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Deformed Twistors and Higher Spin Conformal (Super-)Algebras in Four Dimensions

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Abstract: Massless conformal scalar field in $d = 4$ corresponds to the minimal unitary representation (minrep) of the conformal group $SU(2,2)$ which admits a one-parameter family of deformations that describe massless fields of arbitrary helicity. The minrep and its deformations were obtained by quantization of the nonlinear realization of $SU(2,2)$ as a quasiconformal group in arXiv:0908.3624. We show that the generators of $SU(2,2)$ for these unitary irreducible representations can be written as bilinears of *deformed* twistorial oscillators which transform *nonlinearly* under the Lorentz group and apply them to define and study higher spin algebras and superalgebras in AdS_5. The higher spin (HS) algebra of Fradkin-Vasiliev type in AdS_5 is simply the enveloping algebra of $SU(2,2)$ quotiented by a two-sided ideal (Joseph ideal) which annihilates the minrep. We show that the Joseph ideal vanishes identically for the quasiconformal realization of the minrep and its enveloping algebra leads directly to the HS algebra in AdS_5. Furthermore, the enveloping algebras of the deformations of the minrep define a one parameter family of HS algebras in AdS_5 for which certain $4d$ covariant deformations of the Joseph ideal vanish identically. These results extend to superconformal algebras $SU(2,2|N)$ and we find a one parameter family of HS superalgebras as enveloping algebras of the minimal unitary supermultiplet and its deformations. Our results suggest the existence of a family of (supersymmetric) HS theories in AdS_5 which are dual to free (super)conformal field theories (CFTs) or to interacting but integrable (supersymmetric) CFTs in $4d$. We also discuss the corresponding picture in AdS_4 where the $3d$ conformal group $Sp(4,\mathbb{R})$ admits only two massless representations (minreps), namely the scalar and spinor singletons.

Contents

1 Introduction

Motivated by the work of physicists on spectrum generating symmetry groups in the 1960s the concept of minimal unitary representations of noncompact Lie groups was introduced by Joseph in [1]. Minimal unitary representation of a noncompact Lie group is defined over an Hilbert space of functions depending on the minimal number of variables possible. They have been studied extensively in the mathematics literature [2–14]. A unified approach to the construction and study of minimal unitary representations of noncompact groups was developed after the discovery of novel geometric quasiconformal realizations of noncompact groups in [15]. Quasiconformal realizations exist for different real forms of all noncompact groups as well as for their complex forms [15, 16][1].

The quantization of geometric quasiconformal action of a noncompact group leads directly to its minimal unitary representation as was first shown explicitly for the split exceptional group $E_{8(8)}$ with the maximal compact subgroup $SO(16)$ [17]. The minimal unitary representation of three dimensional U-duality group $E_{8(-24)}$ of the exceptional supergravity [18] was similarly obtained in [19]. In [20] a unified formulation of the minimal unitary representations of noncompact groups based on the quasiconformal method was given and it was extended to the minimal representations of noncompact supergroups G whose even subgroups are of the form $H \times SL(2,\mathbb{R})$ with H compact[2] . These supergroups include $G(3)$ with even subgroup $G_2 \times SL(2,\mathbb{R})$, $F(4)$ with even subgroup $Spin(7) \times SL(2,\mathbb{R})$, $D(2,1;\sigma)$ with even subgroup $SU(2) \times SU(2) \times SU(1,1)$ and $OSp(N|2,\mathbb{R})$. These results were further generalized to supergroups of the form $SU(n,m|p+q)$ and $OSp(2N^*|2M)$ in [21–23] and applied to conformal superalgebras in 4 and 6 dimensions. In particular, the construction of the minreps of $5d$ anti-de Sitter or $4d$ conformal group $SU(2,2)$ and corresponding supergroups $SU(2,2|N)$ was given in [21]. One finds that the minimal unitary representation of the group $SU(2,2)$ obtained by quantization of its quasiconformal realization is isomorphic to the scalar doubleton representation that describes a massless scalar field in four dimensions. Furthermore the minrep of $SU(2,2)$ admits a one parameter family (ζ) of deformations that can be identified with helicity, which can be continuous. For a positive (negative) integer value of the deformation parameter ζ, the resulting unitary irreducible representation of $SU(2,2)$ corresponds to a $4d$ massless conformal field transforming in $\left(0\,,\,\frac{\zeta}{2}\right)$ $\left(\left(-\frac{\zeta}{2}\right)\right)$ representation of the Lorentz subgroup, $SL(2,\mathbb{C})$. These deformed minimal representations for integer values of ζ turn out to be isomorphic to the doubletons of $SU(2,2)$ [24–26].

The minimal unitary supermultiplet of $SU(2,2|N)$ is the CPT self-conjugate (scalar) doubleton supermultiplet, and for $PSU(2,2|4)$ it is simply the four dimensional $N=4$ Yang-Mills supermultiplet. One finds that there exists a one-parameter family of deformations of the minimal unitary supermultiplet of $SU(2,2|N)$. The minimal unitary super-

[1]For the largest exceptional group $E_{8(8)}$ the quasiconformal action is the first and only known geometric realization of $E_{8(8)}$ and leaves invariant a generalized light-cone with respect to a quartic distance function in 57 dimensions [15].

[2]We shall be mainly working at the level of Lie algebras and Lie superalgebras and be cavalier about using the same symbol to denote a (super)group and its Lie (super)algebra.

multiplet of $SU(2,2\,|\,N)$ and its deformations with integer ζ are isomorphic to the unitary doubleton supermultiplets studied in [24–26].

The minrep of $7d$ AdS or $6d$ conformal group $SO(6,2) = SO^*(8)$ and its deformations were studied in [22]. One finds that the minrep admits deformations labelled by the eigenvalues of the Casimir of an $SU(2)_T$ subgroup of the little group, $SO(4)$, of massless particles in six dimensions. These deformed minreps labeled by spin t of $SU(2)_T$ are positive energy unitary irreducible representations of $SO^*(8)$ that describe massless conformal fields in six dimensions. Quasiconformal construction of the minimal unitary supermultiplet of $OSp(8^*|2N)$ and its deformations were given in [22, 23]. The minimal unitary supermultiplet of $OSp(8^*|4)$ is the massless conformal $(2,0)$ supermultiplet whose interacting theory is believed to be dual to M-theory on $AdS_7 \times S^4$. It is isomorphic to the scalar doubleton supermultiplet of $OSp(8^*|4)$ first constructed in [27].

For symplectic groups $Sp(2N,\mathbb{R})$ the construction of the minimal unitary representation using the quasiconformal approach and the covariant twistorial oscillator method coincide [20]. This is due to the fact that the quartic invariant operator that enters the quasiconformal construction vanishes for symplectic groups and hence the resulting generators involve only bilinears of oscillators. Therefore the minreps of $Sp(4,\mathbb{R})$ are simply the scalar and spinor singletons that were called the remarkable representations of anti-de Sitter group by Dirac [28]. The minimal unitary supermultiplets of $OSp(N|4,\mathbb{R})$ are the supersingletons [27, 29].

The Kaluza-Klein spectrum of IIB supergravity over the $AdS_5 \times S^5$ space was first obtained via the twistorial oscillator method by tensoring of the CPT self-conjugate doubleton supermultiplet of $SU(2,2\,|\,4)$ with itself repeatedly and restricting to the CPT self-conjugate short supermultiplets [24]. Authors of [24] also pointed out that the CPT self-conjugate doubleton supermultiplet $SU(2,2\,|\,4)$ does not have a Poincaré limit in five dimensions and its field theory lives on the boundary of AdS_5 on which $SU(2,2)$ acts as a conformal group and that the unique candidate for this theory is the four dimensional $N = 4$ super Yang-Mills theory that is conformally invariant. Similarly the Kaluza-Klein spectra of the compactifications of 11 dimensional supergravity over $AdS_4 \times S^7$ and $AdS_7 \times S^4$ were obtained by tensoring of singleton supermultiplet of $OSp(8\,|\,4,\mathbb{R})$ [30] and of scalar doubleton supermultiplet of $OSp(8^*\,|\,4)$ [27], respectively. The authors of [30] and [27] also pointed out that the field theories of the singleton and scalar doubleton supermultiplets live on the boundaries of AdS_4 and AdS_7 as conformally invariant field theories, respectively. As such these works represent some of the earliest work on AdS/CFT dualities within the framework of Kaluza-Klein supergravity theories. Their extension to the superstring and M-theory arena [31–33] started the modern era of AdS/CFT research. The fact that the scalar doubleton supermultiplets of $SU(2,2|4)$ and $OSp(8^*\,|\,4)$ and the singleton supermultiplet of $OSp(8\,|\,4,\mathbb{R})$ turn out to be the minimal unitary supermultiplets show that they are very special from a mathematical point of view as well.

Tensor product of the two singleton representations of the AdS_4 group $Sp(4,\mathbb{R})$ decomposes into infinitely many massless spin representations in AdS_4 as was shown by Fronsdal and collaborators. These higher spin theories were studied by Fronsdal and collaborators [34–37]. In the eighties Fradkin and Vasiliev initiated the study of higher spin theories involving fields of all spins $0 \leq s < \infty$ [38, 39]. A great deal of work was done on higher spin theories since then and for comprehensive reviews on higher spin theories we refer to [40–44] and references therein. The work on higher spin theories has intensified in the last decade since the conjectured duality between Vasiliev's higher spin gauge theory in AdS_4 and $O(N)$ vector models in [45, 46]. The three point functions of higher spin currents were computed directly and matched with those of free and critical $O(N)$ vector models in [47, 48]. Substantial work has also been done in higher spin holography in the last few years and for references we refer to the review [49].

In the early days of higher spin theories it was pointed out in [50] that the Fradkin-Vasiliev higher spin algebra in AdS_4 [38] corresponds simply to the infinite dimensional Lie algebra defined by the enveloping algebra of the singletonic realization of $Sp(4, \mathbb{R})$ and that this can be extended to construction of HS algebras in higher dimensions[3]. Again in [50] it was pointed out that the supersymmetric extensions of the higher spin algebras in AdS_4, AdS_5 and AdS_7 could be similarly constructed as enveloping algebras of the singletonic or doubletonic realizations of the super algebras $OSp(N/4, \mathbb{R})$, $SU(2,2|N)$ and $OSp(8^*|2N)$. Higher spin algebras and superalgebras in AdS_5 and AdS_7 were studied along these lines in [51, 52] using the doubletonic realizations of underlying algebras and superalgebras given in [25–27, 53]. Higher spin superalgebras in dimensions $d > 3$ were also studied by Vasiliev in [54]. However, they do not have the standard finite dimensional AdS superalgebras as subalgebras except for the case of AdS_4. The relation between higher spin algebras and cubic interactions for simple mixed-symmetry fields in AdS space times using Vasiliev's approach was studied in [55].

Mikhailov showed the connection between $AdS_5/Conf_4$ higher spin algebra and the algebra of *conformal Killing vectors and Killing tensors* in $d = 4$ and their relation to to higher symmetries of the Laplacian [56] . The connection between conformal Killing vectors and tensors and higher symmetries of the Laplacian was put on a firm mathematical foundation by Eastwood [57] who gave a realization of the $AdS_{(d+1)}/CFT_d$ higher spin algebra as an explicit quotient of the universal enveloping algebra $\mathscr{U}(\mathfrak{g})$, ($\mathfrak{g} = \mathfrak{so}(d, 2)$), by a two-sided ideal $\mathscr{J}(\mathfrak{g})$. This ideal $\mathscr{J}(\mathfrak{g})$ was identified as the annihilator of the scalar singleton module or the minimal representation and is known as the Joseph ideal in the mathematics literature [58]. This result agrees with the proposal of [50] for AdS_4/CFT_3 higher spin algebras since singletons are simply the minreps of $SO(3, 2)$ and in its singletonic twistorial oscillator realization the Joseph ideal vanishes identically as discussed in section 4.1. The covariant twistorial oscillators have been used extensively in the formulation and study of higher spin AdS_4 algebras since the early work of Fradkin and Vasiliev. For the doubletonic realization of $SO(4, 2)$ and $SO(6, 2)$ in terms of covariant twistorial oscillators the two sided Joseph ideal does not vanish identically as operators and must be quotiented out. However, as will be shown explicitly in section 4.2.2, the Joseph ideal vanishes identically as operators for the minimal unitary realization of $SU(2, 2)$ obtained via quasiconformal approach[4] .

[3] For related work see also [39].

[4] Same result holds true for the $AdS_7/Conf_6$ algebras and will be given elsewhere

One of the key results of this paper is to show that the basic objects for the construction of irreducible higher spin $AdS_5/Conf_4$ algebras are not the covariant twistorial oscillators but rather the deformed twistorial oscillators that transform nonlinearly under the Lorentz group $SL(2,\mathbb{C})$. The minimal unitary representation of $SO(4,2)$ and its deformations obtained by quasiconformal methods [21] can be written as bilinears of these deformed twistors. One parameter family of higher spin $AdS_5/Conf_4$ algebras and superalgebras can thus be realized as enveloping algebras involving products of bilinears of these deformed twistors. Our results for deformed twistorial oscillators extend to higher spin superalgebras in $d = 6$ and their deformations and the detailed calculations and results will be presented in a separate paper [59].

The plan of the paper is as follows: In section 2 we review the covariant twistorial oscillator (singleton) construction of the conformal group in three dimensions $SO(3,2) \sim Sp(4,\mathbb{R})$ and its superextension $OSp(N|4,\mathbb{R})$. Then we review the covariant twistorial oscillator (doubleton) construction for the four dimensional conformal group $SO(4,2) \sim SU(2,2)$ in section 3.1. In section 3.2, we present the minimal unitary representation of $SU(2,2)$ obtained by the quasiconformal approach [21] in terms of certain deformed twistorial oscillators that transform nonlinearly under the Lorentz group. We then define a one-parameter family of these deformed twistors, which we call *helicity deformed* twistorial oscillators and express the generators of a one parameter family of deformations of the minrep given in [21] as bilinears of the helicity deformed twistors. They describe massless conformal fields of arbitrary helicity which can be continuous. In section 3.4, we use the deformed twistors to realize the superconformal algebra $PSU(2,2|4)$ and its deformations in the quasiconformal framework. In section 4, we review the Eastwood's formula for the generator $\mathscr{J}$ of the annihilator of the minrep (Joseph ideal) and show by explicit calculations that it vanishes identically for the singletons of $SO(3,2)$ and the minrep of $SU(2,2)$ obtained by quasiconformal methods. We then present the generator $\mathscr{J}$ of the Joseph ideal in $4d$ covariant indices and use them to define the deformations $\mathscr{J}_\zeta$ that are the annihilators of the deformations of the minrep. In section 4.4, we use the fact that annihilators vanish identically to identify the $AdS_5/Conf_4$ higher spin algebra (as defined by Eastwood [57]) and define its deformations as the enveloping algebras of the deformations of the minrep within the quasiconformal framework. In section 4.4 we discuss the extension of these results to higher spin superalgebras. Finally in section 5 we discuss the implications of our results for higher spin theories of massless fields in AdS_5 and their conformal duals in $4d$.

2 $3d$ conformal algebra $SO(3,2) \sim Sp(4,\mathbb{R})$ and its supersymmetric extension $OSp(N|4,\mathbb{R})$

In this section we shall review the twistorial oscillator construction of the unitary representations of the conformal groups $SO(3,2)$ in $d = 3$ dimensions that correspond to conformally massless fields in $d = 3$ following [24, 30]. These representations turn out to be the minimal unitary representations and are also called the singleton (scalar and spinor singleton) representations of Dirac [28]. As a consequence, the quasiconformal and oscillator representations coincide and thus we will only review the oscillator construction and formulate it in terms of $3d$ Lorentz covariant twistorial oscillators.

2.1 Twistorial oscillator construction of $SO(3,2)$

The covering group of the three (four) dimensional conformal (anti-de Sitter) group $SO(3,2)$ is isomorphic to the noncompact symplectic group $Sp(4,\mathbb{R})$ with the maximal compact subgroup $U(2)$. Commutation relations of its generators can be written as

$$[M_{AB}\,,\,M_{CD}] = i(\eta_{BC}M_{AD} - \eta_{AC}M_{BD} - \eta_{BD}M_{AC} + \eta_{AD}M_{BC}) \tag{2.1}$$

where $\eta_{AB} = \text{diag}(-,+,+,+,-)$ and $A,B = 0,1,\ldots,4$. Spinor representation of $SO(3,2)$ can be realized in terms of four-dimensional gamma matrices γ_μ that satisfy

$$\{\gamma_\mu\,,\,\gamma_\nu\} = -2\eta_{\mu\nu}$$

where $\eta_{\mu\nu} = \text{diag}(-,+,+)$ and $\mu,\nu = 0,1,\ldots,3$ and $\gamma_5 = \gamma_0\gamma_1\gamma_2\gamma_3$ as follows:

$$\Sigma_{\mu\nu} := -\frac{i}{4}\left[\gamma_\mu\,,\,\gamma_\nu\right], \qquad \Sigma_{\mu 4} := -\frac{1}{2}\gamma_\mu \tag{2.2}$$

We adopt the following conventions for gamma matrices in four dimensions:

$$\gamma_0 = \begin{pmatrix} \mathbb{1}_2 & 0 \\ 0 & -\mathbb{1}_2 \end{pmatrix}, \qquad \gamma_m = \begin{pmatrix} 0 & -\sigma_m \\ \sigma_m & 0 \end{pmatrix}, \qquad \gamma_5 = i\begin{pmatrix} \mathbb{1}_2 & 0 \\ 0 & \mathbb{1}_2 \end{pmatrix} \tag{2.3}$$

where σ_m ($m = 1,2,3$ are Pauli matrices. Consider now a pair of bosonic oscillators $a_i, a_i^\dagger$ ($i = 1,2$) that satisfy

$$\left[a_i\,,\,a_j^\dagger\right] = \delta_{ij}. \tag{2.4}$$

and define a twistorial (Majorana) spinor Ψ and its Dirac conjugate in terms of these oscillators $\overline{\Psi} = \Psi^\dagger\gamma_0$

$$\Psi = \begin{pmatrix} a_1 \\ -ia_2 \\ ia_2^\dagger \\ -a_1^\dagger \end{pmatrix}, \qquad \overline{\Psi} = \begin{pmatrix} a_1^\dagger & ia_2^\dagger & ia_2 & a_1 \end{pmatrix} \tag{2.5}$$

Then the bilinears $M_{AB} = 2\overline{\Psi}\Sigma_{AB}\Psi$ satisfy the commutation relations (2.1) of $SO(3,2)$ Lie algebra:

$$\begin{aligned} \left[\overline{\Psi}\Sigma_{AB}\Psi\,,\,\overline{\Psi}\Sigma_{CD}\Psi\right] &= \overline{\Psi}\left[\Sigma_{AB}\,,\,\Sigma_{CD}\right]\Psi \\ &= i\left(\eta_{BC}\overline{\Psi}\Sigma_{AD}\Psi - \eta_{AC}\overline{\Psi}\Sigma_{BD}\Psi - \eta_{BD}\overline{\Psi}\Sigma_{AC}\Psi + \eta_{AD}\overline{\Psi}\Sigma_{BC}\Psi\right) \end{aligned} \tag{2.6}$$

The Fock space of these oscillators decompose into two ireducible unitary representations of $Sp(4,\mathbb{R})$ that are simply the two remarkable representations of Dirac [28] which were called *Di* and *Rac* in [34]. These representations do not have a Poincaré limit in

$4d$ and their field theories live on the boundary of AdS_4 which can be identified with the conformal compactification of three dimensional Minkowski space [60].

2.2 $SO(3,2)$ algebra in conformal three-grading and $3d$ covariant twistors

The conformal algebra in d dimensions can be given a three graded decomposition with respect to the noncompact dilatation generator Δ as follows:

$$\mathfrak{so}(d,2) = K_\mu \oplus (M_{\mu\nu} \oplus \Delta) \oplus P_\mu \tag{2.7}$$

We shall call this conformal 3-grading. The commutation relations of the algebra in this basis are given as follows:

$$\begin{aligned}
[M_{\mu\nu}\,,\,M_{\rho\tau}] &= i\,(\eta_{\nu\rho}M_{\mu\tau} - \eta_{\mu\rho}M_{\nu\tau} - \eta_{\nu\tau}M_{\mu\rho} + \eta_{\mu\tau}M_{\nu\rho})\\
[P_\mu\,,\,M_{\nu\rho}] &= i\,(\eta_{\mu\nu}\,P_\rho - \eta_{\mu\rho}\,P_\nu)\\
[K_\mu\,,\,M_{\nu\rho}] &= i\,(\eta_{\mu\nu}\,K_\rho - \eta_{\mu\rho}\,K_\nu)\\
[\Delta\,,\,M_{\mu\nu}] &= [P_\mu\,,\,P_\nu] = [K_\mu\,,\,K_\nu] = 0\\
[\Delta\,,\,P_\mu] &= +i\,P_\mu \qquad\qquad [\Delta\,,\,K_\mu] = -i\,K_\mu\\
[P_\mu\,,\,K_\nu] &= 2i\,(\eta_{\mu\nu}\,\Delta + M_{\mu\nu})
\end{aligned} \tag{2.8}$$

where $M_{\mu\nu}$ ($\mu, \nu = 0, 1, ..., (d-1)$) are the Lorentz groups generators. P_μ and K_μ are the generators of translations and special conformal transformations. In $d = 3$ dimensions the Greek indices $\mu, \nu, ...$ run over $0, 1, 2$ and dilatation generator is simply

$$D = -M_{34} \tag{2.9}$$

and translations P_μ and special conformal transformations K_μ are given by:

$$P_\mu = M_{\mu 4} + M_{\mu 3} \tag{2.10}$$
$$K_\mu = M_{\mu 4} - M_{\mu 3} \tag{2.11}$$

In order to make connection with higher spin (super-)algebras it is best to write the algebra in $SO(2,1)$ covariant spinorial oscillators. Let us now introduce linear combinations of $a_i, a_i^\dagger$ which we shall call $3d$ twistors [5]:

$$\kappa_1 = \frac{i}{2}\left(a_1 + a_2^\dagger + a_1^\dagger + a_2\right), \qquad \mu^1 = \frac{i}{2}\left(a_1 - a_2^\dagger - a_1^\dagger + a_2\right) \tag{2.12}$$
$$\kappa_2 = \frac{1}{2}\left(a_1 + a_2^\dagger - a_1^\dagger - a_2\right), \qquad \mu^2 = \frac{1}{2}\left(a_1 - a_2^\dagger + a_1^\dagger - a_2\right) \tag{2.13}$$

They satisfy the following commutation relations:

$$\left[\kappa_\alpha\,,\,\mu^\beta\right] = \delta_\alpha^\beta \tag{2.14}$$

[5]Note that the $3d$ twistor variables defined in [61] look slightly different from the ones defined here because the oscillators a_i, a^i are linear combinations of the ones used in [61].

Using these we can write (spinor conventions for $SO(2,1)$ are given in appendix A):

$$P_{\alpha\beta} = (\sigma^\mu P_\mu)_{\alpha\beta} = -\kappa_\alpha\kappa_\beta \tag{2.15}$$

$$K^{\alpha\beta} = (\bar{\sigma}^\mu K_\mu)^{\alpha\beta} = -\mu^\alpha\mu^\beta \tag{2.16}$$

Similarly we can define the Lorentz generators

$$M_\alpha{}^\beta = i\,(\sigma^\mu\ \bar{\sigma}^\nu)_\alpha{}^\beta\, M_{\mu\nu} \tag{2.17}$$

$$= \kappa_\alpha\mu^\beta - \frac{1}{2}\delta^\beta_\alpha\kappa_\gamma\mu^\gamma \tag{2.18}$$

and the dilatation generator

$$\Delta = -\frac{i}{4}\left(\kappa_\alpha\mu^\alpha + \mu^\alpha\kappa_\alpha\right) \tag{2.19}$$

In this basis the conformal algebra becomes:

$$\left[M_\alpha{}^\beta\,,\,M_\gamma{}^\delta\right] = \delta^\delta_\alpha M_\gamma{}^\beta - \delta^\beta_\gamma M_\alpha{}^\delta \tag{2.20}$$

$$\left[P_{\alpha\beta}\,,\,M_\gamma{}^\delta\right] = 2\delta^\delta_{(\alpha}P_{\beta)\gamma} - \delta^\delta_\gamma P_{\alpha\beta} \tag{2.21}$$

$$\left[K^{\alpha\beta}\,,\,M_\gamma{}^\delta\right] = -2\delta^{(\alpha}_\gamma K^{\beta)\delta} + \delta^\delta_\gamma K^{\alpha\beta} \tag{2.22}$$

$$\left[P_{\alpha\beta}\,,\,K^{\gamma\delta}\right] = 4\delta^{(\gamma}_{(\alpha}M_{\beta)}{}^{\delta)} + 4i\delta^{(\gamma}_{(\alpha}\delta^{\delta)}_{\beta)}\Delta \tag{2.23}$$

$$\left[\Delta\,,\,K^{\alpha\beta}\right] = -iK^{\alpha\beta}, \quad \left[\Delta\,,\,M_\alpha{}^\beta\right] = 0, \quad [\Delta\,,\,P_{\alpha\beta}] = iP_{\alpha\beta} \tag{2.24}$$

2.3 Minimal unitary supermultiplet of $OSp(N|4,\mathbb{R})$

In this section we will formulate the minimal unitary representation of $OSp(N|4,\mathbb{R})$ which is the superconformal algebra with N supersymmetries in three dimensions. The superalgebra $\mathfrak{osp}(N|4)$ can be given a five graded decomposition with respect to the noncompact dilatation generator Δ as follows:

$$\mathfrak{osp}(N|4) = K^{\alpha\beta} \oplus S^{I\alpha} \oplus (O^I_J \oplus M_\alpha{}^\beta \oplus \Delta) \oplus Q_\alpha{}^I \oplus P_{\alpha\beta} \tag{2.25}$$

We shall call this superconformal 5-grading. The bosonic conformal generators are the same as given in previous section. In order to realize the R-symmetry algebra $SO(N)$ and supersymmetry generators, we introduce Euclidean Dirac gamma matrices γ^I $(I, J = 1, 2, \ldots, N)$ which satisfy

$$\left\{\gamma^I\,,\,\gamma^J\right\} = \delta^{IJ} \tag{2.26}$$

The R-symmetry generators are then simply given as follows:

$$O^{IJ} = \frac{1}{2}\left(\gamma^I\gamma^J - \gamma^J\gamma^I\right) \tag{2.27}$$

and supersymmetry generators are the bilinears of $3d$ twistorial oscillators and γ^I:

$$Q_\alpha{}^I = \kappa_\alpha \gamma^I, \qquad S^{I\alpha} = \gamma^I \mu^\alpha \tag{2.28}$$

They satisfy the following commutation relations:

$$\{Q_\alpha{}^I \, , \, Q_\beta{}^J\} = \delta^{IJ} P_{\alpha\beta}, \qquad \left\{S^{I\alpha} \, , \, S^{J\beta}\right\} = \delta^{IJ} K^{\alpha\beta} \tag{2.29}$$

$$\left\{Q_\alpha{}^I \, , \, S^{J\beta}\right\} = M_\alpha{}^\beta \delta^{IJ} + 2O^{IJ}\delta^\beta_a + i\left(\Delta - \frac{i}{2}\right)\delta^{IJ}\delta^\beta_\alpha \tag{2.30}$$

The action of conformal group generators on supersymmetry generators is as follows:

$$\left[M_\alpha{}^\beta \, , \, Q_\gamma{}^I\right] = -\delta^\beta_\gamma Q_\alpha{}^I + \frac{1}{2} Q_\gamma{}^I, \qquad \left[M_\alpha{}^\beta \, , \, S^{I\gamma}\right] = \delta^\alpha_\gamma S^{I\beta} - \frac{1}{2} S^{I\gamma} \tag{2.31}$$

$$\left[\Delta \, , \, Q_\alpha{}^I\right] = \frac{i}{2} Q_\alpha{}^I, \qquad \left[\Delta \, , \, S^{I\alpha}\right] = -\frac{i}{2} S^{I\alpha} \tag{2.32}$$

$$\left[P_{\alpha\beta} \, , \, S^{I\gamma}\right] = -2\delta^\gamma_{(\alpha} Q^I_{\beta)}, \qquad \left[K^{\alpha\beta} \, , \, Q_\gamma{}^I\right] = 2\delta^{(\alpha}_\gamma S^{\beta)I} \tag{2.33}$$

$$\left[P_{\alpha\beta} \, , \, Q_\gamma{}^I\right] = \left[K^{\alpha\beta} \, , \, S^{I\gamma}\right] = 0 \tag{2.34}$$

The R-symmetry generators act only on the I, J indices and rotate them as follows:

$$\left[Q_\alpha{}^I \, , \, O^{JK}\right] = \frac{1}{2}\delta^{I[J} Q_\alpha{}^{K]}, \qquad \left[S^{I\alpha} \, , \, O^{JK}\right] = \frac{1}{2}\delta^{I[J} S^{K]\alpha} \tag{2.35}$$

For even N the singleton supermultiplet consists of conformal scalars transforming in a chiral spinor representation of $SO(N)$ and conformal space-time spinors transforming in the opposite chirality spinor representation of $SO(N)$ [30, 62]. There exists another singleton supermultiplet in which the roles of two spinor representations of $SO(N)$ are interchanged. For odd N both the conformal scalars and conformal space-time spinors transform in the same spinor representation of $SO(N)$.

3 Conformal and superconformal algebras in four dimensions

In this section, we present two different realizations of the conformal algebra and its supersymmetric extensions in $d = 4$. We start by reviewing the doubleton oscillator realization [24–26] and its reformulation in terms of Lorentz covariant twistorial oscillators [25, 61]. We then present a novel formulation of the quasiconformal realization of the minimal unitary representation and its deformations first studied in [21] in terms of deformed twistorial oscillators.

3.1 Covariant twistorial oscillator construction of the doubletons of $SO(4,2)$

The covering group of the conformal group $SO(4,2)$ in four dimensions is $SU(2,2)$. Denoting its generators as M_{AB} the commutation relations in the canonical basis are

$$[M_{AB}\,,\,M_{CD}] = i(\eta_{BC}M_{AD} - \eta_{AC}M_{BD} - \eta_{BD}M_{AC} + \eta_{AD}M_{BC})$$

where $\eta_{AB} = \mathrm{diag}(-,+,+,+,+,-)$ and $A,B = 0,\ldots,5$. The spinor representation of $SO(4,2)$ can be realized in terms in four-dimensional gamma matrices γ_μ that satisfy

$$\{\gamma_\mu\,,\,\gamma_\nu\} = -2\eta_{\mu\nu} \tag{3.1}$$

where $\eta_{\mu\nu} = \mathrm{diag}(-,+,+,+)$ ($\mu,\nu = 0,\ldots,3$) as follows:

$$\Sigma_{\mu\nu} := -\frac{i}{4}\left[\gamma_\mu\,,\,\gamma_\nu\right], \qquad \Sigma_{\mu 4} := -\frac{i}{2}\gamma_\mu\gamma_5, \qquad \Sigma_{\mu 5} := -\frac{1}{2}\gamma_\mu, \qquad \Sigma_{45} := -\frac{1}{2}\gamma_5 \tag{3.2}$$

Consider now two pairs of bosonic oscillators $a_i, a_j^\dagger$ $(i,j = 1,2)$ and $b_r, b_s^\dagger$ $(r,s = 1,2)$ that satisfy

$$\left[a_i\,,\,a_j^\dagger\right] = \delta_{ij}, \qquad \left[b_r\,,\,b_s^\dagger\right] = \delta_{rs} \tag{3.3}$$

We form a twistorial Dirac spinor Ψ and its conjugate $\overline{\Psi} = \Psi^\dagger\gamma_0$ in terms of these oscillators:

$$\Psi = \begin{pmatrix} a_1 \\ a_2 \\ -b_1^\dagger \\ -b_2^\dagger \end{pmatrix}, \qquad \overline{\Psi} = \begin{pmatrix} a_1^\dagger & a_2^\dagger & b_1 & b_2 \end{pmatrix} \tag{3.4}$$

Then the bilinears $M_{AB} = \overline{\Psi}\Sigma_{AB}\Psi$ $(A,B = 0,\ldots,5)$ generate the Lie algebra of $SO(4,2)$:

$$\left[\,\overline{\Psi}\Sigma_{AB}\Psi\,,\,\overline{\Psi}\Sigma_{CD}\Psi\right] = \overline{\Psi}\left[\Sigma_{AB}\,,\,\Sigma_{CD}\right]\Psi \tag{3.5}$$

which was called the doubleton realization [24–26][6].

The Lie algebra of $SU(2,2)$ can be given a three-grading with respect to the algebra of its maximal compact subgroup $SU(2)_L \times SU(2)_R \times U(1)$

$$\mathfrak{su}(2,2) = L_{ir} \oplus (L^i_j + R^i_j + E) \oplus L^{ir} \tag{3.6}$$

which is referred to as the compact three-grading. For the doubleton realizations one has

$$L^i_j = a^i a_j - \frac{1}{2}\delta^i_j(a^k a_k), \qquad R^r_s = b^r b_s - \frac{1}{2}\delta^r_s(b^t b_t) \tag{3.7}$$

$$L_{ir} = a_i b_r, \quad E = \frac{1}{2}(a^i a_i + b_r b^r), \quad L^{ir} = a^i b^r \tag{3.8}$$

[6]The term doubleton refers to the fact that we are using oscillators that decompose into two irreps under the action of the maximal compact subgroup. For $SU(2,2)$ that is the minimal set required. For symplectic groups the minimal set consists of oscillators that form a single irrep of their maximal compact subgroups.

where the creation operators are denoted with upper indices, i.e $a_i^\dagger = a^i$. Under the $SU(2)_L \times SU(2)_R$ subgroup of $SU(2,2)$ generated by the bilinears L^i_j and R^r_s oscillators $a_i(a_i^\dagger)$ and $b_r(b_r^\dagger)$ transform in the $(1/2, 0)$ and $(0, 1/2)$ representation. In contrast to the situation in three dimensions, the Fock space of these bosonic oscillators decomposes into an *infinite* set of positive energy unitary irreducible representations (UIRs), called doubletons of $SU(2,2)$. These UIRs are uniquely determined by a subset of states with the lowest eigenvalue (energy) of the $U(1)$ generator and transforming irreducibly under the $SU(2)_L \times SU(2)_R$ subgroup. The possible lowest energy irreps of $SU(2)_L \times SU(2)_R$ for positive energy UIRs of $SU(2,2)$ are of the form

$$a^\dagger_{i_1} a^\dagger_{i_2} \cdots a^\dagger_{i_n} |0\rangle \Leftrightarrow (j_L, j_R) = \left(\frac{n}{2}, 0\right) \qquad \mathrm{E} = 1 + n/2 \tag{3.9}$$

$$b^\dagger_{r_1} b^\dagger_{r_2} \cdots b^\dagger_{r_m} |0\rangle \Leftrightarrow (j_L, j_R) = \left(0, \frac{m}{2}\right) \qquad \mathrm{E} = 1 + m/2 \tag{3.10}$$

To relate the oscillators transforming covariantly under the maximal compact subgroup $SO(4) \times U(1)$ to twistorial oscillators transforming covariantly with respect to the Lorentz group $SL(2, \mathbf{C})$ with a definite scale dimension one acts with the intertwining operator [26, 61]

$$T = e^{\frac{\pi}{4} M_{05}}. \tag{3.11}$$

which intertwines between the compact and the noncompact pictures

$$\begin{aligned} \mathcal{M}_a T &= T L_a \\ \mathcal{N}_a T &= T R_a \\ D T &= T E \end{aligned} \tag{3.12}$$

where L_a and R_a denote the generators of $SU(2)_L$ and $SU(2)_R$, respectively. M_a and N_a are the generators of $SU(2)_{\mathcal{M}}$ and $SU(2)_{\mathcal{N}}$ given by the following linear combinations of the Lorentz group generators $M_{\mu\nu}$

$$\mathcal{M}_a = -\frac{1}{2}\left(\frac{1}{2}\epsilon_{abc} M_{bc} + i M_{0a}\right), \qquad \mathcal{N}_a = -\frac{1}{2}\left(\frac{1}{2}\epsilon_{abc} M_{bc} - i M_{0a}\right) \tag{3.13}$$

They satisfy

$$[\mathcal{M}_a \, , \, \mathcal{M}_b] = i\epsilon_{abc} \mathcal{M}_c, \qquad [\mathcal{N}_a \, , \, \mathcal{N}_b] = i\epsilon_{abc} \mathcal{N}_c, \qquad [\mathcal{M}_a \, , \, \mathcal{N}_b] = 0 \tag{3.14}$$

where $a, b, .. = 1, 2, 3$.

The oscillators that transform covariantly under the compact subgroup $SU(2)_L \times SU(2)_R$ get intertwined into the oscillators that transform covariantly under the Lorentz group $SL(2, \mathbb{C})$. More specifically the oscillators $a_i(a^i)$ and $b_i(b^i)$ that transform in the $(1/2, 0)$ and $(0, 1/2)$ representation of $SU(2)_L \times SU(2)_R$ go over to covariant oscillators transforming as Weyl spinors $(1/2, 0)$ and $(0, 1/2)$ of the Lorentz group $SL(2, \mathbb{C})$. Denoting the components of the Weyl spinors with undotted $(\alpha, \beta, \ldots = 1, 2)$ and dotted Greek indices ($\dot{\alpha}, \dot{\beta}, \ldots = 1, 2$) one finds :

$$
\begin{aligned}
\eta^{\alpha} &= T a_i T^{-1} = \frac{1}{\sqrt{2}}(b_i - a^i) \\
\lambda_{\alpha} &= T a^i T^{-1} = \frac{1}{\sqrt{2}}(b^i + a_i) \\
\tilde{\eta}^{\dot{\alpha}} &= T b_i T^{-1} = \frac{1}{\sqrt{2}}(a_i - b^i) \\
\tilde{\lambda}_{\dot{\alpha}} &= T b^i T^{-1} = \frac{1}{\sqrt{2}}(a^i + b_i)
\end{aligned}
\tag{3.15}
$$

where $\alpha, \beta, \dot{\alpha}, \dot{\beta}, .. = 1, 2$ and the covariant indices on the left hand side match the indices $i, j..$ on the right hand side of the equations above. They satisfy

$$
\begin{aligned}
[\eta^{\alpha}, \lambda_{\beta}] &= \delta^{\alpha}_{\beta} \\
[\tilde{\eta}^{\dot{\alpha}}, \tilde{\lambda}_{\dot{\beta}}] &= \delta^{\dot{\alpha}}_{\dot{\beta}}
\end{aligned}
\tag{3.16}
$$

They lead to the standard twistor relations[7].

$$
P_{\alpha\dot{\beta}} = -(\sigma^{\mu} P_{\mu})_{\alpha\dot{\beta}} = 2\lambda_{\alpha}\tilde{\lambda}_{\dot{\beta}} = T a^i b^r T^{-1} \tag{3.17}
$$

$$
K^{\dot{\alpha}\beta} = -(\bar{\sigma}^{\mu} K_{\mu})^{\dot{\alpha}\beta} = 2\tilde{\eta}^{\dot{\alpha}}\eta^{\beta} = T a_i b_r T^{-1} \tag{3.18}
$$

The dilatation generator in terms of covariant twistorial oscillators takes the form:

$$
\Delta = \frac{i}{2}\left(\lambda_{\alpha}\eta^{\alpha} + \tilde{\eta}^{\dot{\alpha}}\tilde{\lambda}_{\dot{\alpha}}\right) \tag{3.19}
$$

The Lorentz generators $M_{\mu\nu}$ in a spinorial basis can also be written as bilinears of Lorentz covariant twistorial oscillators:

$$
M_{\alpha}{}^{\beta} = -\frac{i}{2}\left(\sigma^{\mu}\bar{\sigma}^{\nu}\right)_{\alpha}{}^{\beta} M_{\mu\nu} = \lambda_{\alpha}\eta^{\beta} - \frac{1}{2}\delta_{\alpha}{}^{\beta}\lambda_{\gamma}\eta^{\gamma} \tag{3.20}
$$

$$
\bar{M}^{\dot{\alpha}}{}_{\dot{\beta}} = -\frac{i}{2}\left(\bar{\sigma}^{\mu}\sigma^{\nu}\right)^{\dot{\beta}}{}_{\dot{\alpha}} M_{\mu\nu} = -\left(\tilde{\eta}^{\dot{\alpha}}\tilde{\lambda}_{\dot{\beta}} - \frac{1}{2}\delta^{\dot{\alpha}}{}_{\dot{\beta}}\tilde{\eta}^{\dot{\gamma}}\tilde{\lambda}_{\dot{\gamma}}\right) \tag{3.21}
$$

[7]Note the overall minus sign in these expressions compared to [61]. This is due to the fact that we are using a mostly positive metric in this paper.

In this basis the conformal algebra becomes:

$$\left[M_\alpha{}^\beta\,,\,M_\gamma{}^\delta\right]=\delta_\gamma{}^\beta M_\alpha{}^\delta-\delta_\alpha{}^\delta M_\gamma{}^\beta,\qquad \left[\bar{M}^{\dot\alpha}{}_{\dot\beta}\,,\,\bar{M}^{\dot\gamma}{}_{\dot\delta}\right]=\delta^{\dot\gamma}{}_{\dot\beta}\bar{M}^{\dot\alpha}{}_{\dot\delta}-\delta^{\dot\alpha}{}_{\dot\delta}\bar{M}^{\dot\gamma}{}_{\dot\beta} \tag{3.22}$$

$$\left[P_{\alpha\dot\beta}\,,\,M_\gamma{}^\delta\right]=-\delta^\delta_\alpha P_{\gamma\dot\beta}+\frac{1}{2}\delta^\delta_\gamma P_{\alpha\dot\beta},\qquad \left[P_{\alpha\dot\beta}\,,\,\bar{M}^{\dot\gamma}{}_{\dot\delta}\right]=\delta^{\dot\gamma}_{\dot\beta}P_{\alpha\dot\delta}-\frac{1}{2}\delta^{\dot\gamma}_{\dot\delta}P_{\alpha\dot\beta} \tag{3.23}$$

$$\left[K^{\dot\alpha\beta}\,,\,M_\gamma{}^\delta\right]=\delta^\beta_\gamma K^{\dot\alpha\delta}-\frac{1}{2}\delta^\delta_\gamma K^{\dot\alpha\beta},\qquad \left[K^{\dot\alpha\beta}\,,\,\bar{M}^{\dot\gamma}{}_{\dot\delta}\right]=-\delta^{\dot\alpha}_{\dot\delta}K^{\dot\gamma\beta}+\frac{1}{2}\delta^{\dot\gamma}_{\dot\delta}K^{\dot\alpha\beta} \tag{3.24}$$

$$\left[\Delta\,,\,K^{\dot\alpha\beta}\right]=-iK^{\dot\alpha\beta},\quad \left[\Delta\,,\,M_\alpha{}^\beta\right]=\left[\Delta\,,\,\bar{M}^{\dot\alpha}{}_{\dot\beta}\right]=0,\quad \left[\Delta\,,\,P_{\alpha\dot\beta}\right]=iP_{\alpha\dot\beta} \tag{3.25}$$

$$\left[P_{\alpha\dot\beta}\,,\,K^{\dot\gamma\delta}\right]=4\left(\delta^\delta_\alpha\bar{M}^{\dot\gamma}{}_{\dot\beta}-\delta^{\dot\gamma}_{\dot\beta}M_\alpha{}^\delta+i\Delta\right) \tag{3.26}$$

The linear Casimir operator $\mathcal{Z}=N_a-N_b=a^ia_i-b^rb_r$ when expressed in terms of $SL(2,\mathbb{C})$ covariant oscillators becomes

$$\mathcal{Z}=N_a-N_b=\tilde\lambda_{\dot\alpha}\tilde\eta^{\dot\alpha}-\lambda_\alpha\eta^\alpha \tag{3.27}$$

which shows that $\frac{1}{2}\mathcal{Z}$ is the helicity operator.

Denoting the lowest energy irreps in the compact basis as $|\Omega(j_L,j_R,E)\rangle$ one can show that the coherent states of the form

$$e^{-ix^\mu P_\mu}T|\Omega(j_L,j_R,E)\rangle\equiv|\Phi^\ell_{j_\mathcal{M},j_\mathcal{N}}(x_\mu)\rangle \tag{3.28}$$

transform exactly like the states created by the action of conformal fields $\Phi^\ell_{j_\mathcal{M},j_\mathcal{N}}(x_\mu)$ acting on the vacuum vector $|0\rangle$

$$\Phi^\ell_{j_\mathcal{M},j_\mathcal{N}}(x^\mu)|0\rangle\cong|\Phi^\ell_{j_\mathcal{M},j_\mathcal{N}}(x_\mu)\rangle$$

with exact numerical coincidence of the compact and the covariant labels (j_L,j_R,E) and $(j_\mathcal{M},j_\mathcal{N},-l)$, respectively, where l is the scale dimension [26]. The doubletons correspond to massless conformal fields transforming in the (j_L,j_R) representation of the Lorentz group $SL(2,\mathbb{C})$ whose conformal (scaling dimension) is $\ell=-E$ where E is the eigenvalue of the $U(1)$ generator which is the conformal Hamiltonian (or AdS_5 energy) [24–26].

3.2 Quasiconformal approach to the minimal unitary representation of $SO(4,2)$ and its deformations

The construction of the minimal unitary representation (minrep) of the $4d$ conformal group $SO(4,2)$ by quantization of its quasiconformal realization and its deformations were given in [21], which we shall reformulate in this section in terms of what we call deformed twistorial oscillators which transform nonlinearly under the Lorentz group.

The group $SO(4,2)$ can be realized as a quasiconformal group that leaves invariant light-like separations with respect to a quartic distance function in five dimensions. The quantization of this geometric action leads to a nonlinear realization of the generators of $SO(4,2)$ in terms of a singlet coordinate x, its conjugate momentum p and two ordinary bosonic oscillators $d,d^\dagger$ and $g,g^\dagger$ satisfying [21]:

$$[x\,,\,p]=i,\qquad \left[d\,,\,d^\dagger\right]=1,\qquad \left[g\,,\,g^\dagger\right]=1 \tag{3.29}$$

The nonlinearities can be absorbed into certain "singular" oscillators which are functions of the coordinate x, momentum p and the oscillators $d, g, d^\dagger, g^\dagger$:

$$A_{\mathcal{L}} = a - \frac{\mathcal{L}}{\sqrt{2}\,x} \qquad\qquad A_{\mathcal{L}}^\dagger = a^\dagger - \frac{\mathcal{L}}{\sqrt{2}\,x} \tag{3.30}$$

where

$$\begin{aligned} a &= \frac{1}{\sqrt{2}}(x + ip) \\ a^\dagger &= \frac{1}{\sqrt{2}}(x - ip) \\ \mathcal{L} &= N_d - N_g - \frac{1}{2} = d^\dagger d - g^\dagger g - \frac{1}{2} \end{aligned} \tag{3.31}$$

They satisfy the following commutation relations:

$$\begin{aligned} [A_{\mathcal{G}}\,,\,A_{\mathcal{K}}] &= -\frac{(\mathcal{G}-\mathcal{K})}{2\,x^2} \\ \left[A_{\mathcal{G}}^\dagger\,,\,A_{\mathcal{K}}^\dagger\right] &= +\frac{(\mathcal{G}-\mathcal{K})}{2\,x^2} \\ \left[A_{\mathcal{G}}\,,\,A_{\mathcal{K}}^\dagger\right] &= 1 + \frac{(\mathcal{G}+\mathcal{K})}{2\,x^2} \end{aligned} \tag{3.32}$$

assuming that $[\mathcal{G}, \mathcal{K}] = 0$.

The realization of the minrep of $SO(4,2)$ obtained by the quasiconformal approach is nonlinear and "interacting" in the sense that they involve operators that are cubic or quartic in terms of the oscillators in contrast to the covariant twistorial oscillator realization, reviewed in section 3.1 [24], which involve only bilinears. The algebra $\mathfrak{so}(4,2)$ can be given a 3-graded decomposition with respect to the conformal Hamiltonian, which is referred to as the compact 3-grading and the generators in this basis are reproduced in Appendix B following [21].

The Lie algebra of $SO(4,2)$ has a noncompact (conformal) three graded decomposition determined by the dilatation generator Δ as well

$$\mathfrak{so}(4,2) = \mathfrak{N}^- \oplus \mathfrak{N}^0 \oplus \mathfrak{N}^+ \tag{3.33}$$

$$= K_\mu \oplus (M_{\mu\nu} \oplus \Delta) \oplus P_\mu \tag{3.34}$$

Remarkably, one can write the generators of quantized quasiconformal action of $SO(4,2)$ as bilinears of deformed twistorial oscillators $Z^\alpha, \widetilde{Z}^{\dot\alpha}, Y_\alpha, \widetilde{Y}_{\dot\alpha}$ $(\alpha, \dot\alpha = 1, 2)$ which are defined as:

$$Z_1 = \frac{A_{\mathcal{L}}}{\sqrt{2}} - i\,g^\dagger, \qquad Y^1 = -\frac{A_{\mathcal{L}}^\dagger}{\sqrt{2}} + i\,g \tag{3.35}$$

$$\widetilde{Z}_{\dot{1}} = \frac{A_{\mathcal{L}}^\dagger}{\sqrt{2}} + i\,g, \qquad \widetilde{Y}^{\dot{1}} = \frac{A_{\mathcal{L}}}{\sqrt{2}} + i\,g^\dagger \tag{3.36}$$

$$Z_2 = -\frac{A_{-\mathcal{L}}^\dagger}{\sqrt{2}} - i\,d, \qquad Y^2 = -\frac{A_{-\mathcal{L}}}{\sqrt{2}} - i\,d^\dagger \tag{3.37}$$

$$\widetilde{Z}_{\dot{2}} = -\frac{A_{-\mathcal{L}}}{\sqrt{2}} + i\,d^\dagger, \qquad \widetilde{Y}^{\dot{2}} = \frac{A_{-\mathcal{L}}^\dagger}{\sqrt{2}} - i\,d \tag{3.38}$$

Using $(\sigma^\mu)_{\alpha\dot{\alpha}} = (\mathbb{1}_2, \vec{\sigma})$ and $(\bar{\sigma}^\mu)^{\dot{\alpha}\alpha} = (-\mathbb{1}_2, \vec{\sigma})$ one finds that the generators of translations and special conformal transformations can be written as follows[8]:

$$P_{\alpha\dot{\beta}} = (\sigma^\mu P_\mu)_{\alpha\dot{\beta}} = -Z_\alpha \widetilde{Z}_{\dot{\beta}} \tag{3.39}$$

$$K^{\dot{\alpha}\beta} = (\bar{\sigma}^\mu K_\mu)^{\dot{\alpha}\beta} = -\widetilde{Y}^{\dot{\alpha}} Y^\beta \tag{3.40}$$

We see that the operators Z and Y in the quasiconformal realization play similar roles as covariant twistorial oscillators λ and η in the doubleton realization. They transform non-linearly under the Lorentz group and their commutations relations are given in Appendix C

The dilatation generator in terms of deformed twistorial oscillators takes the form:

$$\Delta = \frac{i}{4}\left(Z_\alpha Y^\alpha + \widetilde{Y}^{\dot{\alpha}} \widetilde{Z}_{\dot{\alpha}}\right) \tag{3.41}$$

The Lorentz group generators $M_{\mu\nu}$ in a spinorial basis can also be written as bilinears of singular twistors:

$$M_\alpha{}^\beta = -\frac{i}{2}\left(\sigma^\mu \bar{\sigma}^\nu\right)_\alpha{}^\beta M_{\mu\nu} = \frac{1}{2}\left(Z_\alpha Y^\beta - \frac{1}{2}\delta_\alpha{}^\beta Z_\gamma Y^\gamma\right) \tag{3.42}$$

$$\bar{M}^{\dot{\alpha}}{}_{\dot{\beta}} = -\frac{i}{2}\left(\bar{\sigma}^\mu \sigma^\nu\right)^{\dot{\beta}}{}_{\dot{\alpha}} M_{\mu\nu} = -\frac{1}{2}\left(\widetilde{Y}^{\dot{\alpha}} \widetilde{Z}_{\dot{\beta}} - \frac{1}{2}\delta^{\dot{\alpha}}{}_{\dot{\beta}} \widetilde{Y}^{\dot{\gamma}} \widetilde{Z}_{\dot{\gamma}}\right) \tag{3.43}$$

3.3 Deformations of the minimal unitary representation

As was shown in [21], the minimal unitary representation of $SU(2,2)$ that corresponds to a conformal scalar field admits a one-parameter, ζ, family of deformations that correspond to massless conformal fields of helicity $\frac{\zeta}{2}$ in four dimensions, which can be continuous. For non-integer values of the deformation parameter ζ they correspond, in general, to unitary representations of an infinite covering of the conformal group.

The generators of the deformed minrep take the same form as given in section 3.2 with the simple replacement of the singular oscillators $A_{\mathcal{L}}$ and $A_{\mathcal{L}}^\dagger$ by "deformed" singular

[8]Note that in our conventions P^0 is positive definite.

oscillators:

$$A_{\mathcal{L}_\zeta} = a - \frac{\mathcal{L}_\zeta}{\sqrt{2}\,x} \qquad\qquad A^\dagger_{\mathcal{L}_\zeta} = a^\dagger - \frac{\mathcal{L}_\zeta}{\sqrt{2}\,x} \tag{3.44}$$

where

$$\mathcal{L}_\zeta = \mathcal{L} + \zeta = N_d - N_g + \zeta + \frac{1}{2} \tag{3.45}$$

Since $\zeta/2$ labels the helicity we define "helicity deformed twistors" as follows:

$$Z_1(\zeta) = \frac{A_{\mathcal{L}_\zeta}}{\sqrt{2}} - i\,g^\dagger, \qquad Y^1(\zeta) = -\frac{A^\dagger_{\mathcal{L}_\zeta}}{\sqrt{2}} + i\,g \tag{3.46}$$

$$\widetilde{Z}_1(\zeta) = \frac{A^\dagger_{\mathcal{L}_\zeta}}{\sqrt{2}} + i\,g, \qquad \widetilde{Y}^1(\zeta) = \frac{A_{\mathcal{L}_\zeta}}{\sqrt{2}} + i\,g^\dagger \tag{3.47}$$

$$Z_2(\zeta) = -\frac{A^\dagger_{-\mathcal{L}_\zeta}}{\sqrt{2}} - i\,d, \qquad Y^2(\zeta) = -\frac{A_{-\mathcal{L}_\zeta}}{\sqrt{2}} - i\,d^\dagger \tag{3.48}$$

$$\widetilde{Z}_2(\zeta) = -\frac{A_{-\mathcal{L}_\zeta}}{\sqrt{2}} + i\,d^\dagger, \qquad \widetilde{Y}^2(\zeta) = \frac{A^\dagger_{-\mathcal{L}_\zeta}}{\sqrt{2}} - i\,d \tag{3.49}$$

The realization of the minimal unitary representation in terms of deformed twistors carry over to realization in terms of helicity deformed twistors:

$$P_{\alpha\dot{\beta}} = (\sigma^\mu P_\mu)_{\alpha\dot{\beta}} = -Z_\alpha(\zeta)\widetilde{Z}_{\dot{\beta}}(\zeta) \tag{3.50}$$

$$K^{\dot{\alpha}\beta} = (\bar{\sigma}^\mu K_\mu)^{\dot{\alpha}\beta} = -\widetilde{Y}^{\dot{\alpha}}(\zeta)Y^\beta(\zeta) \tag{3.51}$$

The dilatation generator then takes the form:

$$\Delta = \frac{i}{4}\left(Z_\alpha(\zeta)Y^\alpha(\zeta) + \widetilde{Y}^{\dot{\alpha}}(\zeta)\widetilde{Z}_{\dot{\alpha}}(\zeta)\right) \tag{3.52}$$

and the Lorentz generators $M_{\mu\nu}$ also take the same form in terms of helicity deformed twistors:

$$M_\alpha{}^\beta = -\frac{i}{2}\left(\sigma^\mu\bar{\sigma}^\nu\right)_\alpha{}^\beta M_{\mu\nu} = \frac{1}{2}\left(Z_\alpha(\zeta)Y^\beta(\zeta) - \frac{1}{2}\delta_\alpha{}^\beta Z_\gamma(\zeta)Y^\gamma(\zeta)\right) \tag{3.53}$$

$$\bar{M}^{\dot{\alpha}}{}_{\dot{\beta}} = -\frac{i}{2}\left(\bar{\sigma}^\mu\sigma^\nu\right)^{\dot{\beta}}{}_{\dot{\alpha}} M_{\mu\nu} = -\frac{1}{2}\left(\widetilde{Y}^{\dot{\alpha}}(\zeta)\widetilde{Z}_{\dot{\beta}}(\zeta) - \frac{1}{2}\delta^{\dot{\alpha}}{}_{\dot{\beta}}\widetilde{Y}^{\dot{\gamma}}(\zeta)\widetilde{Z}_{\dot{\gamma}}(\zeta)\right) \tag{3.54}$$

The realization of the generators of $SU(2,2)$ in terms of helicity deformed twistors describe positive energy unitary irreducible representations which can best be seen by going over to the compact three-grading.

3.4 Minimal unitary supermultiplet of $SU(2,2|4)$ and its deformations and deformed singular twistors

The construction of the minimal unitary representations of noncompact Lie algebras by quantization of their quasiconformal realizations extends to noncompact Lie superalgebras

[20–23]. In this section we shall reformulate the results for 4d superconformal algebra $SU(2,2|4)$ which is the symmetry superalgebras of IIB supergravity over $AdS_5 \times S^5$ [9]. The generators of the minimal unitary realization of $SU(2,2|4)$ deformed by the continuous parameter ζ obtained by quasiconformal methods can be expressed in terms of bilinears of helicity deformed twistors and ordinary fermionic oscillators. The deformed singular oscillators are of the form:

$$A_{\mathcal{L}^s_\zeta} = a - \frac{\mathcal{L}^s_\zeta}{\sqrt{2}x}, \qquad A^\dagger_{\mathcal{L}^s_\zeta} = a^\dagger - \frac{\mathcal{L}^s_\zeta}{\sqrt{2}x} \tag{3.55}$$

where the deformation "parameter" (operator) $\mathcal{L}^s_\zeta$ is given as:

$$\mathcal{L}^s_\zeta = N_d - N_g + N_\xi + \zeta - \frac{5}{2} \tag{3.56}$$

where $d, d^\dagger$, and $g, g^\dagger$ are the usual bosonic oscillators with $N_d = d^\dagger d$ and $N_g = g^\dagger g$, and ξ_I, ξ^J $(I, J = 1, 2, 3, 4)$ are the fermionic oscillators

$$\{\xi_I\,,\,\xi^J\} = \delta^J_I \tag{3.57}$$

with $N_\xi = \xi^I \xi_J$ being the corresponding number operator. ζ is again the continuous deformation parameter.

Consider the superconformal (noncompact) 5-graded decomposition of the Lie superalgebra $\mathfrak{su}(2,2|4)$ with respect to the dilatations generator Δ:

$$\begin{aligned} \mathfrak{su}(2,2|4) &= \mathfrak{N}^{-1} \oplus \mathfrak{N}^{-1/2} \oplus \mathfrak{N}^{0} \oplus \mathfrak{N}^{+1/2} \oplus \mathfrak{N}^{+1} \\ &= K^{\dot\alpha\beta} \oplus S_I{}^\alpha, \bar S^{I\dot\alpha} \oplus (M_\alpha{}^\beta \oplus \bar M^{\dot\alpha}{}_{\dot\beta} \oplus \Delta \oplus R^I{}_J) \oplus Q^I{}_\alpha, \bar Q_{I\dot\alpha} \oplus P_{\alpha\dot\beta}, \\ &\qquad (I, J = 1, 2, 3, 4) \end{aligned} \tag{3.58}$$

where the grade zero space consists of the Lorentz algebra $\mathfrak{so}(3,1)$ $(M_\alpha{}^\beta, \bar M^{\dot\alpha}{}_{\dot\beta})$, the dilatations (Δ) and R-symmetry $\mathfrak{su}(4)$ $(R^I{}_J)$, grade $+1$ and -1 spaces consist of translations $(P_{\alpha\dot\beta})$ and special conformal transformations $(K^{\dot\alpha\beta})$ respectively, and the $+1/2$ and -1/2 spaces consist of Poincaré supersymmetries $(Q^I{}_\alpha, \bar Q_{I\dot\alpha})$ and conformal supersymmetries $(S_I{}^\alpha, \bar S^{I\dot\alpha})$ respectively.

The helicity deformed twistors for the superalgebra $SU(2,2|4)$ are obtained from the purely bosonic case by replacing $\mathcal{L}_\zeta$ with $\mathcal{L}^s_\zeta$. The expressions for the bosonic generators given in 3.2 get modified as follows

$$P_{\alpha\dot\beta} = -Z^s_\alpha(\zeta)\widetilde Z^s_{\dot\beta}(\zeta) \tag{3.59}$$

$$K^{\dot\alpha\beta} = -\widetilde Y^{s\dot\alpha}(\zeta) Y^{s\beta}(\zeta) \tag{3.60}$$

$$\Delta = \frac{i}{4}\left(Z^s_\alpha(\zeta)Y^{s\alpha}(\zeta) + \widetilde Y^{s\dot\alpha}(\zeta)\widetilde Z^s_{\dot\alpha}(\zeta)\right) \tag{3.61}$$

[9]Actual symmetry of the IIB theory over $AdS_5 \times S^5$ is $PSU(2,2|4)$. We shall however work with $SU(2,2|4)$ which includes the central charge as was done originally in [24].

$$M_\alpha{}^\beta = \frac{1}{2}\left(Z^s_\alpha(\zeta)Y^{s\beta}(\zeta) - \frac{1}{2}\delta_\alpha{}^\beta Z^s_\gamma(\zeta)Y^{s\gamma}(\zeta)\right) \tag{3.62}$$

$$\bar{M}^{\dot\alpha}{}_{\dot\beta} = -\frac{1}{2}\left(\widetilde{Y}^{s\dot\alpha}(\zeta)\widetilde{Z}^s_{\dot\beta}(\zeta) - \frac{1}{2}\delta^{\dot\alpha}{}_{\dot\beta}\widetilde{Y}^{s\dot\gamma}(\zeta)\widetilde{Z}^s_{\dot\gamma}(\zeta)\right) \tag{3.63}$$

and the fermionic generators in terms of the deformed singular twistors and fermionic oscillators are given below:

$$Q^I{}_\alpha = Z^s_\alpha(\zeta)\xi^I, \qquad \bar{Q}_{I\dot\alpha} = -\xi_I\widetilde{Z}^s_{\dot\alpha}(\zeta) \tag{3.64}$$

$$S_I{}^\alpha = -\xi_I Y^{s\alpha}(\zeta), \qquad \bar{S}^{I\dot\alpha} = \widetilde{Y}^{s\dot\alpha}(\zeta)\xi^I \tag{3.65}$$

The $\mathfrak{su}(4)$ generators $R^I{}_J$ are given as follows:

$$R^I{}_J = \xi^I\xi_J - \frac{1}{4}\delta^I{}_J\xi^K\xi_K \tag{3.66}$$

The bosonic conformal generators satisfy the usual conformal algebra and the supersymmetry generators satisfy the following anti-commutation relations:

$$\left\{Q^I{}_\alpha\,,\,\bar{Q}_{J\dot\beta}\right\} = \delta^I_J P_{\alpha\dot\beta} \tag{3.67}$$

$$\left\{\bar{S}^{I\dot\alpha}\,,\,S_J{}^\beta\right\} = \delta^I_J K^{\dot\alpha\beta} \tag{3.68}$$

$$\left\{Q^I{}_\alpha\,,\,S_J{}^\beta\right\} = -2\delta^I_J M_\alpha{}^\beta + 2\delta^\beta_\alpha R^I{}_J + \delta^I_J\delta^\beta_\alpha\,(i\Delta + C) \tag{3.69}$$

$$\left\{\bar{S}^{I\dot\alpha}\,,\,\bar{Q}_{J\dot\beta}\right\} = 2\delta^I_J\bar{M}^{\dot\alpha}{}_{\dot\beta} - 2\delta^{\dot\beta}_{\dot\alpha}R^I{}_J + \delta^I_J\delta^{\dot\beta}_{\dot\alpha}\,(i\Delta - C) \tag{3.70}$$

where $C = \frac{\zeta}{2}$ is the central charge.

The commutators of conformal group generators with supersymmetry generators are as follows:

$$\left[P_{\alpha\dot\beta}\,,\,S_I{}^\gamma\right] = 2\delta^\gamma_\alpha\bar{Q}_{I\dot\beta}, \qquad \left[K^{\dot\alpha\beta}\,,\,Q_\gamma{}^I\right] = -2\delta^\beta_\gamma\bar{S}^{I\dot\alpha} \tag{3.71}$$

$$\left[P_{\alpha\dot\beta}\,,\,\bar{S}^{I\dot\gamma}\right] = 2\delta^{\dot\gamma}_{\dot\beta}Q_\alpha{}^I, \qquad \left[K^{\dot\alpha\beta}\,,\,\bar{Q}_{I\dot\gamma}\right] = -2\delta^{\dot\alpha}_{\dot\gamma}S_I{}^\beta \tag{3.72}$$

$$\left[M_\alpha{}^\beta\,,\,Q^I{}_\gamma\right] = \delta^\beta_\gamma Q^I{}_\alpha - \frac{1}{2}\delta^\beta_\alpha Q^I{}_\gamma, \qquad \left[M_\alpha{}^\beta\,,\,S_I{}^\gamma\right] = -\delta^\alpha_\gamma S_I{}^\beta + \frac{1}{2}\delta^\beta_\alpha S_I{}^\gamma \tag{3.73}$$

$$\left[\bar{M}^{\dot\alpha}{}_{\dot\beta}\,,\,\bar{Q}_{I\dot\gamma}\right] = -\delta^{\dot\alpha}_{\dot\gamma}\bar{Q}_{I\dot\beta} + \frac{1}{2}\delta^{\dot\alpha}_{\dot\beta}\bar{Q}_{I\dot\gamma}, \qquad \left[\bar{M}^{\dot\alpha}{}_{\dot\beta}\,,\,\bar{S}^{I\dot\gamma}\right] = \delta^{\dot\gamma}_{\dot\beta}\bar{S}^{I\dot\alpha} - \frac{1}{2}\delta^{\dot\alpha}_{\dot\beta}\bar{Q}^{I\dot\gamma} \tag{3.74}$$

$$\left[\Delta\,,\,Q^I{}_\alpha\right] = \frac{i}{2}Q^I{}_\alpha, \qquad \left[\Delta\,,\,\bar{Q}_{I\dot\alpha}\right] = \frac{i}{2}\bar{Q}_{I\dot\alpha} \tag{3.75}$$

$$\left[\Delta\,,\,S_I{}^\alpha\right] = -\frac{i}{2}S_I{}^\alpha, \qquad \left[\Delta\,,\,\bar{S}^{I\dot\alpha}\right] = -\frac{i}{2}\bar{S}^{I\dot\alpha} \tag{3.76}$$

The $\mathfrak{su}(4)_R$ generators satisfy the following commutation relations:

$$\left[R^I{}_J\,,\,R^K{}_L\right] = \delta^K_J R^I{}_L - \delta^I_L R^K{}_J \tag{3.77}$$

They act on the R-symmetry indices I, J of the supersymmetry generators as follows:

$$\left[R^I{}_J\,,\,Q^K{}_\alpha\right] = \delta^K_J Q^I{}_\alpha - \frac{1}{4}\delta^I_J Q^K{}_\alpha, \qquad \left[R^I{}_J\,,\,\bar{Q}_{K\dot{\alpha}}\right] = -\delta^I_K \bar{Q}_{J\dot{\alpha}} + \frac{1}{4}\delta^I_J \bar{Q}_{K\dot{\alpha}} \tag{3.78}$$

$$\left[R^I{}_J\,,\,S_K^{\alpha}\right] = -\delta^I_K S_J{}^\alpha + \frac{1}{4}\delta^I_J S_K{}^\alpha, \qquad \left[R^I{}_J\,,\,\bar{S}^{K\dot{\alpha}}\right] = \delta^K_J \bar{S}^{I\dot{\alpha}} - \frac{1}{4}\delta^I_J \bar{S}^{K\dot{\alpha}} \tag{3.79}$$

The minimal unitary representation of $PSU(2,2|4)$ is obtained when the deformation parameter ζ vanishes and the resulting supermultiplet of massless conformal fields in $d = 4$ is the $N = 4$ Yang-Mills supermultiplet [21]. For each value of the deformation parameter ζ one obtains an irreducible unitary representation of $SU(2,2|4)$. For integer values of the deformation parameter these unitary representations are isomorphic to doubleton supermultiplets studied in [24–26].

The unitarity of the representations of $SU(2,2|N)$ may not be manifest in the Lorentz covariant noncompact three grading. It is however manifestly unitary in compact three grading with respect to the subsupergroup $SU(2|N-M)\times SU(2|M)\times U(1)$ as was shown for the the doubletons in [24, 25] and for the quasiconformal construction in [21]. The Lie algebra of $SU(4)$ can be given a 3-graded structure with respect to the Lie algebra of its subgroup $SU(2)\times SU(2)\times U(1)$. Similarly the Lie superalgebra $SU(2,2|4)$ can be given a 3-graded decomposition with respect to its subalgebra $SU(2|2)\times SU(2|2)\times U(1)$. This is the basis that was originally used by Gunaydin and Marcus [24] in constructing the spectrum of IIB supergravity over $AdS_5 \times S^5$ using twistorial oscillators. In this basis, choosing the Fock vacuum as the lowest weight vector leads to CPT-self-conjugate supermultiplets and it is also the preferred basis in applications to integrable spin chains. The corresponding compact 3-grading of the quasiconformal realization of $SU(2,2|4)$ was given in [21]. In Appendix D we review the compact three grading and give the corresponding formulation in terms of deformed twistors.

4 Higher spin (super-)algebras, Joseph ideals and their deformations

In this section we start by reviewing Eastwood's results [57, 63] on defining $HS(\mathfrak{g})$ algebras as the quotient of universal enveloping algebra $\mathscr{U}(\mathfrak{g})$ by their Joseph ideal $\mathscr{J}(\mathfrak{g})$. We will then explicitly compute the Joseph ideal for $SO(3,2)$, $SO(4,2)$ and its deformations, using the Eastwood formula [63] and recast it in a Lorentz covariant formulation.

The universal enveloping algebra $\mathscr{U}(\mathfrak{g})$, $\mathfrak{g} = \mathfrak{so}(d-1,2)$ is defined as follows:

$$\mathscr{U}(\mathfrak{g}) = \mathscr{G}/\mathscr{I} \tag{4.1}$$

where $\mathscr{G}$ is the associative algebra freely generated by elements of $\mathfrak{g}$, and $\mathscr{I}$ is the ideal of $\mathscr{G}$ generated by elements of form $gh - hg - [g\,,\,h]\,(g, h \in \mathfrak{g})$.

It was already noted in [41] that the higher spin algebra $HS(\mathfrak{g})$ must be a quotient of $\mathscr{U}(\mathfrak{g})$ but the relevant ideal was identified in [57] to be the Joseph ideal or the annihilator of the minimal representation (scalar doubleton). The uniqueness of this quadratic ideal in $\mathscr{U}(\mathfrak{g})$ was proved in [63] and an explicit formula for the generators of the ideal was given:

$$\begin{aligned} J_{ABCD} &= M_{AB}M_{CD} - M_{AB} \circledcirc M_{CD} - \frac{1}{2}\left[M_{AB}\,,\,M_{CD}\right] + \frac{n-4}{4(n-1)(n-2)}\langle M_{AB}, M_{CD}\rangle\,\mathbf{1} \\ &= \frac{1}{2}M_{AB}\cdot M_{CD} - M_{AB} \circledcirc M_{CD} + \frac{n-4}{4(n-1)(n-2)}\langle M_{AB}, M_{CD}\rangle\,\mathbf{1} \end{aligned} \tag{4.2}$$

where the dot $\cdot$ denotes the symmetric product

$$M_{AB}\cdot M_{CD} \equiv M_{AB}M_{CD} + M_{CD}M_{AB} \tag{4.3}$$

of the generators and $\langle M_{AB}, M_{CD}\rangle$ is the Killing form of $SO(n-2,2)$. η_{AB} is the $SO(n-2,2)$ invariant metric and the symbol $\circledcirc$ denotes the Cartan product of two generators, which for $SO(n-2,2)$, can be written in the form [64]:

$$\begin{aligned} M_{AB} \circledcirc M_{CD} = &\frac{1}{3}M_{AB}M_{CD} + \frac{1}{3}M_{DC}M_{BA} + \frac{1}{6}M_{AC}M_{BD} \\ &-\frac{1}{6}M_{AD}M_{BC} + \frac{1}{6}M_{DB}M_{CA} - \frac{1}{6}M_{CB}M_{DA} \\ &-\frac{1}{2(n-2)}\left(M_{AE}M^E_C\eta_{BD} - M_{BE}M^E_C\eta_{AD} + M_{BE}M^E_D\eta_{AC} - M_{AE}M^E_D\eta_{BC}\right) \\ &-\frac{1}{2(n-2)}\left(M_{CE}M^E_A\eta_{BD} - M_{CE}M^E_B\eta_{AD} + M_{DE}M^E_B\eta_{AC} - M_{DE}M^E_A\delta_{BC}\right) \\ &+\frac{1}{(n-1)(n-2)}M_{EF}M^{EF}\left(\eta_{AC}\eta_{BD} - \eta_{BC}\eta_{AD}\right) \end{aligned} \tag{4.4}$$

The Killing term is given by

$$\langle M_{AB}, M_{CD}\rangle = h\,M_{EF}M_{GH}(\eta^{EG}\eta^{FH} - \eta^{EH}\eta^{FG})(\eta_{AC}\eta_{BD} - \eta_{AD}\eta_{BC}) \tag{4.5}$$

where $h = \frac{2(n-2)}{n(4-n)}$ is a c-number fixed by requiring that all possible contractions of J_{ABCD} with the the metric vanish. We shall refer to the operator J_{ABCD} as the generator of the Joseph ideal.

In the following sections we will compute the generators J_{ABCD} for $d = 3$ and 4 conformal algebras in various representations discussed in previous sections. We shall also reformulate the formula for generators of Joseph ideal in Lorentz covariant form which also makes the massless nature of these representations explicit along with certain other identities that must be satisfied within the representation in order for it to be annihilated by Joseph ideal. This also allows us to define the annihilators of the deformations of the minrep of $SO(4,2)$ and define a one parameter family of HS algebras corresponding to these deformed ideals.

4.1 Joseph ideal for $SO(3,2)$ singletons

We will now use the twistorial oscillator realization for $SO(3,2)$ described in section 2.1. For $Sp(4,\mathbb{R}) = SO(3,2)$, the generator J_{ABCD} of the Joseph ideal is

$$\begin{aligned} J_{ABCD} &= M_{AB}M_{CD} - M_{AB} \circledcirc M_{CD} - \frac{1}{2}\left[M_{AB}\,,\, M_{CD}\right] - \frac{1}{40}\langle M_{AB}, M_{CD}\rangle \\ &= \frac{1}{2}M_{AB}\cdot M_{CD} - M_{AB} \circledcirc M_{CD} - \frac{1}{40}\langle M_{AB}, M_{CD}\rangle \end{aligned} \tag{4.6}$$

Substituting the realization of $Sp(4,\mathbb{R}) = SO(3,2)$ in terms of a twistorial Majorana spinor Ψ one finds that the operator J_{ABCD} vanishes identically. This implies that both *Rac* and *Di* are minimal unitary representations (minreps) and we shall refer to them as scalar and spinor singletons (minreps) of $Sp(4,\mathbb{R})$.

Considered as the the three dimensional conformal group the minreps of $Sp(4,\mathbb{R})$ (*Di* and *Rac*) correspond to massless scalar and spinor fields which are known to be the only massless representations of the Poincaré group in three dimensions [4].

If instead of a twistorial Majorana spinor one considers a twistorial Dirac spinor corresponding to taking two copies (colors) of the Majorana spinor one finds that the generators J_{ABCD} of the Joseph ideal do not vanish identically and hence they do not correspond to minimal unitary representations. The corresponding Fock space decomposes into an infinite set of irreducible unitary representations of $Sp(4,\mathbb{R})$, which correspond to the massless fields in AdS_4 [60]. Taking more than two colors in the realization of the Lie algebra of $Sp(4,\mathbb{R})$ as bilinears of oscillators leads to representations corresponding to massive fields in AdS_4 [30].

4.1.1 Joseph ideal of $SO(3,2)$ in Lorentz covariant basis

To get a more physical picture of what the vanishing of the Joseph ideal means we shall go to the conformal 3-graded basis defined in equation 2.7. Evaluating the Joseph ideal in this basis, we find that the vanishing ideal is equivalent to the linear combinations of certain quadratic identities, full set of which hold only in the singleton realization. First we have the masslessness conditions:

$$P^2 = P^\mu P_\mu = 0 \quad , \quad K^2 = K^\mu K_\mu = 0 \tag{4.7}$$

The remaining set of quadratic relations that define the Joseph ideal are

$$6\Delta\cdot\Delta + 2M^{\mu\nu}\cdot M_{\mu\nu} + P^\mu\cdot K_\mu = 0 \tag{4.8}$$

$$P^\mu\cdot(M_{\mu\nu} + \eta_{\mu\nu}\Delta) = 0 \tag{4.9}$$

$$K^\mu\cdot(M_{\nu\mu} + \eta_{\nu\mu}\Delta) = 0 \tag{4.10}$$

$$\eta^{\mu\nu}M_{\mu\rho}\cdot M_{\nu\sigma} - P_{(\rho}\cdot K_{\sigma)} + \eta_{\rho\sigma} = 0 \tag{4.11}$$

$$\Delta\cdot M_{\mu\nu} + P_{[\mu}\cdot K_{\nu]} = 0 \tag{4.12}$$

$$M_{[\mu\nu}\cdot P_{\rho]} = 0 \tag{4.13}$$

$$M_{[\mu\nu}\cdot K_{\rho]} = 0 \tag{4.14}$$

The ideal generated by these relations is completely equivalent to equation 4.6 but it sheds light on the massless nature of these representations. The scalar and spinor singleton modules for $SO(3,2)$ are the only minreps and there are no deformations due to vanishing of the quartic invariant of $Sp(4,\mathbb{R})$ but as we have seen in section 3.3, there exists one parameter family of deformations of the minrep for $4d$ conformal group and we will see in following sections that the Lorentz covariant formulation of Joseph ideal is more useful in identifying the annihilators of these deformations.

The Casimir invariants for the singleton or the minrep of $SO(3,2)$ are as follows:

$$C_2 = M_B{}^A M_A{}^B = \frac{5}{2} \tag{4.15}$$

$$C_4 = M_B{}^A M_C{}^B M_D{}^C M_A{}^D = -\frac{35}{8} \tag{4.16}$$

4.2 Joseph ideal for $SO(4,2)$

4.2.1 Twistorial oscillator or doubleton realization

We will use the doubleton realization given in section 3.1 to compute the generator J_{ABCD} of the Joseph ideal for $SO(4,2)$:

$$\begin{aligned} J_{ABCD} &= M_{AB}M_{CD} - M_{AB} \circledcirc M_{CD} - \frac{1}{2}\left[M_{AB}\,,\,M_{CD}\right] - \frac{1}{60}\langle M_{AB}, M_{CD}\rangle \\ &= \frac{1}{2}M_{AB}\cdot M_{CD} - M_{AB} \circledcirc M_{CD} - \frac{1}{60}\langle M_{AB}, M_{CD}\rangle \end{aligned} \tag{4.17}$$

One finds that it does not vanish identically as an operator in contrast to the situation with the singletonic realization of $SO(3,2)$. However for the doubleton realization of $SO(4,2)$, J_{ABCD} has only 15 independent non-vanishing components which turn out to be equal to one of the following terms (up to an overall sign):

$$\Big((a_1b_2 + a_2b_1) \pm (a_2^\dagger b_1^\dagger + a_1^\dagger b_2^\dagger)\Big)\,\mathcal{Z} \tag{4.18}$$

$$\Big((a_1b_2 - a_2b_1) \pm (a_2^\dagger b_1^\dagger - a_1^\dagger b_2^\dagger)\Big)\,\mathcal{Z} \tag{4.19}$$

$$\Big((a_1b_1 + a_2b_2) \pm (a_2^\dagger b_2^\dagger + a_1^\dagger b_1^\dagger)\Big)\,\mathcal{Z} \tag{4.20}$$

$$\Big((a_1b_1 - a_2b_2) \pm (a_2^\dagger b_2^\dagger - a_1^\dagger b_1^\dagger)\Big)\,\mathcal{Z} \tag{4.21}$$

$$\Big(a_1^\dagger a_2 + a_1 a_2^\dagger \pm b_1^\dagger b_2 \pm b_1 b_2^\dagger\Big)\,\mathcal{Z} \tag{4.22}$$

$$\Big(a_1^\dagger a_2 - a_1 a_2^\dagger \pm b_1^\dagger b_2 \mp b_1 b_2^\dagger\Big)\,\mathcal{Z} \tag{4.23}$$

$$\left((N_{a_1} - N_{a_2}) \pm (N_{b_1} - N_{b_2})\right)\mathcal{Z} \tag{4.24}$$

$$(N_a + N_b + 2)\mathcal{Z} \tag{4.25}$$

The operator $\mathcal{Z} = (N_a - N_b)$ commutes with all the generators of $SU(2,2)$ and its eigenvalues label the helicity of the corresponding massless representation of the conformal group [21]. All the components of the generator J_{ABCD} of Joseph ideal vanish on the states that form the basis of UIR of $SU(2,2)$ whose lowest weight vector is the Fock vacuum $|0\rangle$ since

$$\mathcal{Z}|0\rangle = 0 \tag{4.26}$$

The corresponding UIR describes a conformal scalar in four dimensions (zero helicity) and is the true minrep of $SU(2,2)$ annihilated by the Joseph ideal [21].

The Casimir invariants for $SO(4,2)$ in the doubleton representation are as follows:

$$C_2 = M_B{}^A M_A{}^B = \frac{3}{2}\left(4 - \mathcal{Z}^2\right) \tag{4.27}$$

$$C_4 = M_B{}^A M_C{}^B M_D{}^C M_A{}^D = \frac{C_2^2}{6} - 4C_2 \tag{4.28}$$

$$C_6 = M_B{}^A M_C{}^B M_D{}^C M_E{}^D M_F{}^E M_A{}^F = \frac{C_2^3}{36} - 2C_2^2 + 16C_2 \tag{4.29}$$

Thus we see that all the higher order Casimir invariants are functions of the quadratic Casimir C_2 which itself is given in terms of $\mathcal{Z} = N_a - N_b$.

4.2.2 Joseph ideal of the quasiconformal realization of the minrep of $SO(4,2)$

To apply Eastwood's formula to the generator of Joseph ideal in the quasiconformal realization it is convenient to go from the conformal 3-graded basis to the $SO(4,2)$ covariant canonical basis where the generators M_{AB} satisfy the following commutation relations:

$$[M_{AB}\,,\, M_{CD}] = i(\eta_{BC}M_{AD} - \eta_{AC}M_{BD} - \eta_{BD}M_{AC} + \eta_{AD}M_{BC}) \tag{4.30}$$

where the metric $\eta_{AB} = \text{diag}(-,+,+,+,+,-)$ is used to raise and lower the indices $A, B = 0, 1, \ldots, 5$ etc. In addition to Lorentz generators $M_{\mu\nu}$, we need the following combinations to obtain the generators in canonical basis:

$$M_{\mu 4} = \frac{1}{2}\left(P_\mu - K_\mu\right) \tag{4.31}$$

$$M_{\mu 5} = \frac{1}{2}\left(P_\mu + K_\mu\right) \tag{4.32}$$

$$M_{45} = -\Delta \tag{4.33}$$

Substituting the expressions for the quasiconformal realization of the generators of $SO(4,2)$ given in subsection 3.2 into the generator of the Joseph ideal in the canonical basis:

$$J_{ABCD} = \frac{1}{2}M_{AB}\cdot M_{CD} - M_{AB} \circledcirc M_{CD} - \frac{1}{60}\langle M_{AB}, M_{CD}\rangle \tag{4.34}$$

one finds that it vanishes identically as an *operator* showing that the corresponding unitary representation is indeed the minimal unitary representation.

We should stress the important point that the tensor product of the Fock spaces of the two oscillators d and g with the state space of the singular oscillator form a single UIR which is the minrep. In contrast, the Fock space of the twistorial oscillators presented in section 3.1 [24] decomposes into infinitely many UIRs (doubletons) of which only the irreducible representation whose lowest weight vector is the Fock vacuum is the minrep. That is why the J_{ABCD} does not vanish as an operator in the covariant twistorial oscillator construction.

4.2.3 $4d$ Covariant Formulation of the Joseph ideal of $SO(4,2)$

Above we showed that the generator of the Joseph ideal given in equation (4.34) vanishes identically as an operator for the quasiconformal realization of the minrep of $SO(4,2)$ in the canonical basis. To get a more physical picture of what the vanishing of generator J_{ABCD} means we shall go to the conformal three grading defined by the dilatation generator Δ. Evaluating the generator of the Joseph ideal in this basis, we find that the vanishing condition is equivalent to linear combinations of certain quadratic identities, full set of which hold only in the quasiconformal realization of the minrep. First we have the conditions:

$$P^2 = P^\mu P_\mu = 0 \quad , \quad K^2 = K^\mu K_\mu = 0 \tag{4.35}$$

which hold also for the twistorial oscillator realization given in section 3.1. The remaining set of quadratic relations that define the Joseph ideal are

$$4\Delta \cdot \Delta + M^{\mu\nu} \cdot M_{\mu\nu} + P^\mu \cdot K_\mu = 0 \tag{4.36}$$
$$P^\mu \cdot (M_{\mu\nu} + \eta_{\mu\nu}\Delta) = 0 \tag{4.37}$$
$$K^\mu \cdot (M_{\nu\mu} + \eta_{\nu\mu}\Delta) = 0 \tag{4.38}$$

$$\eta^{\mu\nu} M_{\mu\rho} \cdot M_{\nu\sigma} - P_{(\rho} \cdot K_{\sigma)} + 2\eta_{\rho\sigma} = 0 \tag{4.39}$$
$$M_{\mu\nu} \cdot M_{\rho\sigma} + M_{\mu\sigma} \cdot M_{\nu\rho} + M_{\mu\rho} \cdot M_{\sigma\nu} = 0 \tag{4.40}$$
$$\Delta \cdot M_{\mu\nu} + P_{[\mu} \cdot K_{\nu]} = 0 \tag{4.41}$$

$$M_{[\mu\nu} \cdot P_{\rho]} = 0, \qquad M_{[\mu\nu} \cdot K_{\rho]} = 0 \tag{4.42}$$

In four dimensions, using the Levi-Civita tensor one can define the Pauli-Lubanski vector, W^μ and its conformal analogue, V^μ as follows:

$$W^\mu = \frac{1}{2}\epsilon^{\mu\nu\rho\sigma} P_\nu M_{\rho\sigma}, \qquad V^\mu = \frac{1}{2}\epsilon^{\mu\nu\rho\sigma} K_\nu M_{\rho\sigma} \tag{4.43}$$

where $\epsilon_{0123} = +1, \epsilon^{0123} = -1$ and the indices are raised and lowered by the Minkowski metric. For massless fields, W^μ and V^μ are proportional to P^μ and K^μ respectively with the proportionality constant related to helicity of the fields [65, 66]. Equations (4.42) imply that for the minrep both the W^μ and V^μ vanish implying that it describes a zero helicity (scalar) massless field [10].

The Casimir invariants for the minrep of $SO(4,2)$ are as follows (using the definitions

[10] We should note that a similar set of identities (constraints) were discussed in [66] in the context of deriving field equations for particles of all spins (acting on field strengths) where they arise as conformally covariant forms of massless particles.

given in equations 4.27 - 4.29):

$$C_2 = 6, \qquad C_4 = -18, \qquad C_6 = 30. \tag{4.44}$$

4.3 Deformations of the minrep of $SO(4,2)$ and their associated ideals

As was shown in [21], the minimal unitary representation of $SU(2,2)$ that corresponds to a conformal scalar field admits a one-parameter (ζ) family of deformations corresponding to massless conformal fields of helicity $\frac{\zeta}{2}$ in four dimensions, which can be continuous. For non-integer values of the deformation parameter ζ they correspond, in general, to unitary representations of an infinite covering of the conformal group[11].

The generators of the deformed minrep were given in section 3.3. For the deformations of the minrep, one finds that the generator J_{ABCD} of the Joseph ideal computed in the canonical basis does not vanish. One might wonder if there exists deformations of the Joseph ideal that annihilate the deformed minimal unitary representations labelled by the deformation parameter ζ. This is indeed the case. The quadratic identities that define the Joseph ideal in the conformal basis discussed in the previous section go over to identities involving the deformation parameter ζ and define the deformations of the Joseph ideal.

One finds that the helicity conditions are modified as follows:

$$\frac{1}{2}\epsilon^{\mu\nu\rho\sigma} M_{\nu\rho} \cdot P_\sigma = \zeta P^\mu \tag{4.45}$$

$$\frac{1}{2}\epsilon^{\mu\nu\rho\sigma} M_{\nu\rho} \cdot K_\sigma = -\zeta K^\mu \tag{4.46}$$

The identities (4.39), (4.40) and (4.41) get also modified as follows:

$$\eta^{\mu\nu} M_{\mu\rho} \cdot M_{\nu\sigma} - P_{(\rho} \cdot K_{\sigma)} + 2\eta_{\rho\sigma} = \frac{\zeta^2}{2}\eta_{\rho\sigma} \tag{4.47}$$

$$M_{\mu\nu} \cdot M_{\rho\sigma} + M_{\mu\sigma} \cdot M_{\nu\rho} + M_{\mu\rho} \cdot M_{\sigma\nu} = \zeta\epsilon_{\mu\nu\rho\sigma}\Delta \tag{4.48}$$

$$\Delta \cdot M_{\mu\nu} + P_{[\mu} \cdot K_{\nu]} = -\frac{\zeta}{2}\epsilon_{\mu\nu\rho\sigma} M^{\rho\sigma} \tag{4.49}$$

The remaining quadratic identities remain unchanged in going over to the deformed minimal unitary representations .

The Casimir invariants for the deformations of the minrep of $SO(4,2)$ depend only the deformation parameter ζ and are given as follows (using the definitions given in equations 4.27 - 4.29):

$$C_2 = 6 - \frac{3\zeta^2}{2} \tag{4.50}$$

$$C_4 = \frac{3}{8}\left(\zeta^4 + 8\zeta^2 - 48\right) = \frac{C_2^2}{6} - 2C_2 - 12 \tag{4.51}$$

$$C_6 = -\frac{3}{32}\left(\zeta^6 + 26\zeta^4 - 80\zeta^2 - 320\right) = \frac{C_2^3}{36} - \frac{5}{3}C_2^2 - 6C_2 + 120 \tag{4.52}$$

[11] Recently, such a continuous helicity parameter was introduced as a deformation parameter in [67].

We saw earlier in section 4.2.1 that the generator J_{ABCD} of the Joseph ideal did not vanish identically as an operator for the covariant twistorial oscillator realization of $SO(4,2)$. It annihilates only the states belonging to the subspace that form the basis of the true minrep of $SO(4,2)$. By going to the conformal three grading, one finds that the generator J_{ABCD} of the Joseph ideal can be written in a form similar to the deformed quadratic identities above with the deformation parameter replaced by the linear Casimir operator $\mathcal{Z} = N_a - N_b$

$$\eta^{\mu\nu} M_{\mu\rho} \cdot M_{\nu\sigma} - P_{(\rho} \cdot K_{\sigma)} + 2\eta_{\rho\sigma} = \frac{\mathcal{Z}^2}{2}\eta_{\rho\sigma} \quad (4.53)$$

$$M_{\mu\nu} \cdot M_{\rho\sigma} + M_{\mu\sigma} \cdot M_{\nu\rho} + M_{\mu\rho} \cdot M_{\sigma\nu} = -\mathcal{Z}\epsilon_{\mu\nu\rho\sigma}\Delta \quad (4.54)$$

$$\Delta \cdot M_{\mu\nu} + P_{[\mu} \cdot K_{\nu]} = \frac{\mathcal{Z}}{2}\epsilon_{\mu\nu\rho\sigma} M^{\rho\sigma} \quad (4.55)$$

$$\frac{1}{2}\epsilon^{\mu\nu\rho\sigma} M_{\nu\rho} \cdot P_\sigma = -\mathcal{Z} P^\mu \quad (4.56)$$

$$\frac{1}{2}\epsilon^{\mu\nu\rho\sigma} M_{\nu\rho} \cdot K_\sigma = \mathcal{Z} K^\mu \quad (4.57)$$

The Fock space of the oscillators decompose into an infinite set of unitary irreducible representations of $SU(2,2)$ corresponding to massless conformal fields of all integer and half-integer helicities labelled by the eigenvalue of $\mathcal{Z}/2$.

4.4 Higher spin algebras and superalgebras and their deformations

We shall adopt the definition of the higher spin $AdS_{(d+1)}/CFT_d$ algebra as the quotient of the enveloping algebra $\mathscr{U}(SO(d,2))$ of $SO(d,2)$ by the Joseph ideal $\mathscr{J}(SO(d,2))$ and denote it as $HS(d,2)$ [57]:

$$HS(d,2) = \frac{\mathscr{U}(SO(d,2))}{\mathscr{J}(SO(d,2))} \quad (4.58)$$

We shall however extend it to define deformed higher spin algebras as the enveloping algebras of the deformations of the minreps of the corresponding $AdS_{d+1}/Conf_d$ algebras. For these deformed higher spin algebras the corresponding deformations of the Joseph ideal vanish identically as operators in the quasiconformal realization as we showed explicitly above for the conformal group in four dimensions. We expect a given deformed higher spin algebra to be the *unique* infinite dimensional quotient of the universal enveloping algebra of an appropriate covering[12] of the conformal group by the deformed ideal as was shown for the undeformed minrep in [68]. Similarly, we define the higher spin superalgebras and their deformations as the enveloping algebras of the minimal unitary realizations of the underlying superalgebras and their deformations, respectively[13].

In four dimensions ($d = 4$) we have a one parameter family of higher spin algebras

[12] In $d = 4$ deformed minreps describe massless fields with the helicity $\frac{\zeta}{2}$. For non-integer values of ζ one has to go to an infinite covering of the $4d$ conformal group.

[13] We should note that the universal enveloping algebra of a Lie group as defined in the mathematics literature is an associative algebra with unit element. Under the commutator product inherited from the underlying Lie algebra it becomes a Lie algebra.

labelled by the helicity $\zeta/2$:

$$HS(4,2;\zeta) = \frac{\mathscr{U}(SO(4,2))}{\mathscr{J}_\zeta(SO(4,2))} \tag{4.59}$$

where $\mathscr{J}_\zeta(SO(4,2))$ denotes the deformed Joseph ideal of $SO(4,2)$ defined in section 4.3[14]. On the AdS_{d+1} side the generators of the higher spin algebras correspond to higher spin gauge fields, while on the $Conf_d$ side they are related to conserved tensors including the conserved stress-energy tensor. The charges associated with the generators of conformal algebra $SO(d,2)$ are defined by conserved currents constructed by contracting the stress-energy tensor with conformal Killing vectors. Similarly, the higher conserved currents are obtained by contracting the conformal Killing tensors with the stress-energy tensor. These higher conformal Killing tensors are obtained simply by tensoring the conformal Killing vectors with themselves. Though implicit in previous work on the subject [70, 71], explicit use of the language of Killing tensors in describing higher spin algebras seems to have first appeared in the paper of Mikhailov [56]. This connection was put on a rigorous foundation by Eastwood in his study of the higher symmetries of the Laplacian [57] who showed that the undeformed higher spin algebra can be obtained as the quotient of the enveloping algebra of $SO(d,2)$ generated by the conformal Killing tensors quotiented by the Joseph ideal.

The connection between the doubleton realization of $SO(4,2)$ in terms of covariant twistorial oscillators and the corresponding conformal Killing tensors was also studied by Mikhailov [56]. He pointed out that the higher conformal Killing tensors correspond to the products of bilinears of oscillators that generate $SO(4,2)$ in the doubleton realization of [24–26], which are elements of the enveloping algebra.

The supersymmetric extension of the higher spin algebras $HS(4,2;\zeta)$ is given by the enveloping algebra of the deformed minimal unitary realization of the N-extended conformal superalgebras $SU(2,2|N)_\zeta$ with the even subalgebras $SU(2,2) \oplus U(N)$. We shall denote the resulting higher spin algebra as $HS[SU(2,2|N);\zeta]$. These supersymmetric extensions involve odd powers of the deformed twistorial oscillators and the identities that define the Joseph ideal get extended to a supermultiplet of identities obtained by the repeated actions of Q and S supersymmetry generators on the generators of Joseph ideal. On the conformal side the odd generators correspond to the products of the conformal Killing spinors with conformal Killing vectors and tensors. If we denote the resulting deformed super-ideal as

[14] After the main results of this paper was announced at the GGI Conference on higher spin theories in May 2013, E. Skvortsov brought to our attention his work with Boulanger [55] where they studied possible deformations of purely bosonic higher spin algebras in arbitrary dimensions under certain restrictions. In a subsequent work [69] it was pointed out that there is a one parameter family of deformations in AdS_5 of the type studied in [55] and that the higher spin algebra is unique in $d = 4$ and $d \geq 7$ under their assumptions. They also imply that the one parameter family of deformations of [55] must be the same as one parameter family discussed in this paper which is based on earlier work [21] that they cite. However we do not agree with this claim. The results of [55] are based on Young tableaux analysis of gauge fields in AdS_5 whose labels are discrete and hence the deformation parameter in their analysis takes on certain specific set of values for unitary theories and can not be arbitrary in contrast to our deformation parameter which is $4d$ helicity which can take on arbitrary real values. Furthermore, we find a discrete infinite family of higher spin algebras and superalgebras in AdS_7 .

$\mathscr{J}_\zeta[SU(2,2|N)]$ we can formally write

$$HS[SU(2,2|N);\zeta] = \frac{\mathscr{U}(SU(2,2|N)}{\mathscr{J}_\zeta[SU(2,2|N)]} \tag{4.60}$$

The situation is much simpler for $AdS_4/Conf_3$ higher spin algebras. There are only two minreps of $SO(3,2)$, namely the scalar and spinor singletons. The higher spin algebra $HS(3,2)$ is simply given by the enveloping algebra of the singletonic realization of $Sp(4,\mathbb{R})$ [39, 50]. Singletonic realization of the Lie algebra of $Sp(4,\mathbb{R})$ describes both the scalar and spinor singletons. They form a single irreducible supermultiplet of $OSp(1|4,\mathbb{R})$ generated by taking the twistorial oscillators as the odd generators [50]. The odd generators correspond to conformal Killing spinors of $Sp(4,\mathbb{R})$. Its enveloping algebra leads to the higher spin algebra of Fradkin-Vasiliev type involving all integer and half integer spin fields in AdS_4. One can also construct N-extended higher spin superalgebras in AdS_4 as enveloping algebras of the singletonic realization of $OSp(N|4,\mathbb{R})$ [39, 50].

5 Discussion

The existence of a one-parameter family of $AdS_5/Conf_4$ higher spin algebras and superalgebras raises the question as to their physical meaning. Before discussing the situation in four dimensions, let us summarize what is known for $AdS_4/Conf_3$ higher spin algebras. In $3d$, there is no deformation of the higher spin algebra except for the super extension corresponding to $Sp(4;\mathbb{R}) \to OSp(N|4;\mathbb{R})$. For the bosonic AdS_4 higher spin algebras one finds that the higher spin theories of Vasiliev are dual to certain conformally invariant vector-scalar/spinor models in $3d$ [45] in the large N limit. More recently, Maldacena and Zhiboedov studied the constraints imposed on a conformal field theory in $d = 3$ three dimensions by the existence of a single conserved higher spin current. They found that this implies the existence of an infinite number of conserved higher spin currents. This corresponds simply to the fact that the generators of $SO(d,2)$ get extended to an infinite spin algebra when one takes their commutators with an operator which is bilinear or higher order in the generators, except for those operators that correspond to the Casimir elements. They also showed that the correlation functions of the stress tensor and the conserved currents are those of a free field theory in three dimensions, either a theory of N free bosons or a theory of N free fermions [72], which are simply the scalar and spinor singletons.

The distinguishing feature of $3d$ is the fact that there exists only two minimal unitary representations corresponding to massless conformal fields which are simply the Di and Rac representations of Dirac. However in $4d$, we have a one-parameter, ζ, family of deformations of the minimal unitary representation of the conformal group corresponding to massless conformal fields of helicity $\zeta/2$. The same holds true for the minimal unitary supermultiplet of $SU(2,2|N)$. From M/superstring point of view the most important interacting and supersymmetric CFT in $d = 4$ is the $N = 4$ super Yang-Mills theory. It was argued by Witten [73] that the holographic dual of $\mathcal{N} = 4$ super Yang-Mills theory with gauge group $SU(N)$ at $g^2_{YM}N = 0$ for $N \to \infty$ should be a free gauge invariant theory in

AdS_5 with massless fields of arbitrarily high spin and this was supported by calculations in [56]. Moreover the scalar sector of $\mathcal{N} = 4$ super Yang-Mills theory at $g^2_{YM}N = 0$ with $N \to \infty$ should be dual to bosonic higher spin theories in AdS_5 which provides a non-trivial extension of AdS/CFT correspondence in superstring theory [31, 32] to non-supersymmetric large N field theories [15]. The Kaluza-Klein spectrum of IIB supergravity over $AdS_5 \times S^5$ was first obtained by tensoring of the minimal unitary supermultiplet (scalar doubleton) of $SU(2,2|4)$ with itself repeatedly and restricting to CPT self-conjugate sector [24]. The massless graviton supermultiplet in AdS_5 sits at the bottom of this infinite tower. In fact all the unitary representations corresponding to massless fields in AdS_5 can be obtained by tensoring of two doubleton representations of $SU(2,2)$ which describe massless conformal fields on the boundary of AdS_5 [24–26, 50]. As was argued by Mikhailov [56], in the large N limit the correlation functions in the CFT side become products of two point functions which correspond to products of two doubletons. As such they correspond to massless fields in the AdS_5 bulk. At the level of correlation functions the same arguments suggest that corresponding to a one parameter family of deformations of the N=4 Yang-Mills supermultiplet there must exists a family of supersymmetric massless higher spin theories in AdS_5. Turning on the gauge coupling constant on the Yang-Mills side leads to interactions in the bulk and most of the higher spin fields become massive.

The fact that the quasiconformal realization of the minrep of $SU(2,2|4)$ is nonlinear implies that the corresponding higher spin theory in the bulk must be interacting. Since the same minrep can also be obtained by using the doubletonic realization [24], which corresponds to free field realization, suggests that the interacting supersymmetric higher spin theory may be integrable. There are other deformed higher spin algebras corresponding to non-CPT self-conjugate supermultiplets of $SU(2,2|4)$ that contain scalar fields and are deformations of the minrep. The above arguments suggest that they too should correspond to interacting but integrable supersymmetric higher spin theories in AdS_5. One solid piece of the evidence for this is provided by the fact that the symmetry superalgebras of interacting (nonlinear) superconformal quantum mechanical models of [75] furnish a one parameter family of deformations of the minimal unitary representation of the $N = 4$ superconformal algebra $D(2,1;\alpha)$ in one dimension. This was predicted in [76] and shown explicitly in [77].

Most of the work on higher spin algebras until now have utilized the realizations of underlying Lie (super)algebras as bilinears of oscillators which correspond to free field realizations. The quasiconformal approach allows one to give a natural definition of super Joseph ideal and leads directly to the interacting realizations of the superextensions of higher spin algebras. The next step in this approach is to reformulate these interacting quasiconformal realizations in terms of covariant gauge fields and construct Vasiliev type nonlinear theories of interacting higher spins in AdS_5.

Another application of our results will be to reformulate the spin chain models associated with $N = 4$ super Yang-Mills theory in terms of deformed twistorial oscillators and

[15] We should note that it was conjectured that the scalar sector of free N=4 super Yang-Mills theory in $d = 4$ should be dual to a theory of interacting higher spins in the bulk of AdS_5 [51, 74].

study the integrability of corresponding spin chains non-perturbatively. In fact, a spectral parameter related to helicity and central charge was introduced recently for scattering amplitudes in $N = 4$ super Yang-Mills theory [78]. This spectral parameter corresponds to our deformation parameter which is helicity and appears as a central charge in the quasi-conformal realization of the super algebra $SU(2,2|4)$. We hope to address these issues in future investigations.

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Appendices

A Spinor conventions for $SO(2,1)$

We follow [79] for the spinor conventions in $d = 3$ and thus all the $3d$ spinors are Majorana with $\eta_{\mu\nu} = \text{diag}(-,+,+)$. The gamma-matrices in Majorana representation terms of the Pauli matrices σ^i are as follows:

$$\gamma^0 = -i\sigma^2, \quad \gamma^1 = \sigma^1, \quad \gamma^2 = \sigma^3 \tag{A.1}$$

and they satisfy

$$\{\gamma^\mu, \gamma^\nu\}^\alpha{}_\beta = 2\eta^{\mu\nu}\delta^\alpha{}_\beta \quad . \tag{A.2}$$

Thus the matrices $(\gamma^\mu)^\alpha{}_\beta$ are real and the Majorana condition on spinors imply that they are real two component spinors. Spinor indices are raised/lowered by the epsilon symbols with $\epsilon^{12} = \epsilon_{12} = 1$ and choosing NW-SE conventions

$$\epsilon^{\alpha\gamma}\epsilon_{\beta\gamma} = \delta^\alpha_\beta \;, \quad \lambda^\alpha := \epsilon^{\alpha\beta}\lambda_\beta \Leftrightarrow \lambda_\beta = \lambda^\alpha\epsilon_{\alpha\beta} \; . \tag{A.3}$$

Introducing the real symmetric matrices $(\sigma^\mu)_{\alpha\beta} := (\gamma^\mu)^\rho{}_\beta \; \epsilon_{\rho\alpha}$ and $(\bar\sigma^\mu)^{\alpha\beta} := (\epsilon \cdot \sigma^\mu \cdot \epsilon)^{\alpha\beta} = -\epsilon^{\beta\rho} \; (\gamma^\mu)^\alpha{}_\rho$, a three vector in spinor notation writes as a symmetric real matrix as

$$V_{\alpha\beta} := (\sigma^\mu V_\mu)_{\alpha\beta} \quad \Rightarrow \quad V^\mu = \frac{1}{2}\,(\bar\sigma^\mu)^{\alpha\beta}\,V_{\alpha\beta} \; . \tag{A.4}$$

B The quasiconformal realization of the minimal unitary representation of $SO(4,2)$ in compact three-grading

In this appendix we provide the formulas for the quasiconformal realization of generators of $SO(4,2)$ in compact 3-grading following [21]. Consider the compact three graded decomposition of the Lie algebra of $SU(2,2)$ determined by the conformal Hamiltonian

$$\mathfrak{so}(4,2) = \mathfrak{C}^- \oplus \mathfrak{C}^0 \oplus \mathfrak{C}^+$$

where $\mathfrak{C}^0 = \mathfrak{so}(4) \oplus \mathfrak{so}(2)$. Folllowing [21] we shall label the generators in $\mathfrak{C}^\pm$ and $\mathfrak{C}^0$ subspaces as follows:

$$(B_1, B_2, B_3, B_4) \in \mathfrak{C}^- \tag{B.1}$$

$$L_{\pm,0} \oplus H \oplus R_{\pm,0} \in \mathfrak{C}^0 \tag{B.2}$$

$$(B^1, B^2, B^3, B^4) \in \mathfrak{C}^+ \tag{B.3}$$

where $L_{\pm,0}$ and $R_{\pm,0}$ denote the generators of $SU(2)_L \times SU(2)_R$ and H is the $U(1)$ generator.

The generators of $\mathfrak{so}(4,2)$ in the compact 3-grading take on very simple forms when expressed in terms of the singular oscillators introduced in section 3.2:

$$H = \frac{1}{2}\left(A^\dagger_{\mathcal{L}+1}A_{\mathcal{L}+1} + \mathcal{L} + \frac{1}{2}(N_d + N_g) + \frac{5}{2}\right) \tag{B.4}$$

$$L_+ = -\frac{i}{2}A_{\mathcal{L}}d^\dagger, \quad L_- = \frac{i}{2}dA^\dagger_{\mathcal{L}}, \quad L_3 = N_d - \frac{1}{2}(H-1) \tag{B.5}$$

$$R_+ = \frac{i}{2}g^\dagger A_{-\mathcal{L}}, \quad R_- = -\frac{i}{2}A^\dagger_{-\mathcal{L}}g, \quad R_3 = N_g - \frac{1}{2}(H+1) \tag{B.6}$$

$$B_1 = -iA_{\mathcal{L}}A_{-\mathcal{L}}, \quad B_2 = -i\sqrt{2}dA_{-\mathcal{L}}, \quad B_3 = -i\sqrt{2}A_{\mathcal{L}}g, \quad B_4 = -2igd \tag{B.7}$$

$$B^1 = iA^\dagger_{-\mathcal{L}}A^\dagger_{\mathcal{L}}, \quad B^2 = i\sqrt{2}A^\dagger_{-\mathcal{L}}d^\dagger, \quad B^3 = i\sqrt{2}g^\dagger A^\dagger_{\mathcal{L}}, \quad B^4 = 2ig^\dagger d^\dagger \tag{B.8}$$

C Commutation relations of deformed twistorial oscillators

$$\left[Y^1, \widetilde{Y}^1\right] = \frac{1}{2x}(Y^1 - \widetilde{Y}^1), \qquad \left[Y^2, \widetilde{Y}^2\right] = -\frac{1}{2x}(Y^2 - \widetilde{Y}^2) \tag{C.1}$$

$$\left[Y^1, Y^2\right] = \frac{1}{2x}(Y^1 + Y^2), \qquad \left[\widetilde{Y}^1, \widetilde{Y}^2\right] = \frac{1}{2x}(\widetilde{Y}^1 + \widetilde{Y}^2) \tag{C.2}$$

$$\left[Y^1, \widetilde{Y}^2\right] = -\frac{1}{2x}(Y^1 + \widetilde{Y}^2), \qquad \left[\widetilde{Y}^1, Y^2\right] = -\frac{1}{2x}(\widetilde{Y}^1 + Y^2) \tag{C.3}$$

$$\left[Z_1\,,\,\widetilde{Z}_1\right] = -\frac{1}{2x}(Z_1 + \widetilde{Z}_1), \qquad \left[Z_2\,,\,\widetilde{Z}_2\right] = -\frac{1}{2x}(Z_2 + \widetilde{Z}_2) \tag{C.4}$$

$$[Z_1\,,\,Z_2] = -\frac{1}{2x}(Z_1 - Z_2), \qquad \left[\widetilde{Z}_1\,,\,\widetilde{Z}_2\right] = \frac{1}{2x}(\widetilde{Z}_1 - \widetilde{Z}_2) \tag{C.5}$$

$$\left[Z_1\,,\,\widetilde{Z}_2\right] = -\frac{1}{2x}(Z_1 + \widetilde{Z}_2), \qquad \left[\widetilde{Z}_1\,,\,Z_2\right] = \frac{1}{2x}(\widetilde{Z}_1 + Z_2) \tag{C.6}$$

$$\left[Y^1\,,\,Z_1\right] = 2 + \frac{1}{2x}(Y^1 - Z_1), \qquad \left[\widetilde{Y}^1\,,\,\widetilde{Z}_1\right] = 2 - \frac{1}{2x}(\widetilde{Y}^1 + \widetilde{Z}_1) \tag{C.7}$$

$$\left[Y^1\,,\,Z_2\right] = \frac{1}{2x}(Y^1 - Z_2), \qquad \left[\widetilde{Y}^1\,,\,\widetilde{Z}_2\right] = -\frac{1}{2x}(\widetilde{Y}^1 + \widetilde{Z}_2) \tag{C.8}$$

$$\left[Y^1\,,\,\widetilde{Z}_1\right] = \frac{1}{2x}(Y^1 + \widetilde{Z}_1), \qquad \left[\widetilde{Y}^1\,,\,Z_1\right] = -\frac{1}{2x}(\widetilde{Y}^1 - Z_1) \tag{C.9}$$

$$\left[Y^1\,,\,\widetilde{Z}_2\right] = \frac{1}{2x}(Y^1 + \widetilde{Z}_2), \qquad \left[\widetilde{Y}^1\,,\,Z_2\right] = -\frac{1}{2x}(\widetilde{Y}^1 - Z_2) \tag{C.10}$$

$$\left[Y^2\,,\,Z_1\right] = \frac{1}{2x}(Y^2 + Z_1), \qquad \left[\widetilde{Y}^2\,,\,\widetilde{Z}_1\right] = -\frac{1}{2x}(\widetilde{Y}^2 - \widetilde{Z}_1) \tag{C.11}$$

$$\left[Y^2\,,\,Z_2\right] = 2 + \frac{1}{2x}(Y^2 + Z_2), \qquad \left[\widetilde{Y}^2\,,\,\widetilde{Z}_2\right] = 2 - \frac{1}{2x}(\widetilde{Y}^2 - \widetilde{Z}_2) \tag{C.12}$$

$$\left[Y^2\,,\,\widetilde{Z}_1\right] = \frac{1}{2x}(Y^2 - \widetilde{Z}_1), \qquad \left[\widetilde{Y}^2\,,\,Z_1\right] = -\frac{1}{2x}(\widetilde{Y}^2 + Z_1) \tag{C.13}$$

$$\left[Y^2\,,\,\widetilde{Z}_2\right] = \frac{1}{2x}(Y^2 - \widetilde{Z}_2), \qquad \left[\widetilde{Y}^2\,,\,Z_2\right] = -\frac{1}{2x}(\widetilde{Y}^2 + Z_2) \tag{C.14}$$

D The quasiconformal realization of the minimal unitary representation of $SU(2,2|4)$ in compact three-grading

The Lie super algebra $\mathfrak{su}(2,2|4)$ can be given a three-graded decomposition with respect to its compact subalgebra $\mathfrak{su}(2|2) \oplus \mathfrak{su}(2|2) \oplus \mathfrak{u}(1)$

$$\mathfrak{su}(2,2|4) = \mathfrak{C}^- \oplus \mathfrak{C}^0 \oplus \mathfrak{C}^+ \tag{D.1}$$

We shall label the generators in various graded subspaces as follows

$$\mathfrak{C}^0 = [\mathcal{H} \oplus L_{\pm,0} \oplus R_{\pm,0} \oplus \mathcal{A}^r_s \oplus \mathcal{B}^y_z \oplus \mathcal{U}]_B \oplus [\widetilde{\mathfrak{S}}_r \oplus \widetilde{\mathfrak{S}}^r \oplus \widetilde{\mathfrak{Q}}_r \oplus \widetilde{\mathfrak{Q}}^r \oplus \widetilde{\mathfrak{S}}_y \oplus \widetilde{\mathfrak{S}}^y \oplus \widetilde{\mathfrak{Q}}_y \oplus \widetilde{\mathfrak{Q}}^y]_F \tag{D.2}$$

$$\mathfrak{C}^- = [B_i \oplus \mathcal{C}_{rz}]_B \oplus [\mathfrak{S}_r \oplus \mathfrak{S}_y \oplus \mathfrak{Q}_r \oplus \mathfrak{Q}_y]_F, \qquad (i = 1,2,3,4) \tag{D.3}$$

$$\mathfrak{C}^+ = [B^i \oplus \mathcal{C}^{zr}]_B \oplus [\mathfrak{S}^r \oplus \mathfrak{S}^y \oplus \mathfrak{Q}^r \oplus \mathfrak{Q}^y]_F \tag{D.4}$$

where the subscripts B and F indicate bosonic (even) and fermionic (odd) generators. The generators of the R-symmetry group $SU(4)$ are realized as bilinears of two pairs of fermionic oscillators α_r, α^s $(r, s = 1, 2)$ and β_y, β^z $(y, z = 1, 2)$ that satisfy

$$\{\alpha_r, \alpha^s\} = \delta_r^s \tag{D.5}$$

$$\{\beta_x, \beta^y\} = \delta_x^y \tag{D.6}$$

with $N_\alpha = \alpha^r \alpha_r$ and $N_\beta = \beta^y \beta_y$ the corresponding number operators respectively. The Lie algebra $\mathfrak{su}(4)$ has a 3-graded decomposition with respect to its subalgebra $\mathfrak{su}(2)_{\mathcal{A}} \oplus \mathfrak{su}(2)_{\mathcal{B}} \oplus \mathfrak{u}(1)$

$$\mathfrak{su}(4) \supseteq \mathfrak{su}(2)_{\mathcal{A}} \oplus \mathfrak{su}(2)_{\mathcal{B}} \oplus \mathfrak{u}(1) \tag{D.7}$$

The $\mathfrak{su}(2)_{\mathcal{A}}$ and $\mathfrak{su}(2)_{\mathcal{B}}$ generators are given by:

$$\mathcal{A}_s^r = \alpha^r \alpha_s - \frac{1}{2}\delta_s^r N_\alpha, \qquad \mathcal{B}_z^y = \beta^y \beta_z - \frac{1}{2}\delta_z^y N_\beta \tag{D.8}$$

which satisfy:

$$\left[\mathcal{A}_s^r \,,\, \mathcal{A}_u^t\right] = \delta_s^t \mathcal{A}_u^r - \delta_u^r \mathcal{A}_s^t, \qquad \left[\mathcal{B}_x^w \,,\, \mathcal{B}_z^y\right] = \delta_x^y \mathcal{B}_z^w - \delta_z^w \mathcal{B}_x^y \tag{D.9}$$

The remaining $\mathfrak{su}(4)$ generators are given as follows:

$$\mathcal{C}_{ry} = \alpha_r \beta_y, \qquad \mathcal{C}^{yr} = (\mathcal{C}_{ry})^\dagger = \beta^y \alpha^r \tag{D.10}$$

which satisfy the following commutation relations:

$$\left[\mathcal{C}_{ry} \,,\, \mathcal{C}^{zs}\right] = -\delta_y^z \mathcal{A}_r^s - \delta_r^s \mathcal{B}_y^z - \delta_r^s \delta_y^z \mathcal{U} \tag{D.11}$$

where $\mathcal{U} = \frac{1}{2}(N_\alpha + N_\beta) - 1$ is the $\mathfrak{u}(1)$ generator that determines the 3-graded decomposition of $\mathfrak{su}(4)$.

The $\mathfrak{u}(1)$ generator in $\mathfrak{C}^0$ that defines the 3-grading of $\mathfrak{su}(2,2|4)$ is given as follows:

$$\mathcal{H} = A^\dagger_{\mathcal{L}_\zeta^s + 1} A_{\mathcal{L}_\zeta^s + 1} + \mathcal{L}_\zeta^s + \frac{1}{2}(N_d + N_g + N_\alpha + N_\beta) + \frac{3}{2} \tag{D.12}$$

where the deformed singular oscillators are:

$$A_{\mathcal{L}_\zeta^s} = a - \frac{\mathcal{L}_\zeta^s}{\sqrt{2}x}, \qquad A^\dagger_{\mathcal{L}_\zeta^s} = a^\dagger - \frac{\mathcal{L}_\zeta^s}{\sqrt{2}x} \tag{D.13}$$

with the deformation "parameter" (operator) $\mathcal{L}_\zeta^s$ given by:

$$\mathcal{L}_\zeta^s = N_d - N_g + N_\alpha - N_\beta + \zeta - \frac{1}{2} \tag{D.14}$$

The $SU(2)_L$ and $SU(2)_R$ generators in grade zero subspace take the form:

$$L_+ = -\frac{i}{2} A_{\mathcal{L}^s_\zeta} d^\dagger, \quad L_- = \frac{i}{2} d A^\dagger_{\mathcal{L}^s_\zeta}, \quad L_0 = N_d - \frac{1}{2}\left(H - \frac{1}{2}(N_\alpha - N_\beta + \zeta) - 1\right) \tag{D.15}$$

$$R_+ = \frac{i}{2} g^\dagger A_{-\mathcal{L}^s_\zeta}, \quad R_- = -\frac{i}{2} A^\dagger_{-\mathcal{L}^s_\zeta} g, \quad R_0 = N_g - \frac{1}{2}\left(H + \frac{1}{2}(N_\alpha - N_\beta + \zeta) + 1\right) \tag{D.16}$$

where $H = A^\dagger_{\mathcal{L}^s_\zeta+1} A_{\mathcal{L}^s_\zeta+1} + \mathcal{L}^s_\zeta + \frac{1}{2}(N_d + N_g + 1) + \frac{3}{2}$. They satisfy the commutation relations

$$\begin{aligned} &[L_+\,,\,L_-] = L_0 \qquad && [L_0\,,\,L_\pm] = \pm\,L_\pm \\ &[R_+\,,\,R_-] = R_0 \qquad && [R_0\,,\,R_\pm] = \pm\,R_\pm\,. \end{aligned} \tag{D.17}$$

The supersymmetry generators in $\mathfrak{C}^0$ are given below:

$$\widetilde{\mathfrak{Q}}_r = \alpha_r d^\dagger, \qquad \widetilde{\mathfrak{Q}}^r = \alpha^r d, \qquad \widetilde{\mathfrak{Q}}_y = \beta_y g^\dagger, \qquad \widetilde{\mathfrak{Q}}^y = \beta^y g \tag{D.18}$$

$$\widetilde{\mathfrak{S}}_r = \frac{1}{2}\alpha_r A^\dagger_{\mathcal{L}^s_\zeta}, \qquad \widetilde{\mathfrak{S}}^r = \frac{1}{2} A_{\mathcal{L}^s_\zeta}\alpha^r, \qquad \widetilde{\mathfrak{S}}_y = \frac{1}{2} A^\dagger_{-\mathcal{L}^s_\zeta}\beta_y, \qquad \widetilde{\mathfrak{S}}^y = \frac{1}{2}\beta^y A_{-\mathcal{L}^s_\zeta} \tag{D.19}$$

They close into the generators of $SU(2)_L$ and $SU(2)_R$ and the generators of the $SU(2)_\mathcal{A} \times SU(2)_\mathcal{B} \times U(1)$ subgroup of the R-symmetry group $SU(4)$. The bosonic generators in $\mathfrak{C}^-$ are given by:

$$B_1 = -iA_{\mathcal{L}^s_\zeta} A_{-\mathcal{L}^s_\zeta}, \quad B_2 = -i\sqrt{2} d A_{-\mathcal{L}^s_\zeta}, \quad B_3 = -i\sqrt{2} A_{\mathcal{L}^s_\zeta} g, \quad B_4 = -2igd \tag{D.20}$$

and the fermionic generators in $\mathfrak{C}^-$ are:

$$\mathfrak{Q}_r = \alpha_r g, \quad \mathfrak{Q}_y = \beta_y d, \quad \mathfrak{S}_r = \frac{1}{2}\alpha_r A_{-\mathcal{L}^s_\zeta}, \quad \mathfrak{S}_y = \frac{1}{2} A_{\mathcal{L}^s_\zeta}\beta_y \tag{D.21}$$

The bosonic generators in $\mathfrak{C}^+$ are:

$$B^1 = iA^\dagger_{-\mathcal{L}^s_\zeta} A^\dagger_{\mathcal{L}^s_\zeta}, \quad B^2 = i\sqrt{2} A^\dagger_{-\mathcal{L}^s_\zeta} d^\dagger, \quad B^3 = i\sqrt{2} g^\dagger A^\dagger_{\mathcal{L}^s_\zeta}, \quad B^4 = 2ig^\dagger d^\dagger \tag{D.22}$$

and the fermionic generators in $\mathfrak{C}^+$ are given by:

$$\mathfrak{Q}^r = \alpha^r g^\dagger, \quad \mathfrak{Q}^y = \beta^y d^\dagger, \quad \mathfrak{S}^r = \frac{1}{2} A^\dagger_{-\mathcal{L}^s_\zeta}\alpha^r, \quad \mathfrak{S}^y = \frac{1}{2}\beta^y A^\dagger_{\mathcal{L}^s_\zeta}, \tag{D.23}$$

They satisfy the following commutation relations:

$$\begin{aligned} &[\mathcal{H}\,,\,B_i] = -\,B_i \qquad [\mathcal{H}\,,\,B^i] = +\,B^i \qquad \text{where} \quad i = 1,2,3,4 \\ &\left[B_1\,,\,B^1\right] = 8\,H_\odot \qquad \left[B_2\,,\,B^2\right] = 4\,(H + L_3 - R_3) \\ &\left[B_3\,,\,B^3\right] = 4\,(H - L_3 + R_3) \qquad \left[B_4\,,\,B^4\right] = 8\,(H_d + H_g) \end{aligned} \tag{D.24}$$

The supersymmetry generators in $\mathfrak{C}^{(\pm)}$ satisfy the following (anti-)commutation relations:

$$\begin{array}{llll} [\mathcal{H}\,,\,\mathfrak{S}_\mu] = -\,\mathfrak{S}_\mu & [\mathcal{H}\,,\,\mathfrak{Q}_\mu] = -\,\mathfrak{Q}_\mu & [\mathcal{H}\,,\,\mathfrak{S}_y] = -\,\mathfrak{S}_y & [\mathcal{H}\,,\,\mathfrak{Q}_y] = -\,\mathfrak{Q}_y \\ [\mathcal{H}\,,\,\mathfrak{S}^\mu] = +\,\mathfrak{S}^\mu & [\mathcal{H}\,,\,\mathfrak{Q}^\mu] = +\,\mathfrak{Q}^\mu & [\mathcal{H}\,,\,\mathfrak{S}^y] = +\,\mathfrak{S}^y & [\mathcal{H}\,,\,\mathfrak{Q}^y] = +\,\mathfrak{Q}^y \end{array} \tag{D.25}$$

$$\begin{aligned} \{\mathfrak{S}_r\,,\,\mathfrak{S}^s\} &= \delta_r^s\,(H + L_3 - R_3) - \mathcal{A}_r^s - \delta_r^s\left(N_d + \frac{1}{\mathfrak{p}}N_\alpha\right) \\ \{\mathfrak{Q}_r\,,\,\mathfrak{Q}^s\} &= \delta_r^s\,(H_d + H_g) - \mathcal{A}_r^s - \delta_r^s\left(N_d + \frac{1}{\mathfrak{p}}N_\alpha\right) \\ \{\mathfrak{S}_y\,,\,\mathfrak{S}^z\} &= \delta_y^z\,(H - L_3 + R_3) - \mathcal{B}_y^z - \delta_y^z\left(N_g + \frac{1}{\mathfrak{q}}N_\beta\right) \\ \{\mathfrak{Q}_y\,,\,\mathfrak{Q}^z\} &= \delta_\mu^\nu\,(H_d + H_g) - \mathcal{B}_y^z - \delta_y^z\left(N_g + \frac{1}{\mathfrak{q}}N_\beta\right) \end{aligned} \tag{D.26}$$

The anti-commutators between the supersymmetry generators in $\mathfrak{C}^-$ and those in $\mathfrak{C}^+$ take the following form:

$$\begin{aligned} \left\{\widetilde{\mathfrak{S}}_r\,,\,\widetilde{\mathfrak{S}}^s\right\} &= -2\,\delta_r^s\,L_3 + A_r^s + \delta_r^s\left(N_d + \frac{1}{\mathfrak{p}}N_\alpha\right) \\ \left\{\widetilde{\mathfrak{Q}}_r\,,\,\widetilde{\mathfrak{Q}}^s\right\} &= A_r^s + \delta_r^s\left(N_d + \frac{1}{\mathfrak{p}}N_\alpha\right) \\ \left\{\widetilde{\mathfrak{S}}_y\,,\,\widetilde{\mathfrak{S}}^z\right\} &= -2\,\delta_y^z\,R_3 + B_y^z + \delta_y^z\left(N_g + \frac{1}{\mathfrak{q}}N_\beta\right) \\ \left\{\widetilde{\mathfrak{Q}}_y\,,\,\widetilde{\mathfrak{Q}}^z\right\} &= B_y^z + \delta_y^z\left(N_g + \frac{1}{\mathfrak{q}}N_\beta\right) \end{aligned} \tag{D.27}$$

The commutators between bosonic generators in $\mathfrak{C}^-$ and supersymmetry generators in $\mathfrak{C}^+$ are as follows:

$$\begin{array}{llll} [B_1\,,\,\mathfrak{S}^r] = -2i\,\widetilde{\mathfrak{S}}^r & [B_2\,,\,\mathfrak{S}^r] = -2i\,\widetilde{\mathfrak{Q}}^r & [B_3\,,\,\mathfrak{S}^r] = 0 & [B_4\,,\,\mathfrak{S}^r] = 0 \\ [B_1\,,\,\mathfrak{Q}^r] = 0 & [B_2\,,\,\mathfrak{Q}^r] = 0 & [B_3\,,\,\mathfrak{Q}^r] = -2i\,\widetilde{\mathfrak{S}}^r & [B_4\,,\,\mathfrak{Q}^r] = -2i\,\widetilde{\mathfrak{Q}}^r \\ [B_1\,,\,\mathfrak{S}^y] = -2i\,\widetilde{\mathfrak{S}}^y & [B_2\,,\,\mathfrak{S}^y] = 0 & [B_3\,,\,\mathfrak{S}^y] = -2i\,\widetilde{\mathfrak{Q}}^y & [B_4\,,\,\mathfrak{S}^y] = 0 \\ [B_1\,,\,\mathfrak{Q}^y] = 0 & [B_2\,,\,\mathfrak{Q}^y] = -2i\,\widetilde{\mathfrak{S}}^y & [B_3\,,\,\mathfrak{Q}^y] = 0 & [B_4\,,\,\mathfrak{Q}^y] = -2i\,\widetilde{\mathfrak{Q}}^y \end{array} \tag{D.28}$$

The anticommutators of supersymmetry generators in $\mathfrak{C}^0$ and those in $\mathfrak{C}^+$ are

$$\begin{array}{llll}
\left\{\widetilde{\mathfrak{S}}^r\,,\,\mathfrak{S}^s\right\}=0 & \left\{\widetilde{\mathfrak{S}}^r\,,\,\mathfrak{Q}^s\right\}=0 & \left\{\widetilde{\mathfrak{S}}^r\,,\,\mathfrak{S}^z\right\}=-\mathcal{C}^{zr} & \left\{\widetilde{\mathfrak{S}}^r\,,\,\mathfrak{Q}^z\right\}=0 \\
\left\{\widetilde{\mathfrak{Q}}^r\,,\,\mathfrak{S}^s\right\}=0 & \left\{\widetilde{\mathfrak{Q}}^r\,,\,\mathfrak{Q}^s\right\}=0 & \left\{\widetilde{\mathfrak{Q}}^r\,,\,\mathfrak{S}^z\right\}=0 & \left\{\widetilde{\mathfrak{Q}}^r\,,\,\mathfrak{Q}^z\right\}=-\mathcal{C}^{zr} \\
\left\{\widetilde{\mathfrak{S}}^y\,,\,\mathfrak{S}^r\right\}=\mathcal{C}^{yr} & \left\{\widetilde{\mathfrak{S}}^y\,,\,\mathfrak{Q}^r\right\}=0 & \left\{\widetilde{\mathfrak{S}}^y\,,\,\mathfrak{S}^z\right\}=0 & \left\{\widetilde{\mathfrak{S}}^y\,,\,\mathfrak{Q}^z\right\}=0 \\
\left\{\widetilde{\mathfrak{Q}}^y\,,\,\mathfrak{S}^r\right\}=0 & \left\{\widetilde{\mathfrak{Q}}^y\,,\,\mathfrak{Q}^r\right\}=\mathcal{C}^{yr} & \left\{\widetilde{\mathfrak{Q}}^y\,,\,\mathfrak{S}^z\right\}=0 & \left\{\widetilde{\mathfrak{Q}}^y\,,\,\mathfrak{Q}^z\right\}=0
\end{array} \tag{D.29}$$

where $\mathcal{C}^{zr}=\beta^z\alpha^r$ are the $\mathfrak{su}(4)$ generators that belong to the $\mathfrak{C}^+$ subspace.

The Q and S supersymmetry generators given in equations 3.65 take the following form in terms of the singular twistors and fermionic oscillators α and β:

$$Q^1{}_\alpha = Z_\alpha \alpha^1, \qquad \bar{Q}_{1\dot\alpha} = -\alpha_1 \widetilde{Z}_{\dot\alpha} \tag{D.30}$$
$$Q^2{}_\alpha = Z_\alpha \alpha^2, \qquad \bar{Q}_{2\dot\alpha} = -\alpha_2 \widetilde{Z}_{\dot\alpha} \tag{D.31}$$
$$Q^3{}_\alpha = Z_\alpha \beta_1, \qquad \bar{Q}_{3\dot\alpha} = -\beta^1 \widetilde{Z}_{\dot\alpha} \tag{D.32}$$
$$Q^4{}_\alpha = Z_\alpha \beta_2, \qquad \bar{Q}_{4\dot\alpha} = -\beta^2 \widetilde{Z}_{\dot\alpha} \tag{D.33}$$

$$S_1{}^\alpha = -\alpha_1 Y^\alpha, \qquad \bar{S}^{1\dot\alpha} = \widetilde{Y}^{\dot\alpha}\alpha^1 \tag{D.34}$$
$$S_2{}^\alpha = -\alpha_2 Y^\alpha, \qquad \bar{S}^{2\dot\alpha} = \widetilde{Y}^{\dot\alpha}\alpha^2 \tag{D.35}$$
$$S_3{}^\alpha = -\beta^1 Y^\alpha, \qquad \bar{S}^{3\dot\alpha} = \widetilde{Y}^{\dot\alpha}\beta_1 \tag{D.36}$$
$$S_4{}^\alpha = -\beta^2 Y^\alpha, \qquad \bar{S}^{4\dot\alpha} = \widetilde{Y}^{\dot\alpha}\beta_2 \tag{D.37}$$

References

[1] A. Joseph, "Minimal realizations and spectrum generating algebras," *Comm. Math. Phys.* **36** (1974) 325–338.

[2] D. A. Vogan, Jr., "Singular unitary representations," in *Noncommutative harmonic analysis and Lie groups (Marseille, 1980)*, vol. 880 of *Lecture Notes in Math.*, pp. 506–535. Springer, Berlin, 1981.

[3] B. Kostant, "The vanishing of scalar curvature and the minimal representation of $SO(4,4)$," in *Operator algebras, unitary representations, enveloping algebras, and invariant theory (Paris, 1989)*, vol. 92 of *Progr. Math.*, pp. 85–124. Birkhäuser Boston, Boston, MA, 1990.

[4] B. Binegar and R. Zierau, "Unitarization of a singular representation of $so(p,q)$," *Communications in Mathematical Physics* **138** no. 2, (1991) 245–258.

[5] D. Kazhdan and G. Savin, "The smallest representation of simply laced groups," in *Festschrift in honor of I. I. Piatetski-Shapiro on the occasion of his sixtieth birthday, Part I (Ramat Aviv, 1989)*, vol. 2 of *Israel Math. Conf. Proc.*, pp. 209–223. Weizmann, Jerusalem, 1990.

[6] R. Brylinski and B. Kostant, "Lagrangian models of minimal representations of E_6, E_7 and

E_8," in *Functional analysis on the eve of the 21st century, Vol. 1 (New Brunswick, NJ, 1993)*, vol. 131 of *Progr. Math.*, pp. 13–63. Birkhäuser Boston, Boston, MA, 1995.

[7] R. Brylinski and B. Kostant, "Minimal representations, geometric quantization, and unitarity," *Proc. Nat. Acad. Sci. U.S.A.* **91** no. 13, (1994) 6026–6029.

[8] B. H. Gross and N. R. Wallach, "A distinguished family of unitary representations for the exceptional groups of real rank = 4," in *Lie theory and geometry*, vol. 123 of *Progr. Math.*, pp. 289–304. Birkhäuser Boston, Boston, MA, 1994.

[9] J.-S. Li, "Minimal representations & reductive dual pairs," in *Representation theory of Lie groups (Park City, UT, 1998)*, vol. 8 of *IAS/Park City Math. Ser.*, pp. 293–340. Amer. Math. Soc., Providence, RI, 2000.

[10] T. Kobayashi and B. Ørsted, "Analysis on the minimal representation of $O(p, q)$. I. Realization via conformal geometry," *Adv. Math.* **180** no. 2, (2003) 486–512.

[11] T. Kobayashi and B. Ørsted, "Analysis on the minimal representation of $O(p, q)$. II. Branching laws," *Adv. Math.* **180** no. 2, (2003) 513–550.

[12] T. Kobayashi and B. Ørsted, "Analysis on the minimal representation of $O(p, q)$. III. Ultrahyperbolic equations on $\mathbf{R}^{p-1,q-1}$," *Adv. Math.* **180** no. 2, (2003) 551–595.

[13] A. R. Gover and A. Waldron, "The so(d+2,2) Minimal Representation and Ambient Tractors: the Conformal Geometry of Momentum Space," arXiv:0903.1394 [hep-th].

[14] D. Kazhdan, B. Pioline, and A. Waldron, "Minimal representations, spherical vectors, and exceptional theta series. I," *Commun. Math. Phys.* **226** (2002) 1–40, hep-th/0107222.

[15] M. Günaydin, K. Koepsell, and H. Nicolai, "Conformal and quasiconformal realizations of exceptional Lie groups," *Commun. Math. Phys.* **221** (2001) 57–76, hep-th/0008063.

[16] M. Günaydin and O. Pavlyk, "Generalized spacetimes defined by cubic forms and the minimal unitary realizations of their quasiconformal groups," *JHEP* **08** (2005) 101, hep-th/0506010.

[17] M. Günaydin, K. Koepsell, and H. Nicolai, "The minimal unitary representation of $E_{8(8)}$," *Adv. Theor. Math. Phys.* **5** (2002) 923–946, hep-th/0109005.

[18] M. Günaydin, G. Sierra, and P. K. Townsend, "Exceptional supergravity theories and the magic square," *Phys. Lett.* **B133** (1983) 72.

[19] M. Günaydin and O. Pavlyk, "Minimal unitary realizations of exceptional U-duality groups and their subgroups as quasiconformal groups," *JHEP* **01** (2005) 019, hep-th/0409272.

[20] M. Günaydin and O. Pavlyk, "A unified approach to the minimal unitary realizations of noncompact groups and supergroups," *JHEP* **09** (2006) 050, hep-th/0604077.

[21] S. Fernando and M. Gunaydin, "Minimal unitary representation of SU(2,2) and its deformations as massless conformal fields and their supersymmetric extensions," *J.Math.Phys.* **51** (2010) 082301, arXiv:0908.3624 [hep-th].

[22] S. Fernando and M. Gunaydin, "Minimal unitary representation of SO*(8) = SO(6,2) and its SU(2) deformations as massless 6D conformal fields and their supersymmetric extensions," *Nucl.Phys.* **B841** (2010) 339–387, arXiv:1005.3580 [hep-th].

[23] S. Fernando and M. Gunaydin, "SU(2) deformations of the minimal unitary representation of $OSp(8^*|2N)$ as massless 6D conformal supermultiplets," *Nucl.Phys.* **B843** (2011) 784–815, arXiv:1008.0702 [hep-th].

[24] M. Gunaydin and N. Marcus, "The Spectrum of the S^5 Compactification of the Chiral $N = 2, D = 10$ Supergravity and the Unitary Supermultiplets of $U(2, 2/4)$," *Class. Quant. Grav.* **2** (1985) L11.

[25] M. Gunaydin, D. Minic, and M. Zagermann, "4D doubleton conformal theories, CPT and II B string on AdS(5) x S(5)," *Nucl. Phys.* **B534** (1998) 96–120, arXiv:hep-th/9806042.

[26] M. Gunaydin, D. Minic, and M. Zagermann, "Novel supermultiplets of $SU(2, 2|4)$ and the AdS_5/CFT_4 duality," *Nucl. Phys.* **B544** (1999) 737–758, arXiv:hep-th/9810226.

[27] M. Gunaydin, P. van Nieuwenhuizen, and N. P. Warner, "General Construction of the Unitary Representations of Anti-De Sitter Superalgebras and the Spectrum of the S^4 Compactification of Eleven-Dimensional Supergravity," *Nucl. Phys.* **B255** (1985) 63.

[28] P. A. M. Dirac, "A remarkable representation of the 3+ 2 de sitter group," *Journal of Mathematical Physics* **4** (1963) 901.

[29] M. Gunaydin and S. Hyun, "Unitary lowest weight representations of the noncompact supergroup $OSp(2 - n|2 - m, R)$," *J.Math.Phys.* **29** (1988) 2367.

[30] M. Gunaydin and N. P. Warner, "Unitary Supermultiplets of $OSp(8/4, R)$ and the Spectrum of the S^7 Compactification of Eleven-Dimensional Supergravity," *Nucl. Phys.* **B272** (1986) 99.

[31] J. M. Maldacena, "The Large N limit of superconformal field theories and supergravity," *Adv.Theor.Math.Phys.* **2** (1998) 231–252, arXiv:hep-th/9711200 [hep-th].

[32] E. Witten, "Anti-de Sitter space and holography," *Adv.Theor.Math.Phys.* **2** (1998) 253–291, arXiv:hep-th/9802150 [hep-th].

[33] S. Gubser, I. Klebanov, and A. M. Polyakov, "A Semiclassical limit of the gauge / string correspondence," *Nucl.Phys.* **B636** (2002) 99–114, arXiv:hep-th/0204051 [hep-th].

[34] M. Flato and C. Fronsdal, "Quantum field theory of singletons. the rac," *Journal of Mathematical Physics* **22** (1981) 1100.

[35] C. Fronsdal, "The Dirac Supermultiplet," *Phys. Rev.* **D26** (1982) 1988.

[36] M. Flato and C. Fronsdal, "Quantum Field Theory of Singletons: The Rac," *J. Math. Phys.* **22** (1981) 1100.

[37] E. Angelopoulos, M. Flato, C. Fronsdal, and D. Sternheimer, "Massless Particles, Conformal Group and De Sitter Universe," *Phys. Rev.* **D23** (1981) 1278.

[38] E. Fradkin and M. A. Vasiliev, "Cubic Interaction in Extended Theories of Massless Higher Spin Fields," *Nucl.Phys.* **B291** (1987) 141.

[39] S. Konshtein and M. A. Vasiliev, "Massless representations and admissibility condition for higher spin superalgebras," *Nucl.Phys.* **B312** (1989) 402.

[40] M. A. Vasiliev, "Higher spin gauge theories in four-dimensions, three-dimensions, and two-dimensions," *Int.J.Mod.Phys.* **D5** (1996) 763–797, arXiv:hep-th/9611024 [hep-th].

[41] M. A. Vasiliev, "Higher spin gauge theories: Star product and AdS space," arXiv:hep-th/9910096 [hep-th].

[42] X. Bekaert, S. Cnockaert, C. Iazeolla, and M. Vasiliev, "Nonlinear higher spin theories in various dimensions," arXiv:hep-th/0503128 [hep-th].

[43] C. Iazeolla, "On the Algebraic Structure of Higher-Spin Field Equations and New Exact Solutions," arXiv:0807.0406 [hep-th].

[44] A. Sagnotti, "Notes on Strings and Higher Spins," *J.Phys.* **A46** (2013) 214006, arXiv:1112.4285 [hep-th].

[45] I. Klebanov and A. Polyakov, "AdS dual of the critical O(N) vector model," *Phys.Lett.* **B550** (2002) 213–219, arXiv:hep-th/0210114 [hep-th].

[46] E. Sezgin and P. Sundell, "Holography in 4D (super) higher spin theories and a test via cubic scalar couplings," *JHEP* **0507** (2005) 044, arXiv:hep-th/0305040 [hep-th].

[47] S. Giombi and X. Yin, "Higher Spin Gauge Theory and Holography: The Three-Point Functions," *JHEP* **1009** (2010) 115, arXiv:0912.3462 [hep-th].

[48] S. Giombi and X. Yin, "Higher Spins in AdS and Twistorial Holography," *JHEP* **1104** (2011) 086, arXiv:1004.3736 [hep-th].

[49] S. Giombi and X. Yin, "The Higher Spin/Vector Model Duality," *J.Phys.* **A46** (2013) 214003, arXiv:1208.4036 [hep-th].

[50] M. Gunaydin, "Singleton and doubleton supermultiplets of space-time supergroups and infinite spin superalgebras,". Invited talk given at Trieste Conf. on Supermembranes and Physics in (2+1)-Dimensions, Trieste, Italy, Jul 17-21, 1989.

[51] E. Sezgin and P. Sundell, "Doubletons and 5-D higher spin gauge theory," *JHEP* **0109** (2001) 036, arXiv:hep-th/0105001 [hep-th].

[52] E. Sezgin and P. Sundell, "7-D bosonic higher spin theory: Symmetry algebra and linearized constraints," *Nucl.Phys.* **B634** (2002) 120–140, arXiv:hep-th/0112100 [hep-th].

[53] S. Fernando, M. Gunaydin, and S. Takemae, "Supercoherent states of OSp(8* — 2N), conformal superfields and the AdS(7) / CFT(6) duality," *Nucl.Phys.* **B628** (2002) 79–111, arXiv:hep-th/0106161 [hep-th].

[54] M. Vasiliev, "Higher spin superalgebras in any dimension and their representations," *JHEP* **0412** (2004) 046, arXiv:hep-th/0404124 [hep-th].

[55] N. Boulanger and E. Skvortsov, "Higher-spin algebras and cubic interactions for simple mixed-symmetry fields in AdS spacetime," *JHEP* **1109** (2011) 063, arXiv:1107.5028 [hep-th].

[56] A. Mikhailov, "Notes on higher spin symmetries," arXiv:hep-th/0201019 [hep-th].

[57] M. G. Eastwood, "Higher symmetries of the Laplacian," *Annals Math.* **161** (2005) 1645–1665, arXiv:hep-th/0206233 [hep-th].

[58] A. Joseph, "The minimal orbit in a simple lie algebra and its associated maximal ideal," *Ann. Sci. École Norm. Sup.(4)* **9** no. 1, (1976) 1–29.

[59] K. Govil and M. Gunaydin, "Deformed twistors and higher spin (super) conformal algebras in six dimensions," *to appear* .

[60] E. Angelopoulos, M. Flato, C. Fronsdal, and D. Sternheimer, "Massless particles, conformal group, and de sitter universe," *Physical Review D* **23** no. 6, (1981) 1278.

[61] M. Chiodaroli, M. Gunaydin, and R. Roiban, "Superconformal symmetry and maximal supergravity in various dimensions," *JHEP* **1203** (2012) 093, arXiv:1108.3085 [hep-th].

[62] M. Gunaydin, "Unitary highest weight representations of noncompact supergroups," *J.Math.Phys.* **29** (1988) 1275–1282.

[63] M. Eastwood, P. Somberg, and V. Soucek, "The uniqueness of the joseph ideal for the classical groups," arXiv:0512296 [math].

[64] M. Eastwood, "The cartan product," *Bulletin of the Belgian Mathematical Society-Simon Stevin* **11** no. 5, (2005) 641–651.

[65] S. Weinberg, *The quantum theory of fields*, vol. 1. Cambridge university press, 1996.

[66] W. Siegel, "Fields," arXiv:hep-th/9912205 [hep-th].

[67] L. Ferro, T. Lukowski, C. Meneghelli, J. Plefka, and M. Staudacher, "Harmonic R-matrices for Scattering Amplitudes and Spectral Regularization," *Phys.Rev.Lett.* **110** no. 12, (2013) 121602, arXiv:1212.0850 [hep-th].

[68] M. Eastwood, P. Somberg, and V. Souček, "Special tensors in the deformation theory of quadratic algebras for the classical Lie algebras," *J. Geom. Phys.* **57** no. 12, (2007) 2539–2546. http://dx.doi.org/10.1016/j.geomphys.2007.09.004.

[69] N. Boulanger, D. Ponomarev, E. Skvortsov, and M. Taronna, "On the uniqueness of higher-spin symmetries in AdS and CFT," arXiv:1305.5180 [hep-th].

[70] S. Konstein, M. Vasiliev, and V. Zaikin, "Conformal higher spin currents in any dimension and AdS / CFT correspondence," *JHEP* **0012** (2000) 018, arXiv:hep-th/0010239 [hep-th].

[71] M. Vasiliev, "Cubic interactions of bosonic higher spin gauge fields in AdS(5)," *Nucl.Phys.* **B616** (2001) 106–162, arXiv:hep-th/0106200 [hep-th].

[72] J. Maldacena and A. Zhiboedov, "Constraining Conformal Field Theories with A Higher Spin Symmetry," *J.Phys.* **A46** (2013) 214011, arXiv:1112.1016 [hep-th].

[73] E. Witten, "Spacetime reconstruction," *Talk at JHS* **60** (2001) 3–4.

[74] B. Sundborg, "Stringy gravity, interacting tensionless strings and massless higher spins," *Nucl.Phys.Proc.Suppl.* **102** (2001) 113–119, arXiv:hep-th/0103247 [hep-th].

[75] S. Fedoruk, E. Ivanov, and O. Lechtenfeld, "New D(2,1,alpha) Mechanics with Spin Variables," *JHEP* **1004** (2010) 129, arXiv:0912.3508 [hep-th].

[76] M. Günaydin, "Harmonic Superspace, Minimal Unitary Representations and Quasiconformal Groups," *JHEP* **05** (2007) 049, hep-th/0702046.

[77] K. Govil and M. Gunaydin, "Minimal unitary representation of $D(2,1:\lambda)$ and its SU(2) deformations and d=1, N=4 superconformal models," *Nucl.Phys.* **B869** (2013) 111–130, arXiv:1209.0233 [hep-th].

[78] L. Ferro, T. Lukowski, C. Meneghelli, J. Plefka, and M. Staudacher, "Spectral Parameters for Scattering Amplitudes in N=4 Super Yang-Mills Theory," arXiv:1308.3494 [hep-th].

[79] H. Samtleben and R. Wimmer, "N=8 Superspace Constraints for Three-dimensional Gauge Theories," *JHEP* **1002** (2010) 070, arXiv:0912.1358 [hep-th].