{6j\} \equiv N^{-1} Z''(l) + c.c + O(l^{-\infty}), \tag{5.52}$$

such that we can summarize both (5.51) and (5.52) into the following recursion relation for the $6j$ symbol that has been verified numerically for several $6j$ symbols

$$\det\left[\frac{T^1_{ij} + T^{-1}_{ij}}{2}\right] N\{6j\} \equiv 0, \tag{5.53}$$

where T^v_{ij} is defined as in (5.46).

Another useful form is the following

$$\sum_{\sigma\in S_4} \mathrm{sgn}\,\sigma \frac{1}{16} \sum_{\vec{v}\in\{-1,1\}^4} \frac{N(l + v_{i\sigma_i})}{N(l)} \left(\prod_i T^{v_i}_{i\sigma_i}\right)\{6j\} \equiv 0, \tag{5.54}$$

since the expansion of $\frac{N(l+v_{i\sigma_i})}{N(l)}$ is straightforward to compute. We have to point out though that the coefficients in this formula are not rational, yet they allow for nice a asymptotic expansion. Thus they should in principle allow for the computation of the higher order expansions of the $6j$ symbol.

VI. DISCUSSION AND OUTLOOK

Coherent state approaches are the only available tools so far to successfully compute the asymptotic expansion of spin foam models,[14–19] which gives us a first, and yet, very incomplete understanding of the relation of spin foam models to gravity. The strength and beauty of this approach is its clear geometrical interpretation and straightforward computation of the dominating phase of the expansion, which is identified as the Regge action of the examined triangulation. Despite these successes, the approach usually fails in the computation of the determinant of the Hessian matrix, which provides the normalization to the path integral and, more importantly, a measure on the space of geometries.

To overcome this drawback, we have introduced modified coherent states, i.e., states labelled by null eigenvectors with respect to a generator of rotations, smeared perpendicular to the axis of rotation. We have shown that these states allow for the same geometrical interpretation as the usual $SU(2)$ coherent states and presented a method to deal with the (due to the smearing) increased number of stationary points. This allowed us to derive the well-known asymptotic expansion of the $SU(2)$ $6j$ symbol[5] entirely, by computing its amplitude in the stationary phase approximation, first with respect to the smearing parameters and second, after a variable transformation, with respect to the dihedral angles of the tetrahedron. In the process, we have discovered that the resulting amplitude is proportional to the action of the first order formulation of Regge calculus, a result that supports the conjecture given in Ref. 33 that 4D spin foam models can be better described by angle and area variables instead of only edge lengths, the fundamental variables of ordinary Regge calculus. This result could also stimulate new work following the ideas of Ref. 27 to obtain an invariant path integral measure (under Pachner moves[44,45]) for first order Regge calculus and to compare it to spin foam models.

In addition to this result, we also extended the calculation to the next to leading order correction for the $6j$ symbol. We have been able to prove the conjecture presented in Refs. 30 and 31 that the higher order corrections are alternatingly oscillating with the cosine or the sine of the Regge action, and furthermore we can, in principle, calculate the asymptotic expansion up to arbitrary order. Despite this success, we are not able to present the next-to-leading order in a short and concise way. This is a nuisance of all known derivations of next-to-leading order expansion, see for example, Refs. 30 and 31. However, we derived a recursion relation for the $6j$ symbol, very similar in nature to the one in Refs. 36 and 37, that can in principle be used to obtain more concise form of the next to leading order term.

The main goal of this work was not the derivation of known results, but to develop and advertise a new coherent state method, which is capable of challenging the determination of the measure in spin foam models.[14–18] The computation of the full asymptotic expansion (even only up to leading order) would not only increase the understanding of spin foam models, but could also give a measure on the space of geometries, which could be compared to the proposed measure in Ref. 27. Given such a measure, one would be able to examine which geometries dominate the spin foam transition amplitudes in the various models, which could also be used to exclude some of them. Our successful and complete derivation of the asymptotic expansion of the $SU(2)$ $6j$ symbol is a good start, however the method still has to prove itself by tackling more complicated models. Therefore, two issues have to be overcome:

The first problem is to extend the presented coherent state approach to groups with non-unique intertwiners. Our calculations are heavily based on the fact that the intertwiner of three irreducible representations of $SU(2)$ is unique, which simplified the construction of our model. The only 4D spin foam model with unique intertwiners is the Barrett-Crane model,[8] which has already been ruled out as a viable quantum gravity theory. Nevertheless, our calculations presented in this work can be applied and can lead to interesting new insights.[46]

The second problem is common to all coherent state approaches to spin foam models so far; all the known calculations are restricted to one simplex of the triangulation. To extract the asymptotic expansion for larger triangulations and to examine possible invariances under (local) changes of the triangulation like Pachner moves is still an open issue. In this work, before computing the asymptotic expansion of the $6j$ symbol, we have kept the discussion as general as possible. It would be interesting to examine, whether the relation to the first order formulation of Regge calculus can also be found in larger triangulations or whether one obtains modifications, which could be understood as quantum gravity effects.

At the end we would like to point out that the application of our method to the case of the non-compact group $SL(2,\mathbb{R})$ is rather straightforward and we leave the determination of the 3D Lorentzian $6j$ symbol for future investigations.

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APPENDIX A: SPIN NETWORK EVALUATION AND SIGN CONVENTION

This appendix is devoted to the sign issue. We will show how one can determine the total sign of our formula using the prescription of Refs. 38 and 39.

1. Penrose prescription for spherical graph

In this section we will describe a canonical way to evaluate spherical (planar) spin networks. Let us draw it on the 2-sphere such that no edges intersect; if the spin network is 2-line irreducible there are two distinct ways to do so, which differ by orientation. The result of the evaluation does not depend on this choice. For every node of the graph (a face in the dual picture) we have a natural cyclic order inherited from the orientation of the sphere. In the second step we choose any ordering of nodes (faces). This gives a natural orientation of the edges; they start in nodes lower in the order and end in nodes higher in the order. We draw the graph on the plane as shown in Figure 5 such that the order of the nodes is preserved and the order of legs in every node is consistent with the cyclic order obtained above.

In the third step we count the number of crossings s of half-integer edges with each other. The spin network is evaluated by contracting invariants, given for every node, by using the ϵ bilinear form oriented according to the edge orientation inherited from the nodes. The ordering of the legs is as in Figure 6:

$$\prod_e \epsilon^{j_e}_{A_{s(e)e}A_{t(e)e}} \prod_v I_v^{A_{ve_1}A_{ve_2}A_{ve_3}}. \tag{A1}$$

These invariants are described in Refs. 38 and 39 (see also Sec. III D). One can show that the result does not depend on the made choices.

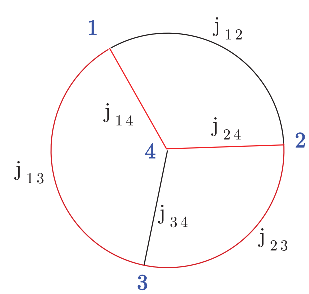

FIG. 5. Orientation of intertwiners inherited from orientation of the sphere (plane). Half-integer spins colored red.

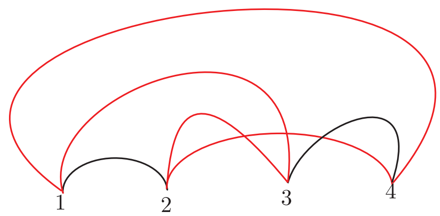

FIG. 6. Nodes are in the right order and for each intertwiner the legs are in the right cyclic order. The number of crossings for half-integer edges is $s = 2$.

2. Sign factors and spin structure

In this section we will show how to compute the sign factor for spherical graphs. First of all, let us notice that in the case that all j are integers, the sign disappears completely. We will prove now that this is also the case in general. Explicitly we will prove that (see Sec. III E for the definitions)

$$s + \sum_f s_f + \sum s_e = 0 \text{ mod } 2. \tag{A2}$$

a. Sign factor in the intertwiner

In this section we compute the sign s_f. In order to do this, we compare our invariant with the one from Refs. 38 and 39 (given for a fixed order of j_1, j_2, j_3). The dual of the latter is given on vectors $\xi_1^{2j_1} \otimes \xi_2^{2j_2} \otimes \xi_3^{2j_3}$ by the formula

$$(-1)^{j_1+j_3-j_2}\, C\, \epsilon(\xi_1, \xi_2)^{j_1+j_2-j_3} \epsilon(\xi_2, \xi_3)^{j_2+j_3-j_1} \epsilon(\xi_3, \xi_1)^{j_1+j_3-j_2} \tag{A3}$$

with normalization $C > 0$.[38,39]

The contraction of (A3) with our invariant is given by

$$(-1)^{j_1+j_3-j_2} C \int \frac{d\phi_1\, d\phi_2\, d\phi_3}{(2\pi)^3} \epsilon(v_{\phi_1}, v_{\phi_2})^{j_1+j_2-j_3} \epsilon(v_{\phi_2}, v_{\phi_3})^{j_2+j_3-j_1} \epsilon(v_{\phi_3}, v_{\phi_1})^{j_1+j_3-j_2}, \tag{A4}$$

where

$$v_\phi = \begin{pmatrix} \cos\phi \\ \sin\phi \end{pmatrix}, \quad \epsilon(v_\phi, v_{\phi'}) = \sin(\phi' - \phi), \tag{A5}$$

and we skipped the integration over U, since (A3) is invariant. Let us recall our notation:

$$\psi_{ij} = \phi_i - \phi_j. \tag{A6}$$

After a change of variables

$$(\phi_1, \phi_2, \phi_3) \to (\psi_{21}, \psi_{32}, \phi_1) \tag{A7}$$

and performing one trivial integration over ϕ_1, (A4) is equal to

$$(-1)^{j_1+j_3-j_2} C \int \frac{d\psi_{21}\, d\psi_{32}}{(2\pi)^2} (\sin\psi_{21})^{j_1+j_2-j_3} (\sin\psi_{32})^{j_2+j_3-j_1} (\sin\psi_{13})^{j_1+j_3-j_2}, \tag{A8}$$

with the constraint $\psi_{21} + \psi_{32} + \psi_{13} = 0$.

As the expression is real (since $j_i + j_k - j_l$ is an integer), in the asymptotic limit it is dominated by the stationary point (maxima of the integral) of the action

$$(j_1 + j_2 - j_3) \ln|\sin\psi_{21}| + (j_2 + j_3 - j_1) \ln|\sin\psi_{32}| + (j_1 + j_3 - j_2) \ln|\sin\psi_{13}| + \rho(\psi_{21} + \psi_{32} + \psi_{13}), \tag{A9}$$

where ρ is a Lagrange multiplier and $\psi_{21}, \psi_{32}, \psi_{13}$ are treated as independent variables. The stationary point condition reads

$$(j_i + j_k - j_l) \cot\psi_{ij} = \rho. \tag{A10}$$

Now we can use the fact that

$$\cot\psi_{32}\cot\psi_{21}+\cot\psi_{13}\cot\psi_{32}+\cot\psi_{21}\cot\psi_{13}=1 \tag{A11}$$

to obtain

$$\rho^2=\frac{(j_1+j_2-j_3)(j_2+j_3-j_1)(j_1+j_3-j_2)}{j_1+j_2+j_3}. \tag{A12}$$

Furthermore, we see that

$$\cot^2\psi_{32}=\frac{(j_1+j_2-j_3)(j_1+j_3-j_2)}{(j_2+j_3-j_1)(j_1+j_2+j_3)}=\frac{j_1^2-(j_2-j_3)^2}{(j_2+j_3)^2-j_1^2}. \tag{A13}$$

Hence, we compute that

$$\cos 2\psi_{32}=\frac{\cot^2\psi_{32}-1}{\cot^2\psi_{32}+1}=\frac{j_1^2-j_2^2-j_3^2}{2j_2j_3}, \tag{A14}$$

$$\sin 2\psi_{32}=\frac{2\cot\psi_{32}}{\cot^2\psi_{32}+1}=\pm\frac{A}{j_2j_3}, \tag{A15}$$

where A is the area of the triangle with edge lengths j_1, j_2, j_3. Thus $\pm 2\psi_{32}$ modulo 2π is the angle in this triangle opposite to the edge j_1. Similar relations hold for ψ_{21} and ψ_{13}. Together with the relation $\psi_{21}+\psi_{32}+\psi_{13}=0$, it gives the condition that

$$2\psi_{21},\ 2\psi_{32},\ 2\psi_{13} \bmod 2\pi \tag{A16}$$

are oriented (i.e., incorporate sign) angles of the triangle on the plane with edges (j_1, j_2, j_3).

In the presence of a function f_f, only one of those stationary points contributes. Since the Jacobian is real, the only contribution to the sign is given by the value of the integral in the stationary point. We know that $\psi_{ij}\in(\pi, 2\pi)$ for consecutive pair of edges (ij) (see Sec. II A 1), thus $\sin\psi_{ij}<0$ and the total sign is

$$(-1)^{j_1+j_3-j_2}\,(-1)^{j_1+j_2-j_3}(-1)^{j_2+j_3-j_1}(-1)^{j_1+j_3-j_2}=(-1)^{2j_2}. \tag{A17}$$

As already discussed above, this is a relative sign of our invariant with respect to the invariant described in Refs. 38 and 39.

b. The sign $\sum s_e$

In the stationary point we can write (see Secs. III A 1 and III A 2 for the derivation)

$$U_{s(e)}^{-1}U_{t(e)}=(-1)^{\tilde{s_e}}O_{s(e)e}e^{-i\tilde{\theta}_{s(e)t(e)}\begin{pmatrix}1&0\\0&-1\end{pmatrix}}\begin{pmatrix}0&-1\\1&0\end{pmatrix}O_{t(e)e}^{-1}, \tag{A18}$$

where we assumed that $\tilde{\theta}_{s(e)t(e)}\in\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$. It is straightforward to check that

$$U_{t(e)}^{-1}U_{s(e)}=(-1)^{\tilde{s_e}+1}O_{t(e)e}e^{-i\tilde{\theta}_{t(e)s(e)}\begin{pmatrix}1&0\\0&-1\end{pmatrix}}\begin{pmatrix}0&-1\\1&0\end{pmatrix}O_{s(e)e}^{-1}, \tag{A19}$$

where $\tilde{\theta}_{t(e)s(e)}=-\tilde{\theta}_{s(e)t(e)}\in\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$. Thus in general we have

$$U_f^{-1}U_{f'}=(-1)^{\tilde{s_e}+c_e}O_{fe}e^{-i\tilde{\theta}_{ff'}\begin{pmatrix}1&0\\0&-1\end{pmatrix}}\begin{pmatrix}0&-1\\1&0\end{pmatrix}O_{f'e}^{-1}, \tag{A20}$$

where

$$c_e = \begin{cases} 0 & f = s(e) \text{ and } f' = t(e) \\ 1 & f = t(e) \text{ and } f' = s(e) \end{cases}. \tag{A21}$$

By a cycle we denote an assignment of a number $\{0, 1\}$ to every edge such that

$$\forall_f \sum_{e \subset f} c_e = 0 \bmod 2. \tag{A22}$$

The set of cycles is denoted by Z_1. Abusing the notation, we will also say that the cycle is formed by edges with $c_e = 1$. Let us notice that such edges form a disjoint sum of loops that we will denote by c_i.

For every cycle c holds

$$\prod_i \left(\prod_j U^{-1}_{f^i_j} U_{f^i_{j+1}} \right) = 1, \tag{A23}$$

where $\{f^i_j f^i_{j+1}\}$ are consecutive pair of faces in the cycle c_i (in the correctly chosen order).

Thus, we can write

$$(-1)^{\tilde{s}(c)} = \prod_{e=[ff'] \subset c} (-1)^{c_e} O_{fe} e^{-i\tilde{\theta}_{ff'} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} O^{-1}_{f'e}, \tag{A24}$$

where we used the same order of multiplications as before. Equation (A23) translate into the set of equations satisfied by $\tilde{s}_e$:

$$\forall\, c \in Z_1 \quad \sum_{e \in c} \tilde{s}_e c(e) = \tilde{s}(c) \bmod 2. \tag{A25}$$

Given a solution for the $\tilde{s}_e$, U_f can be reconstructed up to a U transformation. The solutions $\{\tilde{s}_e\}$ are not completely determined, but the residual symmetry is given by Ran ∂ where

$$\partial : C_0 \to C_1, \tag{A26}$$

is a boundary operator because $C_1 = \ker \partial^* \oplus \operatorname{Ran} \partial$. Those correspond exactly to $-U_f$ transformations.

We are interested in Sec. III E

$$\sum s_e = \sum_{e \in c} \tilde{s}_e c(e), \tag{A27}$$

where c is the cycle formed by all edges that are half-integer.

c. Sign of basic cycles in spherical case

In this section we will compute the sign factor $\tilde{s}(c)$ for cycles consisting of only a single loop. Every other cycle can be uniquely written as a sum (as the $\mathbb{Z}_2$ module) of such disjoint cycles.

Let us take such a cycle. The cycle is described by the sequence of consecutive faces and edges. The value of $(-1)^{\tilde{s}(c)}$ is thus equal to

$$\prod_{\{ff'\} \in c} O_{fe} e^{-i\tilde{\theta}_{ff'} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} O^{-1}_{f'e}. \tag{A28}$$

All parameters (i.e., ϕ_{fe} and $\tilde{\theta}_{ff'}$) can be continuously deformed, i.e., there exists a map

$$[0, 1] \ni t \to \{\phi^t_{fe}, \tilde{\theta}^t_{ff'}\} \tag{A29}$$

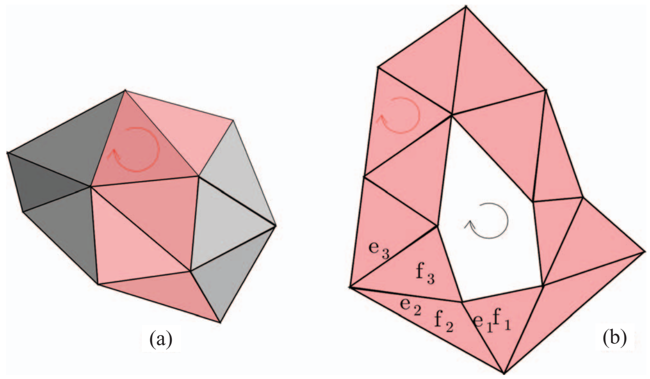

FIG. 7. (a) Cycle before deformation. (b) Cycle after deformation.

such that

$$\forall_{e\subset f}\ \phi^0_{fe} = \phi_{fe}, \quad \forall_{ff'}\ \tilde{\theta}^0_{ff'} = \tilde{\theta}_{ff'}, \tag{A30}$$

that satisfies the following conditions:

- the image of (A28) in $SO(3)$ is always the identity,
- at the end all deformed $SU(2)$ angles $\tilde{\theta}^1_{ff'}$ are equal to 0,
- for every face f with ordered pair of edges e, e' (neighbours in the cycle), the difference $\phi^t_{fe} - \phi^t_{fe'} \in \left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$ modulo 2π during the whole deformation process. In fact, it is larger than π if order of edges agrees with the orientation of the face and smaller if it does not.

The final stage of the deformation will be denoted by

$$\forall_{e\subset f}\ \tilde{\theta}'_{ff'} = \tilde{\theta}^1_{ff'}, \quad \forall_{ff'}\ \phi'_{fe} = \phi^1_{fe}. \tag{A31}$$

Up to 2-dimensional homotopies, there are two possible final stages of such deformations. They differ by orientation of the cycle (loop) drawn on the plane. We assume that the faces are ordered in agreement with total orientation.

The cycle before and after the deformation is shown in Figure 7. The process is shown in Figure 8 as an example of a single-loop cycle around the vertex. In the end we obtain

$$\prod_{\{ff'\}\in c} (-1)^{c_e} O_{fe} e^{-i\tilde{\theta}'_{ff'}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}} \begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix} O^{-1}_{f'e} = (-1)^{C_e} \prod_{\{ff'\}} O'_{fe} O'^{-1}_{f'e} \prod_e \begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}, \tag{A32}$$

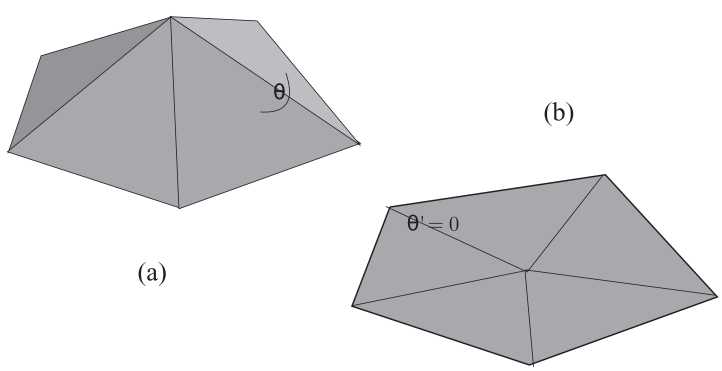

FIG. 8. Example of the single-loop cycle around a vertex. (a) Cycle before deformation: angle θ between two faces depicted. (b) Cycle after deformation $\theta' = 0$, the faces are parallel.

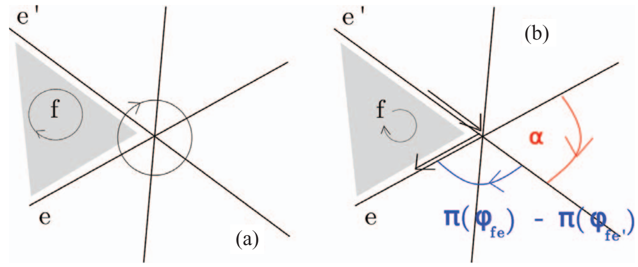

FIG. 9. (a) Example of the cycle with orientations shown, (B) $\pi(\phi'_{fe}) - \pi(\phi'_{fe'})$ and $\alpha = \pi - (\pi(\phi'_{fe}) - \pi(\phi'_{fe'}))$.

where C_e is the number of edges with $c_e = 1$ because O'_{fe} commutes with $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. The images of O'_{fe} (and related $SO(3)$ angles $\pi(\phi'_{fe})$) satisfy (see Figure 9)

$$\sum_e \pi(\phi'_{fe}) - \pi(\phi'_{f'e}) = -\sum_{\{ee'\}\subset f} \pi(\phi'_{fe}) - \pi(\phi'_{fe'}) = -(n-2)\pi, \tag{A33}$$

where n is the number of faces meeting in the cycle c.

Using prescription (see Sec. II A 1) for $\phi_{fe} - \phi_{fe'}$, the fact that $SU(2)$ is the double cover of $SO(3)$ and continuity of the deformation we obtain (modulo 2π)

$$\sum_f \phi'_{fe} - \phi'_{fe'} = \sum_f \frac{\pi(\phi'_{fe}) - \pi(\phi'_{fe'})}{2} - \pi = \left(-n + \frac{n}{2} + 1\right)\pi. \tag{A34}$$

Thus

$$\prod_{\{ff'\}} O'_{fe} O'^{-1}_{f'e} \prod_e \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}^{n-2} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}^{n} = -1. \tag{A35}$$

To sum up, we obtained for a given cycle c

$$\sum_e c(e)\tilde{s}_e = C_e + 1 \bmod 2. \tag{A36}$$

Since the cycle is oriented in the same way as the faces, C_e is the number of edges oriented according to the cycle.

d. Other method of computation

Let us consider an arbitrary cycle c. Let us draw it on the graph G as in Figure 6. We will denote by $s(c)$ the number of crossings in the cycle. For any node (face) f we also denote

$$f(c) = \begin{cases} 0 & \text{if the middle leg edge of } f \text{ does not belong to } c \\ 1 & \text{if the middle leg edge of } f \text{ belongs to } c \end{cases} \tag{A37}$$

In the following, we will present another method of how to compute $\sum_e c_v(e)\tilde{s}_e$ for a basic cycle c. First we will prove

Lemma 4. For a single loop cycle c in a spherical network, the quantity

$$C_e + \sum_f f(c) + s(e) \bmod 2 \tag{A38}$$

does not depend on the choice of a graph G.

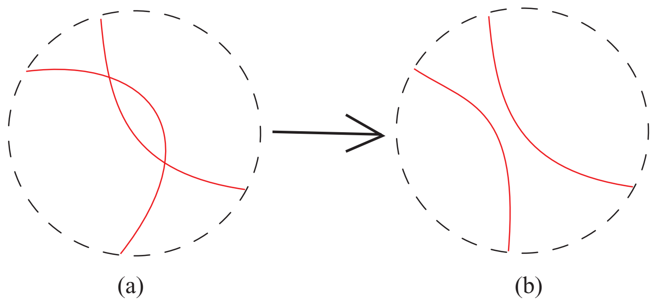

FIG. 10. Part of the graph changed by the move. Example of a Reidemeister move.

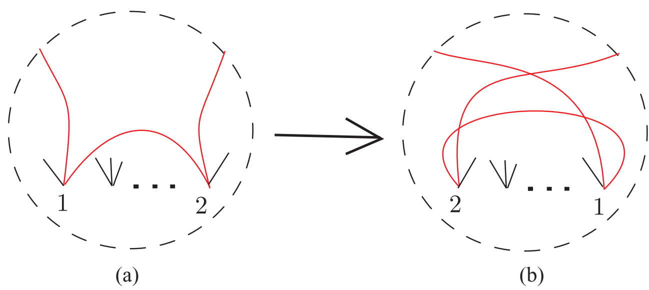

FIG. 11. Part of the graph changed by the move (edges not belonging to the cycle are not drawn). Two consecutive nodes in the cycle transposed.

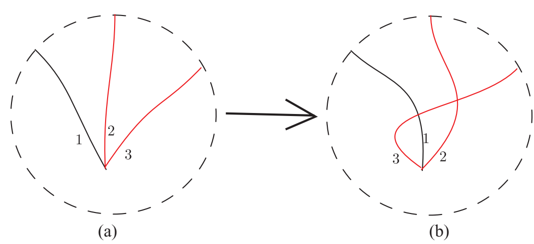

FIG. 12. Part of the graph changed by the move. Cyclic change of the order of legs.

Proof. Any two graphs can be transformed into one another by a sequence of basic moves

- One of the Reidemeister moves[47,48] for the edge (see Figure 10 for example). It only changes $s(c)$ by an even number.
- Transposition of two consecutive nodes belonging to the cycle (see Figure 11). In the move shown in the figure

$$C_e' = C_e \pm 1, \quad s(c)' = s(c) + 3, \tag{A39}$$

  and all $f(c)$ remain unchanged.
- Cyclic permutation of the legs of a node f (Figure 12). In this case

$$f(c) + s(c) \tag{A40}$$

  is preserved.

Thus $C_e + \sum_f f(c) + s(c)$ mod 2 is invariant.

We see that $(C_e + 1) + \sum_f f(c) + s(c)$ does not depend on the chosen graph G, hence, we can choose the most convenient one (see Figure 13). For this particular choice

$$C_e = 1, \quad \forall_{f \subset c} f(c) = 0, \quad s(c) = 0, \tag{A41}$$

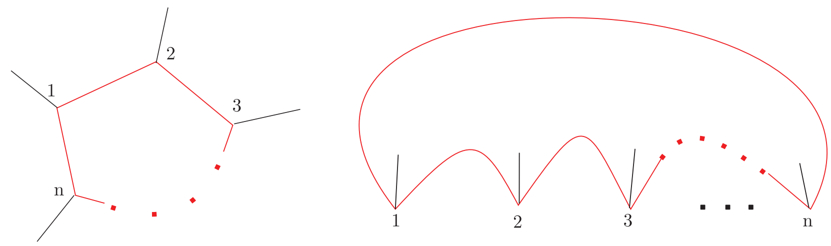

FIG. 13. Convenient choice for the graph G.

and thus $(C_e+1)+\sum_f f(c)+s(c)=(1+1)+0=0 \bmod 2$ and

$$\sum_e c(e)\tilde{s}_e = C_e + 1 = \sum_f f(c) + s(c) \bmod 2. \tag{A42}$$

e. Sign of the general cycle in spherical case

Let us state now a few properties of $f(c)$ and $s(c)$ useful in the sequel.

For two cycles c and c' we denote a cycle by $c+c'$ if it satisfies the following property:

$$\forall_e,\ (c+c')(e) = c(e) + c'(e) \bmod 2. \tag{A43}$$

We have for two disjoint cycles c and c'

$$s(c+c') = s(c) + s(c') \bmod 2, \tag{A44}$$

$$\forall_f\ f(c+c') = f(c) + f(c') \bmod 2. \tag{A45}$$

We can now write every cycle c in the spherical case as a sum of disjoint single-loop cycles c_α, such that $c=\sum_\alpha c_\alpha$:

$$\begin{aligned}\sum_e c(e)\tilde{s}_e &= \sum_e \left(\sum_\alpha c_\alpha(e)\right)\tilde{s}_e = \sum_\alpha \left(\sum_f f(c_\alpha) + s(c_\alpha)\right) = \\ &= \sum_f f\left(\sum_\alpha c_\alpha\right) + s\left(\sum_\alpha c_\alpha\right) = \sum_f f(c) + s(c) \bmod 2.\end{aligned} \tag{A46}$$

f. Final sign formula

Let us notice that in the case when c is the cycle of all half-integer spins we have

$$s = s(c), \quad \forall_f s_f = f(c), \quad \sum s_e = \sum_e c(e)\tilde{s}_e\,, \tag{A47}$$

since if we denote the spin of the middle leg edge of f by j_{f2} then $f(c)=2j_{f2} \bmod 2$. Finally,

$$s + \sum_f s_f + \sum s_e = 2\left(\sum_f f(c) + s(c)\right) = 0 \bmod 2. \tag{A48}$$

APPENDIX B: CHANGES OF VARIABLES AND THEIR JACOBIANS

Let the Lie group G act transitively on the manifold S and let

$$\chi : G \to \mathbb{R} \tag{B1}$$

be a homomorphism. There exists at most one measure μ (up to scaling) on S such that

$$g^*\mu = \chi(g)\mu. \tag{B2}$$

Let

$$H \circlearrowright S_1 \to S_2, \tag{B3}$$

where S_1 is a principal Lie group bundle with the structure group H and the base space S_2. Any (pseudo-)k-form μ_2 on S_2 can be uniquely represented by a (pseudo-) k-form μ_1 on S_1 that satisfies

$$h^*\mu_1 = \mu_1 \quad \forall h \in H, \tag{B4}$$

$$\mu_1 \perp \partial_\xi = 0 \quad \forall\, \partial_\xi \in \mathfrak{h}, \tag{B5}$$

where $\mathfrak{h}$ is the Lie algebra of H and $\perp$ is contraction of the (pseudo-) form with the vector on the first site. Any form μ_1 determines the form μ_2 on S_2. The integration over S_2 is the integration over any section of the projection map $S_1 \to S_2$.

Such a form satisfying conditions (B4) and (B5) can be obtained from the H invariant form μ on S_1 via the formula

$$\mu_2 = \mu \perp \bigwedge_{\xi \text{ basis } \mathfrak{h}} \partial_\xi. \tag{B6}$$

In case of a compact group H it is related to the measure obtained by integration over the fibers, called $\mu_{\int H}$, as follows

$$\mu_2 = (\mu_H \perp \bigwedge \partial_\xi)\mu_{\int H}, \tag{B7}$$

where μ_H is the normalized Haar measure on H.

Let $M \subset S$ be a submanifold described locally by a set of independent equations f_a. For any measure (form) μ on S we can define a measure (form) μ_{f_a} on M by the following integration prescription: Let $g \in C^0(M)$ and $\tilde{g}$ be any continuous extension to S, then

$$\int_M \mu_{f_a} g = \int_M \prod \delta(f_a)\tilde{g}\mu. \tag{B8}$$

Let M be a section of the bundle $H \subset S_1 \to S_2$ described by equations f_a, then we can compare the just described measures on M and S_2 since $M \to S_2$ is a diffeomorphism of M onto S_2:

$$\mu_{f_a} = \left(\det \partial_{\xi_i} f_a\right)^{-1} \mu \perp \bigwedge \partial_\xi. \tag{B9}$$

Indeed, we can choose local coordinates such that $S_1 = M \times H$ and the zero section is described by $f_a = 0$. We have

$$\mu = (\mu \perp \bigwedge \partial_\xi) \wedge \bigwedge d\xi. \tag{B10}$$

By extending the function g constantly along fibers from the zero section, we obtain

$$\begin{aligned} \int_M g\mu_{f_a} = \int_{S_1} \prod \delta(f_a)\tilde{g}\mu = \int_M \tilde{g} \left(\int_H \prod \delta(f_a) \bigwedge d\xi\right) (\mu \perp \bigwedge \partial_\xi) \\ = \int_{S_2} g \left(\det \partial_{\xi_i} f_a\right)^{-1} (\mu \perp \bigwedge \partial_\xi). \end{aligned} \tag{B11}$$

1. Change of variables $u_i \to n_i$ (integrating out gauge)

Let us remind from Sec. III C that

$$S^2 = SU(2)/S^1, \tag{B12}$$

given by the right action of S^1 on $SU(2)$. The sphere S^2 can either be represented by unit vectors $|n_i|^2 = 1$ or traceless 2×2 matrices n_i with the condition

$$\frac{1}{2}\,\mathrm{Tr}\, n_i n_i = 1. \tag{B13}$$

Then the quotient map is given by

$$n_i(u) = uHu^{-1}, \tag{B14}$$

where $H = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}$. The group $SU(2)$ acts on S^2 (as a left action on the quotient) by

$$n_i \to u\, n_i\, u^{-1}. \tag{B15}$$

The Haar measure from $SU(2)$ can be integrated over the fibers giving the invariant measure μ on the sphere with total volume 1.

Another invariant measure is

$$\delta(|n|^2 - 1)\,\mathrm{d}n_1\,\mathrm{d}n_2\,\mathrm{d}n_3. \tag{B16}$$

Since there is only one invariant measure up to scale, both are related by a scaling transformation:

$$\mu = c\delta(|n|^2 - 1)\,\mathrm{d}n_1\,\mathrm{d}n_2\,\mathrm{d}n_3. \tag{B17}$$

The constant is fixed by requiring

$$1 \stackrel{!}{=} \int_{S^2} \mu = c \int_0^{2\pi} \mathrm{d}\phi \int_0^{\pi} \sin\theta\,\mathrm{d}\theta \int_0^{\infty} r^2 \delta(r^2 - 1)\,\mathrm{d}r = 2\pi c, \tag{B18}$$

thus

$$\mu = \frac{1}{2\pi}\delta(|n|^2 - 1)\,\mathrm{d}n_1\,\mathrm{d}n_2\,\mathrm{d}n_3. \tag{B19}$$

2. Variables θ in the flat tetrahedron

Let us consider two sets of variables

$$N = (\vec{n}_1, \ldots, \vec{n}_{m+1}), \tag{B20}$$

where $\vec{n}_i$ are m vectors with exactly one dependency, i.e., every subset of m vectors forms a basis. Let

$$M = N^T N,\ m_{ij} = \vec{n}_i \vec{n}_j,\ i \leq j\ , \tag{B21}$$

where M is a symmetric positive $(m+1) \times (m+1)$ matrix, which is degenerate with exactly one null eigenvector, whose entries are all non vanishing.

On N there exists a left action of $O(m)$, $\vec{n}_i \to O\vec{n}_i$. The matrix M is $O(m)$ invariant, so the parameters of this action can be regarded as supplementary to M. The vector fields of this action will be denoted by L_{ab}. The map $N \to M$ is an $O(m)$ principal bundle.

In the following, our goal is to compare the pseudo-form

$$\mu_1 = \left| \bigwedge_{i=1\ldots m+1, a=1..m} \mathrm{d}n_i^a \perp \bigwedge_{a<b} L_{ab} \right| \tag{B22}$$

with the form

$$\mu_2 = \left| \delta(\det M) \bigwedge_{i \le j} \mathrm{d}\, m_{ij} \right| . \tag{B23}$$

Let us notice that both μ_1 and μ_2 are measures on M.

There are additional transformations parametrized by $U \in GL(m+1)$

$$\tilde{N} = NU, \ \tilde{M} = U^T M U \ , \tag{B24}$$

which commute with the $O(m)$ action on N. The measure μ_1 is χ covariant with respect to this action, where

$$\chi(U) = |\det{}^m|(U) := |\det(U)|^m \quad \forall U \in GL(m+1). \tag{B25}$$

Furthermore, we have

- μ_2 is invariant for $U \in O(m+1)$,
- for transformations of the form $U = \operatorname{diag} \lambda_i$ (diagonal matrix) the measure μ_2 transforms as

$$\Bigg| \underbrace{\frac{1}{\prod_i \lambda_i}}_{\text{from } \det M} \underbrace{\prod_{i \le j} \lambda_i \lambda_j}_{=\prod_i \lambda_i^{m+1}} \Bigg| \mu_2, \tag{B26}$$

so it is also χ covariant as rotations and scaling generate the whole group.

Hence the two measures μ_1 and μ_2 differ by a constant c as $GL(m+1)$ acts transitively on M. This constant can be computed for a special value of N:

$$n_i^a = \begin{cases} \delta_i^a, & i \le m \\ 0, & i = m+1 \end{cases} . \tag{B27}$$

In this choice

$$\mathrm{d}\, m_{ij} = \begin{cases} \mathrm{d}\, n_i^j + \mathrm{d}\, n_j^i, & i, j \le m \\ \mathrm{d}\, n_{m+1}^i, & j = m+1 \\ 0, & i = j = m+1 \end{cases} . \tag{B28}$$

Moreover

$$\mathrm{d}\, n_i^a \perp L_{cd} = (L_{cd} \vec{n}_i)^a = \delta_{ac}\delta_{id} - \delta_{ad}\delta_{ic}. \tag{B29}$$

The following equalities hold

$$2^m \left| \bigwedge_{i=1\ldots m+1, a=1..m} \mathrm{d}\, n_i^a \right| = \left| \bigwedge_{1 \le j \le i \le m+1, j \ne m+1} \underbrace{dn_i^j + dn_j^i}_{dm_{ij}} \wedge \bigwedge_{1 \le j < i \le m+1} dn_i^j \right| , \tag{B30}$$

but since $\mathrm{d}\, m_{ij} \perp L_{ab} = 0$:

$$2^m \left| \bigwedge_{i=1\ldots m+1, a=1..m} \mathrm{d}\, n_i^a \perp \bigwedge_{a<b} L_{ab} \right| = \left| \bigwedge_{1 \le j \le i \le m+1, j \ne m+1} dm_{ij} \right| = \delta(m_{m+1,m+1}) \left| \bigwedge_{(i,j): i \le j} \mathrm{d}\, m_{ij} \right| . \tag{B31}$$

Moreover, in this case

$$M_{ij} = \begin{cases} \delta_{ij}, & i \le m \\ 0, & i = m+1 \end{cases} \tag{B32}$$

and so $\frac{\partial \det M}{\partial m_{m+1,m+1}} = 1$. Eventually,

$$2^m \left| \bigwedge_{i=1...m+1, a=1..m} \mathrm{d}\, n_i^a \perp \bigwedge_{a<b} L_{ab} \right| = \delta(\det M) \left| \bigwedge_{i\leq j} \mathrm{d}\, m_{ij} \right|. \tag{B33}$$

a. Integration over the SO(3) fiber

In the case $m = 3$ we are interested in integrating over the fiber, however not the whole $O(3)$ but only over one connected component with respect to $SO(3)$. This is due to the u transformation symmetry corresponds to $SO(3)$ not $O(3)$ (see also Sec. II C).

In this section we continue to compute the correct constant in front of the measure. We will now consider a fibration with the group $SO(m)$ that can still be described locally by a projection $N \mapsto M$. This is, however, enough because we are only interested in local variables.

In general we have[49]

$$\int_{SO(m)} \left| \bigwedge_{a<b} L^*_{ab} \right| = \prod_{k=2}^{n} \frac{2\pi^{\frac{k}{2}}}{\Gamma\left(\frac{k}{2}\right)}, \tag{B34}$$

so from (B7) our measure integrated over the fibre is equal to

$$\int_{SO(m)} \left| \bigwedge_{i=1...m+1, a=1..m} \mathrm{d}\, n_i^a \right| = \frac{1}{2^m} \prod_{k=2}^{m} \frac{2\pi^{\frac{k}{2}}}{\Gamma\left(\frac{k}{2}\right)} \delta(\det M) \left| \bigwedge_{i\leq j} \mathrm{d}\, m_{ij} \right|. \tag{B35}$$

If we impose the condition $|\vec{n}_i| = 1$, we integrate over a set of unit vectors. This implies for M that we have to skip $\mathrm{d}\, m_{ii}$ in the measure and we define $m_{ij} = \cos\theta_{ij}$. In these new angle variables the measure takes the form

$$\frac{1}{2^m} \prod_{k=2}^{m} \frac{2\pi^{\frac{k}{2}}}{\Gamma\left(\frac{k}{2}\right)} \delta(\det \tilde{G}) \prod_{i<j} |\sin\theta_{ij}| \bigwedge_{i<j} \left|\mathrm{d}\,\theta_{ij}\right| , \tag{B36}$$

where $\tilde{G}$ is the Gram matrix with the convention

$$\tilde{G}_{ij} = \cos\theta_{ij}, \quad \theta_{ii} = 0. \tag{B37}$$

In case $n = 3$ we have

$$\pi^2 \delta(\det \tilde{G}) \prod_{i<j} |\sin\theta_{ij}| \bigwedge_{i<j} \mathrm{d}\,\theta_{ij}. \tag{B38}$$

3. Variables θ/ l in the spherical constantly curved tetrahedron

Let us consider the spaces of matrices

$$\tilde{N} = \{N \in M_n(\mathbb{R}) : \det N > 0\} \tag{B39}$$

and

$$\tilde{M} = \{M \in M_n(\mathbb{R}) : M > 0\}. \tag{B40}$$

We have a fibration with the group $SO(n)$ (via left action on $\tilde{N}$)

$$\tilde{N} \to \tilde{M}, \quad M = N^T N. \tag{B41}$$

We can compare forms

$$\mu_1 = \left|\bigwedge \mathrm{d}\, n_i^a \perp \bigwedge \partial_\xi\right|,$$
$$\mu_2 = (\det M)^{-\frac{1}{2}} \left|\bigwedge_{i<j} \mathrm{d}\, m_{ij}\right|. \tag{B42}$$

As in the Appendix B 2 there is an action of $SL(n)$ by

$$N \to NU, \quad M \to U^T M U. \tag{B43}$$

We can check that both measures are $\chi = |\det^n|$ covariant. Since $SL(n)$ acts transitively on matrices with positive determinant, we have

$$\mu_1 = c\mu_2. \tag{B44}$$

Checking for $N = \mathbb{I}$ gives $c = 1$.

Let us notice that $n_i \cdot n_i = m_{ii}$. On the surface $m_{ii} = 1$ we can introduce angle variables $\cos\theta_{ij} = m_{ij}$ and obtain

$$(\det \tilde{G})^{-\frac{1}{2}} \prod \sin\theta_{ij} \bigwedge \mathrm{d}\,\theta_{ij} = \pm \prod \delta(n_i n_i - 1) \bigwedge \mathrm{d}\, n_i^a. \tag{B45}$$

4. Determinant $\det \frac{\partial\theta}{\partial l}$ for constantly curved simplices

We denote the length Gram matrix by G and the angle Gram matrix by $\tilde{G}$. The dimension is equal to $n - 1$ and we are working in $\mathbb{R}^n$ on the sphere with radius 1.

Our goal is to prove the following formulas for the $n - 1$-dimensional curved simplex. It was first proposed by Dittrich *et al.*[41] and checked using an algebraic manipulator. Now we are presenting the complete derivation.

Lemma 5. The following formulas hold for a spherical $(n - 1)$-simplex:

$$\det \frac{\partial\theta_{ij}}{\partial l'_{km}} = (-1)^n \frac{\prod \sin l'_{ij}}{\prod \sin\theta_{ij}} \left(\frac{\det\tilde{G}}{\det G}\right)^{\frac{n+1}{2}}, \tag{B46}$$

and for $n = 4$

$$\det \frac{\partial\theta_{ij}}{\partial l_{km}} = -\det \frac{\partial\theta_{ij}}{\partial l'_{km}} = -\frac{\det\tilde{G}}{\det G}. \tag{B47}$$

Where we used standard convention that the angle θ_{ij} is the angle on the hinge obtained by leaving out indices i and j. The length of the opposite edge, i.e., the edge connecting vertices i and j, is denoted by l'_{ij} and in 3D, l_{ij} is the length of the edge at which the angle sits.

a. Outline of the proof

We compute how the measure $\bigwedge dl'_{ij}$ transforms under the change of variables

$$\theta_{ij} \to l'_{ij}. \tag{B48}$$

In fact, introducing variables $m_{ij} = \cos\theta_{ij}$ and $m'_{ij} = \cos l_{ij}$, we have (in the right order)

$$\prod \sin\theta_{ij} \bigwedge_{i<j} d\theta_{ij} = \prod_i \delta(m_{ii} - 1) \bigwedge_{i\le j} \mathrm{d}\, m_{ij}, \tag{B49}$$

$$\prod \sin l'_{ij} \bigwedge_{i<j} dl'_{ij} = \prod_i \delta(m'_{ii} - 1) \bigwedge_{i\le j} \mathrm{d}\, m'_{ij}. \tag{B50}$$

Both measures on the left-hand side are on

$$\tilde{M}_1 = \{M \in \tilde{M} : \forall_i m_{ii} = 1\}, \tag{B51}$$

where we introduced the notation $\tilde{M} = GL_+(n)$ for simplicity.

b. Computation

There is an action of the group of diagonal matrices

$$D = \{d \in GL_+(n) : d_{ij} = \lambda_i \delta_{ij}, \quad \lambda_i > 0\} \tag{B52}$$

on $\tilde{M}$ given by

$$M \to d^T M d. \tag{B53}$$

A basis for the Lie algebra $\mathfrak{d}$ of the group D is given by $\partial_{\xi_i} = E_{ii}$ (matrices with only one nonzero entry being the ith element on the diagonal equal to 1).

We have a fibration

$$\tilde{M} \to \tilde{M}/D \tag{B54}$$

and let $\tilde{M}_1 \subset \tilde{M}$ be a cross section given by the equations

$$\forall_i \; m_{ii} = 1. \tag{B55}$$

Let us introduce maps

$$\begin{aligned} \psi_1 &: \tilde{M}_1 \to \tilde{M}/D, \quad M \mapsto [M], \\ \psi_2 &: \tilde{M} \to \tilde{M}, \quad M \mapsto M^{-1}. \end{aligned} \tag{B56}$$

Acting with ψ_2 on matrix transformed as in (B53), we have

$$\psi_2(d^T M d) = (d^{-1})^T \psi_2(M)(d^{-1}), \tag{B57}$$

such that there is a map

$$[\psi_2] : \tilde{M}/D \to \tilde{M}/D. \tag{B58}$$

Let us notice that the composition $\psi := \psi_1^{-1}[\psi_2]\psi_1$ transforms $\tilde{M}_1$ into $\tilde{M}_1$.

We define measures

$$\begin{aligned} \mu_1 &= \prod \delta(m_{ii} - 1) \bigwedge_{i \le j} \mathrm{d}\, m_{ij} = \bigwedge_{i<j} \mathrm{d}\, m_{ij}, \\ \mu &= (\det M)^{-\frac{n+1}{2}} \bigwedge_{i \le j} \mathrm{d}\, m_{ij}, \\ \mu_{\tilde{M}/D} &= \mu \perp \bigwedge_i \partial_{\xi_i}, \end{aligned} \tag{B59}$$

where ∂_{ξ_i} is the basis of the Lie algebra $\mathfrak{d}$

$$\partial_{\xi_i} m_{kl} = (\delta_{ik} + \delta_{il}) m_{kl}. \tag{B60}$$

Let us notice that according to the Appendix B 2 μ is $SL(n)$ invariant (where it acts as D) thus the pullback

$$\psi_2^* \mu = c\mu, \tag{B61}$$

since $SL(n)$ acts transitively on $\tilde{M}$. We can check that $c = 1$ by computing the measures for $M = M^{-1} = \mathbb{I}$.

We have $\psi_2 \partial_{\xi_i} = -\partial_{\xi_i}$ so

$$[\psi_2]^* \mu_{\tilde{M}/D} = (-1)^n \mu_{\tilde{M}/D}. \tag{B62}$$

From basic facts explained in Appendix B we know that (in the right order)

$$\psi_1^* \mu_{\tilde{M}/D} = \left[\det \partial_{\xi_i}(m_{jj} - 1)\right] (\det M)^{\frac{n+1}{2}} \mu_1 = 2^n (\det M)^{\frac{n+1}{2}} \mu_1. \tag{B63}$$

Combining all transformations we obtain

$$\mu_1 = (-1)^n \left(\frac{\det M}{\det \psi(M)}\right)^{\frac{n+1}{2}} \psi^* \mu_1. \tag{B64}$$

Finally, we obtain

$$\bigwedge d\theta_{ij} = (-1)^n \frac{\prod \sin l'_{ij}}{\prod \sin \theta_{ij}} \left(\frac{\det \tilde{G}}{\det G}\right)^{\frac{n+1}{2}} \bigwedge dl'_{ij}. \tag{B65}$$

So eventually

$$\det \frac{\partial \theta_{ij}}{\partial l'_{km}} = (-1)^n \frac{\prod \sin l_{ij}}{\prod \sin \theta_{ij}} \left(\frac{\det \tilde{G}}{\det G}\right)^{\frac{n+1}{2}}. \tag{B66}$$

c. Further simplifications for n = 4

We can simplify the above formula using equalities from Ref. 42:

$$\det \tilde{G} = \frac{(\det G)^{n-1}}{\prod G^*_{ii}}, \ (\sin \theta_{ij})^2 = \frac{\det G \det G_{(ij)}}{G^*_{ii} G^*_{jj}}, \tag{B67}$$

where G^*_{ii} is the ii-element of the minor matrix, $G_{(ij)}$ is the G matrix without ith and jth rows and columns. Eventually, we obtain

$$\det \frac{\partial \theta_{ij}}{\partial l'_{km}} = (-1)^n \frac{\prod \sin l'_{ij} \prod (G^*_{kk})^{\frac{n-1}{2}}}{(\det G)^{\frac{n(n-1)}{4}} \prod \sqrt{\det G_{(ij)}}} \frac{(\det G)^{\frac{(n-1)(n+1)}{2}}}{\prod (G^*_{kk})^{\frac{n+1}{2}}} \frac{1}{(\det G)^{\frac{n+1}{2}}}. \tag{B68}$$

After simplification it is equal to

$$\det \frac{\partial \theta_{ij}}{\partial l'_{km}} = \underbrace{\left(\prod \frac{\sin l'_{ij}}{\sqrt{\det G_{(ij)}}}\right)}_{=1} \frac{\det \tilde{G}}{\det G}. \tag{B69}$$

Note that this simplification only holds for $n = 4$. Since

$$\det \frac{\partial l_{ij}}{\partial l'_{km}} = -1 \tag{B70}$$

we obtain

$$\det \frac{\partial \theta_{ij}}{\partial l_{km}} = -\det \frac{\partial \theta_{ij}}{\partial l'_{km}}. \tag{B71}$$

APPENDIX C: TECHNICAL COMPUTATIONS OF DETERMINANTS

In this section we will prove several technical results.

Let us introduce the notation

$$\det{}' M = \sum_i M^*_{ii}. \tag{C1}$$

It is an invariant of the matrix and, moreover, in the case when the matrix is symmetric and has one null eigenvector, it is the determinant of the matrix restricted to the space perpendicular to that null eigenvector.

Let us also remind some general facts

$$l_{ij}\frac{\partial\theta_{ij}}{\partial l_{kl}}=l_{kl}\frac{\partial\theta_{ij}}{\partial l_{kl}}=0,\quad \frac{\partial\theta_{ij}}{\partial l_{kl}}=\frac{\partial\theta_{kl}}{\partial l_{ij}},\tag{C2}$$

$$l_{ij}=\lambda\frac{\partial\det\tilde{G}}{\partial\theta_{ij}},\quad \lambda=-\frac{2^2\prod S_i^2}{3^5V^5},\tag{C3}$$

$$\det{}'\tilde{G}=\frac{3^4}{2^2}(\sum_i S_i^2)\frac{V^4}{\prod S_i^2},\tag{C4}$$

which are proven for completeness in the Appendix C 1. Our results are (see the Appendix C 3):

Lemma 6. For the flat tetrahedron holds

$$\det{}'\frac{\partial\theta_{ij}}{\partial l_{kl}}=\frac{3^3}{2^5}\frac{|l|^2}{\prod S_i^2}V^3.\tag{C5}$$

Moreover, in the Appendix C 2 we prove

Lemma 7. Let

$$\lambda=-\frac{2^2\prod S_i^2}{3^5V^5}.\tag{C6}$$

For the flat tetrahedron holds

$$\frac{\partial\lambda}{\partial l_{ij}}\frac{\partial\det\tilde{G}}{\partial\theta_{kl}}+\frac{\partial\theta_{ij}}{\partial l_{mn}}\lambda\frac{\partial^2\det\tilde{G}}{\partial\theta_{mn}\partial\theta_{kl}}=\delta_{(ij),(kl)},\tag{C7}$$

$$\exists_c\ \frac{\partial\det\tilde{G}}{\partial\theta_{kl}}c+\lambda\frac{\partial\lambda}{\partial l_{ij}}\frac{\partial^2\det\tilde{G}}{\partial\theta_{ij}\partial\theta_{kl}}=0,\tag{C8}$$

$$\frac{\partial\lambda}{\partial l_{mn}}\frac{\partial\det\tilde{G}}{\partial\theta_{mn}}=1.\tag{C9}$$

1. General knowledge

We know that $\det\tilde{G}=0$ for a geometric set of θ's. Moreover, the null eigenvector is given by

$$(S_1,S_2,S_3,S_4),\tag{C10}$$

where S_i denote the areas of the triangles of the tetrahedron. The computation of $\det'\tilde{G}$ can be found in the Appendix C 1 b. We have

$$\frac{\partial\det\tilde{G}}{\partial\theta_{ij}}=-2\det{}'\tilde{G}\frac{S_iS_j\sin\theta_{ij}}{\sum_k S_k^2}=-\frac{3^5}{2^2}(\sum_k S_k^2)\frac{V^4}{\prod S_k^2}\frac{Vl_{ij}}{\sum_k S_k^2}=-\frac{3^5}{2^2}\frac{V^5}{\prod S_k^2}l_{ij}=\lambda^{-1}l_{ij}.\tag{C11}$$

In addition to that, we also have

$$0=\frac{\partial\det\tilde{G}}{\partial l_{kl}}=\frac{\partial\det\tilde{G}}{\partial\theta_{ij}}\frac{\partial\theta_{ij}}{\partial l_{kl}}=\lambda l_{ij}\frac{\partial\theta_{ij}}{\partial l_{kl}}.\tag{C12}$$

We know that θ has scaling dimension 0, thus

$$l_{kl}\frac{\partial\theta_{ij}}{\partial l_{kl}}=0.\tag{C13}$$

a. Expressing $\frac{\partial\theta_{ij}}{\partial l_{kl}}$ in terms of l_{ij}

Here we recall several well-known facts for flat simplices of arbitrary dimension using the notation of l'_{ij} from the Appendix B 4, see also Refs. 41 and 50 for more details. Let M be the following matrix

$$M = \begin{bmatrix} 0 & 1 & \dots & 1 \\ 1 & l'^2_{11} & \dots & l'^2_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & l'^2_{n1} & \dots & l'^2_{nn} \end{bmatrix}, \tag{C14}$$

where $l'_{11} = \ldots = l'_{nn} = 0$ and $l'_{ij} = l'_{ji}$. Then we have

$$V^2 = \frac{(-1)^{n-1}}{2^n(n-1)!^2}\det M, \quad S_i^2 = \frac{(-1)^{n-2}}{2^{n-1}(n-2)!^2}M^*_{ii}, \tag{C15}$$

$$\cos\theta_{ij} = \frac{M^*_{ij}}{\sqrt{M^*_{ii}M^*_{jj}}}. \tag{C16}$$

In three dimension we also have

$$\sin^2\theta_{ij} = \left(\frac{3}{2}\right)^2 \frac{V^2 l_{ij}^2}{S_i^2 S_j^2}, \tag{C17}$$

so in the case $\theta_{ij} \in (0, \pi)$ we can write

$$\frac{\partial\theta_{ij}}{\partial l_{kl}} = -\frac{1}{\sin\theta_{ij}}\frac{\partial\cos\theta_{ij}}{\partial l_{kl}} = -\frac{2}{3}\frac{S_i S_j}{V l_{ij}}\frac{\partial}{\partial l_{kl}}\frac{M^*_{ij}}{\sqrt{M^*_{ii}M^*_{jj}}}. \tag{C18}$$

This, in principle, allows us to compute $\frac{\partial\theta_{ij}}{\partial l_{kl}}$ and all other derivatives in terms of lengths.

b. Computation of $\det'\tilde{G}$

Let us start with the spherical case, i.e., a tetrahedron with constant non-vanishing (positive) curvature—a curved tetrahedron on the unit sphere. In this case we define $\tilde{l}_{ij} := \epsilon l_{ij}$ and $\theta^\epsilon_{ij} := \theta(\epsilon l_{kl})$ and take the limit $\epsilon \to 0$ in order to reobtain the flat case. The angles (θ^ϵ_{ij}) have a limit as the angles of flat tetrahedron (θ_{ij}) with lengths l_{ij}.

First, let us notice that

$$\det G = \det\begin{pmatrix} 1 & 0 & \cdots \\ 1 & G & \cdots \\ \vdots & \cdots & \cdots \end{pmatrix} = \det\begin{pmatrix} 1 & 0 & \cdots \\ 1 & 1 - \frac{1}{2}\epsilon^2 l_{ij}^2 + O(\epsilon^4) & \cdots \\ \vdots & \cdots & \cdots \end{pmatrix} \tag{C19}$$

$$= \frac{1}{8}\epsilon^6 \det \underbrace{\begin{pmatrix} 0 & 1 & \cdots \\ 1 & l_{ij}^2 & \cdots \\ \vdots & \cdots & \cdots \end{pmatrix}}_{C} + O(\epsilon^8). \tag{C20}$$

We can compute $\det'\tilde{G} = \sum_i \tilde{G}^*_{ii}$ using the following identity from Ref. 42 ($n = 4$, i.e., $D = 3$):

$$\frac{\tilde{G}^*_{ii}}{G^*_{ii}} = \frac{(\det G)^{n-2}}{\prod G^*_{ii}}, \tag{C21}$$

obtaining

$$\det{}'\tilde{G} = \sum_i \tilde{G}^*_{ii} = \left(\sum_i G^*_{ii}\right)\frac{(\det G)^{n-2}}{\prod G^*_{ii}} = \frac{3^4}{2^2}\left(\sum_i S_i^2\right)\frac{V^4}{\prod S_i^2} + O(\epsilon^2). \tag{C22}$$

So in the flat case

$$\det{}'\tilde{G} = \frac{3^4}{2^2}\left(\sum_i S_i^2\right)\frac{V^4}{\prod S_i^2}. \tag{C23}$$

2. Collection of results

Let us prove the following useful formulas:

$$\frac{\partial\lambda}{\partial l_{ij}}\frac{\partial \det\tilde{G}}{\partial\theta_{kl}} + \sum_{m<n}\frac{\partial\theta_{ij}}{\partial l_{mn}}\lambda\frac{\partial^2\det\tilde{G}}{\partial\theta_{mn}\partial\theta_{kl}} = \frac{\partial}{\partial l_{ij}}\left(\lambda\frac{\partial\det\tilde{G}}{\partial\theta_{kl}}\right) = \delta_{(ij),(kl)}, \tag{C24}$$

since we know, due to the Schläfli identity, that $\frac{\partial\theta_{ij}}{\partial l_{kl}} = \frac{\partial\theta_{kl}}{\partial l_{ij}}$. This also implies that

$$\sum_{m<n}\frac{\partial\theta_{mn}}{\partial l_{ij}}\lambda\frac{\partial^2\det\tilde{G}}{\partial\theta_{mn}\partial\theta_{kl}} = \delta_{(ij),(kl)} - \frac{\partial\lambda}{\partial l_{ij}}\frac{l_{kl}}{\lambda}. \tag{C25}$$

We will now prove that there exists such a c that

$$\frac{\partial\det\tilde{G}}{\partial\theta_{kl}}c + \lambda\frac{\partial\lambda}{\partial l_{ij}}\frac{\partial^2\det\tilde{G}}{\partial\theta_{ij}\partial\theta_{kl}} = 0. \tag{C26}$$

Because the range of the matrix $\frac{\partial\theta_{ij}}{\partial l_{kl}}$ is the whole space perpendicular to the vector $\vec{l} = (l_{ij})$ and the vector $\frac{\partial\det\tilde{G}}{\partial\theta_{kl}}$ is proportional to $\vec{l}$, it is enough to compute

$$\begin{aligned}\left(\frac{\partial\det\tilde{G}}{\partial\theta_{kl}}c + \lambda\frac{\partial\lambda}{\partial l_{ij}}\frac{\partial^2\det\tilde{G}}{\partial\theta_{ij}\partial\theta_{kl}}\right)\frac{\partial\theta_{kl}}{\partial l_{mn}} &= \lambda\frac{\partial\lambda}{\partial l_{ij}}\frac{\partial}{\partial l_{mn}}\frac{\partial\det\tilde{G}}{\partial\theta_{ij}} = \\ = \lambda\frac{\partial\lambda}{\partial l_{ij}}\frac{\partial}{\partial l_{mn}}\frac{l_{ij}}{\lambda} &= \lambda\frac{\partial\lambda}{\partial l_{ij}}\left(\frac{\delta_{(ij)(mn)}}{\lambda} - l_{ij}\frac{\partial\lambda}{\partial l_{mn}}\frac{1}{(\lambda)^2}\right).\end{aligned} \tag{C27}$$

On the other hand, we know that, since λ is of scaling dimension 1, $l_{ij}\frac{\partial\lambda}{\partial l_{ij}} = \lambda$, and thus

$$\lambda\frac{\partial\lambda}{\partial l_{ij}}\left(\frac{\delta_{(ij)(mn)}}{\lambda} - l_{ij}\frac{\partial\lambda}{\partial l_{mn}}\frac{1}{(\lambda)^2}\right) = \frac{\partial\lambda}{\partial l_{mn}} - \frac{\partial\lambda}{\partial l_{mn}} = 0. \tag{C28}$$

Let us also remind that

$$\frac{\partial\lambda}{\partial l_{mn}}\frac{\partial\det\tilde{G}}{\partial\theta_{mn}} = \frac{\partial\lambda}{\partial l_{mn}}\frac{l_{mn}}{\lambda} = 1. \tag{C29}$$

3. Computation of $\det{}'\frac{\partial\theta_{ij}}{\partial l_{ij}}$

In this section we will prove that

$$\det{}'\frac{\partial\theta_{ij}}{\partial l_{ij}} = \frac{3^3}{2^5}\frac{|l|^2}{\prod S_i^2}V^3. \tag{C30}$$

We will start from the formula valid for a spherical tetrahedron (Lemma 5):

$$\det\frac{\partial\theta_{ij}}{\partial\tilde{l}_{ij}} = -\frac{\det\tilde{G}}{\det G}. \tag{C31}$$

As mentioned above we set $\tilde{l}_{ij} = \epsilon l_{ij}$ and $\theta^{\epsilon}_{ij} = \theta(\epsilon l_{kl})$ and take the limit $\epsilon \to 0$ in the end. In this limit the angles converge to the angles of a flat tetrahedron with lengths l_{ij}.

Let us remind that

$$\det G = \frac{1}{8}\epsilon^6 \det \underbrace{\begin{pmatrix} 0 & 1 & \cdots \\ 1 & l_{ij}^2 & \cdots \\ \vdots & \cdots & \cdots \end{pmatrix}}_{C} + O(\epsilon^8). \tag{C32}$$

Let us notice that because G (in the spherical case) is a function of $\cos \epsilon l_{ij}$, its expansion around $\epsilon = 0$ is an analytic function in ϵ^2 and not only in ϵ. The same holds for the matrix $\tilde{G}$ since it is

$$\tilde{G}_{ij} = \frac{1}{\sqrt{G^*_{ii}}} G^*_{ij} \frac{1}{\sqrt{G^*_{jj}}}, \tag{C33}$$

where G^*_{ij} is the cofactor matrix of G and $\sqrt{G^*_{ii}}$ is ϵ^3 times an analytic function in ϵ^2.

Hence we know that for the vector $\vec{S} = (S_1, S_2, S_3, S_4)$ (the single null eigenvector in the limit $\epsilon = 0$)

$$\tilde{G}\vec{S} = O(\epsilon^2)\ ,\ (\vec{S}, \tilde{G}\vec{S}) = O(\epsilon^2)\ , \tag{C34}$$

then also $\det \tilde{G} = O(\epsilon^2)$ and

$$\det \tilde{G} = \det{}' \tilde{G} \frac{(\vec{S}, \tilde{G}\vec{S})}{|S|^2} + O(\epsilon^4). \tag{C35}$$

Moreover,

$$\epsilon \frac{\partial}{\partial \epsilon} \frac{(\vec{S}, \tilde{G}\vec{S})}{|S|^2} = 2 \frac{(\vec{S}, \tilde{G}\vec{S})}{|S|^2} + O(\epsilon^3). \tag{C36}$$

We have

$$\epsilon \frac{\partial}{\partial \epsilon} \frac{(\vec{S}, \tilde{G}\vec{S})}{|S|^2} = \sum_{(ij)} l_{ij} \partial_{l_{ij}} \frac{(\vec{S}, \tilde{G}\vec{S})}{|S|^2} = \frac{(\vec{S}, \sum_{(ij)} l_{ij} \partial_{l_{ij}} \tilde{G}\vec{S})}{|S|^2} = \tag{C37}$$

$$= -2 \frac{\sum_{(km)(ij)} S_k S_m \sin\theta_{km} l_{ij} \frac{\partial \theta^{\epsilon}_{km}}{\partial l_{ij}}}{|S|^2} = -3 \frac{V}{|S|^2} \sum_{(km)(ij)} l_{km} l_{ij} \frac{\partial \theta^{\epsilon}_{km}}{\partial l_{ij}}. \tag{C38}$$

Similarly, we know that $\sum_{ij} \frac{\partial \theta^{\epsilon}_{km}}{\partial l_{ij}} l_{ij} = O(\epsilon^2)$ and $\sum_{ijkm} l_{km} \frac{\partial \theta^{\epsilon}_{km}}{\partial l_{ij}} l_{ij} = O(\epsilon^2)$, so

$$\det \frac{\partial \theta_{km}}{\partial \tilde{l}_{ij}} = \epsilon^{-6} \det \frac{\partial \theta^{\epsilon}_{km}}{\partial l_{ij}} = \epsilon^{-6} \det{}' \frac{\partial \theta^{\epsilon}_{km}}{\partial l_{ij}} \frac{\sum_{(ij)(km)} l_{km} \frac{\partial \theta^{\epsilon}_{km}}{\partial l_{ij}} l_{ij}}{|l|^2}. \tag{C39}$$

Eventually, we have

$$\epsilon^{-6} \det{}' \frac{\partial \theta^{\epsilon}_{km}}{\partial l_{ij}} \frac{\sum_{ijkm} l_{km} \frac{\partial \theta^{\epsilon}_{km}}{\partial l_{ij}} l_{ij}}{|l|^2} = 12 \epsilon^{-6} \frac{\det{}' \tilde{G}}{\det C} \frac{V}{|S|^2} \sum_{kmij} l_{km} l_{ij} \frac{\partial \theta^{\epsilon}_{km}}{\partial l_{ij}} + O(\epsilon^{-3}). \tag{C40}$$

and so

$$\det{}' \frac{\partial \theta^{\epsilon}_{km}}{\partial l_{ij}} = \frac{12 |l|^2}{|S|^2} \frac{V}{\det C} \det{}' \tilde{G} + O(\epsilon). \tag{C41}$$

Now we can use the identities from the Appendix C 1 b

$$\det{}' \tilde{G} = \frac{3^4}{2^2} \left(\sum_i S_i^2 \right) \frac{V^4}{\prod S_i^2}, \quad \det C = 8(3!)^2 V^2. \tag{C42}$$

Finally, in the limit $\epsilon \to 0$, ($\theta_{ij} = \lim \theta_{ij}^{\epsilon}$):

$$\det{}' \frac{\partial \theta_{km}}{\partial l_{ij}} = \frac{3^3}{2^5} \frac{|l|^2}{\prod S_i^2} V^3. \tag{C43}$$

APPENDIX D: TECHNICAL COMPUTATIONS

In this appendix we give some explicit computations needed in the main body of the paper.

1. Weak equivalences

In the following we will use the notation introduced in Sec. V B 2. We can compute

$$\begin{aligned} 0 \equiv L_+ A_+^k &= k(L_+ A_+) A_+^{k-1} + A_+^{k+1} \\ &= \left(\frac{k}{2} + 1\right) A_+^{k+1} + \frac{k}{2} A_-^2 A_+^{k-1} - k \cos 2\tilde{\theta}\ A_+^{k-1}, \end{aligned} \tag{D1}$$

such that

$$A_-^2 A_+^{k-1} \equiv -\frac{k+2}{k} A_+^{k+1} + 2\cos 2\tilde{\theta}\ A_+^{k-1}. \tag{D2}$$

Similarly, we can derive an identity by acting on A_+^k with L_-^*:

$$0 \equiv L_-^* A_+^k = (k+1) A_- A_+^k - k A_+^{k-1}, \tag{D3}$$

$$\Rightarrow\ A_- A_+^k \equiv \frac{k}{k+1} A_+^{k-1}. \tag{D4}$$

By acting again on (D4) we obtain

$$L_-^*(A_- A_+^k) = \frac{1}{2} A_+^{k+2} + \left(k + \frac{3}{2}\right) A_-^2 A_+^k - k A_- A_+^{k-1} - \cos 2\tilde{\theta}\ A_+^k \equiv 0. \tag{D5}$$

Hence using (D2) and (D4) we have

$$\begin{aligned} 0 &\equiv \frac{1}{2} A_+^{k+2} + \left(k + \frac{3}{2}\right)\left(-\frac{k+3}{k+1} A_+^{k+2} + 2\cos 2\tilde{\theta}\ A_+^k\right) - k\frac{k+1}{k} A_+^k - \cos 2\theta\ A_+^k \\ &= -\frac{(k+2)^2}{k+1} A_+^{k+2} + 2(k+1)\cos 2\tilde{\theta}\ A_+^k - (k-1) A_+^{k-2}. \end{aligned} \tag{D6}$$

2. Proof of the lemma

In this section we will prove the following lemma:

Lemma 3. For every $m \geq 0$

$$\sum_{k \leq m} (2\beta_{m+1-k}^k + \beta_{m-k}^k) A_k i^{m+1-k} \sin\left(\theta - \frac{\pi}{2}(m-k)\right) = 0, \tag{D7}$$

where

$$\beta_m^k = \frac{(-k-\frac{1}{2})_m}{m!} \in \mathbb{R}, \tag{D8}$$

and

$$(a)_m = a \cdot (a-1) \cdot \ldots \cdot (a-m+1), \quad (a)_0 = 1. \tag{D9}$$

To do so, we need

Lemma 8. The following equality holds

$$\frac{1}{(l\pm 1)^{k+\frac{1}{2}}}=\sum_{m\geq k}\frac{(\pm 1)^{m-k}\beta^k_{m-k}}{l^{m+\frac{1}{2}}}. \tag{D10}$$

Proof.

$$\frac{1}{(l\pm 1)^{k+\frac{1}{2}}}=\sum_{n=0}^{\infty}\binom{-k-\frac{1}{2}}{n}l^{-k-\frac{1}{2}-n}(\pm 1)^n=\sum_{n=0}^{\infty}\frac{(-k-\frac{1}{2})_n}{n!}\frac{1}{l^{k+n+\frac{1}{2}}}(\pm 1)^n$$

$$\stackrel{m:=k+n}{=}\sum_{m\geq k}(\pm 1)^{m-k}\frac{(-k-\frac{1}{2})_{m-k}}{(m-k)!}=\sum_{m\geq k}(\pm 1)^{m-k}\frac{\beta^k_{m-k}}{l^{m+\frac{1}{2}}}.$$

□

In the following we will use Lemma 8 to prove Lemma 3:

Proof of Lemma 3. In any stationary point we have by Lemma 8

$$\tilde{P}_{l\pm 1}\equiv\sum_{k\geq 0}\frac{e^{i(l\pm 1)\theta}}{(l\pm 1)^{k+\frac{1}{2}}}A_k(\theta)\equiv\sum_{k\geq 0}e^{il\theta}\sum_{m\geq k}\frac{(\pm 1)^{m-k}}{l^{m+\frac{1}{2}}}\beta^k_{m-k}A_k(\theta)e^{\pm i\theta}. \tag{D11}$$

We thus have

$$l(\tilde{P}_{l+1}+\tilde{P}_{l-1}-2\cos\theta\,\tilde{P}_l)\equiv\sum_{k\geq 0}\frac{e^{il\theta}}{l^{k-\frac{1}{2}}}A_k\left(\sum_{m\geq k}\frac{\beta^k_{m-k}}{l^{m-k}}(e^{i\theta}+(-1)^{m-k}e^{-i\theta})-2\cos\theta\right). \tag{D12}$$

A simple algebraic manipulation gives

$$e^{i\theta}+(-1)^{m-k}e^{-i\theta}=\begin{cases}2\cos\theta & \text{if } m=k\\ 2i^{m-k}\cos\left(\theta-\frac{\pi}{2}(m-k)\right) & \text{if } m>k\end{cases}, \tag{D13}$$

such that we obtain

$$l(\tilde{P}_{l+1}+\tilde{P}_{l-1}-2\cos\theta\,P_l)\equiv\sum_{k\geq 0}e^{il\theta}A_k\sum_{m\geq k}\frac{2\beta^k_{m+1-k}}{l^{m+\frac{1}{2}}}i^{m+1-k}\underbrace{\cos\left(\theta-\frac{\pi}{2}(m-k+1)\right)}_{\sin\left(\theta-\frac{\pi}{2}(m-k)\right)}. \tag{D14}$$

We also have

$$\frac{1}{2}(\tilde{P}_{l+1}-\tilde{P}_{l-1})=\sum_{k\geq 0}e^{il\theta}A_k\sum_{m\geq k}\frac{\beta^k_{m-k}}{l^{m+\frac{1}{2}}}i^{m+1-k}\sin\left(\theta-\frac{\pi}{2}(m-k)\right). \tag{D15}$$

By combining (D14) and (D15), we obtain for the full recursion relation (5.22):

$$\sum_{k\geq 0}e^{il\theta}A_k\left(\sum_{m\geq k}\frac{2\beta^k_{m+1-k}+\beta^k_{m-k}}{l^{m+\frac{1}{2}}}i^{m+1-k}\sin\left(\theta-\frac{\pi}{2}(m-k)\right)\right)$$

$$=\sum_{m\geq 0}\frac{e^{il\theta}}{l^{m+\frac{1}{2}}}\left(\sum_{k\leq m}\left(2\beta^k_{m+1-k}+\beta^k_{m-k}\right)i^{m+1-k}A_k\,\sin\left(\theta-\frac{\pi}{2}(m-k)\right)\right)=O(l^{-\infty}). \tag{D16}$$

Thus every single term must be zero. That ends the proof. □

a. Expanding C_j

In the following we will expand the normalization factor $C_j = \frac{1}{4^j}\binom{2j}{j}$ up to $O(\frac{1}{j})$. Therefore we use Stirling's series for the logarithm of the factorial:

$$\ln n! = n \ln n - n + \frac{1}{2}\ln(2\pi n) + O\left(\frac{1}{n}\right). \tag{D17}$$

Hence

$$\ln(C_j) = \ln\left(\frac{1}{\sqrt{\pi j}}\right) + O\left(\frac{1}{j}\right). \tag{D18}$$

Therefore, we obtain

$$C_j = \frac{1}{\sqrt{\pi j}} e^{O\left(\frac{1}{j}\right)} = \frac{1}{\sqrt{\pi j}}\left(1 + O\left(\frac{1}{j}\right)\right). \tag{D19}$$

Moreover, since $\ln n!$ admits a complete expansion (neglecting the first terms) in powers of $\frac{1}{n}$, also C_j can be completely expanded in powers of $\frac{1}{j}$. The same is true for an expansion in l.

3. Theta graph

In this section we explain the result that the theta graph $(C_{000}^{j_1 j_2 j_3})^2$ is equal to

$$\frac{1}{2\pi S}\left(1 + O\left(\frac{1}{l^2}\right)\right). \tag{D20}$$

From Ref. 40 we have

$$C_{000}^{j_1 j_2 j_3} = (-1)^g \frac{g!}{(g-j_1)!(g-j_2)!(g-j_3)!}\sqrt{\frac{(2g-2j_1)!(2g-2j_2)!(2g-2j_3)!}{(2g+1)!}}, \tag{D21}$$

where $2g = j_1 + j_2 + j_3$. We compute the expansion of $\ln C_{000}^{j_1 j_2 j_3}$ using the Stirling's formula:

$$\ln(n!) = n \ln n - n + \frac{1}{2}\ln n + \frac{1}{2}\ln 2\pi + \frac{1}{12n} + O(n^{-2}), \tag{D22}$$

obtaining

$$\ln\left((-1)^g C_{000}^{j_1 j_2 j_3}\right) = -\frac{1}{4}\ln\left(\frac{(2\pi)^2}{16}(l_1+l_2+l_3)(-l_1+l_2+l_3)(l_1-l_2+l_3)(l_1+l_2-l_3)\right) + O(l^{-2}). \tag{D23}$$

This is exactly

$$-\frac{1}{4}\ln 4\pi^2 S^2 + O(l^{-2}), \tag{D24}$$

where S is the area of the triangle with edge lengths l_i. We conclude that the theta graph $(C_{000}^{j_1 j_2 j_3})^2$ is equal to

$$\frac{1}{2\pi S}\left(1 + O\left(\frac{1}{l^2}\right)\right). \tag{D25}$$

4. Kinetic term in equilateral case

Let us introduce

$$M_\lambda = \begin{pmatrix} 0 & a & a & a & a & a & a \\ a & b-\lambda & c-\lambda & c-\lambda & c-\lambda & c-\lambda & -\lambda \\ a & c-\lambda & b-\lambda & c-\lambda & c-\lambda & -\lambda & c-\lambda \\ a & c-\lambda & c-\lambda & b-\lambda & -\lambda & c-\lambda & c-\lambda \\ a & c-\lambda & c-\lambda & -\lambda & b-\lambda & c-\lambda & c-\lambda \\ a & c-\lambda & -\lambda & c-\lambda & c-\lambda & b-\lambda & c-\lambda \\ a & -\lambda & c-\lambda & c-\lambda & c-\lambda & c-\lambda & b-\lambda \end{pmatrix}, \tag{D26}$$

where $a = -\sqrt{2}\frac{64}{81}$, $b = \frac{\sqrt{3}}{4}$ and $c = \frac{1}{2\sqrt{3}}$. In the equilateral case (all l equal to 1) the kinetic term is of the form

$$-iM_0. \tag{D27}$$

Let us note that

$$\det M_\lambda = \det M_0 \neq 0, \tag{D28}$$

and all M_λ are symmetric. Thus all of them have the same number of positive and negative eigenvalues. Matrix M_λ for $\lambda = c$ is similar (have the same determinant) by simultaneous permutation of rows and columns to the matrix

$$M' = \begin{pmatrix} b-c & -c & 0 & 0 & 0 & 0 & a \\ -c & b-c & 0 & 0 & 0 & 0 & a \\ 0 & 0 & b-c & -c & 0 & 0 & a \\ 0 & 0 & -c & b-c & 0 & 0 & a \\ 0 & 0 & 0 & 0 & b-c & -c & a \\ 0 & 0 & 0 & 0 & -c & b-c & a \\ a & a & a & a & a & a & 0 \end{pmatrix}. \tag{D29}$$

The matrix M' restricted to its first 6 rows and columns has 3 positive and 3 negative eigenvalues. Applying the min-max principle[51] to M' and $-M'$ shows that M' has at least three positive and three negative eigenvalues. Together with the fact that determinant is positive it shows that there are 4 positive and 3 negative eigenvalues.

Hence, the matrix of kinetic term has 4 $-i\mathbb{R}_+$ eigenvalues and 3 $i\mathbb{R}_+$ and the same is true for matrix $(-\mathcal{H}^{-1})$.

APPENDIX E: DUPUIS-LIVINE FORM AND STATIONARY POINTS

In this section we will prove the following lemma:

Lemma 9. Suppose that the integral is of the form as

$$\int d\theta \frac{e^{i\eta}}{l^\lambda} e^S, \tag{E1}$$

where $S(\theta_i)$ has an asymptotic expansion around the isolated stationary point of $S_{-1}(\theta)$ of the form

$$S = S_{-1} + S_0 + S_1 + \ldots, \tag{E2}$$

and $i^k S_k \in \mathbb{R}$ is a homogeneous function of order $-k$ in l. Then the contribution to the expansion of the integral from this stationary point has the DL property.

Proof. Let us consider the contribution from the isolated stationary point of S_{-1}. They are of the form

$$\frac{1}{\sqrt{\mathcal{H}}} e^{\tilde{S}}, \tag{E3}$$

where $\tilde{S}$ is given by the contraction of all connected Feynman diagrams. They are made up of vertices, given by the derivatives of $S_{\geq 0}$, connected by the propagator H, which is the inverse to (-1) times the matrix of second derivatives of S_{-1}.

$$H = \left(-\partial^2 S_{-1}\right)^{-1}, \quad \mathcal{H} = \det\left(-\partial^2 S_{-1}\right). \tag{E4}$$

Their contribution is computed by contracting the vertices V_k with propagators H. Since vertices are obtained from derivatives of S_m, $m \geq 0$, the homogeneous degree $\deg V_k$ of this vertex is thus m and the matrix elements of $i^{\deg V_k} V_k$ are real. Similarly iH is a real matrix and is of degree 1.

To conclude, the total contraction is thus of degree

$$\sum_k \deg V_k + n, \tag{E5}$$

where n is the number of propagators in the diagram. Moreover, the complete contraction multiplied by

$$i^{\sum_k \deg V_k + n} \tag{E6}$$

is again real as a contraction of real matrices. This proves that expansion is still of DL form. □

APPENDIX F: STATIONARY POINT ANALYSIS

In this paper we use an advanced version of the stationary point analysis. This appendix is intended to explain the details of this method.

Lemma 10. Let $S(x) = i(S_{-1} + S_0) + S_0' + \sum_{i>0} S_i$ be an asymptotic expansion of the action such that

- *S_i is of homogeneous degree $-i$ in j,*
- *S_0 and S_{-1} are real,*
- *$S_{-1} + S_0$ is homogeneous in $l = j + \frac{1}{2}$*

and let x_0 be an isolated stationary point of S_{-1}. Then there is an asymptotic expansion of the contribution to the integral

$$\int dx e^S \tag{F1}$$

from the neighbourhood of x_0 given as follows:

We can write the asymptotic expansion of S in homogeneous terms in l as

$$S = i\tilde{S}_{-1} + S_0' + \sum_{i>0} \tilde{S}_i, \tag{F2}$$

where $\tilde{S}_{-1} = S_{-1} + S_0$. Let x_1 be the stationary point of $\tilde{S}_{-1}$ obtained by perturbation of $x_0 x_0$ (there is exactly one such stationary point if the matrix of second derivatives of S_{-1} is non-degenerate). The asymptotic expansion of the integral is equal to

$$\frac{1}{\sqrt{\det(-H)}} e^{\sum_{i\geq -1} A_i}, \tag{F3}$$

where H is the matrix of second derivatives of $\tilde{S}_{-1}$ and $A_{-1} = \tilde{S}_{-1}$ evaluated on x_1. The terms A_i for $i \geq 0$ are homogeneous functions of order $-i$ in l and can be obtained from the Feynman diagram expansion with the propagator $(-H)^{-1}$ and interaction vertices given by derivatives of $\tilde{S}_i$ for $i \geq 0$.

The same fact applies when the isolated point is replaced by the isolated orbit of the symmetry group of the action.

The second fact concerns with integration over only a part of the variables:

Lemma 11. Let $S(x, y) = iS_{-1}(x, y) + \sum_{i\geq 0} S_i$ has an isolated stationary point (x_0, y_0) with a non-degenerate matrix of second derivatives H with the property

$$H = \begin{pmatrix} H_{xx} & H_{xy} \\ H_{xy} & H_{yy} \end{pmatrix}, \quad H_{yy} \text{ invertible}. \tag{F4}$$

Then there exists a function $y(x)$ such that (in the neighbourhood of stationary point)

$$\nabla_x S_{-1} + \frac{\partial y}{\partial x} \nabla_y S_{-1} = 0 \tag{F5}$$

and the asymptotic expansion of $\int dxdy\, e^S$ is equal to asymptotic expansion of

$$\int dx\, e^{\bar{S}}, \tag{F6}$$

where $\bar{S}$ is obtained by asymptotic expansion of the integral $e^{\bar{S}} = \int dy\, e^S$.

APPENDIX G: FEYNMAN DIAGRAMS

In this appendix we are interested in the next to leading order in the expansion of the $6j$ symbol. We will derive expressions for S_1 in terms of Feynman diagrams. Vertices in this expansion consist of derivatives of

$$-\sum_i \frac{1}{2} \ln \sin \theta_{ij}, \quad \frac{-i}{8l_{ij}} \cot \theta_{ij}, \tag{G1}$$

and higher than second derivatives of $|l|\rho \det \tilde{G}$ with respect to ρ and θ_{ij}. Each propagator contributes a weight $|l|^{-1}$.

We only evaluate closed diagrams, so if the diagram is made up of vertices of valency n_k, i.e., the n_k-th derivative of a function with weight $|l|^{\alpha_k}$, then the scaling behaviour of the whole diagram is as

$$|l|^{\sum_k (\alpha_k - \frac{n_k}{2})}. \tag{G2}$$

The only vertices that can contribute up to order $|l|^{-1}$ are thus

Vertex	$-\frac{1}{2}\ln\sin\theta_{ij}$	$-\frac{1}{2}\cot\theta_{ij}$	$-\frac{1}{2}\frac{\partial}{\partial\theta_{ij}}\cot\theta_{ij}$	$-\frac{i}{8l_{ij}}\cot\theta_{ij}$	$i\|l\|\partial^3\rho\det\tilde{G}$	$i\|l\|\partial^4\rho\det\tilde{G}$
Valency	0	$1_{(ij)}$	$2_{(ij)(ij)}$	0	3	4
Order	$\|l\|^0$	$\|l\|^{-1/2}$	$\|l\|^{-1}$	$\|l\|^{-1}$	$\|l\|^{-1/2}$	$\|l\|^{-1}$

Note that the only diagram that is real (up to the order $|l|^{-1}$) is just the first vertex (being of order 0). Furthermore, this is also the only contribution of order $|l|^0$. All other diagrams are purely imaginary and of order $|l|^{-1}$.

APPENDIX H: RELATION TO SPIN-NETWORK KERNEL FORMULA

We will prove that (for $j \in \mathbb{Z}$)

$$\int_0^\pi \frac{d\phi_1}{\pi} \left(e^{i\theta}\cos\phi_1\cos\phi_2 + e^{-i\theta}\sin\phi_1\sin\phi_2\right)^{2j} = \frac{1}{4^j}\binom{2j}{j}\left(e^{i2\theta}\cos^2\phi_2 + e^{-i2\theta}\sin^2\phi_2\right)^j. \tag{H1}$$

It is straightforward to check that

$$e^{i2\theta}\cos^2\phi_2 + e^{-i2\theta}\sin^2\phi_2 = \cos 2\theta + i\sin 2\theta\cos 2\phi_2. \tag{H2}$$

In this way we obtain the formula from Ref. 29.

To prove (H1), we use the following formulas:

$$2\int_0^{\pi/2} d\phi\ \sin^{2\alpha}\phi\cos^{2\beta}\phi = \frac{\Gamma\left(\alpha+\frac{1}{2}\right)\Gamma\left(\beta+\frac{1}{2}\right)}{\Gamma(\alpha+\beta+1)}, \tag{H3}$$

$$\Gamma(n+1) = n!,\quad \Gamma\left(n+\frac{1}{2}\right) = \frac{(2n)!}{4^n n!}\sqrt{\pi}. \tag{H4}$$

In the case that $k, l \in \mathbb{N}$, these formulas can be simplified to

$$\int_0^{\pi} d\phi\ \sin^{2k}\phi\cos^{2l}\phi = \frac{(2k)!(2l)!\pi}{4^{k+l}k!l!(k+l)!}, \tag{H5}$$

$$\int_0^{\pi} d\phi\ \sin^{2k+1}\phi\cos^{2l+1}\phi = 0. \tag{H6}$$

Expanding the left-hand side of (H1) and using these formulas we have

$$\int_0^{\pi}\frac{d\phi_1}{\pi}\left(e^{i\theta}\cos\phi_1\cos\phi_2 + e^{-i\theta}\sin\phi_1\sin\phi_2\right)^{2j} = \tag{H7}$$

$$\sum_{n=0}^{2j}\binom{2j}{n}e^{i(2j-2n)\theta}\cos^{2j-n}\phi_2\sin^n\phi_2\int_0^{\pi}\frac{d\phi_1}{\pi}\cos^{2j-n}\phi\sin^n\phi. \tag{H8}$$

This is equal to $(k := 2n)$

$$\sum_{k=0}^{j}\binom{2j}{2k}e^{i(j-2k)2\theta}\cos^{2j-2k}\phi_2\sin^{2k}\phi_2\frac{1}{\pi}\frac{(2k)!(2j-2k)!\pi}{4^j k!(j-k)!j!}. \tag{H9}$$

The factors in j and k can be rewritten in terms of binomial coefficients

$$\binom{2j}{2k}\frac{(2k)!(2j-2k)!}{4^j k!(j-k)!j!} = \frac{(2j)!}{4^j k!(j-k)!j!} = \frac{1}{4^j}\binom{2j}{j}\binom{j}{k}, \tag{H10}$$

such that we obtain the final result

$$\begin{aligned}&\frac{1}{4^j}\binom{2j}{j}\sum_{k=0}^{j}\binom{j}{k}\left(e^{i2\theta}\cos^2\phi_2\right)^{j-k}\left(e^{-i2\theta}\sin^2\phi_2\right)^{k} = \\ &= \underbrace{\frac{1}{4^j}\binom{2j}{j}}_{C_j}\left(e^{i2\theta}\cos^2\phi_2 + e^{-i2\theta}\sin^2\phi_2\right)^{j}.\end{aligned} \tag{H11}$$

This explains the occurrence of C_j in our formulas, which is absent in integral kernel approach.[29]

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