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Lattice Laughlin states of bosons and fermions at filling fractions 1/q

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Abstract
We introduce a two-parameter family of strongly-correlated wave functions for bosons and fermions in lattices. One parameter, q, is connected to the filling fraction. The other one, η, allows us to interpolate between the lattice limit ($\eta = 1$) and the continuum limit ($\eta \to 0^+$) of families of states appearing in the context of the fractional quantum Hall effect or the Calogero–Sutherland model. We give evidence that the main physical properties along the interpolation remain the same. Finally, in the lattice limit, we derive parent Hamiltonians for those wave functions and in 1D, we determine part of the low-energy spectrum.

Keywords: fractional quantum Hall effect, Luttinger liquid, conformal field theory, matrix product state, entanglement, topological phase

1. Introduction

The fractional quantum Hall (FQH) effect has attracted a longstanding interest in physics. 2D electrons displaying such an effect form incompressible quantum liquids with a bulk gap, gapless edge states, and quasiparticle excitations with fractional charge and fractional statistics. Their properties are not amenable to the conventional Ginzburg–Landau theory; however, they can be thoroughly analyzed thanks to the discovery of analytical wave functions, which provide good approximations to some of the quantum states responsible for the FQH effect.

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An important family of such states is the Laughlin states [1]

$$\Psi_q(\{Z\}) = \prod_{i<j} \left(Z_i - Z_j\right)^q \exp\left(-\sum_l |Z_l|^2 / 4\right), \quad (1)$$

where Z_i is the position in the complex plane of the ith electron and $\nu = 1/q$ is the filling fraction, i.e. the ratio between the number of electrons and the number of flux quanta. From a modern viewpoint, the Laughlin states belong to the so-called topological phases [2, 3], an exotic class of gapped phases whose full classification is still an outstanding open problem.

In the FQH setups, the Laughlin states arise due to the strong interactions between the electrons in the fractionally filled lowest Landau level. In that case, the size of the electron wave packets is at least one order of magnitude larger than the lattice spacing and thus the lattice effects are usually negligible [4]. A natural question is whether Laughlin states (or their variants) can appear in lattice models without Landau levels. In the late eighties, Kalmeyer and Laughlin (KL) proposed a state [5–7] that is a lattice version of the bosonic Laughlin state with $q = 2$. This state has been shown to share some of the most defining properties of its continuum counterpart, like the fractional statistics of quasiparticle excitations [8] and the presence of chiral edge states [9]. Thus, the continuum and lattice version of the bosonic Laughlin state with $q = 2$ seem to be closely connected, although it is not clear what such a connection is. In [10], it has been shown that an interpolation Hamiltonian between a $q = 2$ Laughlin-like lattice state and the continuum $q = 2$ Laughlin state can be obtained by choosing bases that allow both states to be expressed in the same Hilbert space, although with different base kets. A more direct interpolation, in which the lattice spacing is continuously changed, has been considered in [11], but was found to be valid only for sufficiently small lattice filling factors. A similar situation is encountered in 1D, where the Calogero–Sutherland (CS) model [12, 13], which is defined in the continuum, seems to be closely related to the Haldane–Shastry lattice model [14, 15], although it is not obvious how to transform one into the other.

A very useful description of FQH wave functions in the continuum has been introduced by Moore and Read in [16], where they wrote selected FQH wave functions in terms of correlators of the corresponding edge conformal field theories (CFTs). Recently, for certain lattice systems in 1D and 2D, strongly correlated spin wave functions have also been written in terms of CFT correlators [17–20]. This, in particular, has made it possible to construct parent Hamiltonians and to build in a systematic form simple wave functions with topological properties. We note also that parent Hamiltonians of the KL state have been found in [19, 21–25].

In this paper, we provide an explicit connection between the continuum Laughlin/CS states on the one side and a set of lattice Laughlin/CS states on the other for all filling factors $1/q$. We do this by introducing a family of *lattice* wave functions for hardcore bosons and fermions, which is defined on arbitrary lattices in 1D and 2D and allows us to continuously interpolate between the two limits. We also provide numerical evidence that the states remain within the same phase for all values of the interpolation parameter, so that the interpolation is meaningful. In 1D, we show that the states are critical and describe Tomonaga–Luttinger liquids (TLLs) with Luttinger parameter $K = 1/q$, and in 2D we find that the states have topological entanglement entropy (TEE) [26, 27] $-\ln(q)/2$. The wave functions are constructed from conformal fields, and we use the CFT properties of the states to derive parent Hamiltonians for the wave functions in the lattice limit in both 1D and 2D and for general q. In 1D, the parent

Hamiltonians are closely related to Haldane's inverse-square model [14], and we find that *part* of the spectrum is given by integer eigenvalues described by a simple formula.

2. CFT wave functions

Let us consider a lattice with lattice sites at the positions z_j, $j = 1, 2, \ldots, N$, in the complex plane. The local basis at site j is labeled by $\left| n_j \right\rangle$, where $n_j \in \{0, 1\}$ is the number of particles at the site. The family of wave functions we propose (later on referred to as CFT states) take the form of the following chiral correlators of vertex operators:

$$\Psi(n_1, \ldots, n_N) \propto \left\langle V_{n_1}(z_1) \ldots V_{n_N}(z_N) \right\rangle, \tag{2}$$

where

$$V_{n_j}(z_j) = \chi_j^{n_j} e^{i\pi \sum_{k(<j)} \eta_k n_j} : e^{i\left(q n_j - \eta_j\right)\phi(z_j)/\sqrt{q}} : . \tag{3}$$

Here, $\phi(z)$ is a chiral bosonic field from the $c = 1$ free-boson CFT, $:\ldots:$ denotes normal ordering, χ_j are phase factors that do not depend on n_j, q is a positive integer, and η_j are positive parameters with average $N^{-1}\sum_j \eta_j = \eta \in (0, 1]$. The charge neutrality condition $\sum_i (q n_i - \eta_i) = 0$ of the CFT correlators fixes the number of particles to $\sum_{i=1}^N n_i = \eta N/q \equiv M$, which must hence be an integer, and it follows that η/q is the lattice filling fraction. η is therefore the parameter that interpolates between the continuum limit ($\eta \to 0^+$), with infinitely many lattice sites per particle, and the lattice limit ($\eta = 1$), in which the lattice filling fraction η/q equals the Laughlin filling fraction $1/q$. When varying η, we shall always take all η_j to scale linearly with η, such that η_j/η_l remain constant. Evaluating the vacuum expectation value of the product of vertex operators in (2) [28] yields a Jastrow wave function

$$\Psi(n_1, \ldots, n_N) \propto \delta_n \prod_{i<j} (z_i - z_j)^{q n_i n_j} \prod_l f_N(z_l)^{n_l}, \tag{4}$$

where $\delta_n = 1$ if $\sum_{i=1}^N n_i = \eta N/q$ and zero otherwise and

$$f_N(z_l) \equiv \chi_l \prod_{j(\neq l)} (z_l - z_j)^{-\eta_j} = \chi_l \exp\left[-\sum_{j(\neq l)} \eta_j \ln(z_l - z_j)\right]. \tag{5}$$

3. Relation to the CS and Laughlin wave functions

Let us demonstrate how the CFT states are related to several familiar wave functions in the continuum. We first consider the 1D periodic chain, where the lattice sites are uniformly distributed on a unit circle, i.e. $z_j = e^{2\pi i j/N}$, and we choose $\eta_j = \eta \ \forall j$. In this case, we obtain analytically that $f_N(z_l) \propto \chi_l z_l^{\eta}$, and we can therefore write the state (4) as a product of the wave function $\Psi_{CS} \propto \delta_n \prod_{i<j} (z_i - z_j)^{q n_i n_j} \prod_l z_l^{-q(M-1)n_l/2}$ and the gauge factor $\prod_l \left(\chi_l z_l^{\eta + q(M-1)/2}\right)^{n_l}$. In the continuum limit, where $N \to \infty$, $\eta \to 0^+$, and ηN stays fixed to keep the number of particles M

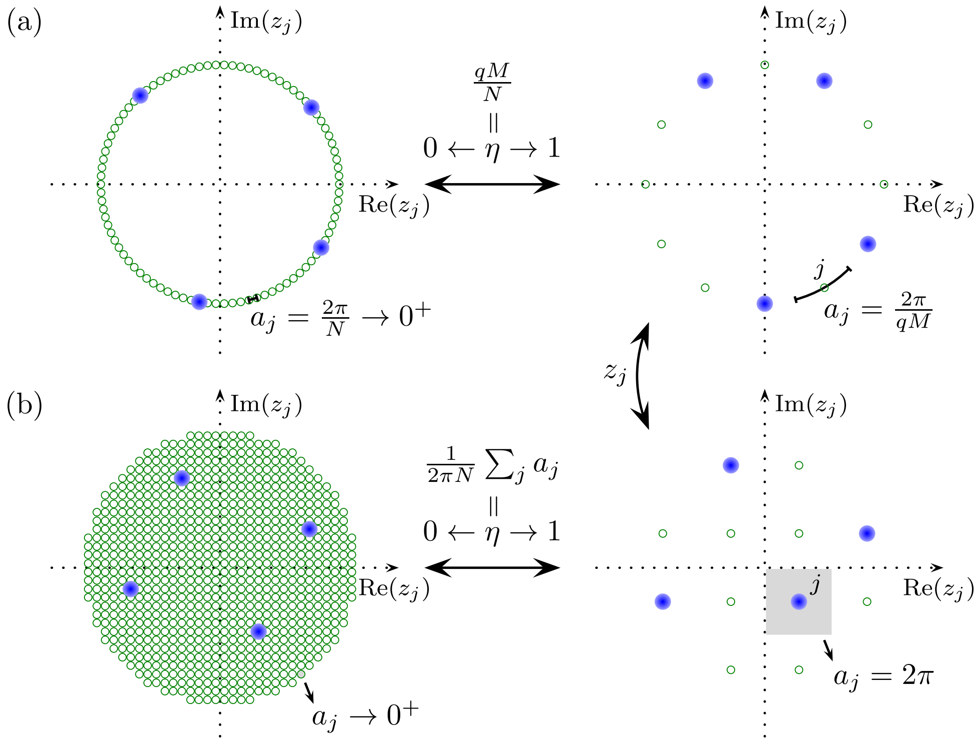


Figure 1. Illustration of the interpolation between the lattice limit ($\eta = 1$) and the continuum limit ($\eta \to 0^+$) for a uniform lattice in 1D and a square lattice in 2D. The interpolation is done, while keeping the area per particle aN/M fixed, where a is the average area per site. (a) In 1D, the lattice is defined by $z_j = e^{2\pi i j/N}$, which fixes the area of site j to $a_j = 2\pi/N\ \forall j$, so that $a \equiv N^{-1}\sum_j a_j = 2\pi/N$. The scaling parameter is therefore $\eta = qM/N = qMa/(2\pi)$. (b) In 2D, the lattice is defined on a disk with radius $R_{\mathcal{D}} \to \infty$, and we choose $a = 2\pi qM/N$, since this fixes the area per particle to $2\pi q$ as in the Laughlin wave functions. The scaling parameter is therefore $\eta = qM/N = a/(2\pi)$. Transformations between different lattices, including the two displayed on the right, is obtained by transforming z_j.

and the area of the lattice constant (see figure 1(a)), the lattice spacing goes to zero, and $\Psi_{\rm CS}$ turns into the ground-state wave function of the CS model [12, 13] for bosons (even q) and fermions (odd q). The gauge factor can be set to unity by choosing $\chi_l = z_l^{-\eta-q(M-1)/2}$ if we like, but we note that its presence does not affect properties such as the particle–particle correlation function and the entanglement entropy. The CFT states thus allow us to define a lattice version of the CS wave functions and to interpolate between the lattice and the continuum limit of the model.

We next consider an arbitrary lattice in 2D, which is defined on a disk $\mathcal{D}$ of radius $R_{\mathcal{D}} \to \infty$. We define the area a_j of site j to be the area of the region consisting of all points in $\mathcal{D}$ that are closer to z_j than to any of the other lattice sites. Let us note that $\left|f_N(z_l)\right| = \exp\left[-\sum_{j(\neq l)}\eta_j \ln\left(\left|z_l - z_j\right|\right)\right]$. If we choose $\eta_j = a_j/(2\pi)$ and consider the continuum limit $\eta \to 0^+$ (as illustrated for a square lattice in figure 1(b)), we can replace the sum over j by

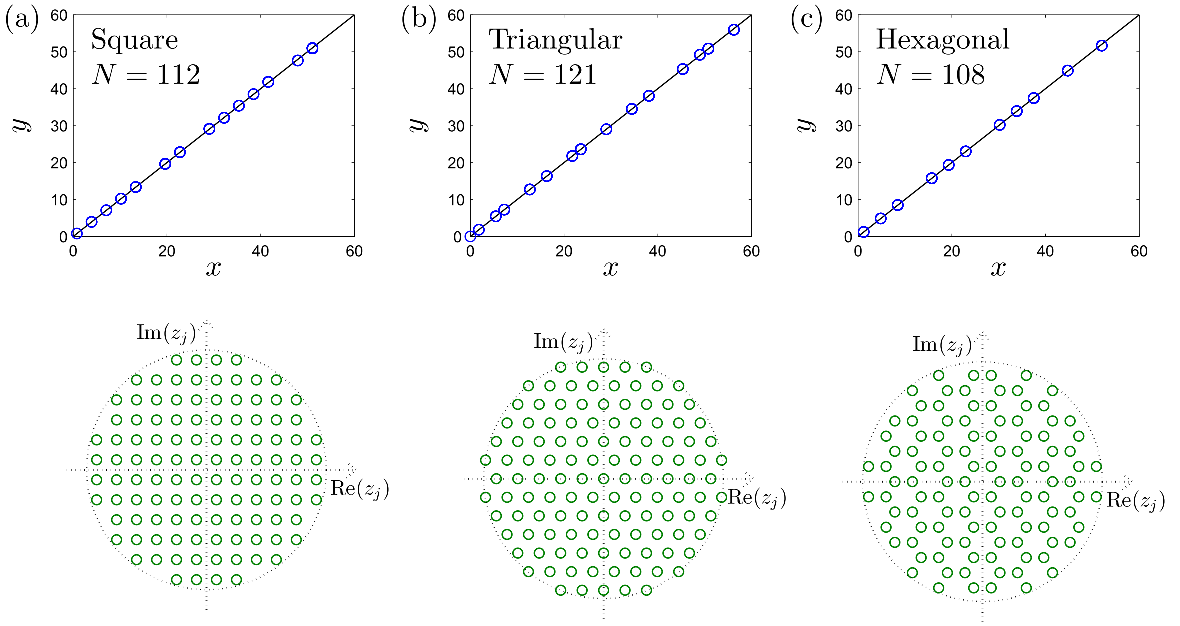


Figure 2. Numerical demonstration that (6) is approximately valid even for a moderate number of lattice sites N for the square (a), the triangular (b), and the hexagonal (c) lattice with a circular edge. The lattices are illustrated in the lower part of the figure, and the upper part show $x = |z_j|^2/4$ versus $y = -\ln\left[|f_N(z_j)|\right] + \text{constant}$, where $-\ln\left[|f_N(z_j)|\right] = \sum_{l(\neq j)} \eta_l \ln\left(|z_j - z_l|\right)$ and we choose the constant to make y (approximately) homogeneously linear in x (each data point corresponds to one value of j and the black lines in the background are the curve $y = x$). The scale of the lattices is chosen such that $\eta_j = 1$ for all j. We note, however, that this choice is unimportant since the transformation $z_j \to \kappa z_j$ just takes $x \to \kappa^2 x$ and $y \to \kappa^2 y + \text{constant}$.

the integral $\int_{\mathcal{D}} d^2z \ln\left(|z_l - z|\right)/(2\pi)$. In the thermodynamic limit $R_{\mathcal{D}} \to \infty$ this integral evaluates to $|z_l|^2/4 + \text{constant}$, where the constant does not depend on z_l. Note, however, that $\sum_{j(\neq l)} \eta_j \ln\left(|z_l - z_j|\right)$ and $\kappa^{-2}\sum_{j(\neq l)} \kappa^2 \eta_j \ln\left(|\kappa z_l - \kappa z_j|\right)$, where κ is a positive constant, only differ by a z_l-independent constant for $R_{\mathcal{D}} \to \infty$. If η_j is not small, we can choose κ very small, transform the resulting sum into an integral, and again conclude that $\sum_{j(\neq l)} \eta_j \ln\left(|z_l - z_j|\right) = |z_l|^2/4 + \text{constant}$. For all 2D lattices in the thermodynamic limit, we therefore obtain

$$f_N(z_l) \propto \chi_l e^{-ig_l} e^{-|z_l|^2/4} (N \text{ large}), \tag{6}$$

where $g_l \equiv \text{Im}\left[\sum_{j(\neq l)} \eta_j \ln(z_l - z_j)\right]$ is a real number. In figure 2, we find numerically for different lattices that (6) is an accurate approximation even if N is only moderately large. Choosing $\chi_l = e^{ig_l}$ and inserting (6) into (4), we observe that the CFT states coincide with the Laughlin states (1), except that the possible particle positions are restricted to the coordinates of the lattice sites. By changing the number of lattice sites per particle, we can thus interpolate between the Laughlin states in the continuum and Laughlin-like states on lattices.

4. Continuous interpolation

We next demonstrate that important properties of the states (4) stay the same as a function of the interpolation parameter, which indicates that the states remain within the same phase when interpolated between the lattice limit and the continuum limit. We first consider the uniform lattice in 1D and show that (4) is well-described by the TLL theory in this case. The Rényi entropy $S_L^{(\alpha)} = \ln\left(\mathrm{Tr}\left(\rho_L^{\alpha}\right)\right)/(1-\alpha)$ of a TLL, where ρ_L is the reduced density operator of L successive sites in the chain, is expected to be [29]

$$S_L^{(\alpha)} = S_{L,\mathrm{CFT}}^{(\alpha)} + \frac{f_\alpha \cos\left(2Lk_{\mathrm{F}}\right)}{\left|2\sin\left(k_{\mathrm{F}}\right)\sin\left(\pi L/N\right)N/\pi\right|^{2K/\alpha}} \tag{7}$$

for $\ln\left(\left|2\sin\left(k_{\mathrm{F}}\right)\sin\left(\pi L/N\right)N/\pi\right|\right) \gg \alpha$, where K is the Luttinger parameter, k_{F} is the Fermi momentum,

$$S_{L,\mathrm{CFT}}^{(\alpha)} = (c/6)(1+1/\alpha)\ln\left(\sin\left(\pi L/N\right)N/\pi\right) + c'_\alpha, \tag{8}$$

c is the central charge, and f_α and c'_α are nonuniversal constants. In our case, $k_{\mathrm{F}} = \eta\pi/q$. The Rényi entropy with index $\alpha = 2$ of the state (4) can be computed numerically by using the method described in [17, 30]. In brief, the idea is to note that $S_L^{(2)} = -\ln\left(\mathrm{Tr}\left(\rho_L^2\right)\right)$ can be rewritten into

$$e^{-S_L^{(2)}} = \sum_{n_1,\ldots,n_N,n'_1,\ldots,n'_N} \frac{\Psi\left(n'_1,\ldots,n'_L,n_{L+1},\ldots,n_N\right)\Psi\left(n_1,\ldots,n_L,n'_{L+1},\ldots,n'_N\right)}{\Psi\left(n_1,\ldots,n_N\right)\Psi\left(n'_1,\ldots,n'_N\right)} \times\left|\Psi\left(n_1,\ldots,n_N\right)\right|^2\left|\Psi\left(n'_1,\ldots,n'_N\right)\right|^2, \tag{9}$$

where n_j and n'_j are summed over 0 and 1 for all j and we assume $\Psi\left(n_1,\ldots,n_N\right)$ to be normalized. One can then interpret $\left|\Psi\left(n_1,\ldots,n_N\right)\right|^2\left|\Psi\left(n'_1,\ldots,n'_N\right)\right|^2$ as a probability distribution and compute the right-hand side of (9) using the Metropolis algorithm as detailed further in appendix A. The result is shown for $\eta = 1$ in figure 3(a). Fixing $c = 1$ and using f_2, K, and c'_2 as fitting parameters, we find that the entanglement entropy of (4), indeed, follows (7).

The expected TLL behavior of the particle–particle correlation function $C(k) = \left\langle n_i n_{i+k}\right\rangle - \left\langle n_i\right\rangle\left\langle n_{i+k}\right\rangle$ is [31]

$$C(k) = \frac{A\cos\left(2kk_{\mathrm{F}}\right)}{\left|\sin\left(\pi k/N\right)N/\pi\right|^{2K}} + \frac{K}{2\pi^2\left|\sin\left(\pi k/N\right)N/\pi\right|^2} \tag{10}$$

for large k, where A is a nonuniversal constant. For the state (4), we have $\left\langle n_i\right\rangle = M/N = \eta/q$. We compute the two-body expectation value

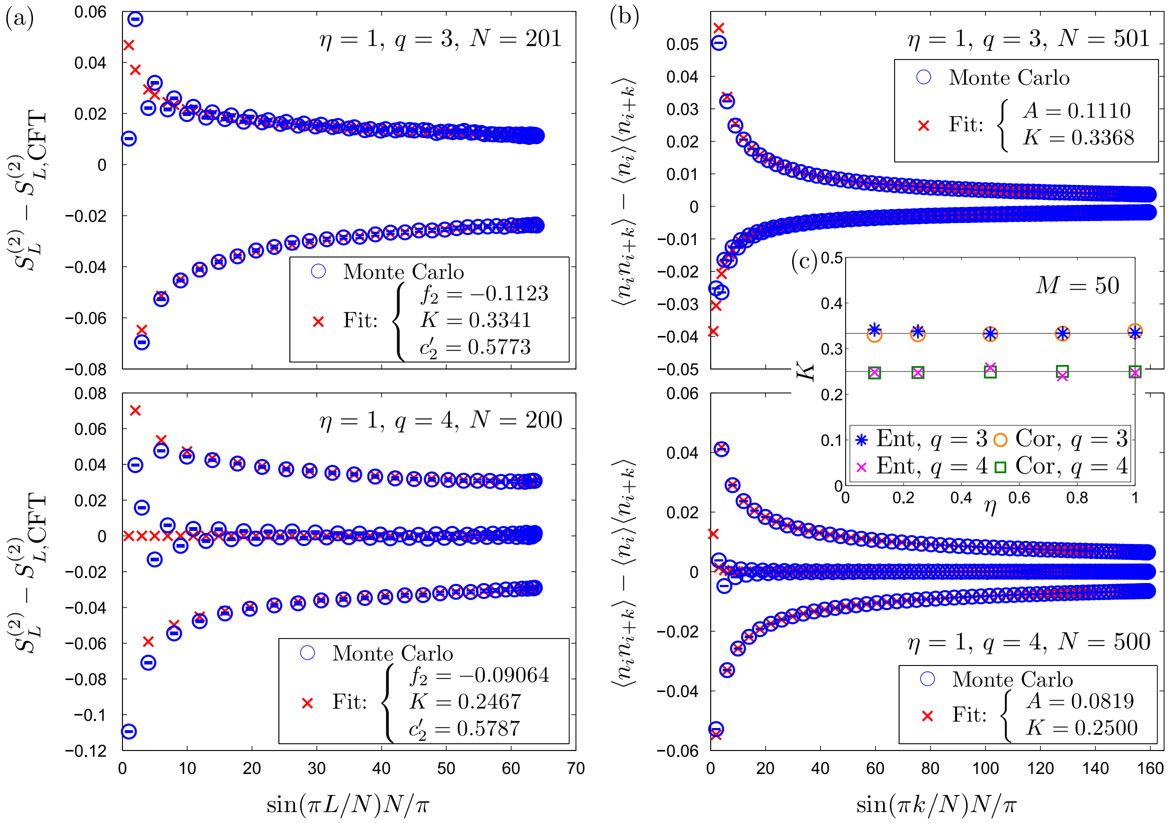


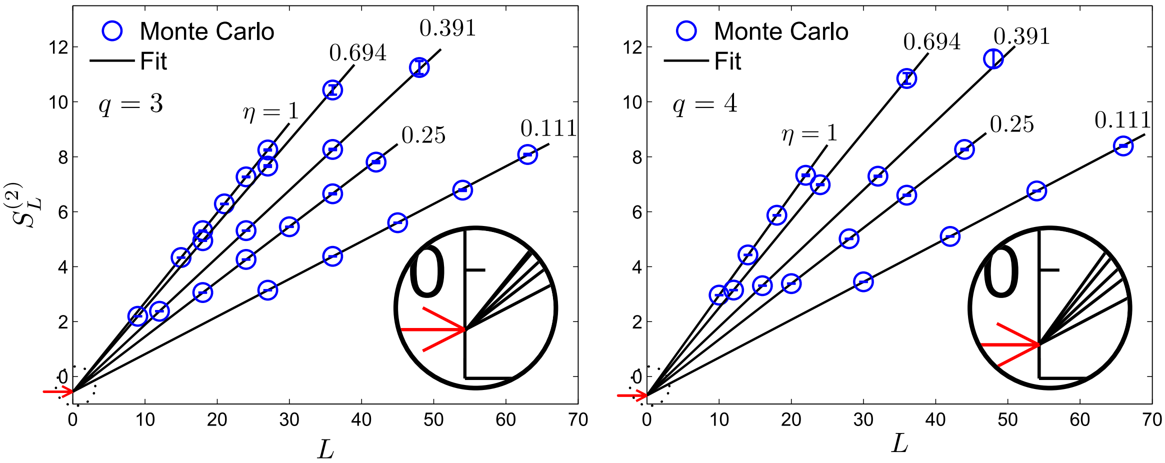
Figure 3. (a) Deviation of the Rényi entropy with index $\alpha = 2$ of a block of L consecutive sites from the lowest order CFT expression (8) and (b) particle–particle correlation function of the CFT state (4) for a uniform 1D lattice in the lattice limit for $q = 3$ (top) and $q = 4$ (bottom) obtained from Monte Carlo simulations. The fits are based on equations (7) and (10), respectively, and allow us to extract the Luttinger parameter K, which is shown for $M = 50$ as a function of the interpolation parameter η in inset (c) ('Ent' ('Cor') means extracted from the entropy (correlator) fit). Since (7) and (10) are valid for large L and k, respectively, we exclude the first $2q/\eta$ points when computing the fits.

$$\langle n_i n_{i+k} \rangle = \sum_{n_1, n_2, \ldots, n_N} n_i n_{i+k} \left| \Psi \left(n_1, n_2, \ldots, n_N \right) \right|^2 \tag{11}$$

by interpreting $\left| \Psi \left(n_1, n_2, \ldots, n_N \right) \right|^2$ as a probability distribution and using the Metropolis algorithm (see appendix A for details). Numerical results for the correlation function are shown for $\eta = 1$ in figure 3(b), and we find that (10) provides a good fit.

The values of K extracted from the entropy and correlation function computations are shown as a function of the interpolation parameter η in figure 3(c), and these results suggest that $K = 1/q$ independent of η. We note that the observed behavior coincides with the properties of the free-boson CFT with radius $R = \sqrt{q}$, which is the low-energy effective theory for the CS model with rational coupling constant q [32].

The Laughlin states in the continuum are topological states with TEE $-\ln(q)/2$. To compute the TEE of the lattice models, we consider the state defined on an $R \times L$ lattice on the cylinder as suggested in [33]. The state on the cylinder is obtained by choosing



η	1.000	0.694	0.391	0.250	0.111	1.000	0.694	0.391	0.250	0.111
γ	0.558	0.555	0.558	0.549	0.551	0.680	0.691	0.676	0.676	0.685
$\delta\gamma$	0.014	0.004	0.004	0.005	0.006	0.007	0.015	0.019	0.013	0.011

Figure 4. Rényi entropy with $\alpha = 2$ of the CFT state (4) with $q = 3$ (left) and $q = 4$ (right) obtained from Monte Carlo simulations. The state is defined on an $R \times L$ square lattice on the cylinder, and the cut divides it into two $R/2 \times L$ lattices, where L is the number of lattice sites in the periodic direction. The fits are weighted linear least squares fits of the form $S_L^{(2)} = \xi L - \gamma$, where ξ and γ are fitting parameters, and the weight of each point is taken to be the inverse of the square of the error bar. Starting from above, η and R are, respectively, $\eta = 1, 0.694, 0.391, 0.25, 0.111$ and $R = 10, 12, 16, 20, 30$ for the five data sets, and the number of particles is $M = \eta RL/q$. The TEE values $-\gamma \pm \delta\gamma$ extracted from the fits are given in the table below the figures, and these values should be compared to the TEE $-\gamma_q \equiv -\ln(q)/2$ of the Laughlin states in the continuum, i.e. to $\gamma_3 = 0.549$ and $\gamma_4 = 0.693$, respectively. When interpreting the error bars $\delta\gamma$ (one standard deviation), one should keep in mind, however, that the expression $S_L^{(2)} = \xi L - \gamma$ is only valid asymptotically for large L and R and that finite size effects may therefore give rise to small errors biased in a particular direction. The relevant size of these errors may be judged from the graphs. The insets are enlarged views, and the red arrows point at the value $-\gamma_q$.

$z_j = \exp\left(2\pi\left(r_j + \mathrm{i}l_j\right)/L\right)$ in (4), where $r_j \in \{-R/2 + 1/2, -R/2 + 3/2, \ldots, R/2 - 1/2\}$ and $l_j \in \{1, 2, \ldots, L\}$. When cut in two halves in the direction perpendicular to the cylinder axis, the entanglement entropy behaves as $S_L^{(2)} = \xi L - \gamma$ for large R and L, where ξ is a nonuniversal constant and $-\gamma$ is the TEE [33]. The techniques mentioned above to compute $S_L^{(2)}$ can also be applied here, and in figure 4, we plot the entanglement entropy as a function of L for different values of the interpolation parameter for $q = 3$ and $q = 4$. In all cases the extracted values of the TEE are compatible with the value $-\ln(q)/2$. This provides further evidence that the CFT states in the lattice limit are continuously connected to the Laughlin states in the continuum. Finally, we note that TEE values close to $-\ln(q)/2$ have also been obtained for another set of Laughlin-like states on lattices in [34].

5. Parent Hamiltonian

For $\eta_j = 1 \; \forall \; j$, the vertex operators constructing the wave functions (2) can be identified as primary fields of a free-boson CFT compactified on a circle of radius $R = \sqrt{q}$. For $q = 2$, the CFT is the $SU(2)_1$ Wess–Zumino–Witten (WZW) model. For $q = 3$, the CFT has a hidden supersymmetry and can be identified as the $\mathcal{N} = 2$ superconformal field theory [16]. For integer q, the rationality of these CFTs ensures the existence of null fields. This is very useful, because null fields can be used for deriving parent Hamiltonians as demonstrated for the case of WZW models in [18]. Here, we identify a suitable set of null fields from which we derive decoupling equations. After some algebra (see appendix B), this procedure gives us a set of operators, which annihilate the wave functions (2) at $\eta_j = 1$. These operators include $\Upsilon = \sum_{i=1}^{N} \tilde{d}_i$, where $\tilde{d}_i = \chi_i^{-1} d_i$ and d_i denotes the fermionic (hardcore bosonic) annihilation operator for odd (even) q, and

$$\Lambda_i = (q-2)\,\tilde{d}_i + \sum_{j\,(\neq i)} w_{ij} \left[\tilde{d}_j - \tilde{d}_i \left(q n_j - 1 \right) \right], \tag{12}$$

where $w_{ij} \equiv \left(z_i + z_j \right) / (z_i - z_j)$. Since $\Upsilon |\Psi\rangle = \Lambda_i |\Psi\rangle = 0 \;\; \forall \; i$, the positive semi-definite Hermitian operators $\Upsilon^{\dagger}\Upsilon$ and $\Lambda_i^{\dagger}\Lambda_i$ $(i = 1, \ldots, N)$ have the wave functions (2) with $\eta_j = 1$ and z_j arbitrary as their zero-energy ground state. Thus, these operators can be used to construct both 1D and 2D parent Hamiltonians for which the wave functions (2) with $\eta_j = 1$ are exact ground states. For the states with $\eta_j \neq 1$, we have not achieved to construct parent Hamiltonians, which is still an interesting open problem.

In the following, we focus on a 1D parent Hamiltonian obtained for $z_j = e^{2\pi i j/N}$, which turns out to have a particularly simple form. Specifically, we consider $H_{1D} = \frac{1}{2}\sum_i \left(\Lambda_i^{\dagger}\Lambda_i - q\Gamma_i^{\dagger}\Gamma_i \right) + \frac{q-2}{2}\Upsilon^{\dagger}\Upsilon + E_0$, where $\Gamma_i = \tilde{d}_i \Lambda_i = \sum_{j\,(\neq i)} w_{ij} \tilde{d}_i \tilde{d}_j$ and $E_0 = -\frac{q-1}{6q} N\,[3N + (q-8)]$ is the eigenenergy of (4). This choice yields a parent Hamiltonian with purely two-body interactions (see appendix C)

$$H_{1D} = \sum_{i \neq j} \left[(q-2)\, w_{ij} - w_{ij}^2 \right] \tilde{d}_i^{\dagger} \tilde{d}_j - \frac{q\,(q-1)}{2} \sum_{i \neq j} w_{ij}^2 n_i n_j. \tag{13}$$

While the $q = 2$ Hamiltonian recovers the spin-1/2 Haldane–Shastry model [14, 15], the Hamiltonians with $q \geqslant 3$ differ from the Haldane's inverse-square Hamiltonians [14] by an extra hopping term. By diagonalizing the Hamiltonian (13) numerically for small N, we confirm that the wave functions (2) are indeed their unique ground states. Additionally, we observe that H_{1D} always has integer eigenvalues besides noninteger ones, which is an interesting feature already arising in Haldane's model [14]. Motivated by Haldane's results, we have found that, after subtracting a constant, *part of* the integer eigenvalues take the form $E = \sum_{\{m_k\}} 2m_k \left(m_k + q - 2 - N \right)$, where $\{m_k\}$ is a set of M pseudomomenta (M: number of particles) satisfying $m_k \in [0, N-1]$ and $m_{k+1} \geqslant m_k + q$. This formula captures the essential low-lying part of the energy spectrum. Similar to Haldane's model, one can prove analytically that the Jastrow wave functions $\Psi_{\mathrm{1D}}^J(n_1, \ldots, n_N) = \delta_n \prod_{i<j} (z_i - z_j)^{qn_in_j} \prod_l \left(\chi_l z_l^J\right)^{n_l}$, where $\delta_n = 1$ for $\sum_i n_i = M$ and zero otherwise and $1 \leqslant J \leqslant N - q(M-1) - 1$, are exact eigenstates of (13) and are a subclass of those eigenstates with integer eigenvalues.

6. Conclusion

The present work combines several known models into a common framework with an underlying CFT structure and shows how the Laughlin states and the CS wave functions can be continuously transformed into lattice wave functions with similar properties. The CFT structure provides useful tools for deriving properties of the states analytically, and, in particular, enables us to derive parent Hamiltonians of the states in the lattice limit. Analytical wave functions play an important role in the investigation of the FQH effect in the continuum, and the model proposed here may similarly be used for analyzing FQH properties in lattice systems. Our present work also provides a method to discretize continuum FQH states in a way that is amenable to projected entangled-pair state description [35, 36], and thus it provides an alternative approach to the one recently introduced based on infinite matrix product states using discrete Landau level orbitals [37–39].

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Appendix A. Details of the Monte Carlo simulations

To compute

$$\left\langle n_i n_{i+k} \right\rangle = \sum_{n_1, n_2, \ldots, n_N} n_i n_{i+k} \left| \Psi(n_1, n_2, \ldots, n_N) \right|^2 \tag{A.1}$$

by use of Monte Carlo simulations, we start from a random configuration $n_1, n_2, \ldots, n_N$ of the occupation numbers fulfilling $\sum_{j=1}^N n_j = M$. In each step of the algorithm we randomly choose a particle from one of the occupied sites and move it randomly to one of the empty sites. We denote the new configuration by $\tilde{n}_1, \tilde{n}_2, \ldots, \tilde{n}_N$ and compute $\Delta = P\left(\tilde{n}_1, \tilde{n}_2, \ldots, \tilde{n}_N\right) / P(n_1, n_2, \ldots, n_N)$, where $P(n_1, n_2, \ldots, n_N) \equiv \left| \Psi(n_1, n_2, \ldots, n_N) \right|^2$. If $\Delta \geqslant 1$, we keep $\tilde{n}_1, \tilde{n}_2, \ldots, \tilde{n}_N$ as our input configuration in the next step,and if $\Delta < 1$, we choose the input configuration in the next step to be $\tilde{n}_1, \tilde{n}_2, \ldots, \tilde{n}_N$ with probability Δ and $n_1, n_2, \ldots, n_N$ with probability $1 - \Delta$. After a warm up period, we run the algorithm for N_S steps. The visited configurations represent the probability distribution $P(n_1, n_2, \ldots, n_N)$, and one can therefore compute $\left\langle n_i n_{i+k} \right\rangle$ from

$$\left\langle n_i n_{i+k} \right\rangle = \frac{1}{N_S} \sum_{\{n_1, n_2, \ldots, n_N\} \in S} n_i n_{i+k}, \tag{A.2}$$

where S is the tuple of visited configurations. Since it follows from (4) with $z_j = \mathrm{e}^{2\pi \mathrm{i} j/N}$ that $\left\langle n_i n_{i+k} \right\rangle$ is independent of i, we also average (A.2) over i. It is important to note that it is not necessary to normalize the wave function $\Psi(n_1, n_2, \ldots, n_N)$ since we only need ratios of probabilities to compute Δ.

The Monte Carlo simulation of (9) is done in the same way, except that there are now two configurations $n_1, n_2, \ldots, n_N$ and $n_1', n_2', \ldots, n_N'$ to keep track of. We therefore choose randomly in each step, whether we move a particle in $n_1, n_2, \ldots, n_N$ or in $n_1', n_2', \ldots, n_N'$.

Appendix B. Operators annihilating the lattice Laughlin states

In this section, we derive operators that annihilate the state (2) for $\eta = 1$. We first assume $\chi_j = 1$ and consider the CFT wave functions defined by

$$\Psi_{n_1, \ldots, n_N}(z_1, \ldots, z_N) = \left\langle V_{n_1}(z_1) V_{n_2}(z_2) \cdots V_{n_N}(z_N) \right\rangle, \tag{B.1}$$

where

$$V_{n_j=1}(z_j) = \mathrm{e}^{\mathrm{i}\pi(j-1)} V_+(z_j), \qquad V_{n_j=0}(z_j) = V_-(z_j). \tag{B.2}$$

Here $V_+(z) = \mathrm{e}^{\mathrm{i}(q-1)\phi(z)/\sqrt{q}}$ and $V_-(z) = \mathrm{e}^{-\mathrm{i}\phi(z)/\sqrt{q}}$.

For the $c = 1$ free-boson CFT with compactification $R = \sqrt{q}$, it is convenient to define two chiral currents,

$$G^{\pm}(z) = \mathrm{e}^{\pm \mathrm{i}\sqrt{q}\phi(z)}, \tag{B.3}$$

besides the U(1) current $J(z) = \frac{\mathrm{i}}{\sqrt{q}} \partial\phi(z)$. For $q = 2$, these currents form the $\mathrm{SU}(2)_1$ Kac–Moody algebra. For $q = 3$, together with the energy–momentum tensor, the currents form the $\mathcal{N} = 2$ superconformal current algebra.

To construct the parent Hamiltonian of (B.1), we need to derive decoupling equations satisfied by the CFT correlator (B.1) using null fields. Let us first consider the null field

$$\begin{aligned}
\chi_1(w) &= \oint_w \frac{dz}{2\pi \mathrm{i}} \frac{1}{z-w} \left[G^+(z) V_-(w) - qJ(z) V_+(w) \right] \\
&= \oint_w \frac{dz}{2\pi \mathrm{i}} \frac{1}{z-w} \left[\mathrm{e}^{\mathrm{i}\sqrt{q}\phi(z)} \mathrm{e}^{-\mathrm{i}\phi(w)/\sqrt{q}} - \mathrm{i}\sqrt{q}\, \partial\phi(z) \mathrm{e}^{\mathrm{i}(q-1)\phi(w)/\sqrt{q}} \right] \\
&= \oint_w \frac{dz}{2\pi \mathrm{i}} \frac{1}{z-w} \left[\frac{1}{z-w} \mathrm{e}^{\mathrm{i}\sqrt{q}\phi(z) - \mathrm{i}\phi(w)/\sqrt{q}} - \mathrm{i}\sqrt{q}\, \partial\phi(w) \mathrm{e}^{\mathrm{i}(q-1)\phi(w)/\sqrt{q}} \right] \\
&= \oint_w \frac{dz}{2\pi \mathrm{i}} \frac{1}{z-w} \left[\mathrm{i}\sqrt{q}\, \partial\phi(w) \mathrm{e}^{\mathrm{i}(q-1)\phi(w)/\sqrt{q}} - \mathrm{i}\sqrt{q}\, \partial\phi(w) \mathrm{e}^{\mathrm{i}(q-1)\phi(w)/\sqrt{q}} \right] \\
&= 0.
\end{aligned} \tag{B.4}$$

By replacing the vertex operator at site i by the null field $\chi_1(z_i)$, the chiral correlator vanishes

$$\begin{aligned}0 &= \left\langle V_{n_1}(z_1)\,\cdots\chi_1(z_i)\,\cdots\,V_{n_N}(z_N)\right\rangle \\ &= \oint_{z_i}\frac{dz}{2\pi\mathrm{i}}\frac{1}{z-z_i}\left\langle V_{n_1}(z_1)\cdots\left[G^+(z)V_-(z_i)-qJ(z)V_+(z_i)\right]\cdots V_{n_N}(z_N)\right\rangle \\ &= -\sum_{j=1(\neq i)}^{N}\oint_{z_j}\frac{dz}{2\pi\mathrm{i}}\frac{1}{z-z_i}\left\langle V_{n_1}(z_1)\cdots\left[G^+(z)V_-(z_i)-qJ(z)V_+(z_i)\right]\cdots V_{n_N}(z_N)\right\rangle,\end{aligned} \tag{B.5}$$

where we have deformed the integral contour in the last step. To proceed we use the operator product expansions (OPEs)

$$G^+(z)V_n(w) \sim \frac{\sum_{n'}(d)_{nn'}}{z-w}V_{n'}(w), \tag{B.6}$$

$$J(z)V_n(w) \sim \frac{1}{q}\frac{\sum_{n'}(qd^\dagger d-1)_{nn'}}{z-w}V_{n'}(w), \tag{B.7}$$

where the particle annihilation and creation operators are defined as

$$d=\begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix}\quad\text{and}\quad d^\dagger=\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}, \tag{B.8}$$

respectively. Applying the OPEs, the chiral correlator with null field $\chi_1(z_i)$ yields the following decoupling equation:

$$\begin{aligned}0 &= \left\langle V_{n_1}(z_1)\,\cdots\chi_1(z_i)\,\cdots\,V_{n_N}(z_N)\right\rangle \\ &= -\sum_{j=1(\neq i)}^{N}\oint_{z_j}\frac{dz}{2\pi\mathrm{i}}\frac{1}{z-z_i}\left\langle V_{n_1}(z_1)\cdots\left[G^+(z)V_-(z_i)-qJ(z)V_+(z_i)\right]\cdots V_{n_N}(z_N)\right\rangle \\ &= -\sum_{j=1(\neq i)}^{N}\oint_{z_j}\frac{dz}{2\pi\mathrm{i}}\frac{1}{z-z_i}\frac{\sum_{n_j'}(d)_{n_jn_j'}}{z-z_j}\left\langle V_{n_1}(z_1)\cdots V_{n_j'}(z_j)\cdots V_-(z_i)\cdots V_{n_N}(z_N)\right\rangle \\ &\quad+\sum_{j=1(\neq i)}^{N}\oint_{z_j}\frac{dz}{2\pi\mathrm{i}}\frac{1}{z-z_i}\frac{\sum_{n_j'}(qd^\dagger d-1)_{n_jn_j'}}{z-z_j}\left\langle V_{n_1}(z_1)\cdots V_{n_j'}(z_j)\cdots V_+(z_i)\cdots V_{n_N}(z_N)\right\rangle \\ &= \sum_{j=1(\neq i)}^{N}\frac{1}{z_i-z_j}\sum_{n_j'}(d)_{n_jn_j'}\left\langle V_{n_1}(z_1)\cdots V_{n_j'}(z_j)\cdots V_-(z_i)\cdots V_{n_N}(z_N)\right\rangle \\ &\quad-\sum_{j=1(\neq i)}^{N}\frac{1}{z_i-z_j}\sum_{n_j'}(qd^\dagger d-1)_{n_jn_j'}\left\langle V_{n_1}(z_1)\cdots V_{n_j'}(z_j)\cdots V_+(z_i)\cdots V_{n_N}(z_N)\right\rangle.\end{aligned} \tag{B.9}$$

Based on the above decoupling equation, we obtain an operator Λ_i'

$$\Lambda_i' = \sum_{j=1(\neq i)}^{N}\frac{1}{z_i-z_j}\left[d_i^\dagger d_j-n_i\left(qn_j-1\right)\right], \tag{B.10}$$

where $n_j=d_j^\dagger d_j$, and which annihilates the wave function (B.1), i.e. $\Lambda_i'|\Psi\rangle=0\ \forall\ i=1,\ldots,N$.

Similarly, decoupling equations can be derived from another two null fields

$$\chi_2(w) = \oint_w \frac{dz}{2\pi i} \frac{1}{z-w} G^+(z) V_+(w) = 0, \tag{B.11}$$

$$\chi_3(w) = \oint_w \frac{dz}{2\pi i} G^+(z) V_+(w) = 0, \tag{B.12}$$

and we obtain two additional operators annihilating the wave function (B.1)

$$\Lambda_i'' = \sum_{j=1(\neq i)}^{N} \frac{1}{z_i - z_j} n_i d_j, \tag{B.13}$$

$$\Upsilon = \sum_{i=1}^{N} d_i. \tag{B.14}$$

These operators can be combined into new operators annihilating (B.1)

$$d_i \Lambda_i' + \Lambda_i'' = \sum_{j=1(\neq i)}^{N} \frac{1}{z_i - z_j} \left[d_j - d_i \left(q n_j - 1 \right) \right], \tag{B.15}$$

$$d_i \Lambda_i'' = \sum_{j=1(\neq i)}^{N} \frac{1}{z_i - z_j} d_i d_j. \tag{B.16}$$

Defining $w_{ij} = \frac{z_i + z_j}{z_i - z_j}$, the operator

$$\Lambda_i = (q-2) d_i + \sum_{j=1(\neq i)}^{N} w_{ij} \left[d_j - d_i \left(q n_j - 1 \right) \right] \tag{B.17}$$

can be written as

$$\begin{aligned}
\Lambda_i &= (q-2) d_i + \sum_{j=1(\neq i)}^{N} \left(\frac{2z_i}{z_i - z_j} - 1 \right) \left[d_j - d_i \left(q n_j - 1 \right) \right] \\
&= (q-2) d_i + 2z_i \left(d_i \Lambda_i' + \Lambda_i'' \right) - \sum_{j=1(\neq i)}^{N} \left[d_j - d_i \left(q n_j - 1 \right) \right] \\
&= (q-2) d_i + 2z_i \left(d_i \Lambda_i' + \Lambda_i'' \right) - \left(\Upsilon - d_i \right) + d_i \left[\sum_{j=1}^{N} \left(q n_j - 1 \right) - \left(q n_i - 1 \right) \right] \\
&= 2z_i \left(d_i \Lambda_i' + \Lambda_i'' \right) - \Upsilon + d_i \sum_{j=1}^{N} \left(q n_j - 1 \right).
\end{aligned} \tag{B.18}$$

Note that the wave function (B.1) has filling fraction $\nu = 1/q$, i.e. $\sum_{j=1}^{N} \left(q n_j - 1 \right) |\Psi\rangle = 0$. Thus, we have proven that $\Lambda_i |\Psi\rangle = 0 \ \forall \ i = 1, \ldots, N$. Since $\Lambda_i |\Psi\rangle = 0$, it is straightforward to prove that $\Gamma_i |\Psi\rangle = 0$, where Γ_i is given by $\Gamma_i = d_i \Lambda_i = \sum_{j=1(\neq i)}^{N} w_{ij} d_i d_j$.

The wave function (2) with $\eta = 1$ differs from (B.1) by the factor $\prod_j \chi_j^{n_j}$. This can, however, easily be taken into account by multiplying the above operators with $\prod_j \chi_j^{-n_j}$ from the right and $\prod_j \chi_j^{n_j}$ from the left, which amounts to replacing d_j by $\tilde{d}_j = \chi_j^{-1} d_j$.

Appendix C. 1D parent Hamiltonian

In this section, we use Λ_i to construct a 1D uniform Hamiltonian, where the lattice sites form a unit circle, i.e. $z_j = \mathrm{e}^{\mathrm{i}2\pi j/N}$.

Since $\sum_{j\,(\neq i)} w_{ij} = 0$ in 1D uniform case, the form of Λ_i can be simplified as

$$\Lambda_i = (q-2)\, d_i + \sum_{j\,(\neq i)} w_{ij} \left(d_j - q d_i n_j \right). \tag{C.1}$$

Then, the positive-semidefinite operators annihilating the wave functions are given by

$$\begin{aligned} \Lambda_i^\dagger \Lambda_i =\; & (q-2)^2 d_i^\dagger d_i + (q-2) \sum_{j\,(\neq i)} w_{ij} \left(d_i^\dagger d_j - q n_i n_j \right) - (q-2) \sum_{j\,(\neq i)} w_{ij} \left(d_j^\dagger d_i - q n_i n_j \right) \\ & - \sum_{j\,(\neq i)} w_{ij}^2 \left(d_j^\dagger - q d_i^\dagger n_j \right) \left(d_j - q d_i n_j \right) - \sum_{j \neq l\,(\neq i)} w_{ij} w_{il} \left(d_l^\dagger - q d_i^\dagger n_l \right) \left(d_j - q d_i n_j \right) \\ =\; & (q-2)^2 n_i + (q-2) \sum_{j\,(\neq i)} w_{ij} \left(d_i^\dagger d_j - d_j^\dagger d_i \right) - \sum_{j\,(\neq i)} w_{ij}^2 \left(n_j + q^2 n_i n_j \right) \\ & - \sum_{j \neq l\,(\neq i)} w_{ij} w_{il} \left[d_l^\dagger d_j - q \left(d_j^\dagger d_i + d_i^\dagger d_j \right) n_l + q^2 n_i n_j n_l \right]. \end{aligned} \tag{C.2}$$

By using the useful identities

$$\sum_{i\,(\neq j)} w_{ij}^2 = -\frac{(N-1)(N-2)}{3}, \tag{C.3}$$

$$\sum_{i\,(\neq j,l)} w_{ij} w_{il} = (N-2) + 2 w_{jl}^2, \tag{C.4}$$

and fixing the filling fraction $\sum_i n_i = N/q$ in the system, we obtain

$$\begin{aligned} \sum_i \Lambda_i^\dagger \Lambda_i =\; & (q-2)^2 \frac{N}{q} + 2(q-2) \sum_{i \neq j} w_{ij} d_i^\dagger d_j + \frac{(N-1)(N-2)}{3} \sum_j n_j - q^2 \sum_{i \neq j} w_{ij}^2 n_i n_j \\ & - \sum_{j \neq l} \left[(N-2) + 2 w_{jl}^2 \right] d_l^\dagger d_j + q \sum_{i \neq j \neq l} w_{ij} w_{il} \left[\left(d_j^\dagger d_i + d_i^\dagger d_j \right) n_l - q n_i n_j n_l \right] \\ =\; & (q-2)^2 \frac{N}{q} + \frac{N(N-1)(N-2)}{3q} + 2(q-2) \sum_{i \neq j} w_{ij} d_i^\dagger d_j - q^2 \sum_{i \neq j} w_{ij}^2 n_i n_j \\ & - \sum_{j \neq l} \left[(N-2) + 2 w_{jl}^2 \right] d_l^\dagger d_j + q \sum_{i \neq j \neq l} w_{ij} w_{il} \left[\left(d_j^\dagger d_i + d_i^\dagger d_j \right) n_l - q n_i n_j n_l \right]. \end{aligned} \tag{C.5}$$

The above expression can be further simplified by using

$$\sum_{j\neq l} d_l^{\dagger} d_j = \Upsilon^{\dagger}\Upsilon - \frac{N}{q} \tag{C.6}$$

and

$$\begin{aligned}\sum_{i\neq j\neq l} w_{ij} w_{il} n_i n_j n_l &= \frac{1}{3}\sum_{i\neq j\neq l}\left(w_{ij}w_{il} + w_{ji}w_{jl} + w_{li}w_{lj}\right) n_i n_j n_l = \frac{1}{3}\sum_{i\neq j\neq l} n_i n_j n_l \\ &= \frac{N(N-q)(N-2q)}{3q^3},\end{aligned} \tag{C.7}$$

where we have used the cyclic identity

$$w_{ij}w_{il} + w_{ji}w_{jl} + w_{li}w_{lj} = 1.$$

Then, we obtain

$$\begin{aligned}\sum_i \Lambda_i^{\dagger}\Lambda_i &= (q-2)^2\frac{N}{q} + \frac{N(N-1)(N-2)}{3q} + 2(q-2)\sum_{i\neq j} w_{ij} d_i^{\dagger} d_j - q^2\sum_{i\neq j} w_{ij}^2 n_i n_j \\ &\quad - (N-2)\left(\Upsilon^{\dagger}\Upsilon - \frac{N}{q}\right) - 2\sum_{i\neq j} w_{ij}^2 d_i^{\dagger} d_j + q\sum_{i\neq j\neq l} w_{ij}w_{il}\left(d_j^{\dagger} d_i + d_i^{\dagger} d_j\right) n_l \\ &\quad - q^2\frac{N(N-q)(N-2q)}{3q^3} \\ &= 2\sum_{i\neq j}\left[(q-2)w_{ij} - w_{ij}^2\right] d_i^{\dagger} d_j - q^2\sum_{i\neq j} w_{ij}^2 n_i n_j - (N-2)\Upsilon^{\dagger}\Upsilon \\ &\quad + q\sum_{i\neq j\neq l} w_{ij}w_{il}\left(d_j^{\dagger} d_i + d_i^{\dagger} d_j\right) n_l + \frac{N}{3q}\left[3qN + (q^2 - 12q + 8)\right].\end{aligned} \tag{C.8}$$

Now we construct positive-semidefinite operators from the operator

$$\Gamma_i = \sum_{j\,(\neq i)} w_{ij} d_i d_j \tag{C.9}$$

by forming

$$\Gamma_i^{\dagger}\Gamma_i = -\sum_{j,l\,(\neq i)} w_{ij}w_{il} d_l^{\dagger} d_j n_i = -\sum_{j\,(\neq i)} w_{ij}^2 n_i n_j - \sum_{j\neq l\,(\neq i)} w_{ij}w_{il} d_l^{\dagger} d_j n_i, \tag{C.10}$$

and

$$\sum_i \Gamma_i^{\dagger}\Gamma_i = -\sum_{i\neq j} w_{ij}^2 n_i n_j - \sum_{i\neq j\neq l} w_{lj}w_{li} d_i^{\dagger} d_j n_l. \tag{C.11}$$

Note that $\sum_i \Lambda_i^{\dagger}\Lambda_i$ and $\sum_i \Gamma_i^{\dagger}\Gamma_i$ both contain three-body interaction terms. However, we observe that, the following combination eliminates the three-body terms by using the cyclic identity:

$$
\begin{aligned}
\sum_i \Lambda_i^\dagger \Lambda_i - q \sum_i \Gamma_i^\dagger \Gamma_i
&= 2\sum_{i\neq j}\left[(q-2)w_{ij} - w_{ij}^2\right] d_i^\dagger d_j - (q^2-q)\sum_{i\neq j} w_{ij}^2 n_i n_j - (N-2)\Upsilon^\dagger\Upsilon \\
&\quad + q\sum_{i\neq j\neq l}\left(w_{ij}w_{il} + w_{ji}w_{jl} + w_{lj}w_{li}\right) d_i^\dagger d_j n_l + \frac{N}{3q}\left[3qN + (q^2 - 12q + 8)\right] \\
&= 2\sum_{i\neq j}\left[(q-2)w_{ij} - w_{ij}^2\right] d_i^\dagger d_j - (q^2-q)\sum_{i\neq j} w_{ij}^2 n_i n_j - (N-2)\Upsilon^\dagger\Upsilon \\
&\quad + q\sum_{i\neq j\neq l} d_i^\dagger d_j n_l + \frac{N}{3q}\left[3qN + (q^2 - 12q + 8)\right] \\
&= 2\sum_{i\neq j}\left[(q-2)w_{ij} - w_{ij}^2\right] d_i^\dagger d_j - (q^2-q)\sum_{i\neq j} w_{ij}^2 n_i n_j - (q-2)\Upsilon^\dagger\Upsilon \\
&\quad + \frac{q-1}{3q}N\left[3N + (q-8)\right],
\end{aligned}
\tag{C.12}
$$

where we have used

$$
\begin{aligned}
\sum_{i\neq j\neq l} d_i^\dagger d_j n_l &= \sum_{i\neq j} d_i^\dagger d_j \left(\frac{N}{q} - n_i - n_j\right) \\
&= \left(\frac{N}{q} - 1\right)\sum_{i\neq j} d_i^\dagger d_j = \left(\frac{N}{q} - 1\right)\Upsilon^\dagger\Upsilon - \frac{N}{q}\left(\frac{N}{q} - 1\right).
\end{aligned}
\tag{C.13}
$$

Finally, we define the 1D parent Hamiltonian as

$$
\begin{aligned}
H_{\mathrm{1D}} &= \frac{1}{2}\sum_i \Lambda_i^\dagger \Lambda_i - \frac{q}{2}\sum_i \Gamma_i^\dagger \Gamma_i + \frac{q-2}{2}\Upsilon^\dagger\Upsilon - E_0 \\
&= \sum_{i\neq j}\left[(q-2)w_{ij} - w_{ij}^2\right] d_i^\dagger d_j - \frac{1}{2}(q^2-q)\sum_{i\neq j} w_{ij}^2 n_i n_j,
\end{aligned}
\tag{C.14}
$$

where E_0 is the ground-state energy of H_{1D},

$$
E_0 = -\frac{q-1}{6q}N\left[3N + (q-8)\right].
\tag{C.15}
$$

If $\chi_j \neq 1$, d_j should be replaced by $\tilde{d}_j = \chi_j^{-1} d_j$.

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