Stable isoperimetric surfaces in super-extreme Reissner-Nordström

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Abstract

We study isoperimetric surfaces in the Reissner-Nordström spacetime, with emphasis on the cuasilocal inequality between area and charge. We analyze the stability of the isoperimetric spheres and we found that there is a lower bound on the area in terms of the charge, and that the inequality is saturated in the transition from the superextremal to the subextremal case. We also derive a general inequality between area and charge for stable isoperimetric surfaces in maximal electro-vacuum initial data.

1 Introduction

An important method to obtain physically relevant properties of General Relativity is through geometrical inequalities. They relate quantities of physical interest and tell us what type of phenomena is allowed within the theory. Particularly fruitful have been the search for geometrical inequalities for axially symmetric black holes (for a recent review see [5] and references therein), where attention to the angular momentum has been paid. There are two important possible generalizations of these kind of inequalities. The first one is for non-axially symmetric spacetimes. Axial symmetry is used in a crucial way to define angular momentum. In order to study general spacetimes a model problem is to replace angular momentum by electric charge, this has been done recently in [6]. The second, and more difficult, generalization is to consider geometrical inequalities for general objects (i.e. not only black holes). Remarkably, in this kind of inequalities the black hole trapped surfaces are replaced by stable isoperimetric surfaces. In particular in [6] the following cuasilocal geometrical inequality has been obtained,

$$A \geq \frac{4}{3}\pi(Q_E^2 + Q_M^2), \tag{1}$$

where A is the area of a stable isoperimetric surface Σ in an electro-vacuum, maximal initial data, with non-negative cosmological constant and Q_E and Q_M are the electric and magnetic charges of Σ. This inequality tells us that it is not possible to put an arbitrarily large quantity of charge inside an isoperimetric surface. The requirement of Σ being isoperimetrical can not be dropped without further requirements. This can be seen by looking at the spacetime presented by Bonnor [2]. There, a spacetime is constructed where a spheroidal distribution of charge is surrounded by electro-vacuum. The solution is such that the quotient A/Q^2 for the surface of the spheroid can be made arbitrarily small.

Taking into account how (1) is obtained it is possible to conjecture that the inequality is not sharp. To investigate this relation we consider the Reissner-Nordström spacetime, which can be considered the simplest non-trivial electro-vacuum solution of Einstein equations. We found that in this case the inequality (1) is not sharp, and that the bound of the area in terms of the charge is obtained in the transition from superextremal to subextremal. We also isolate the possible cause of (1) not being sharp and present a new sharp inequality.

2 Main results

Let us consider a spherically symmetric 3-dimensional metric, written in the form

$$ds^2 = f(r)dr^2 + r^2 d\Omega^2, \qquad d\Omega^2 = d\theta^2 + \sin^2\theta\, d\phi^2. \tag{2}$$

We are interested in the Reissner-Nordström metric, in which case

$$f(r) = \left(1 - \frac{2m}{r} + \frac{Q^2}{r^2}\right)^{-1}, \tag{3}$$

where m is the mass and Q the charge. According to the range of m and Q we have three cases, sub-extreme, $m^2 > Q^2$, extreme, $m^2 = Q^2$ and super-extreme, $m^2 < Q^2$. In the first two cases, the coordinate r has range $r_0 \leq r \leq \infty$, where $r_0 = m + \sqrt{m^2 - Q^2}$. In the super-extreme case the coordinate has range $0 \leq r \leq \infty$.

A surface is called isoperimetric if its area is an extreme with respect to nearby surfaces that enclose the same volume. This implies that its mean curvature is constant. It is also called stable if its area is a minimum. For further discussion on isoperimetric surfaces in this context we refer to [6] [5] and for the concept of stability see [1]. We have the following condition on an isoperimetric surface Σ to be stable [1],

$$F(\alpha) > 0, \tag{4}$$

where

$$F(\alpha) = \int_\Sigma \left[-\alpha\Delta_\Sigma\alpha - \alpha^2\left(\chi_{AB}\chi^{AB} + R_{ab}n^a n^b\right)\right] dA_\Sigma, \tag{5}$$

and α is any function such that

$$\int_\Sigma \alpha \, dA_\Sigma = 0. \tag{6}$$

In (5) R_{ab} is the three-dimensional Ricci tensor and n^a the normal, χ_{AB} the second fundamental form and dA_Σ the volume element of Σ.

It is known that for $m^2 > Q^2$ all spheres of revolution $r = constant$ are isoperimetric stable surfaces (see [4]). We want to analyze the case $m^2 < Q^2$.

Theorem 2.1. *Consider the spheres $r = constant$ in the Reissner-Nordström metric given by equations* (2) (3)*. Then we have the following result:*

1. *For $0 \leq |Q| \leq m$ all these surfaces are isoperimetric stable.*

2. *For $m < 0$ all these surfaces are isoperimetric but not stable.*

3. *For $0 < m < |Q|$ the surfaces with radius $r > r_c$ are isoperimetric stable. The surfaces with $r < r_c$ are unstable, where*

$$r_c = \frac{2Q^2}{3m}. \tag{7}$$

In particular, all stable isoperimetric surfaces satisfy the bound

$$A \geq \frac{16}{9}\pi Q^2. \tag{8}$$

4. *There is not a sphere in Reissner-Nordström where the inequality* (8) *is saturated. The inequality is saturated in the limit for the sphere $r = r_c$ when the extreme case is approached from the superextreme case.*

Remark: we also prove that the stability operator is not positive for test functions that do not satisfy the volume preserving condition (6) (for example the constants).

Note that the bound (8) is higher than the one obtained in [6]. We expect this bound to be optimal. Following the analysis in [6] which is based on [3] we get the following result.

Theorem 2.2. *Consider an electro-vacuum, maximal initial data, with a non-negative cosmological constant. Assume that Σ is a stable isoperimetric sphere. Then*

$$\left(1 - \frac{1}{16\pi}\chi^2 A\right) A \geq \frac{4\pi}{3}(Q_E^2 + Q_M^2), \tag{9}$$

where Q_E and Q_M are the electric and magnetic charges of Σ and χ is its mean curvature.

Moreover, the surfaces r_c in super-extreme Reissner-Nordström satisfy the equality in (9).

Remark: The inequality (9) in this theorem is a straight forward consequence of previous results [3]. The interesting and new part of this theorem is that equality is achieved for this limit surface in Reissner-Nordström, showing that previously neglected terms are of consequence.

It is an interesting open problem to study the same problem for super-extreme Kerr. In that case the problem is much more complex because the location of the isoperimetric surfaces is known only numerically (see [7]).

Let us discuss the different regimes for the solution and the relation with respect to the area-charge inequalities. An appropriate quantity to consider for an isoperimetric stable surface in this context is

$$\frac{A}{4\pi Q^2} \tag{10}$$

as a function of $\epsilon = Q^2/m^2$. The parameter ϵ is the natural parameter for distinguishing the different regimes, where $\epsilon < 1$ corresponds to subextremal, $\epsilon = 1$ to extremal and $\epsilon > 1$ to superextremal. For the subextreme case we have

$$\frac{A_{sub}}{4\pi Q^2} = \frac{r^2}{Q^2} \geq \frac{r_0^2}{Q^2} = -1 + \frac{2}{\epsilon}\left(1 + \sqrt{1-\epsilon}\right). \tag{11}$$

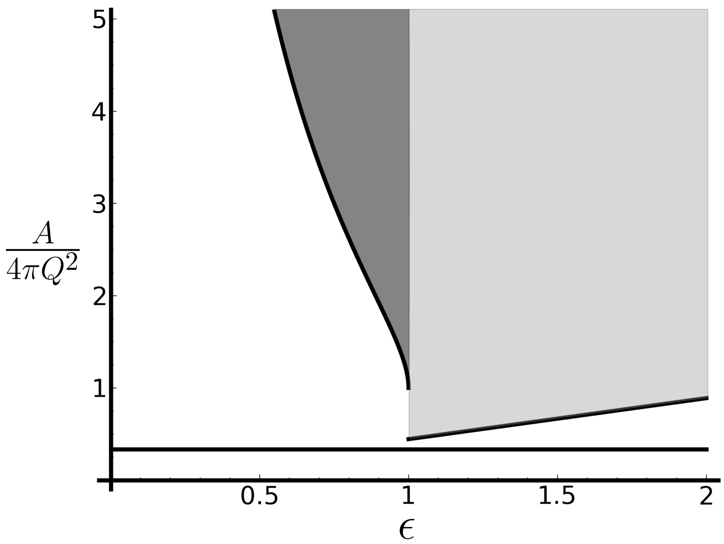
For the superextreme case

$$\frac{A_{super}}{4\pi Q^2} = \frac{r^2}{Q^2} \geq \frac{r_c^2}{Q^2} = \frac{4}{9}\,\epsilon. \tag{12}$$

For comparison, the inequality previously obtained in [6] is

$$\frac{A}{4\pi Q^2} \geq \frac{1}{3}. \tag{13}$$

It is interesting to note that the bounds in (11) and (12) appear because there is a limiting inner sphere. In the subextremal case this is the boundary of the manifold corresponding to the event horizon, while in the superextreme case it is the transition from stable to unstable surfaces. We put these inequalities together in the following graph, where the dark gray region are spheres in the subextremal case, the light gray region is the superextremal case and the bottom line is the previously obtained bound. Here it is worth noticing that the inequality gets close to equality as one approaches the extreme case, both from the subextremal and the superextremal cases. Also, there is a gap between the inequality (13) and the lower bound, suggesting that in general it is not optimal.



3 Proof of the theorems

Proof of theorem 2.1. Let us consider the surface $\Sigma = \{r = constant\}$. From (2) and (3) is a direct calculation to show that the mean curvature is

$$\chi = \frac{2}{r\sqrt{f}} \tag{14}$$

and therefore the surface is isoperimetric. Considering (5),

$$\begin{aligned} F(\alpha) &= \int_\Sigma \left[-\alpha\Delta_\Sigma\alpha - \alpha^2\left(\frac{2}{r^2 f} + \frac{f'}{rf^2}\right)\right] d\Sigma & (15)\\ &= \int_{\Sigma_0} \left[-\alpha\Delta_0\alpha - \alpha^2\frac{1}{f}\left(2 + \frac{rf'}{f}\right)\right] d\Sigma_0 & (16)\\ &= \int_{\Sigma_0} \left[-\alpha\Delta_0\alpha - \alpha^2\left(2 - \frac{6m}{r} + \frac{4Q^2}{r^2}\right)\right] d\Sigma_0, & (17)\end{aligned}$$

where Σ_0 is the unit sphere and Δ_0 is the Laplacian on it. The lowest non-zero eigenvalue $\lambda_1 = 2$ of the laplacian on the sphere, $\Delta_0\alpha = -\lambda\alpha$, can be written in the following variational form

$$\lambda_1 = \inf_{\int \alpha\, dS_0 = 0} \frac{\int |D\alpha|^2 dS_0}{\int \alpha^2 dS_0}. \tag{18}$$

From (18) we deduce

$$\int -\alpha\Delta_0\alpha dS_0 = \int |D\alpha|^2 dS_0 \geq 2\int \alpha^2 dS_0, \tag{19}$$

and hence

$$F(\alpha) \geq \frac{2}{r}\left(3m - \frac{2Q^2}{r}\right)\int_{S_0} \alpha^2 dS_0 \tag{20}$$

where we have restricted to functions that satisfy (6). In particular, equality is obtained in (20) when the function α is an eigenfunction corresponding to λ_1. Using this test function α and the equality in (20) we see that if $m < 0$ then all spheres are unstable. On the other hand, if $|Q| < m$ from the inequality (20) we deduce that all spheres are stable, as in this case $r \geq r_0$. If $0 < m < |Q|$ we can define a critical radius

$$r_c = \frac{2Q^2}{3m}, \tag{21}$$

such that if $r < r_c$ then the sphere is unstable and if $r > r_c$ it is stable.

The proof of (8) comes from the analysis of (11) and (12). The minimum of the r.h.s. in the range of applicability of (11) is attained for the extremal case, that is, $\epsilon = 1$, and gives

$$\frac{A_{sub}}{4\pi Q^2} \geq 1. \tag{22}$$

For (12) we have also that the minimum is obtained for $\epsilon = 1$, although in this case the minimum is obtained as a limit,

$$\frac{A_{super}}{4\pi Q^2} \geq \frac{4}{9}. \tag{23}$$

Comparing the last two inequalities we obtain the bound (8). That there is no sphere that actually saturates the inequality is because if we take $\epsilon = 1$ then we are in the extremal case and then $r_0 > r_c$. $\square$

Proof of theorem 2.2. The proof follows the proof of (1) in [6]. If one follows [3] it is possible to see that for a stable isoperimetric surface

$$12\pi \geq \frac{1}{2}\int_\Sigma \left(R + \frac{3}{2}\chi^2 + \bar{\chi}_{AB}\bar{\chi}^{AB}\right) dA_\Sigma, \tag{24}$$

where $\bar{\chi}_{AB}$ is the trace-free part of the second fundamental form of Σ and R is the three-dimensional Ricci scalar. The constraint equations in the three-dimensional manifold imply

$$R + K^2 - K_{ab}K^{ab} - 2\Lambda = 2(E^2 + B^2). \tag{25}$$

If we consider now that the data is maximal, $K = 0$, and that we assume $\Lambda \geq 0$, combining the previous equations we have

$$12\pi - \frac{1}{2}\int_\Sigma \left(\frac{3}{2}\chi^2 + \bar{\chi}_{AB}\bar{\chi}^{AB}\right) dA_\Sigma \geq \int_\Sigma \left(E^2 + B^2\right) dA_\Sigma \tag{26}$$

As shown in [6],

$$\int_\Sigma \left(E^2 + B^2\right) dA_\Sigma \geq \frac{16\pi^2}{A}(Q_E^2 + Q_M^2). \tag{27}$$

Using this, neglecting the term $\bar{\chi}_{AB}\bar{\chi}^{AB}$ as it is always positive and can be zero, and remembering that $\chi = constant$, we obtain (9). It is important to notice that discarding the term $\bar{\chi}_{AB}\bar{\chi}^{AB}$ does not pose a risk to the inequality being sharp, as this term is zero if the surface is umbilical.

If we evaluate (9) for Reissner-Nordström, we have

$$4\pi(2mr - Q^2) \geq \frac{4}{3}\pi Q^2, \tag{28}$$

which is saturated in the superextremal case for $r = r_c$ and is never saturated in the subextremal case. $\square$

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