Higher-order spin effects in the dynamics of compact binaries. II. Radiation field

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Motivated by the search for gravitational waves emitted by binary black holes, we investigate the
gravitational radiation field of point particles with spins within the framework of the multipolar-post-
Newtonian wave generation formalism. We compute: (i) the spin-orbit (SO) coupling effects in the
binary’s mass and current quadrupole moments one post-Newtonian (1PN) order beyond the dominant
effect, (ii) the SO contributions in the gravitational-wave energy flux, and (iii) the secular evolution of the
binary’s orbital phase up to 2.5PN order. Crucial ingredients for obtaining the 2.5PN contribution in the
orbital phase are the binary’s energy and the spin-precession equations, derived in paper I of this series.
These results provide more accurate gravitational-wave templates to be used in the data analysis of rapidly
rotating Kerr-type black-hole binaries with the ground-based detectors LIGO, Virgo, GEO 600, and
TAMA300, and the space-based detector LISA.

I. INTRODUCTION

The aim of this paper is to derive the spin-orbit coupling
terms in the gravitational radiation field of compact binary
systems one post-Newtonian (1PN) order beyond the
dominant effect. Paper I of this series [1] dealt with the
problem of the spin-orbit contributions in the compact
binary equations of motion at 1PN relative order.

Our motivation is the ongoing search for gravitational
waves (GWs) emitted by inspiralling binary systems of
spinning, and possibly maximally spinning, black holes in
the network of detectors LIGO (Laser Interferometer
Gravitational Wave Observatory), Virgo, GEO 600, and
TAMA300, and the future search with the space-based
detector LISA. When the Kerr black holes are maximally
spinning (or close to maximal), the GW templates need to
take into account the effects of spins, not only for an
accurate parameter estimation [2–7], but also for a suc-
cessful detection [8–19]. Furthermore, spin effects should
be included at PN orders beyond the currently known
dominant spin-orbit and spin-spin terms. The contribu-
tions of spins are added to the templates developed for the
case of nonspinning binary black holes or neutron stars, and
which are currently known at 3.5PN order [20–23]. The
spins represent some of the possible effects depending on
the internal structure of the bodies which can be numeri-
cally important in the LIGO/Virgo bandwidth. This is true
even if we observe only the inspiral phase of moderate-
mass black holes with individual mass less than 10 M⊙.
Within the PN formalism the compact objects are treated as
point particles. It is then natural to model spinning black
holes as point particles with spins.

The equations of motion including the spin-orbit (SO)
effect were obtained in paper I at 1PN relative order, which
 corresponds formally to the 2.5PN order beyond the
Newtonian force law in the case of maximally rotating
compact objects. Paper I essentially confirmed the equa-
tions of motion derived previously by Tagoshi, Ohashi, and
Owen [24]. Furthermore, paper I derived the complete set
of Noetherian conserved integrals of motion at that order
(namely, 2.5PN for the spins). In the present paper, we
tackle the problem of the gravitational radiation field at the
same 2.5PN order, using the multipolar PN wave gener-
ation formalism of Refs. [25–30]. More precisely, we shall
compute here the SO contributions in the compact binary’s
mass-type and current-type quadrupole moments, both of
them with 1PN relative accuracy. These moments, together
with some easily computed higher multipole moments
which necessitate only the lowest-order precision, are nec-
necessary to compute the total GW energy flux F. The com-
putation of the current quadrupole moment was previously
attempted in Ref. [31], but we shall point out two important
flaws in that reference (see below for details). Our result for
the current quadrupole moment is substantially different
from that of [31]. Concerning the mass quadrupole
moment, it is computed here for the first time. Having in hand
the total energy flux, using the center-of-mass energy E
computed in paper I, we deduce (by energy balance argu-
ments) the equation of secular evolution of the binary’s or-
bital frequency. The latter is the crucial ingredient needed
to build GW templates for spinning compact binaries.

To describe particles with spins, we use the formalism
originally developed in Refs. [32–35] (an effective field
theory has recently been proposed [36–38]). This formal-
ism has already been successfully applied to the problem of
spinning compact binaries in Refs. [24,31,39–43], and, in
the mass-type case, in Refs. [44,45]. In particular,
Kidder, Will, and Wiseman derived in Refs. [39,41] the lowest-order spin-coupling effects—at 1.5PN order in the case of maximal Kerr black holes—and the first spin-spin effect, quadratic in the spins—appearing at 2PN order—in the equations of motion and the gravitational radiation field. As we shall see below, we find complete agreement at that order with their results. The present paper, together with paper I, extends therefore the works [39,41] to include the next-order spin effects. Since those effects, of 2.5PN order, are linear in the spins (the next-order spin-term coming along at 3PN order), we complete the derivation of all the spin contributions in the GW form up to 2.5PN order.

The paper is organized as follows. In Sec. II we review the general formalism for wave generation from arbitrary PN sources. In Sec. III we compute the multipole moments of compact binary systems at the lowest PN level in the spins (which means 1.5PN for mass moments, and 0.5PN for current ones). Section IV constitutes the core of the paper. We compute there the mass and current quadrupole moments at 1PN relative order, i.e. 2.5PN and 1.5PN for mass moments, and 0.5PN of compact binary systems at the lowest PN level in the gravitational field. As we shall see below, we find complete agreement with their results. The present paper, together with paper I, extends therefore the works [39,41] to include the next-order spin effects. Since those effects, of 2.5PN order, are linear in the spins (the next-order spin-term coming along at 3PN order), we complete the derivation of all the spin contributions in the GW form up to 2.5PN order.

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II. POST-NEWTONIAN SOURCE MULTIPOLAR MOMENTS

We start with a brief review of the PN multipole moment formalism at the basis of this approach (full details can be found in Refs. [21,28–30]). This formalism is valid for general localized material sources, satisfying the usual PN requirements of weak self-gravity, slow motion, and weak internal stresses. In particular, the size of the source $a$ has to be small with respect to the typical (reduced) wavelength $\lambda$ of the gravitational radiation this source produces, i.e. $a/\lambda = O(\epsilon)$, with $\epsilon \sim \omega/c$ being the slowness PN parameter. We shall abbreviate it as $\epsilon = 1/c$ and denote the PN remainder terms by $O(1/c^n)$ henceforth.

Let $x^\mu = (ct, \mathbf{x})$ be an harmonic coordinate system covering the whole material source. We pose $h^{\mu\nu} = \sqrt{-g} g^{\mu\nu} - \eta^{\mu\nu}$ where $g^{\mu\nu}$ and $g$ are the inverse and the determinant of the usual covariant metric $g_{\mu\nu}$, and where $\eta^{\mu\nu}$ denotes an auxiliary Minkowskian background metric (Greek letters represent space-time indices; our signature is $+2$). The Einstein field equations, relaxed by the stress-energy pseudotensor $\tau^{\mu\nu}$ of the matter and gravitational fields in harmonic coordinates. Here $T^{\mu\nu}$ is the stress-energy tensor of the matter fields and $\Lambda^{\mu\nu}[h]$ represents the gravitational source term, namely, a complicated nonlinear functional of $h^{\mu\sigma}$ and its space-time derivatives $\partial_\nu h^{\mu\nu}$ (see e.g. [21] for the expression). We shall see that $\Lambda^{\mu\nu}$ gives a crucial contribution to the multipole moments at the relative 1PN order in the spins. The stress-energy pseudotensor is conserved by virtue of the harmonic-coordinate condition,

$$\partial_\nu h^{\mu\nu} = 0 \Rightarrow \partial_\nu \tau^{\mu\nu} = 0.$$  

The multipole moments of the source are generated by the components of the pseudotensor $\tau^{\mu\nu}$ or, more precisely, of its formal PN expansion $\tau^{\mu\nu} \equiv PN[\tau^{\mu\nu}]$. In this sense, the formalism is physically valid for PN sources only. The PN expansion $\tau^{\mu\nu}$ has a special structure which can be matched to the exterior multipolar field of a PN source, allowing one to define an appropriate notion of PN multipole moments [28–30]. It is convenient to define (Latin letters representing space indices)

$$\Sigma_{ij} = c^{-2} \left[ \tilde{\tau}_{0ij} + \tilde{\tau}_{ij} \right]$$  

where $\tilde{\tau}_{ij} = \delta_{ij} \tilde{\tau}^{ij},$  

$$\Sigma_{ij} = c^{-1} \tilde{\tau}^{ij},$$  

$$\Sigma_{ij} = \tilde{\tau}^{ij}.$$  

The mass-type moments $I_L(t)$ and current-type ones $J_L(t)$ are referred to as the source multipole moments in order to distinguish them from the so-called radiative moments, seen at infinity and generally denoted $U_L(t)$ and $V_L(t)$. They are given by

$$I_L(t) = \int d^3x \rho(x,t) \delta(x-x(t)), \quad J_L(t) = \int d^3x \mathbf{J}(x,t) \delta(x-x(t)).$$
These expressions are general, in the sense that they are formally valid at any PN order, so that there are no remainder terms \( O(e^{-n}) \) involved. The dots indicate the time derivatives and \( e_{abc} \) is the Levi-Civita symbol in 3 dimensions with \( e_{123} = 1 \). Notice the peculiar feature that, besides the usual spatial integration, the moments involve an extra integral over a variable \( z \) defining a “cone” of integration \( u = t + z|\mathbf{x}|/c \); hence, the sources \( \Sigma_{\mu} \) depend on the point \((\mathbf{x}, u)\) as indicated. The “weighting” function associated with the \( z \)-integral, \( \delta_{\ell}(z) = \frac{2(\ell + 1)}{\pi^2} \left( 1 - z^2 \right) \), is normalized in such a way that \( \int_{-1}^{1} d\xi \delta_{\ell}(\xi) = 1 \). When performing explicitly the PN expansion of the moments, the \( z \)-integration is to be transformed into an infinite local series (in the sense that \( z \) is integrated out), which constitutes the basis of the practical evaluation of the multipole moments. Namely, we have

\[
\int_{-1}^{1} d\zeta \delta_{\ell}(\zeta) \Sigma(\mathbf{x}, t + z|\mathbf{x}|/c) = \sum_{k=0}^{+\infty} \frac{(2\ell + 1)!!}{(2k)!!(2\ell + 2k + 1)!!} \left( \frac{|\mathbf{x}|}{c} \frac{\partial}{\partial t} \right)^{2k} \Sigma(\mathbf{x}, t). \tag{2.5}
\]

The above expression is then inserted into the right-hand side (RHS) of Eqs. (2.4), and truncated to fit with the PN order of the calculation.

A crucial finite part (FP) procedure is involved in the definition of the source multipole moments (2.4). It consists of (i) multiplying the integrand of the moments by a regularization factor \(|\mathbf{x}|^B\), where \( B \) is a complex number,\(^3\) (ii) performing the Laurent expansion when \( B \) tends to the “physical” value \( B = 0 \), and (iii) picking up its finite part FP, namely, the coefficient of the zeroth power of \( B \). The finite part regularization is therefore equivalent to removing the poles \( B^{-1}, B^{-2}, \ldots \) (in the analytically continued \( B \)-dependent integral) before taking the limit \( B \to 0 \). The FP procedure is needed to compute the nonlinear contributions to the moments (generated by the gravitational source term \( \Lambda^{\mu} \)), which have a noncompact support extending up to spatial infinity. Notice that no assumption nor physical restriction (in principle) is involved, in the latter FP procedure, which has been proved \([28–30]\) to yield the correct expression of the multipole moments for general extended PN sources. It is precisely the FP that guarantees this within the present formalism; the divergent terms \( B^{-1}, B^{-2}, \ldots \) have no direct physical significance. Such FP when \( B \to 0 \) is to be carefully distinguished from the self-field regularization (e.g., Hadamard’s or dimensional regularization) which is to be invoked when treating the application of the general formalism to singular point-particle sources.

We emphasize that the expressions (2.4) constitute \textit{a priori} only a definition of the source multipole moments. The point is that such definition is fully related to the physical asymptotic wave form at infinity from the source, which is computed using the multipolar post-Minkowskian formalism \([25,26]\). In particular, the \textit{radiative} multipole moments \( U_{\ell}(t) \) and \( V_{\ell}(t) \) which parametrize the asymptotic wave form are given by some nonlinear functionals of the source moments (2.4), made of many nonlinear interactions between them, including the famous GW tails corresponding to the interaction between these moments and the total monopole mass \( M \) of the source. The tails have been computed within this approach in Refs. \([46,47]\). However, for deriving the spin terms at 1PN relative order, all these nonlinear multipole interactions, notably all the tails, are negligible (see further discussion below). It is therefore sufficient to consider only the source multipole moments of Eqs. (2.4).

### III. LOWEST-ORDER SPIN EFFECTS IN THE MULTIPOLE MOMENTS

The multipole moments discussed above can be applied to any source. Here, we specialize them to binary systems of point particles with spins. The formalism to describe particles with spins was developed in Refs. \([32–35]\), and constitutes the basis of most subsequent computations in this field \([24,31,39–41,44,45]\). We reviewed this formalism in paper I and refer the reader to this paper for details and notation. The spin contribution (marked by the underneath label S) to the stress-energy tensor of the particles reads

\[
T_{\mu \nu}^{S}(t, \mathbf{x}) = -\frac{1}{c} \nabla_{\rho} \left[ S_{1}^{\rho \mu} v_{1}^{\nu} \frac{\delta(\mathbf{x} - \mathbf{y}_{1})}{\sqrt{-g_{1}}} \right] + 1 \leftrightarrow 2, \tag{3.1}
\]

where \( \delta \) is the Dirac three-dimensional delta function, \( \nabla_{\rho} \) denotes the covariant derivative, \( v_{1}^{\mu}(t) = (c, v_{1}^{i}) \) with \( v_{1}^{i} = dy_{1}^{i}/dt \) being the coordinate velocity of particle 1, \( g_{1} \) is the determinant of the metric evaluated at the location of particle 1 (following Hadamard’s self-field regularization), and \( 1 \leftrightarrow 2 \) means the same expression as preceding, but for

\(^3\) Generally, the regularization factor is taken to be \(|\mathbf{x}|/r_{0}\)^{B}, where \( r_{0} \) denotes some arbitrary constant length scale. Here we set \( r_{0} = 1 \) for convenience.
particle 2. The antisymmetric spin tensor $S_i^{\mu\nu}(t)$ is introduced in Sec. 2 of paper I. The covariant four-vector $S'_\mu$ is defined by $S_i^{\mu\nu} = -\frac{1}{\sqrt{g}} \epsilon_i^{\mu\nu\rho\sigma} u^\rho S^\sigma_{\nu}$; it is transverse to the particle’s four-velocity, $S'_\mu u^\mu = 0$. All results below are expressed in terms of some particular spacelike contravariant spin variables for each of the particles, namely $S'_i$ and $S'_i$, the definition of which can be found in Eq. (2.19) of paper I.

Similarly to the quantities $\Sigma_{\mu\nu}$ introduced in Eq. (2.3), we define the following matter-source densities, depending on the components of the spin stress-energy tensor (3.1):

$$\sigma_s = c^{-2}[T_{00}^s + T_{ij}^s] \quad \text{with} \quad T_{ij}^s = \delta_{ij}T_{ij}^s,$$  
$$\sigma_{ij}^s = c^{-1}T_{ij}^s,$$  
$$\sigma_{ij}^s = T_{ij}^s.$$  

They are such that their “nonspin” counterparts (say $\Sigma_{\mu\nu}$) admit a finite nonzero limit when $c \rightarrow +\infty$. They read

$$\sigma_s = -\frac{2}{c} \epsilon_{ijk} v_1^i S'_1 j_{k}\delta_1 + 1 \leftrightarrow 2 + O\left(\frac{1}{c^3}\right),$$  
$$\sigma_{ij}^s = -\frac{1}{2c} \epsilon_{ijk} S'_k \partial_{k}\delta_1 + 1 \leftrightarrow 2 + O\left(\frac{1}{c^3}\right),$$  
$$\sigma_{ij}^s = -\frac{1}{c} \epsilon_{ijkl} v_1^j S'_k j_{l}\delta_1 + 1 \leftrightarrow 2 + O\left(\frac{1}{c^3}\right),$$  

where we denote $\delta_1 = \delta(x - y_1)$ and where $\partial_{k}\delta_1$ means the gradient of $\delta_1$ with respect to the field point $x = (x^i)$.

As shown by Eqs. (3.3), the leading order of the vector and tensor densities $\sigma_s$ and $\sigma_{ij}^s$ is $0.5\text{PN} \sim 1/c$. However, the scalar density $\sigma_s$ starts at a higher level, being at least $1.5\text{PN} \sim 1/c^2$. At leading order in the spins, the $\Sigma_{\mu\nu}$’s, which depend on the contributions of both matter and gravitational fields according to Eqs. (2.1) and (2.3), will reduce to their compact-support material parts, namely, the $\Sigma_{\mu\nu}$’s given by Eqs. (3.3). Indeed, the noncompact support gravitational part, whose origin lies in the source term $A_{\mu\nu}$ present in the RHS of the field equations (2.1), always appears at a subdominant level, $1/c^2$ beyond the leading PN order. Hence,

$$\Sigma_s = \sigma_s + O\left(\frac{1}{c^3}\right),$$  
$$\Sigma_{ij} = \sigma_{ij}^s + O\left(\frac{1}{c^3}\right),$$  
$$\Sigma_{ij} = \sigma_{ij}^s + O\left(\frac{1}{c^3}\right).$$  

The noncompact-support gravitational source terms play a role in our computations at the next-to-leading order only (see Sec. IV). We conclude that the dominant contribution to the multipole moments (2.4) due to the spins is given by

$$I_{ij}^L = \int d^3\mathbf{x} \left[ \hat{x}_{L} \sigma_s - \frac{4(2\ell + 1)}{c^2} \hat{x}_{L} \sigma_{ij}^s \right] + O\left(\frac{1}{c^3}\right).$$  

Thus, the dominant order is $1.5\text{PN} \sim 1/c^2$ for spins in the mass-type moments $I_{ij}$, but only $0.5\text{PN} \sim 1/c$ in the current-type ones $J_{ij}$. It is then evident (mathematically and physically) that the spin part of the current moments always dominate over that of the mass moments. We insert the explicit values (3.3) into Eqs. (3.5), integrate in a straightforward way (resorting to an integration by parts and using the properties of the delta function), and get

$$I_{ij}^L = \frac{2\ell}{c^2(\ell + 1)} \left[ (\ell + 1) S'_i \epsilon_{ij} S'_j y_1^{(L - 1)} \right] + 1 \leftrightarrow 2 + O\left(\frac{1}{c^3}\right).$$  

It is worth mentioning that in this calculation, limited to the lowest PN order, the spins can be considered as constant since their time variations, as given by the precessional equations (see paper I), are always smaller by a factor $1/c^2$ at least.

**IV. HIGHER-ORDER SPIN EFFECTS IN THE MULTIPOLe MOMENTS**

For the present purpose, we need the spin contributions to the mass-quadrupole moment $I_{ij}$ and the current-quadrupole moment $J_{ij}$ one PN order beyond the leading terms obtained in Eqs. (3.6). This means at $2.5\text{PN}$ order for $I_{ij}$ and at $1.5\text{PN}$ order for $J_{ij}$. As said previously, the nonlinear gravitational source terms, with noncompact support, start playing a role at the $1\text{PN}$ relative order.4 Therefore, they do make a net contribution to the spin parts of both $I_{ij}$ at $2.5\text{PN}$ order and $J_{ij}$ at $1.5\text{PN}$ order. We can note here that the authors of Ref. [31] computed $J_{ij}$ at $1.5\text{PN}$ order but neglected all the noncompact source terms in their calculation, thus obtaining an incorrect result.

We now reduce the multipole moments to the required PN order by inserting the expansion formula (2.5) into the general expressions (2.4). Neglecting PN terms of higher

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4The rule admits some exceptions, though. For instance, in the nonspinning part of the mass multipole moments $I_L$, one may have expected the nonlinear noncompact gravitational terms to appear at $1\text{PN}$ order, but these terms turn out to be in the form of a pure divergence and can be integrated out to zero. As a result, the noncompact terms of the nonspinning parts contribute to $I_L$ at $2\text{PN}$ order only (and at $1\text{PN}$ order to $J_L$). See Refs. [27,28] for details.
order, as indicated, we have

\[
I_{ij} = \frac{FP}{b-0} \int d^3x|x|^\beta \left\{ \dot{x}_{ij} \Sigma + \frac{1}{14c^2} \dot{x}_{ij} |x|^2 \Sigma + \frac{1}{504c^4} \dot{x}_{ij} |x|^4 \Sigma - \frac{20}{21c^2} \dot{x}_{ijk} \dot{\Sigma}_k - \frac{10}{189c^4} \dot{x}_{ijk} |x|^2 \dot{\Sigma}_k + \frac{5}{54c^6} \dot{x}_{ijkl} \dot{\Sigma}_{kl} \right\} + O(\frac{1}{c^6}),
\]

\[
J_{ij} = \frac{FP}{b-0} e_{abij} \int d^3x|x|^\beta \left\{ \ddot{x}_{ab} \Sigma_b + \frac{1}{14c^2} \ddot{x}_{ab} |x|^2 \Sigma_b - \frac{5}{28c^2} \ddot{x}_{ab} |x|^2 \dot{\Sigma}_b + O(\frac{1}{c^6}) \right\},
\]

(4.1a)

(4.1b)

where FP denotes the essential process of extracting the finite part, as explained in Sec. II. We can further reduce Eqs. (4.1) by substituting the appropriate expressions of the source terms \( \Sigma_{\mu\nu} \) as given by Eqs. (11.11) of Ref. [28]. For completeness, we list all the necessary expressions below:

\[
\Sigma = \left[ 1 + \frac{4V}{c^2} - \frac{2}{c^4} (\ddot{W} + V^2) \right] \sigma - \frac{1}{\pi G c^2} \partial_i V \partial_i V + \frac{1}{\pi G c^4} \left\{ -V \partial_i^2 V - 2V_i \partial_i V \dot{W}_ij \partial_j V - \frac{1}{2} (\partial_i V)^2 + 2V_i \partial_i V \dot{V}_i \right\} - \partial_i V \partial_j \dot{W} = \frac{7}{2} V \partial_i V \partial_j V + O(\frac{1}{c^6}),
\]

\[
\Sigma = \left[ 1 + \frac{4V}{c^2} \right] \dot{\sigma} + \frac{1}{\pi G c^2} \left[ \dot{\partial}_i V \dot{\partial}_i V - \dot{\partial}_i V \dot{\partial}_j V + \frac{3}{4} \partial_i V \dot{\partial}_j V \right] + O(\frac{1}{c^6}).
\]

(4.2a)

(4.2b)

(4.2c)

where the material source densities are given by \( \sigma = c^{-2} [T^{00} + T^{ij}] \), \( \dot{\sigma} = c^{-1} T^{0i} \), and \( \dot{\sigma} = T^{ij} \). The noncompact support terms in Eqs. (4.2) are parametrized by a particular set of “elementary” potentials \( V, V_i, \dot{W}_ij, \ldots \), which enter the harmonic coordinate near zone metric at the 2PN order computed in Ref. [48]. Their complete expressions are given in Sec. III of paper I.

We can simplify \( \Sigma \) substantially by using some identities of the type \( \partial_i \partial_j \partial_k \partial_l = \frac{1}{2} \left\{ \partial_i \partial_j \partial_k \partial_l - \partial_k \partial_l \partial_i \partial_j \right\} \), the Laplacians \( \Delta A \) and \( \Delta B \) by their PN sources, and disregarding the pure Laplacian term \( \frac{1}{2} \Delta (AB) \) because it makes zero contribution to the moment. This last point comes essentially from the fact that, after integration by parts, a pure Laplacian term in the moment sources is equivalent to a source term proportional to some \( \Delta \dot{x}_i \). Beware that, because of the presence of the finite part FP, such a “proof” is not correct in general and, in fact, the pure Laplacian terms do generally contribute at high PN orders. However, for the terms under concern, merely at the 1PN relative order, the argument can be made rigorous and shown to work, so that we can indeed discard these pure Laplacians in the present computation (see Ref. [28] for the proof). Hence, we get the simpler formula,

\[
\Sigma = \sigma + \frac{4V}{c^4} \sigma_{ij} + \frac{1}{\pi G c^2} \left\{ -V \partial_i^2 V - 2V_i \partial_i V \dot{W}_ij \partial_j V - \frac{1}{2} (\partial_i V)^2 + 2V_i \partial_i V \dot{V}_i \right\} + O(\frac{1}{c^6}).
\]

(4.3)

\[\Delta(AB) \]. We have checked that the two different forms \( \Sigma \) and \( \Sigma' \), Eqs. (4.2a) and (4.3), lead to the same final result.

A. Compact-support contribution

Having now the general setup for our computation, we consider first the compact-support part of the multipole moments, i.e., that part proportional to the material source densities \( \sigma_{\mu\nu} \), and given by the first terms in Eqs. (4.2) and (4.3). For these terms we make two computations. The first one consists of (i) evaluating all the components of \( g T^\mu\nu \) to the correct PN order [extending thus Eqs. (3.3)], (ii) computing their time derivatives by making use of the usual replacement of accelerations by the equations of motion, and of the time derivatives of the spins by the precessional equations, (iii) transforming the time derivatives, when applied to delta functions, into spatial ones using the formula \( \delta t \delta_i = -\dot{\delta}_i \delta_1 \), (iv) operating by parts the spatial derivatives of delta functions to finally integrate thanks to the basic property of delta functions.

Normally, such a basic property of delta functions reads

\[
\int d^3x F(x) \delta_i(x) = F_1 \delta_i, \quad \text{where} \quad F_1 \text{ is simply the value of the function at the point } 1.
\]

When the function is regular, there is no problem and we have \( F_1 = F(y_1) \), for instance \( \delta^i = \delta_1^i \). However, when the function \( F \) is singular at the point 1 (i.e., when \( x \to y_1 \)), a choice must be made for a “self-field” regularization, able to subtract the infinities in a consistent way. There are various possibilities. In the present work we adopt, following paper I, the Hadamard self-field regularization. At the order we are working (relative 1PN order) the various possible choices are equivalent. For instance, one can show that dimensional regularization would give the same result as Hadamard’s regularization, essentially because at such low PN order.

5The potential called \( \dot{W}_ij \) in Ref. [28] differs from the present \( \dot{W}_ij \) (whose definition is given in paper I) according to the formula \( \dot{W}_ij = \dot{\dot{W}}_ij - \frac{1}{2} \delta_{ij} \dot{W} \), hence \( \dot{W} = \dot{W}_ii = -\frac{1}{2} \dot{W} \).
there are no poles in the dimension of space [say, \( \propto (d-3)^{-1} \)], which correspond to logarithmic divergences in Hadamard’s regularization. We then define \((F)\) to be given by the partie finie of the function \(F\) at point 1 in the sense of Hadamard (see e.g. [49] for a full account of this regularization). Suppose, for example, that \(F = U\hat{x}_L\) where \(U = \frac{Gm_1}{r_1} + \frac{Gm_2}{r_2}\) is the Newtonian potential of point particles, singular at the location of the particles. Then we easily compute that Hadamard’s partie finie is \((F)_1 = \frac{Gm_2}{r_2} t\).

Our alternative computation of the compact-support terms in the moments is the same as the one performed by Owen et al. [31]. It consists of applying the following formula (derived in [31]):

\[
\int d^3x F(x, t) T^\mu_{\nu}(t, x) = -\frac{d}{dt} \left[ S_0(\mu, \nu)^{(\nu)} \left( \frac{F_1}{\sqrt{-g_1}} \right) \right] + S_0^{(\mu, \nu)} v^\nu \left( \frac{\partial_\mu F_1}{\sqrt{-g_1}} \right) + \left[ \frac{\partial_\mu S_0^{(\nu, \rho)}}{\sqrt{-g_1}} \right] \left( \frac{F_1}{\sqrt{-g_1}} \right) + 1 \leftrightarrow 2, \tag{4.4}
\]

which is valid for any function \(F(x, t)\), and where \((F)\) and \((\partial_\mu F)_1\) have to be understood as the Hadamard partie finie of the function and its derivative. This formula is very useful but must be handled with care. In particular, when computing \((\partial_\mu F)_1\) in the second term of the RHS of (4.4), we notice that the gradient is to be taken first, and only then should one deduce the value at point 1. The result can be different if one permutes the order of operations. Suppose for instance that one is computing \((\partial_\nu F)_1\) where \(F = \hat{x}_L\). Clearly, since \(\partial_\nu \hat{x}_L = 0\) is the case, the derivative is zero. However, if one computes \(\frac{d}{dt}(\hat{x}_L)\) instead of \((\partial_\nu F)_1\), one obtains an incorrect nonzero result, which is equal in this case to \(\frac{d}{dt}(\hat{x}_L) = \ell y_{(1-1)} v^i\). We found that this error, i.e. computing \(\frac{d}{dt}(\hat{x}_L)\) instead of \((\partial_\nu \hat{x}_L)\) = 0, is committed in the evaluation of the (compact-support part of the current quadrupole moment \(J_{ij}\) in Ref. [31].

### B. Noncompact-support contribution

We now derive the noncompact support part of the multipole moments. Inspection of the expressions (4.2) and (4.3) shows that we need only the elementary potentials \(V, V_i, \tilde{W}_{ij}\) and \(\tilde{W} = \tilde{W}_{ij}\) at their lowest PN order in the spins. They read

\[
V = -\frac{2G}{c^3} e_{ijk} v^j_1 S^i_1 \partial_k \frac{1}{r_1} + 1 \leftrightarrow 2 + \mathcal{O}\left(\frac{1}{c^3}\right), \tag{4.5a}
\]

\[
V_i = -\frac{G}{2c} e_{ijk} S^i_1 \partial_k \frac{1}{r_1} + 1 \leftrightarrow 2 + \mathcal{O}\left(\frac{1}{c^3}\right), \tag{4.5b}
\]

\[
\tilde{W}_{ij} = -\frac{G}{c} e_{kl(i)j} S^i_1 \partial_k \frac{1}{r_1} + \frac{1}{c} \delta_{ij} e_{klm} v^k_1 S^j_1 \partial_m \frac{1}{r_1},
\]

\[
\tilde{W} = \frac{2G}{c} e_{klm} v^k_1 S^j_1 \partial_m \frac{1}{r_1} + 1 \leftrightarrow 2 + \mathcal{O}\left(\frac{1}{c^3}\right). \tag{4.5c}
\]

The latter spin contributions enter only the “nonspin” parts of Eqs. (4.2) and (4.3). We need now to perform the integrations in Eq. (4.1). The calculation becomes simple if we use the following trick. In the multipole moment’s integrands we transform all the gradients evaluated at the field point \(x\) into gradients evaluated at the source points \(y_1\) or \(y_2\) using e.g. \(\frac{\partial}{\partial y_1}(1/r_1) = -\frac{\partial}{\partial y_1}(1/r_1)\). Then, we put the source-type gradients \(\frac{\partial}{\partial y_1}\) and \(\frac{\partial}{\partial y_2}\) outside the integrals and we express the result solely in terms of the function

\[
Y_L(y_1, y_2) = \frac{-1}{2\pi B_0} \int d^3x |x|^p \frac{\hat{x}_L}{r_1 r_2}, \tag{4.6a}
\]

which is known to admit the analytically closed form [28]

\[
Y_L = \frac{r_1}{\ell + 1} \sum_{p=0}^{\ell} \frac{\ell!}{\ell-p} y_1^{(L-p)} y_2^p. \tag{4.6b}
\]

Thus, the closed form expressions of the noncompact (NC) parts of the spin multipole moments (they depend on the function \(Y_L\) for \(\ell = 2, 3\) are

\[
J^{(NC)}_{ij} = \frac{2Gm_2}{c^5} \left[ e_{nnn} v^j_1 S^i_1 \partial_{1\mu} \partial_{2\nu} Y_{ij} - e_{nnn} v^j_1 S^i_1 \partial_{1\nu} \partial_{2\mu} Y_{ij} - e_{kmn} v^j_2 S^i_1 \partial_{1\mu} \partial_{2\nu} Y_{ij} - e_{kmn} v^j_2 S^i_1 \partial_{1\nu} \partial_{2\mu} Y_{ij} - 2e_{kmn} v^j_2 S^i_1 \partial_{1\mu} \partial_{2\nu} Y_{ij} \right] + 1 \leftrightarrow 2 + \mathcal{O}\left(\frac{1}{c^3}\right), \tag{4.7a}
\]

\[
J^{(NC)}_{ij} = \frac{Gm_2}{c^5} \left[ e_{kl(i)j} \left( -e_{kmn} S^i_1 \partial_{2\nu} \partial_{1\mu} Y_{jk} + e_{kmn} S^i_1 \partial_{2\nu} \partial_{1\mu} Y_{jk} \right) + 1 \leftrightarrow 2 + \mathcal{O}\left(\frac{1}{c^3}\right). \tag{4.7b}
\]

where \(\partial_{1\mu} \equiv \partial/\partial y_1^\mu\) and \(\partial_{2\nu} \equiv \partial/\partial y_2^\nu\). The final computation of the moments using formula (4.6b) is straightforward.

### V. RESULTS FOR THE MULTIPLE MOMENTS AND FLUX

The expressions for the multipole moments \(S^{(NC)}_{ij}\) and \(S^{(NC)}_{ij}\), including both compact and noncompact contributions as computed in Sec. IV, are quite long if written in a general frame. They can be substantially simplified by going to the frame

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of the center-of-mass (CM). When working in the CM frame it is convenient to use the following spin variables:

$$
S = S_1 + S_2, \quad \Sigma = \frac{m}{m_2-m_1} S_2 - \frac{S_1}{m_1}. \quad (5.1a)
$$

These spin variables were initially introduced by Kidder [39] except that here we denote $\Sigma$ what he calls $\Delta$. Mass parameters will be denoted by $m = m_1 + m_2$, $\delta m = m_2 - m_1$, and $\nu = m_1 m_2 / m^2$ for a mass ratio such that $\nu = 1/4$ for equal masses and $\nu \rightarrow 0$ in the test-mass limit.

The CM frame is defined by the nullity of the binary’s dipole moment or equivalently the CM vector $G$. At 2.5PN order including spin effects, it can easily be determined using the vector $G$ evaluated in paper I. However, here we need only the lowest-order term (1.5PN in the spins) together with the 1PN nonspin correction; the 2.5PN term in the spins cancels out. To the needed order we have [see e.g. Eq. (5.13) in [24]]

$$
y_1 = \left[ \frac{m_1}{m} + \frac{\nu \delta m}{m c^2} \left( \frac{v^2 - \frac{G m}{r}}{m} \right) \right] x + \frac{\nu}{m c} v \times \Sigma. \quad (5.2a)
$$

$$
y_2 = \left[ - \frac{m_1}{m} + \frac{\nu \delta m}{m c^2} \left( \frac{v^2 - \frac{G m}{r}}{m} \right) \right] x + \frac{\nu}{m c} v \times (\Sigma \Sigma) \quad (5.2b)
$$

which gives the CM positions of the particles, $y_1$ and $y_2$, in terms of the relative position and velocity, $x = y_1 - y_2$ and $v = dx / dt = v_1 - v_2$ (we pose $r = |x|$ and $v^2 = v \cdot v$).

### A. The multipole moments

Our final result for the spin part of the mass-quadrupole moment at 2.5PN order (1PN order beyond the dominant SO term), for general orbits and in the CM frame, reads

$$
I_{ij} = \frac{\nu}{c^3} \left\{ \frac{8}{3} x^{ij}(v \times S)^{\dagger} - \frac{4}{3} v^{ij}(x \times (v \times S))^{\dagger} + \frac{8}{3} \frac{\delta m}{m} x^{ij}(v \times \Sigma)^{\dagger} - \frac{4}{3} \frac{\delta m}{m} v^{ij}(x \times \Sigma)^{\dagger} \right\}
$$

$$
+ \frac{\nu}{c^3} \left\{ \left[ \frac{26}{7} - \frac{16}{7} \nu \right] \frac{G}{m r^3} \frac{\delta m}{m} (xv) x^{ij}(x \times \Sigma)^{\dagger} + \left( \frac{19}{28} + \frac{13}{28} \nu \right) \frac{G}{m r^3} \frac{\delta m}{m} x^{ij}(v \times \Sigma)^{\dagger} \right\}
$$

$$
+ \left( - \frac{4}{7} + \frac{31}{14} \nu \right) \frac{\delta m}{m} \left( xS \right) x^{ij}(x \times v)^{\dagger} + \frac{1}{c^3} \mathcal{O}(1).
$$

(5.3)

The scalar product of ordinary Euclidean vectors is indicated by parenthesis, e.g. $(vS) = v \cdot S$, the cross product by the usual cross symbol, $(x \times S)^{\dagger} = e^{ijk} x^i S^j k$, and the mixed product of three vectors by $(S, x, v) = S \cdot (x \times v) = e^{ijk} S^i x^j v^k$.

We recall also that the STF projection is denoted using carets surrounding indices, i.e. $(ij)$. Next, the spin part of the current quadrupole moment at 1.5PN order (also 1PN order beyond the leading term) is

$$
J_{ij} = \frac{\nu}{c} \left\{ \frac{3}{2} x^{ij}(\Sigma \Sigma)^{\dagger} + \frac{\nu}{c^3} \left[ \frac{3}{7} - \frac{16}{7} \nu \right] (xv) v^{ij}(\Sigma \Sigma)^{\dagger} + \frac{3}{7} \frac{\delta m}{m} (xv) v^{ij}(S \Sigma)^{\dagger} + \left( \frac{27}{14} - \frac{109}{14} \nu \right) (vS) + \frac{27}{14} \frac{\delta m}{m} (vS) \right\} x^{ij} v^{\dagger}
$$

$$
+ \left( - \frac{11}{14} + \frac{47}{14} \nu \right) \left( S \Sigma \right) - \frac{11}{14} \frac{\delta m}{m} \left( xS \right) v^{ij} v^{\dagger} + \left( \frac{19}{28} + \frac{13}{28} \nu \right) \frac{G}{m r^3} \frac{\delta m}{m} x^{ij}(x \times v)^{\dagger} + \frac{1}{c^3} \mathcal{O}(1). \quad (5.4)
$$

Notice that the 1.5PN current quadrupole moment $J_{ij}$ was also computed in Ref. [31], see Eq. (4.18) there. However, our result (5.4) differs from their result. There are two reasons for this discrepancy. The main reason is that Ref. [31] completely neglected the noncompact support terms, which originate from the nonlinearities of the Einstein field equations via the term $\Lambda^{\mu \nu}$ in Eq. (2.1), and physically represent the gravitational field acting as a source for the multipole moment. As we have seen in Sec. IV these terms are not negligible. Their contribution
to the 1.5PN order current moment $I_{ij}$ has been computed in Eq. (4.7b). The second reason for the difference between Eq. (5.4) and the result of [31] is a computational error in [31] when they apply the integration formula (4.4) for computing the compact-support terms. We have already commented upon this error after Eq. (4.4) above. These two errors fully account for the discrepancy between our result and the one of Eq. (4.18) in Ref. [31].

Next, in order to derive the GW flux at 2.5PN order, we need the spin parts in the mass octupole and current octupole moments, but only at the lowest order in the spins. They can be obtained from our previous computation leading to Eqs. (3.6), after the CM reduction. We find

$$J_{ijk} = \frac{\nu}{c^3} \left[ -\frac{9}{2m} x^{i(j} x^{k)} + \frac{3}{2} (3-11\nu) x^{i(j} x^{k)} - \frac{3}{2} (3-11\nu) x^{i(j} x^{k)} \right]$$

$$J_{ijkl} = \frac{\nu}{c^3} \left[ -\frac{27}{42} x^{i(j} x^{k)} + \frac{3}{2} (3-11\nu) x^{i(j} x^{k)} \right]$$

B. The gravitational-wave energy flux

With all these moments, Eqs. (5.3), (5.4), and (5.5), and only with those, we can compute the 2.5PN spin part of the GW flux. Indeed, recall that the spins start at 1.5PN order in the mass moments and at 0.5PN order in the current ones, so one can easily see that in higher multipoles spins will enter the flux at higher PN order. On the other hand, one can check that it is not necessary to include the effects of tails of GWs, and more generally of any nonlinear multipole interaction. Indeed, the tails give a correction to each of the source-type multipole moments $I_{ij}$ and $J_{ij}$ at the relative order 1.5PN $\sim 1/c^3$ (see e.g. [28]). For the mass quadrupole $I_{ij}$ the spin itself is at order 1.5PN so the tail will arise only at order 3PN $\sim 1/c^5$ in the flux. For the current quadrupole $J_{ij}$ the spin is at 0.5PN but $J_{ij}$ comes in the flux at 1PN order, so again we see that the corresponding tail will only be at 3PN order in the flux. In conclusion, for this problem it is sufficient to express the flux solely in terms of the source multipole moments $I_{ij}$ and $J_{ij}$; all multipole interactions built in the radiative moments seen at infinity, namely $U_L$ and $V_L$, are negligible. Furthermore, as we have seen, only four multipole contributions are important for this application. Therefore [cf. Eq. (4.28) in [28]]

$$J_{ijkl} = -\nu \delta m x^{i} x^{j} e^{k(i} e^{l)} \left[ x^{j)} + \frac{1}{c^2} \left[ \frac{27}{14} + \frac{15}{4} \nu \right] \frac{Gm}{r} \right]$$

$$J_{ijkl} = -\nu \delta m x^{i} x^{j} e^{k(i} e^{l)} \left[ x^{j)} + \frac{1}{c^2} \left[ \frac{27}{14} + \frac{15}{4} \nu \right] \frac{Gm}{r} \right]$$

Note that for both $I_{ij}$ and $J_{ij}$ there are some contributions at 1.5PN order which depends on the spin variables and are generated from the Newtonian term evaluated in the CM; these contributions have been included in the results (5.3) and (5.4) above.

Finally, we obtain for the flux (in the general orbit case but in the CM frame) the structure

$$\mathcal{F} = \frac{8}{15} \frac{G^3 m^4 \nu^2}{c^3 r^4} \left[ f_N + \frac{1}{c} f_{1PN} + \frac{1}{c} f_{1.5PN} + \frac{1}{c} f_{1.5PN} + \frac{1}{c} f_{2.5PN} + \frac{1}{c} f_{2.5PN} \right]$$

The nonspin pieces $f_N, f_{1PN}, f_{1.5PN},$ and $f_{2.5PN}$ are already known and we shall need below the Newtonian and 1PN terms which are given by [50,52].
\[ f_N = 12v^2 - 11(nv)^2, \quad \text{(5.10a)} \]

\[ f_{1PN} = \left( \frac{785}{28} - \frac{213}{7} v \right) v^4 + \left( -\frac{1487}{14} + \frac{696}{7} v \right) (nv)^2 v^2 + \left( \frac{2061}{28} - \frac{465}{7} v \right) (nv)^4 + \left( -\frac{680}{7} + \frac{40}{7} v \right) \frac{GM}{r} v^2 + \left( \frac{734}{28} - \frac{30}{7} v \right) \frac{GM}{r} (nv)^2 + \left( \frac{4}{7} - \frac{16}{7} v \right) \left( \frac{GM}{r} \right)^2. \quad \text{(5.10b)} \]

Notice also that the nonspin terms \( f_{1.5PN} \) and \( f_{2.5PN} \) include the contributions of GW tails. Here we do not deal with the spin-spin (SS) term at 2PN order which is given in Refs. [39,41]. We obtain the SO coupling part at 1.5PN order as

\[ f_{1.5PN} = \frac{(S,n,v)}{mr} \left[ 78(nv)^2 - 8 \frac{GM}{r} - 80v^2 \right] + \frac{(\Sigma,n,v)}{mr} \times \left[ 51(nv)^2 + 4 \frac{GM}{r} - 43v^2 \right] \delta m \; m, \quad \text{(5.11)} \]

and for this part we find perfect agreement with Kidder et al. [39,41]. Finally, for the next-order SO part our result is

\[ f_{2.5PN} = \frac{(S,n,v)}{mr} \left[ (nv)^4 \left( \frac{597}{7} - \frac{13269}{7} v \right) + \frac{Gm^2}{r^2} \left( 104 + \frac{898}{7} v \right) + \frac{GM}{r} (nv)^2 \left( \frac{2158}{7} - \frac{1422}{7} v \right) + (nv)^2 v^2 \left( -\frac{4896}{7} \right) \right] + \frac{(\Sigma,n,v)}{mr} \left[ (nv)^4 \left( -\frac{2613}{28} - \frac{5079}{7} v \right) \right] \left( \frac{(nv)^4}{7} - \frac{2613}{28} - \frac{5079}{7} v \right) \delta m \; m. \quad \text{(5.12)} \]

VI. FREQUENCY AND PHASE EVOLUTION BY RADIATION REACTION

In this section we compute the time evolution of the binary’s orbital frequency \( \omega \), which results from the gravitational radiation reaction damping force. Instead of computing directly the reaction force, we use the standard energy balance argument

\[ \mathcal{F} = -\frac{dE}{dt}, \quad \text{(6.1)} \]

where \( \mathcal{F} \) is the total emitted GW energy flux computed in Sec. V, and \( E \) denotes the binary’s center-of-mass energy, namely, the integral of the motion associated with the conservative part of the equations of motion, and which has been computed in paper I.

A. Flux, energy, and angular momentum for circular orbits

From now on, we assume that when the binary enters the frequency bandwidth of the LIGO/Virgo/LISA detectors the orbit has been circularized by the gravitational radiation reaction effect. By circular orbit we mean an orbit which is circular when the gradual radiation reaction inspiral can be neglected, and when the effects of spins are averaged over time. With such proviso there is a well-defined notion of a circular orbit (see Refs. [39,41] and paper I).

For circular orbits the orbital frequency \( \omega \) is linked to the distance \( r \) between particles in harmonic coordinates by a relativistic extension of Kepler’s law, which has already been given in paper I for what concerns the SO effects. Let us write it again here, but let us also add to it, for the benefit of potential users of these formulas, all the nonspin contributions up to the 2.5PN order, following known results from the literature (e.g. [21] and references therein). The 3PN and 3.5PN nonspin terms, computed in Refs. [20,23], can be added straightforwardly if necessary. However, for convenience in this paper, we shall not display the nonlinear spin-spin (SS) terms. Thus, all formulas of this section will be complete up to 2.5PN order at linear order in the spins (i.e. but for the SS contributions). We have

\[ \omega^2 = \frac{GM}{r^3} \left[ 1 + \gamma(-3 + v) + \frac{\gamma^2(6 + \frac{41}{4} \gamma + v^2)}{v^2} \right] + \frac{\gamma^{3/2}}{Gm^2} \left[ -5S_{\ell} - 3 \frac{\delta m}{m} \Sigma_{\ell} \right] + \frac{\gamma^{5/2}}{Gm^2} \left[ \left( \frac{39}{2} - 23 \gamma \right) S_{\ell} \right] + \left( \frac{21}{2} - \frac{11}{2} \frac{1}{v} \right) \frac{\delta m}{m} \Sigma_{\ell} \right] + O(\frac{1}{\gamma^3}), \quad \text{(6.2)} \]

in which \( \gamma \equiv \frac{GM}{r^3} = O(c^{-2}) \) denotes the harmonic-coordinate PN parameter. We recognize the lowest-order (1.5PN \( \sim \gamma^{3/2} \)) spin-orbit term and its 1PN correction at the 2.5PN \( \sim \gamma^{5/2} \) level. Here, as in paper I, we introduce an orthonormal triad \( \{ \mathbf{n}, \mathbf{\lambda}, \mathbf{\ell} \} \) defined by \( \mathbf{n} = \mathbf{x}/r, \; \mathbf{\ell} = \mathbf{L}_N/|\mathbf{L}_N|, \) where \( \mathbf{L}_N = \mu \mathbf{x} \times \mathbf{v} \) denotes the Newtonian angular momentum, and \( \mathbf{\lambda} = \mathbf{\ell} \times \mathbf{n} \). The quantities \( S_\ell \) and \( \Sigma_\ell \) in Eq. (6.2) are the components of the spin vectors (5.1) perpendicular to the orbital plane, namely \( S_\ell = \mathbf{S} \cdot \mathbf{\ell} \) and \( \Sigma_\ell = \mathbf{\Sigma} \cdot \mathbf{\ell} \). The relation (6.2) can be inverted to give \( \gamma \) in terms of an alternative PN parameter \( x = \gamma^{(GM/c^3)} \). As usual, it is better to express the PN formulas in terms of
the frequency-dependent PN parameter $x$ rather than $\gamma$ because they are invariant under a large class of gauge transformations. Hence,

$$
\gamma = x\left[1 + x\left(1 - \frac{\nu}{3}\right) + x^2\left(1 - 65\frac{\nu}{3}\right) + \frac{x^3/2}{Gm_1^2} \left(3 - \frac{\nu}{3}\right) \delta m \Sigma_{\ell} \right] + \frac{x^{5/2}}{Gm_1^2} \left[-\frac{13}{3} + \frac{2}{9} \nu\right] S_{\ell} + \left(3 - \frac{\nu}{3}\right) \frac{\delta m}{m} \Sigma_{\ell} + O\left(\frac{1}{c^6}\right).
$$

(6.3)

The SO term at order $1.5\text{PN} \sim x^{3/2}$ is in agreement with Eq. (16) in [41].

The reduction of the GW flux $F$, given by Eqs. (5.11) and (5.12), to circular orbits is straightforward, but care has to be taken from the fact that the nonspin parts of the flux at Newtonian and 1PN orders yield crucial contributions to the SO terms for circular orbits [beside the ones given by straightforward reduction of Eqs. (5.11) and (5.12)]. Such contributions are generated by replacement of Eq. (6.2) into the 1PN flux given for general orbits by Eq. (5.10). Finally, we obtain

$$
F = \frac{32}{5} \frac{c^5}{G} \gamma^2 \nu^2 \left[1 + \gamma\left(-\frac{2927}{336} - \frac{5}{4}\nu\right) + 4\pi \gamma^{3/2} + \gamma^2\left(-\frac{380}{9}\nu\right) + \pi \gamma^{5/2}\left(-\frac{25,663}{672}\right)ight] 
$$

$$
- \frac{125}{8}\nu + \frac{2\gamma^{3/2}}{Gm_1^2}\left[-\frac{37}{3} S_{\ell} - \frac{25}{4} \frac{\delta m}{m} \Sigma_{\ell}\right] 
$$

$$
+ \frac{\gamma^{5/2}}{Gm_1^2}\left[-\frac{14,493}{5} S_{\ell} + \frac{5}{3} \frac{\delta m}{m} \Sigma_{\ell}\right] 
$$

$$
+ \left(\frac{657}{112} - \frac{269}{168}\nu\right) \frac{\delta m}{m} \Sigma_{\ell} + O\left(\frac{1}{c^6}\right),
$$

(6.4)

or, equivalently, in terms of the PN parameter $x$,

$$
F = \frac{32}{5} \frac{c^5}{G} x^2 \nu^2 \left[1 + x\left(-\frac{1247}{336} - \frac{35}{12}\nu\right) + 4\pi x^{3/2} + x^2\left(-\frac{9271}{9072} + \frac{504}{18}\nu\right) + \frac{x^{3/2}}{Gm_1^2}\left[-4S_{\ell}\right]
$$

$$
- \frac{5}{4} \frac{\delta m}{m} \Sigma_{\ell} + \frac{x^{5/2}}{Gm_1^2}\left[-\frac{95}{28} + \frac{239}{63}\nu\right] S_{\ell} + \left(\frac{31}{16} - \frac{109}{28}\nu\right) \frac{\delta m}{m} \Sigma_{\ell} + O\left(\frac{1}{c^6}\right).
$$

(6.5)

All the nonspin terms are included up to $2.5\text{PN}$ order. Notice in particular the nonspin terms, proportional to $\pi$, which are at the same $1.5\text{PN}$ and $2.5\text{PN}$ orders as the SO effects; these terms are due to GW tails [20,21]. For the leading SO term at order $1.5\text{PN} \sim x^{3/2}$ we find perfect agreement with Eq. (17b) in [41].

The reduction of the center-of-mass energy $E$ (computed in Sec. VII of paper I) to circular orbits is straightforward, and we simply report here the final result, completing it by the known nonspin terms [53]. We have

$$
E = -\frac{\mu c^2 x^2}{2}\left[1 + x^2\left(-\frac{3}{4} - \frac{3}{12}\nu\right) + \gamma^2\left(-\frac{27}{8} + \frac{19}{8}\nu - \frac{\nu^2}{24}\right)
$$

$$
+ \frac{x^{3/2}}{Gm_1^2}\left[\frac{14}{3} S_{\ell} + 2 \frac{\delta m}{m} \Sigma_{\ell}\right] + \frac{\gamma^{5/2}}{Gm_1^2}\left[-\frac{13}{4} - \frac{9}{4}\nu\right] S_{\ell}
$$

$$
+ \left(\frac{5}{8} - \frac{3}{4}\nu\right) \frac{\delta m}{m} \Sigma_{\ell} + O\left(\frac{1}{c^6}\right).\right]
$$

(6.6)

or, equivalently,

$$
E = -\frac{\mu c^2 x^2}{2}\left[1 + x\left(-\frac{3}{4} - \frac{3}{12}\nu\right) + x^2\left(-\frac{27}{8} + \frac{19}{8}\nu - \frac{\nu^2}{24}\right)
$$

$$
+ \frac{x^{3/2}}{Gm_1^2}\left[\frac{14}{3} S_{\ell} + 2 \frac{\delta m}{m} \Sigma_{\ell}\right] + \frac{\gamma^{5/2}}{Gm_1^2}\left[-\frac{13}{4} - \frac{9}{4}\nu\right] S_{\ell}
$$

$$
+ \left(\frac{5}{8} - \frac{3}{4}\nu\right) \frac{\delta m}{m} \Sigma_{\ell} + O\left(\frac{1}{c^6}\right).\right]
$$

(6.7)

Alternatively, in terms of the single-spin variables the spin-dependent part of the above equation reads

$$
E = -\frac{\mu c^2 x^2}{2}\sum_{i=1,2} \chi_i \kappa_i \left[\frac{8}{3} m_i^2 \epsilon_i^2 + 2 \nu\right]
$$

$$
+ \frac{\gamma^{3/2}}{Gm_1^2}\left[\frac{8}{3} \frac{m_i^2}{m^2} \left(\frac{25}{9}\nu + \nu^2\right)\right],
$$

(6.8)

where we denote by $\kappa_i = \vec{S}_i \cdot \vec{\ell}$ for $i = 1, 2$ the orientation of the spins with respect to the Newtonian angular momentum.

**TABLE I.** Energy and angular frequency at the ICO for equal-mass ($\nu = \frac{3}{2}$) binary systems. The spins are maximal ($\chi_i = 1$) and have different orientations ($\kappa_i = 0, \pm 1$). In three cases, indicated by dots, there is no ICO, i.e., the energy function admits no real minimum. Spin-spin effects at 2PN order are included.

<table>
<thead>
<tr>
<th>PN Order</th>
<th>$\kappa_i$</th>
<th>$m_{\text{ICO}}$</th>
<th>$E_{\text{ICO}}/m$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1PN</td>
<td>$\kappa_i = 0$</td>
<td>0.522</td>
<td>-0.0405</td>
</tr>
<tr>
<td></td>
<td>$\kappa_i = +1$</td>
<td>$\cdots$</td>
<td>$\cdots$</td>
</tr>
<tr>
<td></td>
<td>$\kappa_i = -1$</td>
<td>0.111</td>
<td>-0.0163</td>
</tr>
<tr>
<td>2PN</td>
<td>$\kappa_i = 0$</td>
<td>0.137</td>
<td>-0.0199</td>
</tr>
<tr>
<td></td>
<td>$\kappa_i = +1$</td>
<td>0.318</td>
<td>-0.0390</td>
</tr>
<tr>
<td></td>
<td>$\kappa_i = -1$</td>
<td>0.0733</td>
<td>-0.0130</td>
</tr>
<tr>
<td>2.5PN</td>
<td>$\kappa_i = 0$</td>
<td>0.137</td>
<td>-0.0199</td>
</tr>
<tr>
<td></td>
<td>$\kappa_i = +1$</td>
<td>$\cdots$</td>
<td>$\cdots$</td>
</tr>
<tr>
<td></td>
<td>$\kappa_i = -1$</td>
<td>0.060</td>
<td>-0.0117</td>
</tr>
<tr>
<td>3PN</td>
<td>$\kappa_i = 0$</td>
<td>0.129</td>
<td>-0.0193</td>
</tr>
<tr>
<td></td>
<td>$\kappa_i = +1$</td>
<td>$\cdots$</td>
<td>$\cdots$</td>
</tr>
<tr>
<td></td>
<td>$\kappa_i = -1$</td>
<td>0.059</td>
<td>-0.0116</td>
</tr>
</tbody>
</table>

For the nonspin tail term at 2.5PN order we take into account the published erratum to [20].
momentum, and by $\chi_i$ their magnitude defined in the standard way by $S_i = Gm_i^2 \chi_i \hat{S}_i$.

Assuming nonprecessing orbits, we list in Table I the energy and the frequency at the so-called innermost circular orbit (ICO) [54]. The ICO is defined by the minimum of the center-of-mass energy for circular orbits expressed as a function of the orbital frequency $\omega$, and is computed from Eq. (6.7).

The orbital angular momentum (computed in paper I) in the case of circular orbits reads

$$
\mathbf{L} = \frac{Gm^2}{c} \nu^{-1/2} \left\{ \left[ 1 + 2\gamma \right] + \left( -3S_\ell - \Sigma_\ell \frac{\delta m}{m} \right) \frac{\gamma^{3/2}}{Gm^2} + \left( \frac{5}{2} - \frac{9}{2} \nu \right) \gamma^2 + \left( -\frac{59}{8} + \frac{25}{8} \nu \right) S_\ell + \left( \frac{27}{8} + \frac{3}{2} \nu \right) \frac{\delta m}{m} \right\} \frac{\gamma^{5/2}}{Gm^2} + \frac{\gamma^{3/2}}{Gm^2} \left\{ \left( -\frac{3}{2} S_\lambda - \frac{1}{2} \Sigma_\lambda \frac{\delta m}{m} \right) \left[ 1 - \frac{9}{8} \nu \right] S_\lambda + \left( -\frac{9}{8} - \frac{15}{4} \nu \right) \frac{\delta m}{m} \right\} \frac{\gamma^{5/2}}{Gm^2} + \frac{\gamma^{3/2}}{Gm^2} \left\{ \frac{5}{2} S_n + \frac{3}{2} \Sigma_n \frac{\delta m}{m} \right\} \frac{\gamma^{5/2}}{Gm^2} + \frac{\gamma^{3/2}}{Gm^2} \left\{ \left( -\frac{13}{8} - \frac{13}{8} \nu \right) S_n + \left( \frac{15}{8} - \frac{3}{4} \nu \right) \frac{\delta m}{m} \right\} \frac{\gamma^{3/2}}{Gm^2},
$$

(6.9)
or equivalently

$$
\mathbf{L} = \frac{Gm^2}{c} \nu^{-1/2} \left\{ \left[ 1 + 2\gamma \right] + \left( -3S_\ell - \Sigma_\ell \frac{\delta m}{m} \right) \frac{\gamma^{3/2}}{Gm^2} + \left( \frac{27}{8} - \frac{19}{8} \nu \right) \gamma^2 + \left( -\frac{77}{8} + \frac{259}{72} \nu \right) S_\ell + \left( \frac{33}{8} + \frac{21}{4} \nu \right) \frac{\delta m}{m} \right\} \frac{\gamma^{5/2}}{Gm^2} + \frac{\gamma^{3/2}}{Gm^2} \left\{ \left( -\frac{3}{2} S_\lambda - \frac{1}{2} \Sigma_\lambda \frac{\delta m}{m} \right) \left[ 1 - \frac{9}{8} \nu \right] S_\lambda + \left( -\frac{9}{8} - \frac{43}{12} \nu \right) \frac{\delta m}{m} \right\} \frac{\gamma^{5/2}}{Gm^2} + \frac{\gamma^{3/2}}{Gm^2} \left\{ \frac{7}{8} S_n + \frac{59}{24} \nu S_n + \frac{27}{8} - \frac{5}{4} \nu \right\} \frac{\delta m}{m} \frac{\gamma^{3/2}}{Gm^2},
$$

(6.10)

For future use we give here the precessional equations evaluated in paper I, but reduced to circular orbits:

$$
\frac{d\mathbf{S}}{dt} = \nu \mathbf{\omega} \left[ \left[ -4S_\ell - \frac{2}{m} \delta \Sigma_\ell \right] \mathbf{n} + \left[ 3S_n + \frac{2}{m} \delta \Sigma_n \right] \mathbf{\lambda} + \left[ \frac{20}{3} \nu S_\lambda + \left( -2 - \frac{10}{3} \nu \right) \frac{\delta m}{m} \right] \mathbf{\Sigma} \right] + \mathcal{O}\left( \frac{1}{\epsilon^5} \right),
$$

(6.11a)

$$
\frac{d\mathbf{\Sigma}}{dt} = \mathbf{\omega} \left[ \left[ -2 + 4\nu \right] S_\ell - \frac{2}{m} \delta \Sigma_\lambda \right] \mathbf{n} + \left[ \left( 1 - \nu \right) \Sigma_n + \frac{2}{m} \Sigma_n \right] \mathbf{\lambda} + \frac{2}{3} \left[ \frac{17}{3} \nu + \frac{20}{3} \nu^2 \right] \Sigma_\lambda \mathbf{n} + \left[ \frac{16}{3} \nu - \frac{16}{3} \nu^2 \right] \Sigma_\ell \mathbf{n} + \mathcal{O}\left( \frac{1}{\epsilon^5} \right).
$$

(6.11b)

We recall our notation (5.1) for the spin variables. We denote by $S_n, \Sigma_n$ and $S_\ell, \Sigma_\lambda$ the components of the spins along the vectors $\mathbf{n}$ and $\mathbf{\lambda}$, respectively. As before we have neglected the SS terms.

To compare easily with previous results in the literature, we have also computed the precessing equations for the spin variables $\mathbf{S}_1$ and $\mathbf{S}_2$. They read

$$
\frac{d\mathbf{S}_1}{dt} = \omega \nu \left[ S_{1n} \mathbf{\lambda} \left( 2 + \frac{m_2}{m_1} \right) - 2S_{1n} \mathbf{n} \left( 1 + \frac{m_2}{m_1} \right) + \frac{xS_{1n} \mathbf{\lambda} \left( 9 \frac{m_1^2}{m_2^2} + \frac{37}{3} \frac{m_1 m_2}{m_2^2} + \frac{11}{3} \frac{m_2^2}{m_2^2} \right)}{\epsilon^5} \right] + \mathcal{O}\left( \frac{1}{\epsilon^6} \right),
$$

(6.12a)

$$
\frac{d\mathbf{S}_2}{dt} = \omega \nu \left[ S_{2n} \mathbf{\lambda} \left( 2 + \frac{m_1}{m_2} \right) - 2S_{2n} \mathbf{n} \left( 1 + \frac{m_1}{m_2} \right) + \frac{xS_{2n} \mathbf{\lambda} \left( 9 \frac{m_1^2}{m_2^2} + \frac{37}{3} \frac{m_1 m_2}{m_2^2} + \frac{11}{3} \frac{m_2^2}{m_2^2} \right)}{\epsilon^5} \right] + \mathcal{O}\left( \frac{1}{\epsilon^6} \right).
$$

(6.12b)

where $S_{1n}, S_{1n}$ and $S_{2n}, S_{2n}$ are the projections of the single-spin variables along $\mathbf{\lambda}$ and $\mathbf{n}$. We notice that with our choice of spin variables $\mathbf{S}_1$ and $\mathbf{S}_2$, the magnitude of the spin is not constant even when restricting Eqs. (6.12) to the 1.5PN order. In Sec. VII we define some alternative spin variables $\mathbf{S}_i^\prime$ and $\mathbf{S}_2^\prime$, such that the magnitude of these spin vectors remains constant, i.e. $\mathbf{S}_i^\prime \cdot d\mathbf{S}_i^\prime / dt = 0$ with $i = 1, 2$. The spin vectors $\mathbf{S}_i^\prime$ agree with Kidder’s [39] spin vari-
ables at the 1PN order, and generalize them to the next 2PN order. The main advantage of the definition $S_i^x$ is that the precession equations can be then written in the form $dS_i^x/dt = \Omega_i \times S_i^x$, where $\Omega_i$ are the precession angular frequency vectors (given in Sec. VII).

B. Phase evolution and accumulated number of GW cycles

Finally, having in hand both the flux function $\mathcal{F}(x)$ and the energy function $E(x)$, we deduce the time evolution of the orbital frequency $\omega$ from the energy balance Eq. (6.1), which is equivalent to

$$\frac{d\omega}{\omega} = \frac{3}{2x} \left( \frac{dE(x)}{dx} \right)^{-1} \mathcal{F}(x). \quad (6.13)$$

During this computation the standard PN approximation is applied, i.e. we expand both the numerator and the denominator of Eq. (6.13) in the usual PN way, and finally express the result as a Taylor series in $x$. Other ways of addressing the computation, using particular PN resummation techniques, can be found in Refs. [14,55] and references therein. We give the end result for the parameter $\xi = \omega/\omega^2$, which can be viewed as the dimensionless adiabatic parameter associated with the gradual inspiral, and which is dominantly of 2.5PN order (namely, the order of radiation reaction). In the final result, as everywhere else, the SO effects are at the 1.5PN and 2.5PN orders beyond the dominant approximation. We get

$$\omega \approx \frac{96}{5} \nu x^{5/2} \left[ 1 + x \left( 743 - 336 \right) \right] + 4 \pi x^{3/2}
$$

$$+ x^2 \left( 34103 \right) 1844 \nu + \frac{13661}{16} \nu^2 \right)
$$

$$+ \frac{\pi}{2} \left( 4519 \right) 672 \nu + \frac{189}{8} \nu^2 \right)
$$

$$+ \frac{25}{4} \delta \left[ \frac{40}{1008} \nu + \frac{1465}{28} \nu \right] S_{\ell}
$$

$$+ \left( \frac{583}{42} \right) 3049 \nu \delta \left[ \frac{3049}{168} \nu \right] S_{\ell}
$$

$$+ \mathcal{O} \left( \frac{1}{\nu} \right). \quad (6.14)$$

If necessary the nonspin contributions at orders 3PN and 3.5PN can be straightforwardly added [20,23].

In the general case, taking into account the effect of precession of the orbital plane induced by spin modulations, the GW phase $\Phi_{GW}$ is given by $\Phi_{GW} = \phi_{GW} + \delta \phi_{GW}$, where $\phi_{GW}$ is the “carrier phase,” defined by $\phi_{GW} = 2\phi$ with $\phi = \int \alpha dt$, and $\delta \phi_{GW}$ is a standard precessional correction, arising from the changing orientation of the orbital plane. The precessional correction $\delta \phi_{GW}$ can be computed by standard methods using numerical integration, see Ref. [9]. Thus, the carrier phase $\phi_{GW}$ constitutes the main theoretical output to be provided for the templates, and can directly be computed numerically from our main result, Eq. (6.14). In the absence of the orbital-plane’s precession, e.g., for spins aligned or anti-aligned with the orbital angular momentum, the GW phase reduces to $\phi_{GW}$, and the latter can be obtained by integrating analytically $\phi_{GW}$. We get

$$\phi = \phi_0 - \frac{1}{32\nu} \left[ x^{-5/2} + x^{-3/2} \right]
$$

$$+ \frac{x^{-1}}{Gm^2} \left( \frac{325}{6} S_{\ell} + 125 \delta m \right) \left[ \frac{125}{\nu} \right] - 10\pi x^{-1}
$$

$$+ \frac{x^{-1/2}}{Gm^2} \left[ \frac{1016064}{1529365} + \frac{27145}{1008} \nu + \frac{3085}{144} \nu^2 \right]
$$

$$+ \frac{\pi}{2} \left[ \frac{38645}{1344} - \frac{65}{16} \nu \right] + \frac{\ln \left[ \left( \frac{549845}{2016} \right) - \frac{14215}{168} \nu \right]}{\ln \left[ \left( - \frac{549845}{2016} \right) - \frac{14215}{168} \nu \right]}. \quad (6.15)$$

where $\phi_0$ denotes some constant phase. In terms of the single-spin variables $S_1$ and $S_2$, the spin-dependent part of the above equation reads

$$\phi_s = - \frac{1}{32\nu} \sum_{i=1,2} \chi_i \left[ \frac{565}{24} \nu + \frac{125}{8} \nu \right] x^{-1}
$$

$$+ \left[ \frac{68145}{4032} - \frac{925}{21} \nu \right] \left[ \frac{m_i^2}{m} \right]
$$

$$+ \left[ - \frac{68150}{448} - \frac{6815}{168} \nu \right] \ln]. \quad (6.16)$$

The number of accumulated GW cycles between some minimal and maximal frequencies is

$$\mathcal{N}_{GW} = \frac{\phi_{max} - \phi_{min}}{\pi}. \quad (6.17)$$

We list in Tables II and III the number of accumulated GW cycles (6.17) for typical binary masses in the most sensitive frequency band of ground-based and space-based detectors. For comparison, we also show the contribution due to spin-spin terms at 2PN order evaluated in Ref. [41], as well as those due to the nonspin 3PN and 3.5PN orders computed in [20,23]. We denote $\xi = \dot{S}_1 \cdot \dot{S}_2$. From Tables II and III, we deduce two important results of this paper. First, we see that at 2.5PN order, if spins are maximal, i.e. $\chi_i = 1$, the number of GW cycles due to spin couplings is comparable to the number of GW cycles due to nonspin terms. Second, we find that, for small mass-ratio binaries, the number of GW cycles due to linear spins at 2.5PN order can be much larger than the number of GW cycles due to spin-spin terms at 2PN order. These results thus show that the 2.5PN spin terms evaluated in the present paper have to be included in the GW templates if we want to extract accurately the binary parameters.

The number of accumulated GW cycles can be a useful diagnostic to understand the importance of spin effects, but taken alone it provides incomplete information. First, $\mathcal{N}_{GW}$ is related only to the number of orbital cycles of
the binary within the orbital plane, but it does not reflect the precession of the plane, which modulates the wave form in both amplitude and phase. These modulations are important effects. In fact, it has been shown [13, 14, 16] that neither the standard nonspinning-binary template (which do not have built-in modulations) nor the original Apostolatos templates [10] (which add only modulations to the phase) can reproduce satisfactorily the detector response to the GWs emitted by precessing binaries. Modulations both in the phase and the amplitude of the wave form must be included [14, 15, 17, 19]. Second, even if two signals have $N_{GW}$s that differ by $\sim 1$, one can always shift their arrival times to obtain higher overlaps, but at the cost of introducing systematic errors in the binary parameters. To quantify the impact of the 2.5PN spin terms in detecting GWs from spinning, precessing binaries, one should evaluate the maximized overlap (fitting factor) between the 2.5PN template family and the 2PN template family used in Refs. [14, 15, 17, 19]. Those template families are defined by the GW signal computed along the binary evolution together with the spin and angular-momentum precession equations. We expect that the maximized overlap between the 2.5PN and 2PN templates could be high, because the spins and the directional parameters entering the template families provide much leeway to compensate for nontrivial variations of the phasing (see e.g., Table II in Ref. [15] where maximized overlaps between several PN templates were computed). This study goes beyond the goal of this paper and will be tackled in future work.

VII. SPIN VARIABLES WITH CONSTANT MAGNITUDE

In this paper and paper I, we found convenient to use some specific spin variables $S_1$ and $S_2$, defined in Sec. II of paper I. However, as discussed in paper I, other papers in the literature use a definition of the spin variables different from ours. For example, the spin-precession equations at 1PN order in Ref. [39] read

$$\frac{dS_1}{dt} = \omega \nu \times (S_1^c \times S_1^c) \left( 2 + \frac{3 m_2}{2 m_1} \right) \tag{7.1a}$$

$$\frac{dS_2}{dt} = \omega \nu \times (S_2^c \times S_2^c) \left( 2 + \frac{3 m_1}{2 m_2} \right) \tag{7.1b}$$

where the superscript $c$ stands for constant; in fact, the spin variables $S_1^c$, $S_2^c$ are such that their norm or magnitude remains constant. Indeed one can readily check from Eqs. (7.1) that $S_i \cdot dS_i^c/dt = 0$ with $i = 1, 2$. Our spin variables are related to the 1PN order to the constant-spin ones (in the center-of-mass and for circular orbits) as

$$S_1^c = \left( 1 + \frac{Gm_2}{c^2 r} \right) S_1 - \frac{m_2^2}{2c^2 m_1} S_{1a} r^2 \omega^2 \mathbf{a}. \tag{7.2a}$$

$$S_2^c = \left( 1 + \frac{Gm_1}{c^2 r} \right) S_2 - \frac{m_1^2}{2c^2 m_2} S_{2a} r^2 \omega^2 \mathbf{a}. \tag{7.2b}$$
We can check that, by taking the time derivative of the RHS of Eqs. (7.2), plugging in Eqs. (6.12) at 1PN order, we recover Eqs. (7.1). Note that the total angular momentum is invariant, since

$$ J = L + \frac{1}{c} S_1 + \frac{1}{c} S_2 = L + \frac{1}{c} S_1^i + \frac{1}{c} S_2^i. \quad (7.3) $$

Let us now define, at the 2PN order, in a general frame and for general orbits, some spin variables reducing to $S_i^i$ and $S_i^j$ at the 1PN order, and such that the magnitude of these spins remains constant. We shall still denote the latter 2PN spins as $S_i^i$, $S_i^j$; thus, we shall have, at the 2PN order, $S_i \cdot dS_j/dt = 0$ with $i = 1, 2$. First of all, we find that the new spin variables are related to the ones used in previous sections (and in the whole of paper I) by

$$ S_i^i = S_i + \frac{1}{c^4} \left[ -\frac{1}{2} (v_i S_i^i) v_i + \frac{G m_i}{r_{ij}} S_i^i \right] + \frac{1}{c^4} \left[ n_{ij} \frac{G m_i}{r_{ij}} (n_{ij} v_i) - \frac{4}{3} \frac{G m_i}{r_{ij}} S_i^i \right] \times \left( -\frac{1}{2} (v_i S_i^i) v_i + \frac{G m_i}{r_{ij}} S_i^i \right) + \frac{G m_i}{r_{ij}} S_i^i \left( -\frac{1}{2} (v_i S_i^i) v_i + \frac{G m_i}{r_{ij}} S_i^i \right) + \frac{1}{2} \left( -\frac{1}{v_i S_i^i} + \frac{G m_i}{r_{ij}} \left( -\frac{5}{2} (v_i S_i^i) + 4 (v_i S_i^i) \right) \right) + 2 \frac{G m_i}{r_{ij}} S_i^i \left( v_i S_i^i \right), \quad (7.4) $$

together with the expression for $S_i^j$ obtained by exchanging all the particle’s labels $1 \leftrightarrow 2$. In Eq. (7.4) the notation is exactly the same as in paper I. The main advantage of such definition (7.4) is that the precession equations can now be written into the form

$$ \frac{dS_i^i}{dt} = \Omega_1 \times S_i^i, \quad (7.5a) $$
$$ \frac{dS_i^j}{dt} = \Omega_2 \times S_i^j, \quad (7.5b) $$

showing that the spins precess around the directions of $\Omega_1$ and $\Omega_2$, and at the rates $|\Omega_1|$ and $|\Omega_2|$. The precession angular frequency vectors $\Omega_1$, $\Omega_2$ can be computed up to the 2PN order by using the precession Eqs. (6.4), (6.2), and (6.3) of paper I. We find

$$ \Omega_1 = \frac{G m_i}{c r_{ij}^2} \left[ \frac{3}{2} n_{ij} \times v_i - 2 n_{ij} \times v_j \right] + \frac{G m_j}{c r_{ij}^2} \left[ n_{ij} \times v_i \left( -\frac{9}{4} (n_{ij} v_j) + \frac{1}{8} v_j (v_i v_j) \right) + v_j^2 + \frac{7}{2} \frac{G m_j}{r_{ij}} \left( -\frac{1}{2} \frac{G m_j}{r_{ij}} \right) + n_{ij} \times v_j \left( 3 (n_{ij} v_j) \right) + 2 (v_i v_j) - 2 v_i^2 + \frac{9}{2} \frac{G m_j}{r_{ij}} \left( -\frac{1}{2} \frac{G m_j}{r_{ij}} \right) + 2 \frac{G m_j}{r_{ij}} \left( v_i v_j \right) \right] + \frac{\Omega}{c^4}, \quad (7.6) $$

where we denote $\kappa_i^j = S_i^j \cdot \ell$ for $i = 1, 2$, and $\chi_i^j$ is defined by $S_i^j = G m_i \chi_i^j \hat{S}_i^j$. Using the energy (7.9) as function of the constant spin variables, we have computed the ICO. For an equal-mass binary with spins antialigned with the orbital angular momentum, at 2.5PN (3PN) order, we get $E_{\text{ICO}}/m = -0.0122$ ($E_{\text{ICO}}/m = -0.0119$) and $m \omega_{\text{ICO}} = 0.064$ ($m \omega_{\text{ICO}} = 0.061$) to be compared with the numbers listed in Table II. The difference is not negligible.

We also computed the spin-dependent part of the orbital angular momentum, the flux and $\dot{\omega}$ in terms of the spin variables with constant magnitude,
Equations (7.5), (7.8), (7.9), and (7.12) with nonspin terms added through 3.5PN order and spin-spin terms included, together with the equation describing the rate of change of the orbital angular-momentum direction (deduced from \( L^c = -\frac{Gm^2}{c} \nu x^{-1/2} \left\{ \left( -\frac{35}{6} S^c_x - \frac{5}{2} \delta m m \Sigma c \right) x^{3/2} + \left( \frac{11}{8} - \frac{19}{24} \nu \right) S^c_n + \left( \frac{11}{8} - \frac{5}{12} \nu \right) \frac{\delta m m \Sigma c}{Gm^2} x \right. \)

\[ (7.10) \]

\[ T_s = \frac{32}{5} \frac{c^5}{G} x^s \nu^2 \left[ 1 + \frac{x^{3/2}}{Gm^2} \left( -4 S^c - \frac{5 \delta m m \Sigma c}{4 m} \right) + \frac{x^{5/2}}{Gm^2} \left( \frac{65}{14} + \frac{428}{63} \nu \right) S^c_n + \left( \frac{51}{16} - \frac{67}{28} \nu \right) \frac{\delta m m \Sigma c}{Gm^2} \right] + \mathcal{O} \left( \frac{1}{c^6} \right). \]

\[ (7.11) \]

\[ \left( \frac{\omega}{\omega_s} \right) = \frac{96}{5} \frac{c^5}{G} \nu^2 \left[ 1 + \frac{x^{3/2}}{Gm^2} \left( -4 S^c - \frac{5 \delta m m \Sigma c}{4 m} \right) + \frac{x^{5/2}}{Gm^2} \left( \frac{65}{14} + \frac{428}{63} \nu \right) S^c_n + \left( \frac{51}{16} - \frac{67}{28} \nu \right) \frac{\delta m m \Sigma c}{Gm^2} \right] + \mathcal{O} \left( \frac{1}{c^6} \right). \]

\[ (7.12) \]

Using Eq. (7.14) we have computed the number of GW cycles in LIGO/Virgo frequency band for some binary mass configurations. We find that the number of GW cycles computed using Eq. (6.15) is of the same order of magnitude as computed using Eq. (7.13). However, the difference is not negligible. For example, using constant spin variables, the spin contribution at 2.5PN order is \( 32.1 \kappa_1^2 \chi_1^2 + 2.8 \kappa_2 \chi_2^2 \) for a \((10 + 1.4)M_\odot\) binary \([5.6 \kappa_1^0 \chi_1^2 + 5.6 \kappa_2 \chi_2^2]\) for a \((10 + 10)M_\odot\) binary. Those numbers should be compared with the numbers listed in Table II.

Finally, let us comment on the fact that there are different possible choices for the spin variables, and correlatively different possible choices for the GW templates of spinning binaries. We may for instance use the templates based on our original definition for the spin vectors, satisfying the covariant spin supplementary condition (SSC) \( \hat{S}^\mu \hat{u}_\mu = 0 \) (see paper I); or the templates defined from the constant spin variables, which satisfy some more complicated SSC. In these two cases, the GW phase would be given either by (6.16) or (7.14). Definitely the templates using different spin variables satisfying different SSC are different. However, they are expected to carry exactly the same physics. Indeed, in the description of extended spinning bodies, it is known that different SSC correspond in fact to different choices for the central line of the extended body, with respect to which the spin angular momentum is defined (see [39] for a discussion). Here, the extended spinning body is modeled by a spinning “point particle” though this model differs from that of a genuine pointlike object because of the appearance of dipolar-type terms in the stress-energy tensor. In such a model, there is no notion of the central world line of the body (since there is only one trajectory—that of the particle). Nonetheless, it has been shown [34] that, in the description of spinning point particles, the arbitrariness in the choice of the SSC reflects the freedom in the choice of the central world line of an extended body. Practically speaking, this means that whatever definitions of the template the experimenters use to...
measure spinning binaries, for instance (6,16) or (7,14), the results should be physically equivalent, i.e. should correspond to the same physical system. As we have seen, the values measured for the spins can be significantly different, depending on the choice made for the SSC. The meaningful result of the data analysis of spinning binaries consists of the measurement of some particular spin values, having specified which particular SSC they correspond to, with the SSC being encoded through the definition of the templates used for the measurement.

VIII. CONCLUSIONS

Within the multipole-moment formalism developed in Refs. [25–30], we obtained the SO couplings, 1PN order beyond the dominant effect, in the binary’s mass and current-quadrupole moments, as well as in the GW energy flux. The current quadrupole moment with SO couplings at 1.5PN order was derived in Ref. [31], but our result differs from the expression computed there for two reasons. (i) The authors of Ref. [31] neglected the noncompact-support terms which originate from the nonlinearities of the Einstein field equations and are not negligible at this order. (ii) Their result for the compact-support terms is affected by a computational error. The mass-quadrupole moment with SO couplings at 2.5PN order (including all compact and noncompact-support terms) is computed here for the first time.

The binary’s energy and the spin-precession equations including SO couplings through 2.5PN order were computed in paper I. They were used to derive the secular evolution of the binary’s orbital phase through 2.5PN order in the spins. We found that the 2.5PN terms give a relevant contribution to the number of accumulated GW cycles within the binary’s orbital plane. In Tables II and III, we listed the number of GW cycles for typical binaries detectable with ground-based and space-based detectors, such as LIGO/Virgo and LISA. When spins are maximal, the SO contribution at 2.5PN order is comparable to that of the nonspin part at the same 2.5PN order. For some binary mass configurations, the SO contribution at 2.5PN order can be larger than that of SS couplings at 2PN order.

In order to extract accurately the parameters of maximally or mildly spinning binaries with ground-based detectors of first generation, having typical signal-to-noise ratio (SNR) of the order of 10, we expect the spin corrections through 3.5PN order to be sufficient. With space-based detectors having SNR of the order of $10^2$–$10^3$, we would need a priori to compute nonspin and spin corrections at much higher PN order (for parameter estimation). For what concerns the impact of the SO couplings at 2.5PN order on the actual detection, it would be relevant to evaluate the maximized overlaps between templates that include SO effects through 2.5PN order against templates that include SO and SS effects through 2PN order. We anticipate that, at the cost of introducing systematic errors in the estimation of the binary parameters, the maximized overlaps could be high. In fact, the binary and directional parameters may compensate variations in the PN phasing.

For future applications, we listed in Sec. VII the relevant equations defining the spinning dynamics and the GW phasing in terms of the constant spin variables. Such formulation is broadly used in the literature to define spinning, precessing templates for compact binaries [9–19,39,41].

Finally, we computed the contributions of the spin terms to the location of the innermost circular orbit in the case of black-hole binaries. The results for equal-mass objects with maximal spins are summarized in Table I. Spin couplings at 1.5PN and 2.5PN orders can give significant contributions to the energy and frequency at the ICO (and nearby).

HIGHER-ORDER SPIN EFFECTS IN THE DYNAMICS...

PHYSICAL REVIEW D 74, 104034 (2006)