Fate of classical tensor inhomogeneities in pre-big-bang string cosmology

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In pre-big-bang string cosmology one uses a phase of dilaton-driven inflation to stretch an initial (microscopic) spatial patch to the (much larger) size of the big-bang fireball. We show that the dilaton-driven inflationary phase does not naturally iron out the initial classical tensor inhomogeneities unless the initial value of the string coupling is smaller than $g_{\rm in} \lesssim 10^{-35}$.

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I. INTRODUCTION

The pre-big-bang (PBB) scenario [1] is an attempt to use the kinetic energy of the string-theory dilaton to drive a period of inflation of the universe. The basic motivations of the PBB scenario are: (i) the existence of exact (spatially homogeneous) dilaton-driven inflationary solutions following from the *T*-duality symmetries of string-theory [2], and (ii) the need to bypass the fact that a tree-level dilaton essentially destroys [3] the usual (potential-driven) inflationary mechanism. In the ''stochastic'' version of the PBB scenario [4] one envisages the birth of an ensemble of pre-big-bang bubbles from the gravitational instability of a *generic* string vacuum made of a stochastic bath of classical incoming gravitational and dilatonic waves. In this approach the only needed condition for the blistering (in string units) of a PBB bubble (of size $H_{\rm in}^{-1}$, where $H_{\rm in}$ is the initial Hubble expansion rate of a patch of space) is similar to the corresponding condition in ''chaotic'' inflation [5] (see below). Namely, locally, the inhomogeneous contributions (of wavelengths smaller than $H_{\rm in}^{-1}$) to the local Friedmann equation should be fractionally smallish (say by a factor of 5) compared to the homogeneous contribution $\dot{\varphi}_{\rm in}^2 \sim H_{\rm in}^2$. This ''stochastic'' PBB approach, together with other studies of inhomogeneous versions of PBB [6–8], was intended to answer (or at least to soothe) the concerns about fine tuning [9,10] in the PBB scenario. However, as far as we are aware, no complete study of the effectiveness of the PBB dilaton-driven inflation (DDI) in smoothing out initial homogeneities has been performed. [Note that this smoothing out of *classical* inhomogeneities is the *prerequisite* for the discussion of the irrepressible *quantum* fluctuations that might be the seed of the large-scale structure of the universe.] Reference [11] discussed the fate of (quantum) inhomogeneities during the DDI phase and concluded that their growth, when they get out of the horizon, was only logarithmic, but they did not analyze the smoothing properties of the entire pre-big-bang plus post-big-bang scenario. The recent discovery of the generic appearance of an inhomogeneous chaos, ultimately leading to a string-scale foam near a big crunch [12], prompted us to reexamine in detail the fate of initial classical inhomogeneities during the entire evolution of the simplest PBB scenario (comprising an initial DDI phase matched onto a subsequent ordinary big-bang evolution).

In this paper we consider the ''stochastic'' version of the PBB scenario, and study the evolution of the tensor inhomogeneities present in a *generic* PBB inflationary bubble. Our conclusions is that the PBB scenario is not very effective in smoothing out initial classical inhomogeneities (we limit ourselves to inhomogeneities small enough for not developing into a turbulent chaos *before* reaching the string scale). Indeed, analyzing tensor inhomogeneities, we find that they need to be initially unnaturally small, *except* in the case where the initial value of the string coupling is parametrically smaller than the (already very small) minimal value $g_{\rm in}^{\rm min} \simeq 10^{-26}$ needed to solve the horizon problem, i.e. to generate a space at least as large as our horizon from an initial patch of size $H_{\rm in}^{-1}$ [1,9,10]. More precisely, we find that if we wish generic, coarsely homogeneous, bubbles to evolve into our (globally very homogeneous) universe we need to require $g_{\rm in} \lesssim (10^{-10})^{\sqrt{3}/2} g_{\rm in}^{\rm min} \simeq 10^{-35}$. We note that the necessity (for solving this ''homogeneity problem'') of having more inflation than the minimal amount needed for solving the horizon problem applies also to the standard inflationary models (see below).

II. TENSOR PERTURBATIONS IN PRE-BIG-BANG COSMOLOGY

We restrict our investigation to the simplest version of the PBB scenario, which is described in the string frame by the four-dimensional low-energy string-effective action

$$\Gamma_S = \frac{1}{\lambda_s^2} \int d^4x \sqrt{-g_S}\, e^{-\varphi} [R(g_S) + g_S^{\mu\nu}\, \partial_\mu \varphi\, \partial_\nu \varphi], \tag{2.1}$$

where φ is the dilaton field, related to the string coupling by $g = e^{\varphi/2}$, and λ_s is the string scale. In the following, we shall systematically use the string metric $g_{\mu\nu}^{\rm S}$ to measure physical lengths or frequencies. However, it will also be technically useful to introduce the Einstein metric $g_{\mu\nu}^{\rm E}$. The string and Einstein metrics are related (in 4 dimensions) by $g_{\mu\nu}^{\rm S} = e^{\varphi - \varphi_0} g_{\mu\nu}^{\rm E}$. Indicating the tensor perturbations as

$\delta g^{\rm S}_{\mu\nu}=h^{\rm S}_{\mu\nu}$, and working in the synchronous gauge ($g^{\rm S}_{00}=-1, g^{\rm S}_{0i}=0, g^{\rm S}_{ij}=a^2_{\rm S}\,\delta_{ij}$ and $h^{\rm S}_{00}=0, h^{\rm S}_{0i}=0, g^{ij}_{\rm S}\,h^{\rm S}_{ij}=0, \partial_j h_i^{{\rm S}\ j}=0$), it is easily checked that $h_i^{{\rm S}\ j}=h_i^{{\rm E}\ j}$. Henceforth, we denote the tensor perturbations by $h_i^{\ j}\equiv h_i^{{\rm S}\ j}=h_i^{{\rm E}\ j}$. Introducing the conformal time $d\eta=dt_{\rm E}/a_{\rm E}=dt_{\rm S}/a_{\rm S}$ and working in Fourier space we have

$$h_i^j(\boldsymbol{x},\eta)=\int\frac{d^3k}{(2\pi)^3}e^{i\boldsymbol{k}\cdot\boldsymbol{x}}\sum_{\sigma=\pm 2}\epsilon_i^{(\sigma)j}(\boldsymbol{k})h^{(\sigma)}(\boldsymbol{k},\eta), \tag{2.2}$$

where $\epsilon^{(\sigma)j}{}_i$ is the polarization tensor, which satisfies the usual relations

$$\epsilon_i^{(\sigma_1)j*}(\boldsymbol{k})\epsilon_i^{(\sigma_2)j}(\boldsymbol{k})=\delta^{\sigma_1\sigma_2},$$

$$\sum_\sigma\epsilon_i^{(\sigma)j*}(\boldsymbol{k})\epsilon_k^{(\sigma)l}(\boldsymbol{k})=\delta_{ik}^{TT\,jl}(\boldsymbol{k}). \tag{2.3}$$

In the following to ease the notation we shall drop the superscript σ over h in Eq. (2.2).

A. Evolution of tensor fluctuations

During the dilaton-driven inflationary (DDI) phase[1] the Fourier transform of the tensor fluctuations satisfies the equation:

$$h''+2\mathcal{H}_{\rm E}h'+k^2h=0, \tag{2.4}$$

where $\mathcal{H}_{\rm E}=a'_{\rm E}/a_{\rm E}$. Introducing the canonical variable $\psi=a_{\rm E}\,h$, we obtain

$$\psi''+[k^2-V(\eta)]\psi=0,\qquad V(\eta)=\frac{a''_{\rm E}}{a_{\rm E}}. \tag{2.5}$$

From the above equation it is straightforward to derive that the perturbations propagating inside the horizon ($k^2\gg V(\eta)$, i.e. $|k\eta|\gg 1$), during the DDI phase, evolve simply as $\psi\equiv a_{\rm E}h\simeq{\rm const}\times\exp(\pm ik\eta)$, so that (modulo a phase factor)

$$h_{\text{in hor.}}(k,\eta)=\frac{a_{\rm E}(\eta_{\rm in})}{a_{\rm E}(\eta)}h(k,\eta_{\rm in}), \tag{2.6}$$

where $\eta_{\rm in}$ is some initial time. Note that the scale factor $a_{\rm E}$ decreases in time during the DDI era $[a_{\rm E}\propto(-\eta)^{1/2}$ and $\eta\to 0^-]$; therefore, as long as it is within the horizon, $h(k,\eta)$ increases in time during the DDI phase. A generic fluctuation exits the horizon for the first time during the DDI era at $|\eta_{\rm ex}|\equiv 1/k$. Later on, while outside the horizon ($|k\eta|\ll 1$), its evolution is given by $h''+2a'_{\rm E}/a_{\rm E}h'\simeq 0$, so that $h\simeq c_1+c_2\int d\eta\, a_{\rm E}^{-2}$. As $a^2_{\rm E}\propto\eta$, one gets a logarithmic growth

$$h_{\text{out hor.}}(k,\eta)\simeq\log\left(\frac{\eta}{\eta_{\rm ex}}\right)\frac{a_{\rm E}(\eta_{\rm in})}{a_{\rm E}(\eta_{\rm ex})}h(k,\eta_{\rm in}). \tag{2.7}$$

Outside the horizon $h(k,\eta)$ undergoes a logarithmic growth while the physical wavelength in the string frame, $\hat{l}^s=a_S/k$, is stretched during this DDI phase. In the following, we refer to physical quantities with a hat, e.g., $\hat{k}_i=k/a_{S\,i}$, where i refers to the instant of time t_i at which we evaluate the physical quantity.

Later on, if $a_{\rm E}$ starts to increase while the fluctuation is still outside the horizon the fluctuation $h\simeq c_1+c_2\int d\eta\, a_{\rm E}^{-2}\simeq$const. During the radiation and matter eras the amplitude of the tensor fluctuations, after reentering the horizon, decreases as $\sim 1/a_{\rm E}$, notably as $\sim 1/\eta$ during the radiation-dominated (RD) phase and as $\sim 1/\eta^2$ during the matter-dominated (MD) era.

Let us now introduce several (dimensionless) quantities that play a crucial role in our analysis: the coefficient $\mathcal{A}$ that measures the *amplification* of (tensor) fluctuations from the initial time until today, the coefficient $\mathcal{B}\ll 1$ (the inverse of the redshift factor) which keeps track of the stretching of physical frequencies and length scales between the initial patch and now, and a coefficient $\mathcal{C}$ whose meaning will be described below:

$$\mathcal{A}(\hat{k}_0)\equiv\frac{h(\hat{k}_0,\eta_0)}{h(\hat{k}_{\rm in},\eta_{\rm in})},\qquad\mathcal{B}\equiv\frac{\hat{k}_0}{\hat{k}_{\rm in}},$$

$$\mathcal{C}(\hat{k}_0)\equiv\left(\frac{\hat{H}_{\rm in}}{\hat{H}_0}\right)^2\mathcal{B}^2\mathcal{A}^2(\hat{k}_0). \tag{2.8}$$

Here, given a comoving wave-number k, $\hat{k}_0=k/a_{S0}$ and $\hat{k}_{\rm in}=k/a_{S\rm in}$. The index 0 refers to the present time, $\eta=\eta_0$, while the index, in, refers to the initial time, $\eta=\eta_{\rm in}$. The couple of functions of $\hat{k}_0$, $\{\mathcal{A}(\hat{k}_0),\mathcal{C}(\hat{k}_0)\}$, and the constant $\mathcal{B}$, exhaust the description of the ''transfer function'' between the classical initial inhomogeneities and the present ones.

For simplicity, we restrict our attention in this paper to the simplest PBB scenario in which there is not any intermediate phase between the DDI era and the standard Friedmann-Lemaitre-Gamow one. We denote by η_1 the conformal time at which the evolution of the universe (which is always expanding in the string frame) changes from the DDI expansion phase to a big-bang fireball. We assume that this transition takes place when the expansion rate reaches the string-scale, $\hat{H}_1=\dot{a}_{S1}/a_{S1}\simeq\lambda_s^{-1}$ and when the string coupling $g_1=e^{\varphi_1/2}$ equals its present value $g_1\equiv g_0\simeq 0.1$. For times $\eta>\eta_1$ we assume that the dilaton has become effectively fixed so that $a_E=a_S$.

If a fluctuation reenters the horizon before the MD era, i.e. during the RD phase, we have

$$\mathcal{A}(\hat{k}_0)=\frac{a_{\rm E}(\eta_{\rm in})}{a_{\rm E}(\eta_{\rm ex})}\log(k\,\eta_1)\frac{a_{\rm E}(\eta_{\rm re})}{a_{\rm E}(\eta_{\rm eq})}\,\frac{a_{\rm E}(\eta_{\rm eq})}{a_{\rm E}(\eta_0)}, \tag{2.9}$$

while if reentry occurs during the MD phase we get

[1]Many of the results below were already derived in [11] and other places. It is, however, simpler to give a self-contained presentation.

$$\mathcal{A}(\hat{k}_0)=\frac{a_{\rm E}(\eta_{\rm in})}{a_{\rm E}(\eta_{\rm ex})}\log(k\ \eta_1)\frac{a_{\rm E}(\eta_{\rm re})}{a_{\rm E}(\eta_0)}, \tag{2.10}$$

where $\eta_{\rm eq}$ stands for the time at which there is equality in the universe between radiation and matter density.

Assuming homogeneity and isotropy, the background fields in the string frame evolve as

$$a_S(\eta)=\left(\frac{\eta}{\eta_1}\right)^{-(\sqrt{3}-1)/2}, \qquad \varphi(\eta)=\varphi_1-\sqrt{3}\log\left(\frac{\eta}{\eta_1}\right)$$
$$-\infty<\eta<\eta_1,$$
$$a_S(\eta)=a_E(\eta)=\left(\frac{\eta}{\eta_1}\right), \qquad \varphi(\eta)=\varphi_1 \qquad \eta_1<\eta<\eta_{\rm eq},$$
$$a_S(\eta)=a_E(\eta)=\left(\frac{\eta^2}{\eta_1\ \eta_{\rm eq}}\right), \qquad \varphi(\eta)=\varphi_1$$
$$\eta_{\rm eq}<\eta<\eta_0. \tag{2.11}$$

Henceforth, to ease the notation, when referring to the scale factor in the string frame, we shall drop the subscript S. Using the above equations and neglecting the logarithmic growth in Eqs. (2.9), (2.10), we derive

$$\mathcal{A}(\hat{k}_0)=\left(\frac{g_1}{g_{\rm in}}\right)\left(\frac{\hat{H}_1}{\hat{k}_{\rm in}}\right)\left(\frac{\hat{k}_0}{\hat{\omega}_0^1}\right)^{3/2} \qquad \hat{k}_0\ll\hat{H}_0, \tag{2.12}$$

$$\mathcal{A}(\hat{k}_0)=\left(\frac{g_1}{g_{\rm in}}\right)\left(\frac{\hat{\omega}_0^{\rm eq}}{\hat{k}_{\rm in}}\right)\left(\frac{\hat{\omega}_0^1}{\hat{\omega}_0^{\rm eq}}\right)^{1/2}\left(\frac{\hat{\omega}_0^{\rm eq}}{\hat{k}_0}\right)^{1/2},$$
$$\hat{H}_0\ll\hat{k}_0\ll\hat{\omega}_0^{\rm eq}, \tag{2.13}$$

$$\mathcal{A}(\hat{k}_0)=\left(\frac{g_1}{g_{\rm in}}\right)\left(\frac{\hat{\omega}_0^1}{\hat{k}_{\rm in}}\right)\left(\frac{\hat{k}_0}{\hat{\omega}_0^1}\right)^{1/2}, \qquad \hat{\omega}_0^{\rm eq}\ll\hat{k}_0\ll\hat{\omega}_0^1, \tag{2.14}$$

where $\hat{\omega}_0^1=\omega_1/a_0$, $\hat{\omega}_0^{\rm eq}=\omega_{\rm eq}/a_0$. Here ω_1 and $\omega_{\rm eq}$ are the constant comoving wave numbers whose physical counterparts coincide with the Hubble expansion rates at time η_1 and $\eta_{\rm eq}$, respectively. More explicitly,

$$\frac{\omega_{\rm eq}^2}{a_{\rm eq}^2}\equiv\hat{H}_{\rm eq}^2=\frac{8\pi G}{3}\rho_c(t_{\rm eq}), \qquad \frac{\omega_1^2}{a_1^2}\equiv\hat{H}_1^2=\frac{8\pi G}{3}\rho_c(t_1), \tag{2.15}$$

$$\frac{\hat{\omega}_0^1}{\hat{H}_1}=\frac{1}{1+z_1}\simeq10^{-30}, \qquad \frac{\hat{\omega}_0^{\rm eq}}{\hat{H}_0}=\sqrt{1+z_{\rm eq}}\simeq10^2, \tag{2.16}$$

where we defined the redshift factor z as $a/a_0\equiv1/(1+z)$. Correspondingly, we get

$$\mathcal{B}=\frac{\hat{k}_0}{\hat{k}_{\rm in}}=\left(\frac{a_{\rm in}}{a_1}\right)\left(\frac{a_1}{a_0}\right)=\left(\frac{g_{\rm in}}{g_1}\right)^{2/(3+\sqrt{3})}\left(\frac{\hat{H}_0}{\hat{H}_1}\right)\left(\frac{\hat{H}_1}{\hat{\omega}_0^{\rm eq}}\right)^{1/2}. \tag{2.17}$$

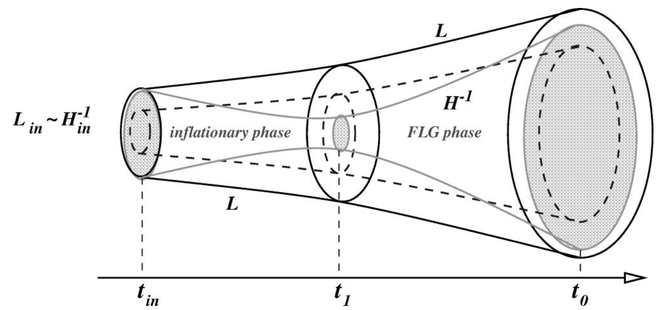

FIG. 1. Schematic representation, in the string frame and for the nonminimal version of the PBB scenario, of the evolution of: the Hubble horizon H^{-1}, an intermediate physical wavelength (dashed line) and the comoving size L (continuous line) corresponding to the initial patch H^{-1}.

It has been derived in Refs. [9,7] that in order to solve the horizon (and flatness) problems in the PBB model, one has to require that

$$g_{\rm in}\lesssim g_{\rm in}^{\rm min}\simeq10^{-26}, \qquad \hat{H}_{\rm in}^{-1}\gtrsim(\hat{H}_{\rm in}^{\rm min})^{-1}\simeq10^{18}\lambda_s. \tag{2.18}$$

Indeed, defining the total amount of inflation as the ratio between the comoving Hubble length at the end and beginning of the PBB inflationary phase,

$$\mathcal{Z}=\frac{a_1\hat{H}_1}{a_{\rm in}\hat{H}_{\rm in}}, \tag{2.19}$$

the horizon problem is solved if we impose that

$$\mathcal{Z}\geqslant\frac{\hat{l}_0(t_1)}{\hat{l}_c(t_1)}, \tag{2.20}$$

where $\hat{l}_0(t_1)=\hat{H}_0^{-1}a_1/a_0$ and $\hat{l}_c(t_1)=\hat{H}_1^{-1}\sim\lambda_s$. The equality sign in Eq. (2.20) refers to the *minimal* PBB scenario, in which the horizon volume today has evolved from an initial (Hubble) patch of size $\hat{H}_{\rm in}^{-1}$. The minimal and nonminimal scenarios are illustrated in Fig. 1. Note that, in the nonminimal scenario, the Hubble scale at present time $\hat{H}_0^{-1}$ is strictly smaller than the comoving scale $L(t_0)=L_{\rm in}a_0/a_{\rm in}$. Using the isotropic and homogeneous PBB background solutions (2.11), it is easily derived that $a_{\rm in}/a_1=(g_{\rm in}/g_1)^{2/(3+\sqrt{3})}$ and $\hat{H}_{\rm in}/\hat{H}_1=(g_{\rm in}/g_1)^{2\sqrt{3}/(3+\sqrt{3})}$. Imposing Eq. (2.20) with the equality sign, we find the minimal initial conditions as

$$g_{\rm in}^{\rm min}\equiv g_1\left(\frac{\hat{H}_1}{\hat{H}_0}\right)^{-\sqrt{3}/2}\left(\frac{a_1}{a_0}\right)^{-\sqrt{3}/2}=g_1\left(\frac{\hat{H}_1}{\hat{\omega}_0^{\rm eq}}\right)^{-\sqrt{3}/4}, \tag{2.21}$$

and

$$\hat{H}_{\rm in}^{\rm min} \equiv \hat{H}_1 \left(\frac{\hat{H}_1}{\hat{H}_0}\right)^{-\sqrt{3}/(\sqrt{3}+1)} \left(\frac{a_1}{a_0}\right)^{-\sqrt{3}/(\sqrt{3}+1)} = \hat{H}_1 \left(\frac{\hat{\omega}_0^{\rm eq}}{\hat{H}_1}\right)^{\sqrt{3}/2(\sqrt{3}+1)}, \tag{2.22}$$

where we used $a_1/a_0 = \hat{\omega}_0^1/\hat{H}_1 = (\hat{H}_0/\hat{H}_1)(\hat{H}_1/\hat{\omega}_0^{\rm eq})^{1/2}$. Inserting in Eqs. (2.21), (2.22) the numerical values $\hat{H}_0 = 10^{-18}$ Hz, $\hat{\omega}_0^{\rm eq} = 10^{-16}$ Hz and $\hat{H}_1 \simeq \lambda_s^{-1} \sim 10^{42}$ Hz, we obtain Eq. (2.18).

Introducing the notation $\overline{g}_{\rm in} \equiv g_{\rm in}/g_{\rm in}^{\rm min} \equiv 10^{26} g_{\rm in}$ ($\overline{g}_{\rm in} \leqslant 1$, with equality in the minimal scenario), we can rewrite Eqs. (2.13), (2.14) in the form

$$\mathcal{A}(\hat{k}_0) = \frac{1}{(\overline{g}_{\rm in})^{1/\sqrt{3}}} \left(\frac{\hat{H}_0}{\hat{k}_0}\right)^{3/2}, \qquad \hat{H}_0 \leqslant \hat{k}_0 \leqslant \hat{\omega}_0^{\rm eq}, \tag{2.23}$$

$$\mathcal{A}(\hat{k}_0) = \frac{1}{(\overline{g}_{\rm in})^{1/\sqrt{3}}} \left(\frac{\hat{H}_0}{\hat{k}_0}\right)^{1/2} \frac{1}{\sqrt{1+z_{\rm eq}}}, \qquad \hat{\omega}_0^{\rm eq} \leqslant \hat{k}_0 \leqslant \hat{\omega}_0^1. \tag{2.24}$$

The above equations can also be recast in a unique formula which interpolates between the two frequency regions:

$$\mathcal{A}(\hat{k}_0) = \frac{1}{(\overline{g}_{\rm in})^{1/\sqrt{3}}} \left(\frac{\hat{H}_0}{\hat{k}_0}\right)^{1/2} \left[\frac{\hat{H}_0}{\hat{k}_0} + \frac{1}{\sqrt{1+z_{\rm eq}}}\right]. \tag{2.25}$$

The most striking consequence of Eq. (2.25) concerns tensor fluctuations on the present horizon scale $\hat{k}_0 \sim \hat{H}_0$. For these scales, the amplification coefficient connecting them to the initial fluctuations is $\mathcal{A}(\hat{H}_0) \simeq (\overline{g}_{\rm in})^{-1/\sqrt{3}}$. In the minimal scenario ($\overline{g}_{\rm in} = 1$) this is $\mathcal{A}(\hat{H}_0) \simeq 1$ which means that horizon-scale tensor fluctuations today just reproduce (modulo a logarithmic amplification factor that we neglected) the corresponding initial horizon-scale fluctuations. [Note that this preservation of the amplitude of horizon-scale fluctuations in the minimal, horizon-solving, case applies equally well to a standard potential-driven inflation scenario.] On the other hand, the amplification properties of PBB inflation look worse in the nonminimal scenario ($\overline{g}_{\rm in} \leqslant 1$) for which horizon-scale tensor fluctuations today are *parametrically* amplified compared to the corresponding initial fluctuations. This behavior is different in the PBB model than in ordinary inflation. For example, if inflation is implemented by a de Sitter phase ($\hat{H}_{\rm dS} \simeq$ const), starting at t_i, and ending at t_1 when the transition to RD phase occurs, then the amplification factor for fluctuations that are just outside the horizon ($\hat{k}_0 \sim \hat{H}_0$) is

$$\mathcal{A}(\hat{k}_0) = e^{-\mathcal{N}} \left(\frac{\hat{H}_{\rm dS} a_1}{\hat{H}_o\, a_0}\right), \tag{2.26}$$

where $\mathcal{N} \equiv \log(a_1/a_i)$ is the total number of e foldings. Hence, differently from the PBB scenario, where tensor perturbations increase during the PBB era, in ordinary inflation as $\mathcal{N}$ increases (longer inflationary era) $\mathcal{A}$ decreases parametrically.

However, this *amplification* of initial tensor fluctuations by PBB cosmology (instead of the usual *deamplification* mechanism of potential-driven inflation in the nonminimal case), though paradoxical, does not, by itself, imply that the initial value of the tensor inhomogeneities must be fine tuned to an unnaturally small value. Indeed, the classical quantity that needs to be smallish for the dilaton-driven inflation to start is not the *amplitude* of tensor waves, but their *energy density* (compared to $\dot{\varphi}_{\rm in}^2$). We shall postpone the study of the latter quantity to the next section.

For the quantity $\mathcal{B}$, defined by Eq. (2.8), we find for the PBB scenario under investigation that

$$\mathcal{B} = \frac{\hat{k}_0}{\hat{k}_{\rm in}} = \frac{\hat{H}_0}{\hat{H}_{\rm min}^{\rm in}} (\overline{g}_{\rm in})^{2/(3+\sqrt{3})} \simeq 10^{-42} (\overline{g}_{\rm in})^{2/(3+\sqrt{3})}, \tag{2.27}$$

where in the last equation we used the fact that $\hat{H}_{\rm min}^{\rm in} \sim 10^{24}$ Hz. Here the behavior is similar to what happens in standard inflation, e.g., with a de Sitter phase we derive

$$\mathcal{B} = e^{-\mathcal{N}} \left(\frac{a_1}{a_0}\right). \tag{2.28}$$

In PBB cosmology, as in ordinary inflation, the wavelength of the tensor perturbations always gets stretched, and the amount of stretching is parametrically larger in the nonminimal case ($\overline{g}_{\rm in} \leqslant 1$; or $\mathcal{N} > \mathcal{N}_{\rm min}$) than in the minimal one.

It is important to notice that the formulas given above for the *classical* "transfer function" $\mathcal{A}(\hat{k}_0)$ and the (inverse) redshift factor $\mathcal{B}$ are physically meaningful only when it concerns a present spatial frequency $\hat{k}_0$ such that the corresponding blueshifted frequency $\mathcal{B}^{-1}\hat{k}_0$ (which represents the initial frequency) is smaller than the string scale $\hat{\omega}_s = 1/\lambda_s$. When this is not the case, this classical transfer function does not apply, and one must consider the problem of quantum-normalized fluctuations (as studied, e.g., in Ref. [13]). The results for $\mathcal{A}$ and $\mathcal{B}$ provided by Eqs. (2.23), (2.24) and (2.27) are summarized in Fig. 2.

B. Power spectrum for tensor fluctuations

The "bare" power spectrum is defined by the relation:

$$\langle h^{(\sigma_1)}(\boldsymbol{k}_1, \eta) h^{(\sigma_2)*}(\boldsymbol{k}_2, \eta) \rangle = \delta^{\sigma_1 \sigma_2}\, \delta^{(3)}(\boldsymbol{k}_1 - \boldsymbol{k}_2) \times \mathcal{P}_h^{bare}(\boldsymbol{k}_1, \eta), \tag{2.29}$$

where $h^{(\sigma)}(\boldsymbol{k}, \eta)$ is given by Eq. (2.2). Using the relations (2.3) and assuming isotropy it is straightforward to derive:

$$\langle h_i^j(\boldsymbol{x}_1,\eta) h_l^{k*}(\boldsymbol{x}_2,\eta)\rangle = \int \frac{dk}{k}\frac{d\Omega_k}{4\pi} e^{i\boldsymbol{k}\cdot(\boldsymbol{x}_1-\boldsymbol{x}_2)} \times \delta_{ik}^{TT\, jl}(\boldsymbol{k}) \frac{k^3}{2\pi^2}\mathcal{P}_h^{bare}(\boldsymbol{k},\eta). \tag{2.30}$$

Hence, the ‘‘physical’’ (per logarithmic interval of spatial frequency) power spectrum is related to the ‘‘bare’’ one through

$$\mathcal{P}_h^{phys.}(\boldsymbol{k},\eta) \equiv \frac{k^3}{2\pi^2}\mathcal{P}_h^{bare}(\boldsymbol{k},\eta). \tag{2.31}$$

[In the following we drop the superscript ‘‘physical’’ on the power spectrum.] Note that the ‘‘physical’’ power spectrum has the same dimensions as $h^2(\boldsymbol{x},\eta)$ (i.e. it is dimensionless). The energy density in gravitational waves is given by:

$$\rho_{\rm GW} = \int \frac{dk}{k}\frac{d\rho_{\rm GW}(k)}{d\log k}, \qquad \frac{d\rho_{\rm GW}(k)}{d\log k} = \frac{1}{16\pi G}\mathcal{P}_{\dot h}(k), \tag{2.32}$$

where $\mathcal{P}_{\dot h}$ is the (physical) power spectrum for the time derivative of the tensor fluctuation dh/dt, i.e. $\mathcal{P}_{\dot h}(k) \simeq \hat k^2 \mathcal{P}_h(k)$. The ratio between the energy density in gravitational waves and the critical energy ρ_c (conventionally defined in all cases[2] by $3\hat H^2 = 8\pi G\rho_c$) reads

$$\Omega_{\rm GW}(\hat k) \equiv \frac{1}{\rho_c}\frac{d\rho_{\rm GW}(\hat k)}{d\log \hat k} \equiv \frac{1}{6}\frac{\hat k^2}{\hat H^2}\mathcal{P}_h(\hat k). \tag{2.33}$$

As explained in the Introduction, in the stochastic PBB model a generic inflating bubble is expected to have initial values of $\Omega_{\rm GW}(\hat k)$ and of similar ratios for the other field inhomogeneities which are smallish (say $\sim 1/5$), but not parametrically small. As we are interested in order-of-magnitude estimates, we shall henceforth consider that generic inhomogeneities should be allowed to be as large as $\Omega_{\rm GW} \sim \Omega_\varphi \sim 1$.

Therefore, while by definition the amplification of the power spectrum of h is given only by the $\mathcal{A}$ factor [defined in Eq. (2.8)],

$$\frac{\mathcal{P}_h(\hat k_0)}{\mathcal{P}_h(\hat k_{\rm in})} = \mathcal{A}^2(\hat k_0), \tag{2.34}$$

the amplification of $\Omega_{\rm GW}$ reads

[2]Here, $\hat H, G$ and ρ_c are all measured in string units. For discussing initial values it would be more accurate to work with Einstein-frame quantities. We neglect the inaccuracy (due to the difference between the physical Einstein and string Hubble expansion rates) introduced by our definition, which is only a factor of order unity.

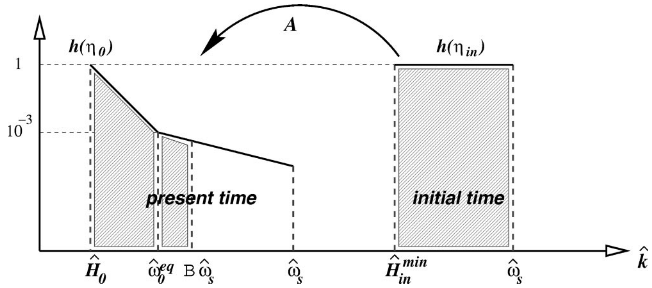

FIG. 2. We show schematically how the ‘‘transfer function’’ between initial and present time acts both on the amplitude, Eq. (2.25), and the frequency, Eq. (2.26), of tensor fluctuations. We have assumed for simplicity that $h(\eta_{\rm in})=1$ and $\overline{g}_{\rm in}=1$. In the minimal PBB scenario the initial string frequency $\hat\omega_s = \lambda_s^{-1}$ corresponds today to $\mathcal{B}\hat\omega_s = 1$ Hz. The present Hubble frequency $\hat H_0 = \mathcal{B}\hat H_{\rm in}^{\rm min} \sim 10^{-18}$ Hz originates from the initial frequency $\hat H_{\rm in}^{\rm min} \sim 10^{24}$ Hz, which corresponds to 1 Fermi. Note that, whereas the colored regions refer to initial classical fluctuations, the white one on the left part of the figure concerns wavelengths that would correspond to initial length scales formally smaller than the string scale, i.e. to quantum fluctuations. For them the classical transfer function does not apply.

$$\frac{\Omega_{\rm GW}(\hat k_0)}{\Omega_{\rm GW}(\hat k_{\rm in})} = \mathcal{C}(\hat k_0), \tag{2.35}$$

where the dimensionless quantity $\mathcal{C}(\hat k_0)$ was defined in Eq. (2.8) above. We can compute the explicit expression of the tranfer function $\mathcal{C}(\hat k_0)$ by using Eq. (2.25) and the useful relation $\mathcal{B}\hat H_{\rm in}/\hat H_0 = (\overline{g}_{\rm in})^{2/\sqrt{3}}$. We find

$$\mathcal{C}(\hat k_0) = (\overline{g}_{\rm in})^{2/\sqrt{3}} \frac{\hat H_0}{\hat k_0}\left(\frac{\hat H_0}{\hat k_0} + \frac{1}{\sqrt{1+z_{\rm eq}}}\right)^2. \tag{2.36}$$

Note the good news that, in this result, the power of $\overline{g}_{\rm in}$ on the right-hand side is now *positive* [contrary to the paradoxical negative power of $\overline{g}_{\rm in}$ entering the amplitude-amplification coefficient $\mathcal{A}(\hat k_0)$, given by Eq. (2.25)]. [The positiveness of the exponent of $\overline{g}_{\rm in}$ in $\mathcal{C}(\hat k_0)$ is due to the positive compensating exponent entering $\mathcal{B}\hat H_{\rm in}/\hat H_0 = (\overline{g}_{\rm in})^{2/\sqrt{3}}$.] Therefore from the point of view of the parametric dependence of the overall decrease of tensor inhomogeneities, PBB inflation is not qualitatively different from ordinary inflation. However, we shall see that, from a quantitative point of view, solving the ‘‘homogeneity problem’’ leads to a more severe constraint for PBB inflation.

Let us now use the existing limits on the amount of gravitational waves generating inhomogeneities in the cosmic microwave background radiation (CMBR) to constrain the initial amount of gravitational waves $\Omega_{\rm GW}(\hat k_{\rm in})$. From [14,15] we read:

$$\Omega_{\rm GW}(\hat k_0) h_{100}^2 < 7\times 10^{-11}\left(\frac{\hat H_0}{\hat k_0}\right)^2, \qquad \hat H_0 < \hat k_0 < 30\hat H_0. \tag{2.37}$$

Using $h_{100}=0.65$ and writing the most stringent consequence of this limit (corresponding to $\hat{k}_0=\hat{H}_0$), we get the "homogeneity constraint"

$$\Omega_{\rm GW}(\hat{k}_{\rm in0})(\overline{g_{\rm in}})^{2/\sqrt{3}}\lesssim 10^{-10}, \tag{2.38}$$

where we denoted by $\hat{k}_{\rm in0}$ the initial wave number that corresponds now to $\hat{H}_0$, i.e. $\hat{k}_{\rm in0}=\mathcal{B}^{-1}\hat{H}_0$. [Note that $\hat{k}_{\rm in0}\geqslant\hat{H}_{\rm in}$.] If we were to restrict ourselves to the minimal PBB scenario ($\overline{g_{\rm in}}=1$), i.e. to the case in which we require the minimum amount of inflation in order to solve the horizon problem (i.e. an initial PBB bubble of size 1 Fermi), Eq. (2.38) would tell us that the initial tensor inhomogeneities must be unnaturally small: $\Omega_{\rm GW}(\hat{H}_{\rm in}^{\rm min})\lesssim 10^{-10}$. This would mean that one looses all the genericity benefits of considering a "stochastic" PBB model.

There is, however, a way to solve this "homogeneity problem," i.e. to relax this unnatural fine tuning of initial inhomogeneities, and to allow for "generic" initial inhomogeneities $\Omega_{\rm GW}(\hat{k}_{\rm in})\sim 1$. Indeed, the fact that $\overline{g_{\rm in}}$ enters Eq. (2.36) with a positive power means that it is enough to impose

$$\overline{g_{\rm in}}\lesssim(10^{-10})^{\sqrt{3}/2}\sim 10^{-9}. \tag{2.39}$$

In terms of the string coupling $g_{\rm in}$, this limit is 9 orders of magnitude smaller than the value given in Eq. (2.18), i.e.

$$g_{\rm in}\lesssim 10^{-35}. \tag{2.40}$$

Note, however, that this inequality applies only if the initial spectrum is not completely redshifted out of the present horizon. The condition for this is $\hat{k}_{\rm in0}<\lambda_s^{-1}$, i.e. $\mathcal{B}\lambda_s^{-1}>\hat{H}_0$. Using $\mathcal{B}=10^{-42}(\overline{g_{\rm in}})^{2/(3+\sqrt{3})}$ this yields:

$$g_{\rm in}\gtrsim g_{\rm in}^{\rm thr.}=10^{-69}. \tag{2.41}$$

In conclusion, we obtain three possible scenarios: (i) if $10^{-35}\lesssim g_{\rm in}\lesssim 10^{-26}$, we must require initially $\Omega_{\rm GW}^{\rm in}\ll 1$ and as a consequence, the PBB scenario suffers from a serious homogeneity problem; (ii) if $10^{-69}\lesssim g_{\rm in}\lesssim 10^{-35}$, there is no need to fine-tune the initial tensor perturbations, $\Omega_{\rm GW}^{\rm in}\sim\mathcal{O}(1)$ [in this case, the tensor fluctuations on very large scales can still, in principle, be seen as classical small fluctuations in the CMBR], and (iii) for $g_{\rm in}\lesssim 10^{-69}$ only initial quantum fluctuations survive.

Before ending this section, it is instructive to discuss, for comparison, the fate of initial inhomogeneities, discussed so far for the PBB scenario, within an ordinary inflation scenario (modelled for simplicity as a simple de Sitter phase). For a de Sitter inflationary phase it is straightforward to derive from Eqs. (2.26), (2.28) that (for $\hat{k}_0\sim\hat{H}_0$),

$$\mathcal{C}(\hat{k}_0)=e^{-4(\mathcal{N}-\mathcal{N}_{\rm min})}, \tag{2.42}$$

where $\mathcal{N}_{\rm min}=\log(\hat{H}_{\rm dS}\,a_1/(\hat{H}_0\,a_0))$ is the minimal amount of e-foldings needed to solve the horizon problem. Applying the CMBR's bound given by Eq. (2.37) to these fluctuations that are re-entering the horizon now we get

$$\Omega_{\rm GW}^{\rm dS}(\hat{k}_{\rm in0})e^{-4(\mathcal{N}-\mathcal{N}_{\rm min})}\lesssim 10^{-10}. \tag{2.43}$$

Therefore, as happens in the minimal PBB scenario, in the minimal (horizon-problem-solving) de Sitter case ($\mathcal{N}=\mathcal{N}_{\rm min}$) one is still facing an "homogeneity problem," i.e. the CMBR's bound forces the initial tensor inhomogeneities to be unnaturally small: $\Omega_{\rm GW}^{\rm dS}(\hat{H}_{\rm in}^{\rm min})\lesssim 10^{-10}$. To solve this homogeneity problem, i.e. to relax this fine tuning and to be able to start with $\Omega_{\rm GW}^{\rm dS}(\hat{k}_{\rm in0})\sim 1$, we must depart from the minimal de Sitter scenario by at least 6 e foldings, i.e. $\mathcal{N}\gtrsim\mathcal{N}_{\rm min}+6$.

III. DISCUSSION AND CONCLUSIONS

We have shown that the dilaton-driven inflationary phase of the pre-big-bang scenario is not very effective in smoothing out the classical inhomogeneities that are expected to be present in a generic, initial patch of space which starts its inflationary evolution. We computed the various "transfer functions" that relate the initial spectrum of inhomogeneities to the present one. Our main conclusion is that the requirement of naturalness of initial inhomogeneities ($\Omega_{\rm GW}\sim 1$) can be satisfied only at the price of a constraint [9,10] on the initial value of the (homogeneous part of the) string coupling, which is much stronger (by a factor $\sim 10^{-9}$) than the previously acknowledged constraint (following from the necessity to solve the horizon and flatness problems).

Ordinary inflation qualitatively faces an analogous homogeneity problem. For example in the de Sitter case we need to require ~ 6 e foldings more than the minimal number needed to solve the horizon (and flatness) problems in order to overcome this initial inhomogeneity issue. Quantitatively, this additional constraint is not very severe for ordinary inflation because, in many inflationary models, the number of e folds is exponentially dependent on some inverse power of the coupling constants of the underlying theory.

This additional "homogeneity" constraint on the PBB model discussed here does not necessarily mean that the basic (elegant) idea of dilaton-driven inflation is to be discarded. There might be other ways of using the kinetic energy of a scalar field to drive a nonfine-tuned inflationary phase. In particular the recently proposed model of "k inflation" [16], which differs from the PBB scenario in making use of *higher-order* kinetic terms to drive an inflationary phase, has been shown to have efficient smoothing properties [17].

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