

On the AdS stability problem.

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Abstract

We discuss the notion of stability and the choice of boundary conditions for AdS-type space-times and point out difficulties in the construction of Cauchy data which arise if reflective boundary conditions are imposed.

1 Introduction

P. Bizoń and A. Rostworowski recently presented a study of the stability of anti-de Sitter space ([6]) which raises some extremely interesting questions concerning solutions to Einstein's field equations

$$\hat{R}_{\mu\nu} - \frac{1}{2} \hat{R} \hat{g}_{\mu\nu} + \lambda \hat{g}_{\mu\nu} = \kappa \hat{T}_{\mu\nu}, \quad (1.1)$$

with cosmological constant $\lambda < 0$ that are subject to conditions on the boundary \mathcal{J} at space-like and null infinity. They analyse the spherically symmetric Einstein-massless-scalar field system with homogeneous Dirichlet asymptotics and Gaussian type initial data and observe the formation of trapped surfaces for (numerically) arbitrarily small initial data. They perform a perturbative analysis, which points into the same direction but also exhibits *small one-mode initial data* which develop into globally smooth solutions. Further, they supply numerical evidence that the development of trapped surfaces results from an energy transfer from low to high frequency modes. Of the follow-up work ([8], [9], [13], [14], [21]) we only mention the observation in [9] that also solutions arising from data sufficiently close to the small one-mode initial data exist (numerically) for all time.

Their results led P. Bizoń and A. Rostworowski to conjecture: *AdS is unstable against the formation of black holes for a large class of arbitrarily small perturbations.* This

conjecture has been formulated in the context of a particular model but it may easily be misunderstood as a statement applying to more general situations. The purpose of the present note is to point out that such a conclusion may possibly be too strong if the class of competing perturbations is reasonably large and it should rather be replaced by: *AdS with reflecting boundary conditions is unstable against the formation of black holes for a large class of arbitrarily small perturbations.* This is still a very interesting conjecture. Understanding in detail how the focussing property inherent in the non-linear equations combines with the, in principle unlimited in time, refocussing effected at the reflecting boundary should give important insights into the evolution process.

By many workers in the field reflecting boundary conditions are so naturally associated with AdS-type solutions that they are sometimes simply referred to as ‘the AdS boundary conditions’. They are clearly convenient because they provide clean initial boundary value problems which exclude any information entering or leaking out of the system. But as a consequence, these systems cannot interact with an ambient cosmos and thus certainly do not represent observable objects as suggested by some of the names given to them in the literature. Their astrophysical interest thus remains unclear. Reflecting boundary conditions are very special and as long as it is not clear what kind of objects the solutions represent from the point of view of physics (and why a negative cosmological constant should be introduced in the first place) it does not appear reasonable to exclude all others possibilities.

If more general boundary conditions are admitted there may still be arbitrarily small perturbation leading to trapped surfaces but the class of these perturbations may be much smaller relative to the complete set of admitted perturbations because radiation can enter and leave the system through the boundary \mathcal{I} at space-like and null infinity over an unlimited length of (conformal) time. It is here where the problem of stability becomes for AdS-type space-times much more difficult than in the case of de Sitter-type or Minkowski-type solutions. In the latter cases the conformal boundary consists of two components: \mathcal{I}^+ through which radiation can leave the system and \mathcal{I}^- across which radiation can enter the system and conveniently be controlled, at least in principle, in its size and form. The possibility of such a clear distinction is not obvious in AdS-type space-times and whether any kind of analogue to this can be established in terms of boundary data/conditions on \mathcal{I} is part of the more general stability problem.

The latter problem will not be considered in this article. Instead, we shall revisit a known existence result for AdS-type space-times which exhibits the full freedom to prescribe boundary data on \mathcal{I} and discuss various aspects pertaining to the given problem in some detail. In particular, reflecting boundary conditions will be reconsidered from this more general perspective. Here comes into play a specific feature of the initial boundary value problems for Einstein’s field equations: If conditions are imposed on the boundary data, the clean separation between the evolution problem and the analysis of the constraints on the initial space-like slice, usual in the standard Cauchy problem, cannot be maintained any longer. As a consequence, reflective boundary conditions lead to a type of problem for the Cauchy data on the initial space-like slice which has not been discussed so far. Besides the standard constraints and the hyperboloidal fall-off behaviour required at space-like infinity the data are constrained by a sequence of additional conditions at space-like infinity. These encode the requirements of reflective boundary conditions in the

structure of the Cauchy data. Whether this may give a way to decide whether the *class of arbitrarily small perturbations which lead to the formation of black holes* is relatively increased by imposing reflective boundary conditions remains to be seen.

2 AdS-type solutions

In four space-time dimensions anti-de Sitter covering space, short AdS, is given by

$$\hat{M} = \mathbb{R} \times \mathbb{R}^3, \quad \hat{g} = -\cosh^2 r dt^2 + dr^2 + \sinh^2 r h_{\mathbb{S}^2},$$

where $r \geq 0$ denotes the standard radial coordinate on \mathbb{R}^3 and $h_{\mathbb{S}^2}$ the standard round metric on \mathbb{S}^2 . It is a static solution to

$$R_{\mu\nu}[\hat{h}] = \lambda \hat{g}_{\mu\nu} \quad \text{with} \quad \lambda = -3.$$

On the space-like slices $t = \text{const.}$ it induces a metric of constant negative curvature which can be conformally compactified and the clearest picture of its global features is obtained by performing the related conformal extension of AdS. Combining the coordinate transformation $r \rightarrow \rho = 2 \arctan(e^r) - \frac{\pi}{2}$ with the rescaling by the conformal factor $\Omega = \frac{1}{\cosh r} = \cos \rho$, both depending only on the spatial coordinates, gives the conformal representation

$$g = \Omega^2 \hat{g} = -dt^2 + d\rho^2 + \sin^2 \rho h_{\mathbb{S}^2}, \quad t \in \mathbb{R}, \quad 0 \leq \rho < \frac{\pi}{2},$$

of AdS which induces on the slices $t = \text{const.}$ the standard round metric on \mathbb{S}^3 . The metric g extends smoothly as $\rho \rightarrow \frac{\pi}{2}$ and then lives on the manifold $M = \mathbb{R} \times \frac{\mathbb{S}^3}{2}$ with g -time-like conformal boundary $\mathcal{J} = \{\rho = \frac{\pi}{2}\} \sim \mathbb{R} \times \mathbb{S}^2$. The boundary points can be understood as endpoints of the space-like and null geodesics and \mathcal{J} as representing space-like and null infinity for AdS.

In the following a solution of Einstein's field equations with negative cosmological constant which admits in a similar way a smooth conformal extension that adds a time-like hypersurface \mathcal{J} representing space-like and null infinity will be referred to as *AdS-type space-time*. (cf. [24] for details). There may be notions of *asymptotically AdS space-times* involving boundaries at space-like and null infinity of lower smoothness but for the sake of our discussion 'AdS-type' as above is convenient. What will be said in the following may apply to more general situations but working out the consequences will also be more difficult.

Two features of the global causal/conformal structure of AdS make global problems involving AdS-type solutions quite different from those for de Sitter-type or Minkowski-type solutions. The first basic observation, emphasized almost everywhere, is that such space-times are not globally hyperbolic; a time-like curve can always decide to escape through the conformal boundary before hitting a given achronal hypersurface. The second observation, almost never mentioned though of equal importance in the stability problem, is that AdS does not admit a finite conformal representation of past/future time-like infinity *which is also smooth*. In this sense AdS is always infinite in time, also in conformal time. (We are not referring here to AdS-type solutions because no completeness in time-like directions has been required for them so far).

3 Well-posed initial boundary value problems for AdS-type solutions

Expanding an AdS-type vacuum solution in the conformal setting formally in terms of a spatial coordinate r which vanishes on \mathcal{J} gives a Taylor expansion in powers of r if the space-time dimension is even and a ‘poly-homogeneous expansion’ in terms of powers of r and $\log r$ if the space-time dimension is odd (cf. [16], [19]). If the data prescribed on \mathcal{J} are real analytic and the formal expansion is Taylor this procedure yields real analytic AdS-type solutions in some neighbourhood of \mathcal{J} . While they may be useful in some contexts, we shall not consider such solutions. Analyticity is not a desirable assumption for our purpose, Cauchy problems for hyperbolic equations with data on time-like hypersurfaces are known to be not well-posed in the category of C^∞ functions, and, in particular, it is not clear under which conditions on the data such solutions will extend so as to define a smooth interior or possibly another component of the conformal boundary.

The natural problem here is the initial boundary value problem with boundary data prescribed on \mathcal{J} and Cauchy data on a space-like slice that extends to the boundary. The basic question then is: How must boundary conditions/data be prescribed so as to obtain well-posed initial boundary value problems for Einstein’s equations, possibly coupled to some matter fields, that produce AdS-type solutions? There exists quite some literature about field equations on AdS- or asymptotically AdS-backgrounds ([3], [4], [7], [20], [22], [25], [26]) in which ill- and well-posed initial boundary value problems for different matter fields are discussed. The boundary conditions and data which can be given on \mathcal{J} and the behaviour of the solutions near \mathcal{J} will clearly depend on the nature of the test field equations and their behaviour under conformal rescalings. For conformally covariant field equations the boundary \mathcal{J} is as good as any other time-like hypersurface and offers the same freedom to prescribe boundary data. The boundary analysis can be much more difficult, however, for ‘conformally ill-behaved’ equations. The huge spectrum of possibilities has hardly been explored so far and very little is known for Einstein’s field equations coupled to matter fields if no symmetries are assumed. We shall thus consider Einstein’s field equations with $\lambda < 0$ in the vacuum case $\hat{T}_{\mu\nu} = 0$. The following results in four space-time dimensions have been obtained by working in the conformal picture and using the *conformal field equations* ([17]). In the following we shall only discuss some aspects of it; for details we refer the reader to the original article.

Theorem 3.1 *Suppose $\lambda < 0$ and $(\hat{S}, \hat{h}_{ab}, \hat{\chi}_{ab})$ is a smooth Cauchy data set for $\text{Ric}[\hat{g}] = \lambda \hat{g}$ so that \hat{S} is an open, orientable, 3-manifold and (\hat{S}, \hat{h}_{ab}) is a complete Riemannian space. Let these data admit a smooth conformal completion*

$$\hat{S} \rightarrow S = \hat{S} \cup \Sigma, \quad \hat{h}_{ab} \rightarrow h_{ab} = \Omega^2 \hat{h}_{ab}, \quad \hat{\chi}_{ab} \rightarrow \chi_{ab} = \Omega \hat{\chi}_{ab},$$

so that (S, h_{ab}) is a Riemannian space with compact boundary $\Sigma = \partial S$, Ω a defining function of Σ , and $W^\mu{}_{\nu\lambda\rho} = \Omega^{-1} C^\mu{}_{\nu\lambda\rho}$ extends smoothly to Σ on S where $C^\mu{}_{\nu\lambda\rho}$ denotes then conformal Weyl tensor determined by the metric h_{ab} and the second fundamental form χ_{ab} .

Consider the boundary $\mathcal{J} = \mathbb{R} \times \partial S$ of $M = \mathbb{R} \times S$ and identify S with $\{0\} \times S \subset M$ and Σ with $\{0\} \times \partial S = S \cap \mathcal{J}$. Let on $\mathcal{J} \equiv \mathbb{R} \times \partial S$ be given a smooth 3-dimensional Lorentzian

conformal structure which satisfies in an adapted gauge together with the Cauchy data the corner conditions implied on Σ by the conformal field equations, where it is assumed that the normals to S are tangent to \mathcal{J} on Σ .

Then there exists for some $t_o > 0$ on the set

$$\hat{W} =] - t_o, t_o[\times \hat{S} \subset \mathbb{R} \times \hat{S} \subset M$$

a unique solution \hat{g} to $Ric[\hat{g}] = \lambda \hat{g}$ which admits with some smooth boundary defining function Ω on M a smooth conformal extension

$$\hat{W} \rightarrow W =] - t_o, t_o[\times S, \quad \hat{g} \rightarrow g = \Omega^2 \hat{g},$$

that induces (up to a conformal diffeomorphism) on S and $\mathcal{J}_o =] - t_o, t_o[\times \partial S$ the given conformal data.

It has been assumed here for convenience that all data are smooth and the corner conditions are satisfied at all orders. This ensures smoothness up to the boundary. If one is willing to accept some finite differentiability it should be observed that this might entail a loss of differentiability at the boundary \mathcal{J} (cf. [5] and the literature given there). Whether this loss does not occur because the fields satisfy besides the evolution equations also constraints has not been analysed.

In the following we give some background information, explain details, and point out particular features of AdS-type vacuum solutions which allow one to obtain this result.

Existence of Cauchy data.

Concerning the construction of initial data it has been observed in [17] that there exists a correspondence between Cauchy data for the equations $Ric[\hat{g}] = \lambda \hat{g}$ with the appropriate behaviour at space-like infinity and hyperboloidal data for the equations $Ric[\hat{g}] = 0$ which are conformally smooth at infinity. In fact, in the case of AdS-type solutions that are time reflection symmetric with respect to the initial slice S , so that the second fundamental form induced on it satisfies $\hat{\chi}_{ab} = 0$, the constraints reduce to

$$R[\hat{h}] = 2\lambda = const. < 0,$$

and the solution must satisfy at space-like infinity an asymptotic behaviour similar to that of hyperboloidal data. On the other hand, there have been constructed in [2] hyperboloidal data which are conformally smooth at infinity under the assumption that $\lambda = 0$ and that the inner metric \hat{h}_{ab} and the second fundamental form $\hat{\chi}_{ab}$ on the initial slice satisfy $\hat{\chi}_{ab} = \frac{\hat{\chi}}{3} \hat{h}_{ab}$, $\hat{\chi} = const. \neq 0$. In this case the constraints reduce to

$$R[\hat{h}] = -2\hat{\chi}^2/3 = const. < 0,$$

and it is seen that the correspondence just requires a reinterpretation of the constants. In [1] the existence of a much more general class of conformally smooth hyperboloidal initial data has been shown and the generalization of the correspondence has been worked out in [23].

It may be mentioned here that in [1], and even more so in [2], there have also been considered hyperboloidal data which only admit a poly-homogeneous expansion at infinity.

This may suggest to generalize the existence theorem cited above so as to include also boundary data on \mathcal{J} with weaker smoothness requirements. The situation then becomes much more difficult, however, and special care must be taken to ensure that no undesired non-smoothness is spreading into the space-time along the characteristic which generates the boundary of the domain of dependence of the data on \hat{S} .

The boundary conditions/data.

Because no constraints are required on the conformal structure on \mathcal{J} the prescription of the boundary data seems to be the easy part. There are, however, hidden subtleties here which are worth a detailed discussion. The initial boundary value problem for the conformal field equations does not immediately lead to the conditions stated in Theorem 3.1. To obtain the covariant formulation specific features of AdS-type solutions must be observed (cf. [18] for a discussion of the problems which arise in other initial boundary value problems for Einstein's field equations).

The first of these special properties is the following. If k_{ab} and κ_{ab} denote the first and second fundamental forms on \mathcal{J} then the conformal field equations imply that the trace free part of κ_{ab} vanishes so that

$$\kappa_{ab} = 0 \quad \text{on } \mathcal{J}, \quad (3.1)$$

in a suitable conformal gauge. This has the consequence that conformal geodesics (cf. [17]) which are tangent to \mathcal{J} at one point stay in \mathcal{J} . Assuming for convenience that the space-like slice S meets the boundary \mathcal{J} orthogonally in the sense that the normal to S is tangent to \mathcal{J} , this allows us to set up the initial boundary value problem in terms of an *adapted conformal Gauss gauge*, which is generated by conformal geodesics that start orthogonally to S with g -unit tangent vector. In particular, the coordinates are defined in terms of a natural parameter $\tau = x^0$ on the conformal geodesics, with $\tau = 0$ on S , and coordinates x^α on S which are extended so that they are constant along these curves. The location of the boundary \mathcal{J} is then determined by the conformal geodesics which start on Σ and the conformal factor Ω , which vanishes on \mathcal{J} and has non-vanishing differential there, is known explicitly in terms of these coordinates.

In the adapted gauge the conformal field equations then imply a hyperbolic system of evolution equations which assumes in Newman-Penrose notation the form

$$\partial_\tau u = F(u, \psi, x^\mu), \quad (3.2)$$

$$(1 + A^0) \partial_\tau \psi + A^\alpha \partial_\alpha \psi = G(u, \psi, x^\mu). \quad (3.3)$$

The unknown u comprises the coefficients of a double null frame field $(e^\mu{}_k)_{k=0,\dots,3} = (l^\mu, n^\mu, m^\mu, \bar{m}^\mu)$ in these coordinates, the connection coefficients with respect to this frame, and the Schouten tensor

$$L_{jk} = \frac{1}{n-2} \left(R_{jk} - \frac{1}{2(n-1)} R g_{jk} \right) \quad (3.4)$$

(with $n = 4$) of the metric g in that frame. The matrices A^μ depend on the frame coefficients and the coordinates and $\psi = (\psi_0, \dots, \psi_4)$ represents the essential components of the symmetric spinor field ψ_{ABCD} corresponding to the tensor field $W^i{}_{jkl}$.

To discuss the boundary conditions it is convenient to choose the frame such that the future directed time-like vector field $l + n$ is tangent to \mathcal{J} and the space-like vector field $l - n$ is normal to \mathcal{J} and inward pointing (this choice has in fact been included in the gauge conditions). Then m and \bar{m} are tangent to \mathcal{J} , l is inward and n is outward pointing on \mathcal{J} . This leads to boundary condition on \mathcal{J} of the form

$$\psi_4 - a\psi_0 - c\bar{\psi}_0 = d, \quad |a| + |c| \leq 1, \quad (3.5)$$

where the smooth complex-valued function d on \mathcal{J} denotes the free boundary data and the smooth complex-valued functions a and c on \mathcal{J} can be chosen freely within the indicated restrictions.

The form of the boundary conditions is made plausible by the following consideration. It follows immediately from (3.2) that no part of u can be prescribed. The Bianchi system $\nabla^A{}_{A'}\psi_{ABCD} = 0$, from which equation (3.3) is extracted, splits into two subsystems. The first one, which is of the form

$$\nabla_n\psi_k - \nabla_m\psi_{k+1} = H_k(u, \psi), \quad k = 0, 1, 2, 3,$$

determines on \mathcal{J} the outward transport of $\psi_0, \psi_1, \psi_2, \psi_3$ in terms of the fields given in the interior and on the boundary. These components of the rescaled Weyl tensor cannot be prescribed $\psi_0, \psi_1, \psi_2, \psi_3$ on \mathcal{J} . The second subset, which is of the form

$$\nabla_l\psi_j - \nabla_{\bar{m}}\psi_{j-1} = K_j(u, \psi), \quad j = 1, 2, 3, 4,$$

describes an inward transport of the fields $\psi_1, \psi_2, \psi_3, \psi_4$ on \mathcal{J} and suggests that one may prescribe the field ψ_4 and possibly feed back into it some information on the other fields. The equations

$$\nabla_{l+n}\psi_j - \nabla_{\bar{m}}\psi_{j-1} - \nabla_m\psi_{j+1} = H_j(u, \psi) + K_j(u, \psi), \quad j = 1, 2, 3,$$

implied by the equations above shows that the fields ψ_1, ψ_2, ψ_3 are governed by interior equations on \mathcal{J} . This implies with a detailed discussion involving energy estimates that only information on ψ_0 can be fed back into ψ_4 and that the restriction given above has to be observed.

The corner conditions

Given the conformal Cauchy data on S and the adapted gauge in which the lines $x^\alpha = \text{const.}$ that start on Σ are tangent to \mathcal{J} , the formal expansion of the unknowns in equations (3.2), (3.3) in terms of the coordinate τ is determined at all orders uniquely by these equations on S and in particular on Σ . Let smooth functions a and c satisfying the restriction on the right hand side of (3.5) be given on \mathcal{J} . Using their formal expansion in terms of τ on Σ , we get a formal expansion of the term on the right hand side of (3.5) on Σ . The corner conditions consist in the requirement that this formal expansion coincides on Σ with the formal expansion of the free boundary datum d . Borel's theorem ([15]) guarantees that there always exist smooth functions d on \mathcal{J} which satisfy this requirement. Away from Σ they are essentially arbitrary.

With Cauchy data as stated in the theorem and boundary conditions of the form (3.5) where d satisfies the corner conditions one obtains a *well-posed initial boundary*

value problem which preserves the constraints and the gauge conditions. This implies the existence and uniqueness of smooth solutions on a domain as indicated in the theorem.

Covariant boundary conditions

The formulation obtained above has a drawback. Condition (3.5) is not covariant, it depends in an implicit way on the choice of $l + n$. This is related to a general problem arising in initial boundary value problems for Einstein's field equations (cf. [18]). In the case of AdS-type solutions this problem can be overcome by making use of a second specific feature of such solutions. They always satisfy the relation

$$w_{ab}^* = \sqrt{3/|\lambda|} B_{ab} \quad \text{on } \mathcal{J}. \quad (3.6)$$

Here w_{ab}^* denotes the \mathcal{J} -magnetic part of $W^i{}_{jkl}$, obtained by contracting the right dual of $W^i{}_{jkl}$ twice with the inward pointing unit normal of \mathcal{J} , and $B_{ab} = \epsilon_a{}^{cd} D_c L_{db}$ is the Cotton tensor of the metric k_{ab} on \mathcal{J} , which is derived from the Schouten tensor L_{ab} of the metric k_{ab} on \mathcal{J} given by (3.4) with $n = 3$ and g replaced by k .

Two further observations are needed. The borderline case

$$\psi_4 - \bar{\psi}_0 = d \quad (3.7)$$

of (3.5) can be written in real notation with $d = d_1 + i d_2$ in the form

$$w_{cd}^* M^{cd}{}_A = d_A, \quad A = 1, 2,$$

with constant real coefficients $M^{cd}{}_A$. Moreover, if the components $B_{cd} M^{cd}{}_A$ of the Cotton tensor are given on \mathcal{J} , the differential identity $D^a B_{ab} = 0$ of the Cotton tensor turns into a hyperbolic system of PDE's for the remaining components if the frame vector fields tangent to \mathcal{J} and the connection defined by k are known. The structural equations of the normal conformal Cartan connection of k_{ab} allow us, however, to deduce transport equations for the latter fields so that the combined system becomes hyperbolic. From the Cauchy data on S the field B_{ab} and the lower order structures can be determined on Σ , which allows us to integrate the conformal structure of k_{ab} on \mathcal{J} . Here and below it is used that because of (3.1) the conformal geodesics with respect to g which are lying in \mathcal{J} are in fact also conformal geodesics with respect to the inner metric k on \mathcal{J} .

It follows that the gauge underlying a boundary condition of the form (3.7) and the functions entering (3.7) can be determined on \mathcal{J} in terms of the intrinsic conformal structure defined by k . PDE uniqueness then shows that the 'physical solution' determined by the boundary condition (3.7) is determined uniquely by the Cauchy data and the conformal structure on \mathcal{J} .

Any smooth AdS-type vacuum solution can be determined locally in time in the future of an initial slice like S in terms of boundary conditions like (3.5) or (3.7). Besides leading to a covariant formulation, the conditions (3.7) are distinguished by another property. Under the time reflection $\tau \rightarrow -\tau$ and the transitions $l \rightarrow -l$, $n \rightarrow -n$ the roles of l and n and of ψ_4 and ψ_0 in the discussion of the boundary conditions are swapped but the structure of the boundary condition (3.7) is preserved. This is not necessarily true for other boundary conditions given by (3.5). It would be interesting to understand whether the freedom to choose the functions a and c on \mathcal{J} has useful applications in the stability problem.

4 Reflecting boundary condition

Any of the boundary conditions (3.5) with $d = 0$ on \mathcal{J} can be regarded as a reflecting boundary condition. To ensure gauge independence we require

$$\psi_4 - \bar{\psi}_0 = 0 \quad \text{on } \mathcal{J}, \quad (4.1)$$

and

$$\psi_3 - \bar{\psi}_1 = 0, \quad \psi_2 - \bar{\psi}_2 = 0 \quad \text{on } \Sigma, \quad (4.2)$$

which imply in particular that $w_{ab}^* = 0$ on Σ and thus with (3.6) that

$$B_{ab}[k] = 0 \quad \text{on } \Sigma.$$

With the PDE implied by (4.1) and the identity $D^a B_{ab} = 0$ it follows then that (4.1), (4.2) are equivalent to the *conformal flatness* condition

$$B_{ab}[k] = 0 \quad \text{on } \mathcal{J}. \quad (4.3)$$

In the following we refer to this condition or the equivalent conditions (4.1) and (4.2) as the *reflecting boundary condition*.

Since (4.1) is by itself a ‘reflecting boundary condition’ one may wonder why conditions (4.2) are required as well. The reason, which is not immediately seen and will not be explained here in detail, is that (4.1) by itself still depends on the choice of the conformal gauge.

The analysis of the standard Cauchy problem admits a clean separation between the evolution problem and the construction of initial data. If restrictions are imposed on the boundary data in an initial boundary value problem this separation cannot be maintained any longer. An extreme example of this situation is provided by reflecting boundary conditions. They do not only prevent a flow of gravitational radiation across \mathcal{J} , they also induce rather strong additional fall-off conditions on the initial data at space-like infinity.

Proposition 4.1 *In the situation considered in Theorem (3.1) the reflecting boundary condition (4.3) imposes on the hyperboloidal Cauchy data set $(S, \Omega, h_{ab}, \chi_{ab})$ and the rescaled Weyl spinor ψ_{ABCD} calculated on S from these data not only the restrictions $\psi_4 = \bar{\psi}_0, \psi_3 = \bar{\psi}_1, \psi_2 = \bar{\psi}_2$ at Σ , but it implies with the corner conditions in addition a sequence of differential conditions on Σ which involve derivatives of ψ_{ABCD} of all orders.*

Proof: The corner conditions imply with (4.1) the relations

$$\partial_\tau^k (\psi_4 - \bar{\psi}_0) = 0, \quad k = 0, 1, 2, \dots \quad \text{on } \Sigma. \quad (4.4)$$

With the evolution equations

$$\begin{aligned} (1 + A^0) \partial_\tau \psi + A^\alpha \partial_\alpha \psi &= G(u, \psi), \\ \partial_\tau u &= F(u, \psi, x^\mu), \end{aligned}$$

and their formal derivatives with respect to τ these conditions translate into inner differential conditions of all orders on the Cauchy data on S .

□

If one is prepared to accept solutions of finite differentiability (and the associated drop of smoothness at the boundary), one could do with a finite number of derivatives in (4.4). The essential problem in the construction of the Cauchy data still remains.

5 Concluding remarks

The requirement on the data in Proposition 4.1 raises the question to what extent data satisfying these conditions can be provided in a systematic way by the known methods to construct hyperboloidal Cauchy data. For simplicity we consider again the case treated in [2]. It is shown there that, given a conformal structure on \hat{S} in terms of a ‘seed metric’ \hat{h} which satisfies certain conditions at infinity, there exists a unique conformal factor Ω which satisfies together with the metric $h = \Omega^{-2} \hat{h}$ the conformal constraints and the required smoothness properties at infinity. Unless the differential conditions above can be expressed for some unexpected reason directly in terms of conditions on the conformal structure defined by \hat{h} it is hard to imagine that this method can be used to construct the data as required by Proposition. P. T. Chruściel and E. Delay used hyperboloidal gluing techniques to show the existence of a class of non-trivial Cauchy data which are diffeomorphic to Schwarzschild-anti-de Sitter data outside some compact set and thus satisfy the corner conditions considered in Proposition 4.1 ([11]). To what extent this result can be generalized remains to be seen.

A second question is what the restriction on the data considered in Proposition 4.1 does to the development in time. Do the additional fall-off conditions increase the likeliness for the formation of trapped surfaces in the domain of dependence of the given Cauchy data? In the case of asymptotically flat data there have been considered data which also satisfy, besides the constraints and the asymptotically flatness condition, further conditions at space-like infinity ([10], [12]). These do not give rise to problems in their development in time if the data were sufficiently small. In that case the additional conditions have been imposed, however, only on the initial slice while everything else was left to the field equations and the solutions were only studied on the domain of dependence of the initial slice (and its conformal extension). An AdS-type solution satisfying reflective boundary conditions it required to develop as a curve in the set of restricted data as considered in Proposition 4.1 and, if possible, to develop beyond the domain of dependence of any space-like slice extending to \mathcal{J} .

If the stability of solutions is to be analysed under less restrictive boundary conditions, there arises the question which boundary conditions/data should be given and what, in fact, should be meant by ‘stability’. As long as there are no physical considerations which could give a clue, the best one could do is perhaps to characterize in a most general way the boundary conditions/data and Cauchy data for which solutions which start close to AdS stay close to AdS for all times. The global causal structure of AdS suggests that it will in general not be reasonable to require anything more.

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