

Metric- and frame-like higher-spin gauge theories in three dimensions

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Abstract

We study the relation between the frame-like and metric-like formulation of higher-spin gauge theories in three space-time dimensions. We concentrate on the theory that is described by an $SL(3) \times SL(3)$ Chern-Simons theory in the frame-like formulation. The metric-like theory is obtained by eliminating the generalised spin connection by its equation of motion, and by expressing everything in terms of the metric and a spin-3 Fronsdal field.

We give an exact map between fields and gauge parameters in both formulations. To work out the gauge transformations explicitly in terms of metric-like variables, we have to make a perturbative expansion in the spin-3 field. We describe an algorithm how to do this systematically, and we work out the gauge transformations to cubic order in the spin-3 field. We use these results to determine the gauge algebra to this order, and explain why the commutator of two spin-3 transformations only closes on-shell.

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1 Introduction

Higher-spin gauge theories have gained a lot of attention in recent years, in particular because of the proposed higher-spin AdS/CFT correspondence in four and three dimensions (see [1, 2] for reviews). Higher-spin gauge fields can either be described by extending the vielbein formalism of gravity to higher-spins [3], or by extending the metric formulation [4]. Although the metric-like description might be the more intuitive ansatz, because one needs less auxiliary fields, it is the frame-like formulation that allowed Vasiliev to construct a consistent non-linear theory of interacting higher-spin gauge fields [5, 6]. In the

metric-like formulation, on the other hand, one only knows how to construct interactions in a perturbative expansion, e.g. one has obtained a classification of consistent cubic terms [7, 8, 9, 10, 11, 12, 13, 14, 15].

It would be desirable to understand the theory also in the metric-like formulation. In particular one would hope that one could get a better geometric understanding of the higher-spin gauge symmetry as generalised diffeomorphisms. This might also improve our understanding of particular solutions of higher-spin theories like higher-spin analogues of black holes [16, 17, 18, 19, 20]. In [21] it was shown how one could use the Wald formula in a metric-like higher-spin formulation to compute the entropy of higher-spin black holes (for other approaches see e.g. [16, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31]).

Higher-spin gauge theories in three dimensions are considerably simpler than in higher dimensions, because they do not contain propagating degrees of freedom and can be written as a Chern-Simons theory [32, 33]. Also, in contrast to higher dimensions, it is possible to truncate the tower of typically infinitely many higher-spin gauge fields to a finite selection – the simplest theory only contains gravity and one spin-3 field. In this case the generalised vielbein $e = e_\mu dx^\mu$ and the generalised spin connection take values in the Lie algebra $sl(3, \mathbb{R})$,

$$e_\mu = e_\mu^A J_A \quad , \quad \omega_\mu = \omega_\mu^A J_A \quad , \quad (1.1)$$

where J_A form a basis of $sl(3, \mathbb{R})$,

$$[J_A, J_B] = f_{AB}^C J_C \quad . \quad (1.2)$$

The gauge sector of this theory is described by the action

$$S = \frac{1}{16\pi G} \int \text{tr} \left(e \wedge R + \frac{1}{3l^2} e \wedge e \wedge e \right) \quad , \quad (1.3)$$

where

$$R = d\omega + \omega \wedge \omega \quad \Leftrightarrow \quad R^A = d\omega^A + \frac{1}{2} f^A_{BC} \omega^B \wedge \omega^C \quad (1.4)$$

is the curvature of the generalised spin connection, G is the gravitational constant and tr is the trace in the fundamental representation of $sl(3, \mathbb{R})$. The parameter l is related to the cosmological constant – a real and positive l coincides with the radius of the AdS solution. This action can be rewritten as a Chern-Simons theory whose gauge group depends on the cosmological constant: e.g. for a negative constant (positive l^2) the gauge group is $SL(3, \mathbb{R}) \times SL(3, \mathbb{R})$.

The frame-like formulation being so simple, it is tempting to try to reformulate it in terms of metric-like fields. First one has to eliminate the spin connection by its equation of motion,

$$D_{[\mu} e_{\nu]} = 0 \quad , \quad (1.5)$$

where D_μ is the covariant derivative including spin connection and the Levi-Civita Christoffel symbols,

$$D_\mu e_\nu^A = \partial_\mu e_\nu^A + f^A_{BC} \omega_\mu^B e_\nu^C - \Gamma^\lambda_{\mu\nu} e_\lambda^A \quad . \quad (1.6)$$

Then one has to express everything in terms of metric-like fields, which have to be expressions in the vielbeins where all $sl(3)$ indices are contracted with invariant tensors. In [34] it was proposed to define the metric and the spin-3 field as

$$g_{\mu\nu} = \kappa_{AB} e_\mu^A e_\nu^B \quad (1.7)$$

and

$$\phi_{\mu\nu\rho} = \frac{1}{3!} d_{ABC} e_\mu^A e_\nu^B e_\rho^C \quad . \quad (1.8)$$

The remaining task is then to rewrite the action (after eliminating ω) in terms of these fields.

Because the vielbein is not invertible (it is not a square matrix), this is rather complicated. In [21] the action was worked out to quadratic order in the spin-3 field by making a general ansatz and then

demanding that explicit solutions of the frame-like theory should map to solutions in the metric-like theory. We will follow here a different approach which was started also in [21].¹

Instead of considering the action and its solutions we concentrate on the gauge transformations. We formulate an exact map of the gauge parameters in the frame- and the metric-like formulation (section 2). Notice that one always has the freedom to reparameterise the gauge transformations, therefore this map is not unique. We then formulate an algorithm that can be used to map any given contraction of frame-like quantities to metric-like quantities in a perturbative expansion in the spin-3 field (section 3). We use this algorithm to explicitly compute the gauge transformations in the metric-like theory to cubic order. We are then in the position to compute commutators of these transformations to better understand the gauge algebra in the metric-like theory (section 4). We also discuss there why the commutator of two spin-3 transformations only closes on-shell.

Our work clarifies a few issues that were left unanswered in [21]. First of all we could show that the perturbative ansatz for the map between gauge parameters in [21] can be used as an exact map without any corrections, such that one has an exact dictionary of fields and gauge parameters. Secondly we can explain why the gauge transformations only close on-shell in the metric-like theory (whereas they close off-shell on the frame-like side). Thirdly we worked out a systematic approach to obtain the explicit expressions on the metric-like side that does not require the use of specific solutions of the theory. Last but not least we worked out the gauge transformations and gauge algebra to one order higher than in [21] in the hope to understand the metric-like theory better.

The expressions that we obtain for the gauge transformations to the order we consider are already quite large, they fill two pages of the appendix. In principle one could now go on and determine the corresponding action (which is a fairly easy task if one uses a powerful computer algebra program), and the result will be of similar size. We have not found any pattern in our expressions that could help to organise them – but without such a pattern it does not make sense to work out the metric-like theory to even higher orders. On the other hand, one might hope that there is a clever redefinition of fields and gauge parameters which makes the theory more manageable.

2 Relating frame- and metric-like gauge transformations

In this section we relate the gauge transformations in the frame- and in the metric-like description. In the frame-like theory there are the two types of gauge transformations; the generalised local Lorentz transformations,

$$\delta_{\Lambda}^L e_{\mu} = [\Lambda, e_{\mu}] \tag{2.1}$$

$$\delta_{\Lambda}^L \omega_{\mu} = D_{\mu} \Lambda \ , \tag{2.2}$$

and the generalised local translations,

$$\delta_{\Xi} e_{\mu} = D_{\mu} \Xi \tag{2.3}$$

$$\delta_{\Xi} \omega_{\mu} = \frac{1}{l^2} [e_{\mu}, \Xi] \ . \tag{2.4}$$

The local Lorentz transformations act trivially on all metric-like fields built from the vielbeins e_{μ} . The generalised local translations, on the other hand, induce non-trivial transformations on them, and they can be interpreted as diffeomorphisms and higher-spin generalisations thereof.

¹For an alternative ansatz for a metric-like description see [35, 36].

Let us first consider pure diffeomorphisms. It is well-known (see e.g. [37]) that a generalised translation, where the parameter Ξ is of the form

$$\Xi^A = e_\mu^A \xi^\mu , \quad (2.5)$$

induces a diffeomorphism generated by the vector-field ξ^μ (up to a local Lorentz rotation) if one imposes the torsion constraint (1.5). The action of such a diffeomorphism (spin-2 gauge transformation) on any metric-like field ϕ built from the vielbeins e is given by

$$\delta_\xi^{(2)} \phi = \mathcal{L}_\xi \phi , \quad (2.6)$$

where \mathcal{L}_ξ denotes the Lie derivative.

For the higher-spin transformations we do not know how they act in general, but only in the linearised theory where they should reproduce the transformations of free Fronsdal fields [4]. The spin-3 transformation should act as

$$\delta_\xi^{(3)} g_{\mu\nu} = 0 + \dots \quad (2.7)$$

$$\delta_\xi^{(3)} \phi_{\mu\nu\rho} = \nabla_{(\mu} (\xi_{\nu\rho)} - \frac{1}{3} g_{\nu\rho} \xi^\lambda) + \dots , \quad (2.8)$$

where $\xi^{\mu\nu}$ is a symmetric tensor that labels the spin-3 gauge transformations, and the dots indicate terms that are at least linear in the spin-3 field. The covariant derivative ∇_μ is defined with respect to the Levi-Civita connection.

We combine the gauge parameters for spin-2 and spin-3 transformations into a single object $\xi = (\xi^\mu, \xi^{\nu\rho})$. We are looking for a map

$$\xi = (\xi^\mu, \xi^{\nu\rho}) \mapsto \Xi(\xi) , \quad (2.9)$$

such that

$$\delta_{\Xi(\xi)} \phi = \delta_\xi^{(2)} \phi + \delta_\xi^{(3)} \phi . \quad (2.10)$$

Note that such a map is not unique, even if we have fixed the expression of the metric-like fields in terms of frame-like ones such that no field redefinitions are possible. We can still redefine the higher-spin gauge parameters by terms that are at least linear in the higher-spin fields, such that the linearised gauge transformations are untouched. In the following we will construct one such map that is valid at all orders in the spin-3 field.

2.1 A proposal for the map

The map $\xi \mapsto \Xi(\xi)$ is linear, so we can write it as

$$\Xi^A(\xi) = \mathcal{S}^A{}_\mu \xi^\mu + \mathcal{S}^A{}_{\nu\rho} \xi^{\nu\rho} , \quad (2.11)$$

with possibly field-dependent matrices \mathcal{S} . The implementation of pure diffeomorphisms is given by (2.5), this fixes the coefficients $\mathcal{S}^A{}_\mu$ to

$$\mathcal{S}^A{}_\mu = e_\mu^A . \quad (2.12)$$

An arbitrary frame-like gauge transformation Ξ^A will induce both a diffeomorphism and a spin-3 transformation, therefore there will be projections P and $(1 - P)$ such that $P\Xi$ induces a pure diffeomorphism, and $(1 - P)\Xi$ a pure spin-3 transformation. Instead of fixing $\mathcal{S}^A{}_{\nu\rho}$ directly, we will rather first attempt to fix the projection P . It should project an arbitrary gauge transformation to a pure diffeomorphism, therefore we demand that

$$\text{for every } \Xi^A \text{ there is a } \xi^\mu \text{ such that } P^A{}_B \Xi^B = \mathcal{S}^A{}_\mu \xi^\mu , \quad (2.13)$$

and $P^2 = P$. A natural requirement for the projector is that it is orthogonal w.r.t. the Killing form, in other words that

$$P^{AB} = P^{BA} , \quad (2.14)$$

where we have raised the indices with the Killing form. This then fixes the projector uniquely to be

$$P^{AB} = e_\mu^A g^{\mu\nu} e_\nu^B . \quad (2.15)$$

Indeed we can easily check that

$$P^A_B P^B_C = e_\mu^A g^{\mu\nu} e_\nu^D \kappa_{BD} e_\rho^B g^{\rho\sigma} e_\sigma^C \kappa_{\mathcal{E}C} \quad (2.16)$$

$$= e_\mu^A g^{\mu\nu} g_{\nu\rho} g^{\rho\sigma} e_\sigma^C \kappa_{\mathcal{E}C} \quad (2.17)$$

$$= e_\mu^A g^{\mu\sigma} e_\sigma^C \kappa_{\mathcal{E}C} = P^A_C , \quad (2.18)$$

where we used the definition of the metric (1.7) to go to the second line. Furthermore, for an arbitrary Ξ^A we have

$$P^A_B \Xi^B = e_\mu^A \left(g^{\mu\nu} e_\nu^C \kappa_{CB} \Xi^B \right) , \quad (2.19)$$

therefore P indeed projects onto pure diffeomorphisms (where we interpret the term in the parentheses as the corresponding vector field). Notice that defining P to be an orthogonal projector was a choice we made, but we will see in the next section that by redefining the gauge parameters it is always possible to bring the projector to the form above.

Having fixed P we can now look for an $\mathcal{S}^A_{\nu\rho}$ that satisfies

$$P^A_B \mathcal{S}^B_{\nu\rho} = 0 . \quad (2.20)$$

In addition we want that $\mathcal{S}^A_{\nu\rho}$ coincides with the free field expression when we set the higher-spin fields to zero, i.e.

$$\mathcal{S}^A_{\nu\rho} = 3 d^A_{bc} e_\nu^b e_\rho^c + \dots \quad (2.21)$$

A natural ansatz for a covariant expression that is consistent with the linearisation and with the projection P is then

$$\mathcal{S}^A_{\nu\rho} = (\delta_{\mathcal{D}}^A - P_{\mathcal{D}}^A) 3 d^{\mathcal{D}}_{BC} e_\nu^B e_\rho^C + \dots \quad (2.22)$$

The linearised gauge transformation does only depend on the traceless part of the gauge parameter $\xi^{\mu\nu}$. If we want this property to hold also at the non-linear level, we have to add the projection to the traceless part,

$$\xi^{\mu\sigma} \mapsto \left(\delta_\nu^\mu \delta_\rho^\sigma - \frac{1}{3} g^{\mu\sigma} g_{\nu\rho} \right) \xi^{\nu\rho} . \quad (2.23)$$

Our final ansatz for $\mathcal{S}^A_{\nu\rho}$ then reads

$$\mathcal{S}^A_{\nu\rho} = (\delta_{\mathcal{D}}^A - P_{\mathcal{D}}^A) 3 d^{\mathcal{D}}_{BC} e_\nu^B e_\rho^C \left(\delta_\nu^\mu \delta_\rho^\sigma - \frac{1}{3} g^{\mu\sigma} g_{\nu\rho} \right) \quad (2.24)$$

$$= 3 \left(d^A_{BC} e_\nu^B e_\rho^C - 6 e_\kappa^A g^{\kappa\lambda} \phi_{\lambda\mu\sigma} \right) \left(\delta_\nu^\mu \delta_\rho^\sigma - \frac{1}{3} g^{\mu\sigma} g_{\nu\rho} \right) , \quad (2.25)$$

where we used our definition for the spin-3 field ϕ in (1.8).

To summarise we propose the following map for the gauge parameters,

$$\Xi^A = \mathcal{S}^A_{\mu} \xi^\mu + \mathcal{S}^A_{\nu\rho} \xi^{\nu\rho} \quad (2.26)$$

$$= e_\mu^A \xi^\mu + 3 (\delta_{\mathcal{D}}^A - P_{\mathcal{D}}^A) d^{\mathcal{D}}_{BC} e_\nu^B e_\rho^C \left(\delta_\nu^\mu \delta_\rho^\sigma - \frac{1}{3} g^{\mu\sigma} g_{\nu\rho} \right) \xi^{\nu\rho} . \quad (2.27)$$

In the following section we will argue that this is a consistent choice to all orders in the higher-spin field.

2.2 The proposed map is exact

Our goal is to obtain an exact map between the gauge parameters on the metric-like side and on the frame-like side,

$$\Xi^{\mathcal{A}} = \mathcal{S}^{\mathcal{A}}{}_M \xi^M . \quad (2.28)$$

Here, M is a collective label for the metric-like labels, e.g. in the $sl(3)$ case $\{M\} = \{\mu, (\nu\rho)\}$, where $(\nu\rho)$ denote symmetric pairs of space-time labels without any trace constraints. The matrix \mathcal{S} is then not a square matrix, and it can depend on the fields and on the vielbein. In the last section we have made a proposal for such a map (see (2.27)). In this section we want to show that there will always be a redefinition of the gauge parameters such that the proposal (2.27) provides the exact map.

Given a frame-like gauge parameter $\Xi^{\mathcal{A}}$ we may ask what the corresponding diffeomorphism and spin-3 transformation are that it induces, in other words we want to have an inverse relation of the form

$$\xi^M = \mathcal{T}^M{}_{\mathcal{A}} \Xi^{\mathcal{A}} , \quad (2.29)$$

such that

$$\mathcal{S}^{\mathcal{A}}{}_M \mathcal{T}^M{}_{\mathcal{B}} = \delta_{\mathcal{B}}^{\mathcal{A}} \quad , \quad \mathcal{T}^M{}_{\mathcal{A}} \mathcal{S}^{\mathcal{A}}{}_N = \mathcal{K}^M{}_N = \begin{pmatrix} \delta_{\nu}^{\mu} & 0 \\ 0 & \mathcal{K}_{\nu_1\nu_2}^{\mu_1\mu_2} \end{pmatrix} . \quad (2.30)$$

Here, \mathcal{K} is a projector: not all components of $\xi^{\mu\nu}$ give rise to independent gauge transformations, and those ξ that are annihilated by \mathcal{K} do not contribute. Therefore \mathcal{K} projects $\xi^{\mu\nu}$ to the part that contributes to non-trivial gauge transformations. In the linearised approximation, \mathcal{K} projects onto traceless tensors, in the full theory \mathcal{K} could act differently. Of course this structure generalises straightforwardly to the situation with more higher-spin fields.

Such a map between frame- and metric-like gauge parameters, specified by \mathcal{S} and \mathcal{T} , is not unique, because we can redefine the gauge parameter on the metric-like side. Suppose we are given \mathcal{S} and \mathcal{T} , and the associated projector \mathcal{K} . Then we can parameterise ξ^M by a new gauge parameter $\tilde{\xi}^M$,

$$\xi^M = \Phi^M{}_N \tilde{\xi}^N , \quad (2.31)$$

with a possibly field-dependent matrix Φ . It does not need to be invertible, but we want that $\tilde{\xi}$ still parametrises the full set of gauge transformations. Therefore we request that

$$\text{Im}(\mathcal{K}\Phi) = \text{Im}(\mathcal{K}) . \quad (2.32)$$

Then there is a map Ψ in the opposite direction,

$$\tilde{\xi}^M = \Psi^M{}_N \xi^N , \quad (2.33)$$

such that

$$\mathcal{K}\Phi\Psi = \mathcal{K} . \quad (2.34)$$

Ψ acts as an inverse after projection by \mathcal{K} .

With such a redefinition of the gauge parameters, the map (2.28) between frame-like and metric-like gauge parameters is changed into

$$\Xi^{\mathcal{A}} = \tilde{\mathcal{S}}^{\mathcal{A}}{}_M \tilde{\xi}^M \quad , \quad \tilde{\mathcal{S}} = \mathcal{S}\Phi . \quad (2.35)$$

Similarly we can introduce a new inverse map $\tilde{\mathcal{T}}$,

$$\tilde{\xi}^M = \tilde{\mathcal{T}}^M{}_{\mathcal{A}} \Xi^{\mathcal{A}} \quad , \quad \tilde{\mathcal{T}} = \Psi\mathcal{T} , \quad (2.36)$$

such that

$$\tilde{\mathcal{S}}\tilde{\mathcal{T}} = \mathbf{1} \quad , \quad \tilde{\mathcal{T}}\tilde{\mathcal{S}} = \Psi\mathcal{K}\Phi =: \tilde{\mathcal{K}} . \quad (2.37)$$

Notice that the prescription is symmetric in the sense that we can also view ξ^M to provide a new parameterisation of $\tilde{\xi}^M$ via Ψ , and the maps Φ and Ψ satisfy

$$\tilde{\mathcal{K}}\Psi\Phi = \tilde{\mathcal{K}} , \quad (2.38)$$

in analogy to (2.34).

Let us now apply this general discussion to the situation we are interested in. We assume that there is an exact map relating the gauge parameters as above with corresponding matrices \mathcal{S} and \mathcal{T} which are a priori unknown. We then show that there is a redefinition of gauge parameters such that the transformed \mathcal{S} coincides with our proposal.

We already know how a pure diffeomorphism is implemented on the frame-like side, therefore $\mathcal{S}^A{}_\mu = e_\mu^A$ is fixed, and should not be altered by a reparameterisation. We can then restrict to matrices Φ and Ψ of the form

$$\Phi = \begin{pmatrix} \mathbf{1} & * \\ 0 & * \end{pmatrix} , \quad \Psi = \begin{pmatrix} \mathbf{1} & * \\ 0 & * \end{pmatrix} . \quad (2.39)$$

Suppose now that we have found a $\hat{\mathcal{T}}^\mu{}_{\mathcal{A}}$ such that

$$\hat{\mathcal{T}}^\mu{}_{\mathcal{A}} \mathcal{S}^{\mathcal{A}}{}_\nu = \delta_\nu^\mu . \quad (2.40)$$

In our case this will be given by

$$\hat{\mathcal{T}}^\mu{}_{\mathcal{A}} = g^{\mu\nu} e_\nu^{\mathcal{B}} \kappa_{\mathcal{B}\mathcal{A}} , \quad (2.41)$$

such that we recover our projector on diffeomorphisms (see (2.15)) as

$$\mathcal{S}^{\mathcal{A}}{}_\mu \hat{\mathcal{T}}^\mu{}_{\mathcal{B}} = P^{\mathcal{A}}{}_{\mathcal{B}} . \quad (2.42)$$

Then we set

$$\Psi = \begin{pmatrix} \mathbf{1} & \Psi^\mu{}_{\rho\sigma} \\ 0 & \mathbf{1} \end{pmatrix} , \quad \Psi^\mu{}_{\rho\sigma} = (\hat{\mathcal{T}}^\mu{}_{\mathcal{A}} - \mathcal{T}^\mu{}_{\mathcal{A}}) \mathcal{S}^{\mathcal{A}}{}_{\rho\sigma} . \quad (2.43)$$

With this transformation one finds

$$\tilde{\mathcal{T}}^\mu{}_{\mathcal{A}} = \mathcal{T}^\mu{}_{\mathcal{A}} + \Psi^\mu{}_{\rho\sigma} \mathcal{T}^{\rho\sigma}{}_{\mathcal{A}} \quad (2.44)$$

$$= \mathcal{T}^\mu{}_{\mathcal{A}} + (\hat{\mathcal{T}}^\mu{}_{\mathcal{B}} - \mathcal{T}^\mu{}_{\mathcal{B}}) \mathcal{S}^{\mathcal{B}}{}_{\rho\sigma} \mathcal{T}^{\rho\sigma}{}_{\mathcal{A}} \quad (2.45)$$

$$= \mathcal{T}^\mu{}_{\mathcal{A}} + (\hat{\mathcal{T}}^\mu{}_{\mathcal{B}} - \mathcal{T}^\mu{}_{\mathcal{B}}) (\delta_{\mathcal{A}}^{\mathcal{B}} - \mathcal{S}^{\mathcal{B}}{}_\nu \mathcal{T}^\nu{}_{\mathcal{A}}) \quad (2.46)$$

$$= \hat{\mathcal{T}}^\mu{}_{\mathcal{A}} , \quad (2.47)$$

so it is possible to transform \mathcal{T} such that the new $\tilde{\mathcal{T}}$ coincides in its μ -components with $\hat{\mathcal{T}}$. This means that it is always possible to redefine the gauge parameters such that the projection on pure diffeomorphisms is indeed given by P as defined in (2.15).

Assume now that we have fixed $\mathcal{S}^{\mathcal{A}}{}_\mu = e_\mu^{\mathcal{A}}$ as well as $\mathcal{T}^\mu{}_{\mathcal{A}}$ as in (2.41). Then we are left with block-diagonal transformation matrices Φ and Ψ with the identity matrix in the $\mu - \nu$ -block. Suppose now that we have found a matrix $\hat{\mathcal{S}}^{\mathcal{A}}{}_{\mu\nu}$ such that

$$\mathcal{T}^\rho{}_{\mathcal{A}} \hat{\mathcal{S}}^{\mathcal{A}}{}_{\mu\nu} = 0 . \quad (2.48)$$

In our case such a $\hat{\mathcal{S}}$ is given by the expression in (2.25). Now we set

$$\Phi = \begin{pmatrix} \mathbf{1} & 0 \\ 0 & * \end{pmatrix} , \quad \Phi^{\mu\nu}{}_{\rho\sigma} = \mathcal{T}^{\mu\nu}{}_{\mathcal{A}} \hat{\mathcal{S}}^{\mathcal{A}}{}_{\rho\sigma} . \quad (2.49)$$

With this map Φ , the matrix \mathcal{S} is transformed to

$$\tilde{\mathcal{S}}^A_{\rho\sigma} = \mathcal{S}^A_{\mu\nu} \Phi^{\mu\nu}_{\rho\sigma} \quad (2.50)$$

$$= \mathcal{S}^A_{\mu\nu} \mathcal{T}^{\mu\nu}_{\mathcal{B}} \hat{\mathcal{S}}^{\mathcal{B}}_{\rho\sigma} \quad (2.51)$$

$$= (\delta^A_{\mathcal{B}} - \mathcal{S}^A_{\mu} \mathcal{T}^{\mu}_{\mathcal{B}}) \hat{\mathcal{S}}^{\mathcal{B}}_{\rho\sigma} \quad (2.52)$$

$$= \hat{\mathcal{S}}^A_{\rho\sigma} . \quad (2.53)$$

We have to make sure that the transformation Φ that we defined is an allowed one, i.e. that it does not reduce the set of gauge transformations. In the case at hand this is clear at least in a perturbative expansion in the higher-spin fields, where we only have to check that the transformation is regular at leading order. The leading terms of $\hat{\mathcal{S}}$ and \mathcal{S} coincide and are given by the linearised expression (2.21), therefore the transformation is regular.

In conclusion we have shown that indeed there is a parameterisation of the metric-like gauge transformations such that the proposed map (2.27) gives an exact relation between metric-like and frame-like gauge parameters.

3 Translating frame- to metric-like quantities

In this section we will discuss an algorithm to translate frame-like to metric-like quantities. We will first outline this algorithm for quantities which do not contain any covariant derivatives and illustrate it by explicitly calculating the cosmological constant term in the metric-like formulation up to quartic order in the spin-3 field. We will then generalise the algorithm appropriately for quantities containing covariant derivatives. This will allow us to explicitly calculate the gauge transformations of the metric and spin-3 field to cubic order. Finally we will discuss why in these cases the mapping between metric-like and frame-like quantities is unique despite the appearance of seemingly free parameters in the metric-like expressions.

3.1 Restricting to vielbeins only

The aim of this section is to describe an algorithm which allows us to rewrite a frame-like expression in terms of metric-like fields. This algorithm is based on a perturbative expansion of all quantities in the spin-3 field ϕ . To this end we split the $sl(3)$ generators into $sl(2)$ generators $\{J_a\}$, labelled by small Latin indices, and the remaining generators $\{J_A\}$, labelled by capital Latin indices and chosen to be orthogonal to the J_a with respect to the Killing form. Using this notation we decompose the vielbein into the following components

$$e^A = (e^a, E^A) . \quad (3.1)$$

We first note that a given order in the spin-3 field ϕ corresponds to the same order of vielbeins E^A ,

$$\mathcal{O}(\phi) = \mathcal{O}(E) . \quad (3.2)$$

This can be seen by expanding (1.8) and (1.7),

$$\phi_{\mu\nu\rho} = \frac{1}{2} d_{Abc} E^A_{(\mu} e^b_{\nu} e^c_{\rho)} + \frac{1}{6} d_{ABC} E^A_{\mu} E^B_{\nu} E^C_{\rho} , \quad (3.3)$$

$$g_{\mu\nu} = \kappa_{ab} e^a_{\mu} e^b_{\nu} + \kappa_{AB} E^A_{\mu} E^B_{\nu} =: \overset{(0)}{g}_{\mu\nu} + \overset{(2)}{g}_{\mu\nu} . \quad (3.4)$$

Given any frame-like expression², with space-time indices $\mu_1 \dots \mu_m$ and all frame indices contracted, we can perturbatively find the metric-like equivalent by making an ansatz consisting of all possible

²For the moment we will not consider terms containing covariant derivatives. But, as discussed in section 3.3, by slightly modifying our algorithm these kind of terms can be dealt with as well.

contractions of metric-like fields up to a certain order n in the spin-3 field ϕ . We then proceed in five steps:

Step 1: Expand both sides in terms of E^A using (3.4) and (3.3) up to order n and subtract them from each other.

Step 2: Isolate different orders in E^A . For each order we obtain an equation of the following form

$$\sum_i c^{(i)} t_{a_1 \dots a_p B_1 \dots B_{l_i}}^{(i)} (\{e\}, \{E\}, \{\epsilon\}, \{g^{-1}\})_{\mu_1 \dots \mu_m}^{a_1 \dots a_p B_1 \dots B_{l_i}} = 0 \quad (3.5)$$

where $(\{e\}, \{E\}, \{\epsilon\}, \{g^{-1}\})$ denotes a contraction of vielbeins of the given index structure containing the inverse zero-order metric, the vielbeins and the invariant space-time tensor $\epsilon^{\mu\nu\rho}$. We will assume that each term has the same number of $\epsilon^{\mu\nu\rho}$, which carries only upper indices. The $t^{(i)}$ are $sl(2)$ -invariant tensors. Furthermore some of the $c^{(i)}$ are understood to be the coefficients of the terms arising from the expansion of the frame-like side and are therefore equal to 1. The next steps are to be performed for each order separately.

Step 3: Replace the E^A by

$$E_\mu^A = E_\sigma^A \Delta_\mu^\sigma, \quad (3.6)$$

where we used the following definition

$$\Delta_\mu^\sigma = g^{\sigma\rho} e_\rho^b e_\mu^c \kappa_{bc}. \quad (3.7)$$

This operation ensures that the spacetime index of E is now contracted with a $sl(2)$ -vielbein e .

Step 4: Impose

$$g^{\mu\nu} e_\mu^a e_\nu^b = \kappa^{ab} \quad (3.8)$$

for all contractions of this type. After this replacement all terms in the sum of (3.5) are of the same form and can therefore be written as

$$\tilde{t}_{a_1 \dots a_p B_1 \dots B_{l_i}}(c^{(i)}) (\{e\}, \{E\}, \{\epsilon\}, \{g^{-1}\})_{\mu_1 \dots \mu_m}^{a_1 \dots a_p B_1 \dots B_{l_i}} = 0. \quad (3.9)$$

This is because the replacement (3.6) will transfer the space-time index of the vielbein E to a $sl(2)$ -vielbein e . If the vielbein E carries a free space-time index it is therefore ensured that the free index is now carried by an $sl(2)$ -vielbein. If however the space-time indices of two vielbeins E are contracted with each other they will be contracted with a $sl(2)$ -vielbein after the substitution (3.6) and imposing (3.8). Finally a vielbein E contracted with an $sl(2)$ -vielbein e will stay invariant under both performing (3.6) and (3.8). Note that the number of $sl(2)$ frame indices in (3.9) might have changed during this step.

Step 5: Solve (3.9) by stripping off the vielbeins. This leads to

$$\mathcal{P} \tilde{t}_{a_1 \dots a_p B_1 \dots B_{l_i}}(c^{(i)}) = 0, \quad (3.10)$$

where \mathcal{P} is a projector imposing the symmetry inherent in the tensor $(\{e\}, \{E\}, \{\epsilon\}, \{g^{-1}\})$. We will explain this aspect in more detail in the next section. But (3.10) is a linear equation in the coefficients $c^{(i)}$ and can therefore easily be solved using a computer algebra program.

We stress again that step 3 to 5 have to be performed for all orders from 0 to n separately.

3.2 Example: cosmological constant term

Let us illustrate the algorithm described in the previous section by an example. We will consider the higher-spin cosmological constant term which is given by

$$\frac{1}{3!^2} \text{tr}(e \wedge e \wedge e) = \frac{1}{3!^2} f_{ABC} \epsilon^{\mu\nu\rho} e_\mu^A e_\nu^B e_\rho^C d^3x . \quad (3.11)$$

We make an ansatz for the metric-like equivalent of this term by writing down all the possible contractions of the spin-3 field with the metric such that the resulting expression is a space-time scalar, i.e.

$$\frac{1}{3!^2} f_{ABC} \epsilon^{\mu\nu\rho} e_\mu^A e_\nu^B e_\rho^C = \frac{2}{l^2} \sqrt{-g} \left(1 + \sum_{n=2}^{\infty} \mathcal{L}_n \right) . \quad (3.12)$$

Here \mathcal{L}_n denotes all possible contractions compatible with the symmetries of the equation's lhs containing n of the ϕ fields and an arbitrary number of metric tensors. In the case of $n = 2$ this is given by

$$\mathcal{L}_2 = c_1 \phi^{\mu\nu\rho} \phi_{\mu\nu\rho} + c_2 \phi^\mu \phi_\mu , \quad (3.13)$$

where ϕ_μ denotes the trace of the spin-3 field. We will now explain how the algorithm described in the previous section allows us to fix the coefficients c_1 and c_2 .

Step 1 and 2: We expand (3.12) up to second order in E^A which corresponds to second order in ϕ as explained in the previous section. For this we have to expand the determinant of the metric which will also depend on E^A . This yields

$$\sqrt{-g} = \frac{1}{3!} f_{uvw} \epsilon^{\chi\delta\epsilon} e_\chi^u e_\delta^v e_\epsilon^w \left(1 + \frac{1}{2} g^{(\mu\nu)} \kappa_{AB} E_\mu^A E_\nu^B \right) + \mathcal{O}(E^4) . \quad (3.14)$$

Subtracting the lhs from the rhs of equation (3.12) and considering only terms of quadratic order we obtain up to an overall factor

$$\begin{aligned} & c_1 f_{uvw} d_{Acd} d_{Bef} \epsilon^{\chi\delta\epsilon} e_\chi^u e_\delta^v e_\epsilon^w e_\rho^c e_{\rho'}^e g^{(\rho\rho')} e_\sigma^d e_{\sigma'}^f g^{(\sigma\sigma')} E_\mu^A E_\nu^B g^{(\mu\nu)} \\ & + 2c_1 f_{uvw} d_{Acd} d_{Bef} \epsilon^{\chi\delta\epsilon} e_\chi^u e_\delta^v e_\epsilon^w e_\rho^c e_{\rho'}^e g^{(\rho\rho')} E_\sigma^A e_{\sigma'}^f g^{(\sigma\sigma')} e_\mu^d E_\nu^B g^{(\mu\nu)} \\ & + \dots \\ & + \mathcal{O}(E^4) = 0 . \end{aligned} \quad (3.15)$$

Here we have only written out two terms explicitly and we will now show how the algorithm transforms them into the same form.

Step 3: Performing the substitution (3.6) leads to

$$\begin{aligned} & c_1 f_{uvw} d_{Acd} d_{Bef} \epsilon^{\chi\delta\epsilon} e_\chi^u e_\delta^v e_\epsilon^w e_\rho^c e_{\rho'}^e g^{(\rho\rho')} e_\sigma^d e_{\sigma'}^f g^{(\sigma\sigma')} E_\kappa^A g^{(\kappa\gamma)} e_\gamma^g e_\mu^h \kappa_{gh} E_\lambda^B g^{(\lambda\tau)} e_\tau^l e_\nu^j \kappa_{lj} g^{(\mu\nu)} \\ & + 2c_1 f_{uvw} d_{Acd} d_{Bef} \epsilon^{\chi\delta\epsilon} e_\chi^u e_\delta^v e_\epsilon^w e_\rho^c e_{\rho'}^e g^{(\rho\rho')} E_\kappa^A g^{(\kappa\gamma)} e_\gamma^g e_\sigma^h \kappa_{gh} e_{\sigma'}^f g^{(\sigma\sigma')} e_\mu^d E_\lambda^B g^{(\lambda\tau)} e_\tau^l e_\nu^j \kappa_{lj} g^{(\mu\nu)} \\ & + \dots \\ & + \mathcal{O}(E^4) = 0 . \end{aligned} \quad (3.16)$$

Step 4: Imposing the relation (3.8) we obtain

$$\begin{aligned} & c_1 f_{uvw} d_{Acd} d_B{}^{cd} \kappa_{lg} \epsilon^{\chi\delta\epsilon} e_\chi^u e_\delta^v e_\epsilon^w E_\kappa^A g^{(\kappa\gamma)} e_\gamma^g E_\lambda^B g^{(\lambda\tau)} e_\tau^l \\ & + 2c_1 f_{uvw} d_{Adc} d_B{}^c{}^c \epsilon^{\chi\delta\epsilon} e_\chi^u e_\delta^v e_\epsilon^w E_\kappa^A g^{(\kappa\gamma)} e_\gamma^f E_\lambda^B g^{(\lambda\tau)} e_\tau^d \\ & + \dots \\ & + \mathcal{O}(E^4) = 0 . \end{aligned} \quad (3.17)$$

Having written all terms in the same form we find up to a global factor

$$\begin{aligned}
& (-108 f_{ABC} \kappa_{gd} \kappa_{he} + f_{cde} (6c_1 d_{Ahf} d_{Bg}^f + 4c_2 d_{Agf} d_{Bh}^f + 3c_1 d_{Asf} d_B^{sf} \kappa_{gh}) \\
& + 18 f_{cde} \kappa_{AB} \kappa_{gh}) \epsilon^{\chi\delta\epsilon} e_\chi^c e_\delta^d e_\epsilon^e E_\gamma^A g^{\alpha\gamma} e_\alpha^g E_\sigma^B g^{\beta\sigma} e_\beta^h = 0 .
\end{aligned} \tag{3.18}$$

Step 5: We can solve this equation by stripping off the vielbeins. The remaining term has to be anti-symmetrised in c, d, e and symmetrised with respect to exchange of the pair g, A with h, B . This operation was denoted by \mathcal{P} in (3.10). The resulting equation is linear in c_1, c_2 and can be easily solved,

$$c_1 = -3, \quad c_2 = \frac{9}{2} . \tag{3.19}$$

This is most conveniently done by choosing an explicit representation for the invariant tensors of $sl(3)$ and solving the resulting equation using a computer algebra program.

Therefore by using the algorithm described in the previous section we have found the metric-like equivalent of the cosmological constant term to quadratic order in the spin-3 field.

By applying the algorithm also to the quartic order we obtain the result

$$\frac{1}{3l^2} \int f_{ABC} e^A \wedge e^B \wedge e^C = \frac{2}{l^2} \int d^3x \sqrt{-g} (1 + \mathcal{L}_2 + \mathcal{L}_4) + \mathcal{O}(\phi^6) , \tag{3.20}$$

where the quadratic terms are given by

$$\mathcal{L}_2 = -3 \phi^{\mu\nu\rho} \phi_{\mu\nu\rho} + \frac{9}{2} \phi^\mu \phi_\mu , \tag{3.21}$$

and the quartic contribution is

$$\begin{aligned}
\mathcal{L}_4 = & (9 + c) \phi_\mu^{\sigma\kappa} \phi^{\mu\nu\rho} \phi_{\nu\sigma}{}^\tau \phi_{\rho\kappa\tau} + c \phi_{\mu\nu}{}^\sigma \phi^{\mu\nu\rho} \phi_\rho^{\kappa\tau} \phi_{\sigma\kappa\tau} \\
& - (54 + 4c) \phi^\nu \phi_\nu{}^{\rho\sigma} \phi_\rho^{\kappa\tau} \phi_{\sigma\kappa\tau} - 9 \phi^\nu \phi_\nu{}^{\rho\sigma} \phi_\rho \phi_\sigma \\
& - (6 + \frac{1}{2}c) \phi_{\mu\nu\rho} \phi^{\mu\nu\rho} \phi_{\sigma\kappa\tau} \phi^{\sigma\kappa\tau} + (\frac{9}{2} + c) \phi^\nu \phi_\nu \phi_{\sigma\kappa\tau} \phi^{\sigma\kappa\tau} \\
& + (81 + 2c) \phi^\nu \phi_\nu{}^{\rho\sigma} \phi_{\rho\sigma}{}^\kappa \phi_\kappa - (\frac{81}{8} + \frac{1}{2}c) \phi^\nu \phi_\nu \phi^\kappa \phi_\kappa .
\end{aligned} \tag{3.22}$$

The sum of all terms term proportional to c is zero due to a dimension dependent identity as will be explained in section 3.5.

Note that we can not build a scalar by contracting an odd number of spin-3 fields and therefore there are no such contributions in (3.20).

3.3 Including covariant derivatives

In the last two sections we did not include terms involving covariant derivatives in our discussion. In principle we can apply our algorithm also to these types of terms, but there is an additional complication. In the frame-like approach covariant derivatives can act both on E^A and e^a . The algorithm described in section 3.1 crucially relies on the fact that we can bring our expressions into the form (3.9). For this to work for quantities involving covariant derivatives we need to be able to express $\mathcal{D}_\mu e_\nu^a$ in terms of $\mathcal{D}_\mu E_\nu^A$. This can be achieved as follows. The metric is covariantly constant,

$$\nabla_\rho g_{\mu\nu} = \mathcal{D}_\rho g_{\mu\nu} = \kappa_{AB} e_\mu^A \mathcal{D}_\rho e_\nu^B + \kappa_{AB} e_\nu^A \mathcal{D}_\rho e_\mu^B = 0 , \tag{3.23}$$

where \mathcal{D}_μ was defined in (1.6).

By summing three permutations of equation (3.23),

$$\begin{aligned} & \kappa_{AB} e_\mu^A \mathcal{D}_\rho e_\nu^B + \kappa_{AB} e_\nu^A \mathcal{D}_\rho e_\mu^B \\ & - \kappa_{AB} e_\rho^A \mathcal{D}_\mu e_\nu^B - \kappa_{AB} e_\nu^A \mathcal{D}_\mu e_\rho^B \\ & + \kappa_{AB} e_\rho^A \mathcal{D}_\nu e_\mu^B + \kappa_{AB} e_\mu^A \mathcal{D}_\nu e_\rho^B = 0 , \end{aligned} \quad (3.24)$$

and using torsion constraint (1.5) we conclude

$$\kappa_{AB} e_\mu^A \mathcal{D}_\rho e_\nu^B = 0 . \quad (3.25)$$

Expanding this we obtain

$$\mathcal{D}_\mu e_\nu^c = -\kappa_{AB} \overset{(0)}{g}{}^{\sigma\rho} e_\sigma^c E_\rho^A \mathcal{D}_\mu E_\nu^B . \quad (3.26)$$

Using this result we can reformulate the algorithm described in section 3.1 such that it is also applicable to expressions involving covariant derivatives. We only need to modify the prescriptions for step 1 and step 3.

Step 1': Expand both sides in terms of E^A using (3.4) and (3.3) up to order n and subtract them from each other. Perform the following substitution

$$\mathcal{D}_\mu e_\nu^c = -\kappa_{AB} \overset{(0)}{g}{}^{\sigma\rho} e_\sigma^c E_\rho^A \mathcal{D}_\mu E_\nu^B . \quad (3.27)$$

This ensures that all the covariant derivatives act on E^A .

Step 3': Replace covariant derivatives of E^A by

$$D_\mu E_\nu^A = \Delta_\mu^\sigma \Delta_\nu^\rho D_\sigma E_\rho^A \quad (3.28)$$

and the E^A without a derivative by

$$E_\mu^A = E_\sigma^A \Delta_\mu^\sigma . \quad (3.29)$$

The notation Δ_μ^ρ was defined in (3.7).

All others steps are unchanged.

3.4 Spin-3 transformations

In this section we determine the spin-3 transformations of both the metric and the spin-3 field perturbatively. For this we again make the most general ansatz for the gauge transformations of the metric-like fields and fix its coefficients by applying the modified algorithm described in the last section. The gauge transformation of the spin-3 field is then given by

$$\delta_{\Xi}^{(3)} \phi_{\alpha\beta\chi} = 3 \nabla_{(\alpha} \hat{\xi}_{\beta\chi)} + (\hat{\xi} \phi \nabla \phi)_{\alpha\beta\chi} + (\nabla \hat{\xi} \phi \phi)_{\alpha\beta\chi} + \mathcal{O}(\phi^4) . \quad (3.30)$$

Here $\hat{\xi}$ denotes the traceless component of ξ , see (A.3). The explicit expressions for $(\hat{\xi} \phi \nabla \phi)_{\alpha\beta\chi}$ and $(\nabla \hat{\xi} \phi \phi)_{\alpha\beta\chi}$ are quite involved and are given in appendix D.

The metric transforms as follows

$$\begin{aligned} \delta_{\Xi}^{(3)} g_{\alpha\beta} &= 6 (2 \hat{\xi}^{\chi\delta} \nabla_\delta \phi_{\alpha\beta\chi} + 4 \hat{\xi}^{\chi\delta} g_{\alpha\beta} \nabla_\delta \phi_\chi + \hat{\xi}_{\alpha\beta} \nabla_\delta \phi^\delta - 2 \hat{\xi}^{\chi\delta} g_{\alpha\beta} \nabla_\epsilon \phi_{\chi\delta}^\epsilon \\ & - 4 \hat{\xi}_{(\alpha}^\chi \nabla_{\beta)} \phi_\chi - 4 \hat{\xi}_{(\alpha}^\chi \nabla_{|\chi|} \phi_{\beta)} + 4 \hat{\xi}_{(\alpha}^\chi \nabla^\delta \phi_{\beta)\chi\delta} \\ & - 4 \hat{\xi}^{\chi\delta} \nabla_{(\alpha} \phi_{\beta)\chi\delta}) + (\hat{\xi} \phi \phi \nabla \phi)_{\alpha\beta} + \mathcal{O}(\phi^5) . \end{aligned} \quad (3.31)$$

The explicit expression for $(\hat{\xi} \phi \phi \nabla \phi)_{\alpha\beta}$ can be found in appendix D.

3.5 Ambiguities

Equation (3.22) contains a free parameter c , seemingly suggesting that the frame-like cosmological constant term does not have a unique metric-like counterpart. However this is not the case. The parameter c is due to a dimensional dependent identity (DDI), which arises by over-antisymmetrisation. An example for a DDI is

$$\delta_{[\sigma}^{\mu} \delta_{\rho}^{\nu} \delta_{\tau]}^{\gamma} \delta_{\tau]}^{\lambda} = 0 , \quad (3.32)$$

which obviously vanishes in three dimensions. A systematic way to construct all DDIs of a set of tensors is described in [38]. For a certain tensor all possible contractions with (3.32) are determined. All identities which arise by over-antisymmetrisation can be constructed in such a way as we can always pull out deltas on the over-antisymmetrised indices. Using the Mathematica package xTras, described in [38], these identities can automatically constructed by this method. For the case of the cosmological constant term at quartic order there is the following relevant DDI,

$$\begin{aligned} & \phi_{\mu}^{\delta\epsilon} \phi^{\mu\nu\chi} \phi_{\nu\delta}^{\phi} \phi_{\chi\epsilon\phi} + \phi_{\mu\nu}^{\delta} \phi^{\mu\nu\chi} \phi_{\chi}^{\epsilon\phi} \phi_{\delta\epsilon\phi} - 4 \phi^{\nu} \phi_{\nu}^{\chi\delta} \phi_{\chi}^{\epsilon\phi} \phi_{\delta\epsilon\phi} - \frac{1}{2} \phi_{\mu\nu\chi} \phi^{\mu\nu\chi} \phi_{\delta\epsilon\phi} \phi^{\delta\epsilon\phi} \\ & + \phi^{\nu} \phi_{\nu} \phi_{\delta\epsilon\phi} \phi^{\delta\epsilon\phi} + 2 \phi^{\nu} \phi_{\nu}^{\chi\delta} \phi_{\chi\delta}^{\epsilon} \phi_{\epsilon} - \frac{1}{2} \phi^{\nu} \phi_{\nu} \phi^{\epsilon} \phi_{\epsilon} \equiv 0 . \end{aligned} \quad (3.33)$$

But the terms proportional to c in (3.22) are exactly given by this DDI and therefore vanish. Thus the cosmological constant term to quartic order is uniquely determined by our calculation.

4 Gauge algebra

We will now discuss the algebra of the gauge transformations for the metric-like fields. While the algebra of the frame-like transformations closes off-shell, in the metric-like formulation the algebra only closes on-shell. We start by explaining this phenomenon and discuss also why the commutators with spin-2 transformations (diffeomorphisms) still close off-shell. We then determine explicitly the gauge algebra to linear order in the spin-3 field ϕ .

4.1 On-shell gauge algebra

Recall from section 2.1 that in the frame-like theory general local translations induce pure diffeomorphisms and spin-3 transformations. General translations are parametrised by Ξ^A . Obviously this corresponds to having 8 degrees of freedom which nicely matches the 3+5 degrees of freedom of the parameter v^{μ} corresponding to pure diffeomorphisms and $\hat{\xi}^{\mu\nu} = \xi^{\mu\nu} - \frac{1}{3} g^{\mu\nu} \xi^{\lambda}{}_{\lambda}$ parameterising the spin-3 transformations. According to (2.3) the fields of the frame-like formalism transform as follows

$$\delta_{\Xi} e_{\mu}^A = D_{\mu} \Xi^A , \quad (4.1)$$

$$\delta_{\Xi} \omega_{\mu}^A = \frac{1}{l^2} [e_{\mu}, \Xi^A] , \quad (4.2)$$

and the gauge algebra closes off-shell.

When we translate the frame-like theory to the metric-like formulation we have to use the torsion constraint (1.5) to express the spin connection in terms of vielbeins, $\omega = \omega(e)$. This implicit dependence induces a gauge transformation of the spin connection that differs from the transformation (4.2), and only coincides with it on-shell, i.e. after using the equation of motion. This can be seen as follows. The induced transformation of the spin connection can be calculated by varying the torsion constraint (1.5),

$$\delta_{\Xi} \left(D_{[\mu} e_{\nu]}^A \right) = \delta_{\Xi} D_{[\mu} e_{\nu]}^A + D_{[\mu} D_{\nu]} \Xi^A = 0 . \quad (4.3)$$

The Christoffel symbol is symmetric in μ and ν and therefore the variation of the covariant derivative in the equation above is given by the transformation of the spin connection. We thus obtain

$$f^A{}_{BC} \delta_{\Xi} \omega_{[\mu}^B e_{\nu]}^C + f^A{}_{BC} R_{\mu\nu}^B \Xi^C = 0 \quad (4.4)$$

with $R_{\mu\nu}^A$ from (1.4). The equation of motion for the vielbein is

$$R_{\mu\nu}^A = -\frac{1}{2l^2} f^A{}_{BC} e_{\mu}^B e_{\nu}^C . \quad (4.5)$$

Using this equation of motion in (4.4), we find that the induced transformation reduces on-shell to (we assume that the vielbein is non-degenerate)

$$\delta_{\Xi} \omega_{\mu}^A = \frac{1}{l^2} f^A{}_{BC} e_{\mu}^B \Xi^C , \quad (4.6)$$

which coincides with the transformation (4.2) in the frame-like theory. Therefore we expect that the metric-like gauge algebra only closes on-shell.

Let us explicitly consider the commutator of two gauge transformations on a vielbein (all metric-like fields are built out of the vielbein). Using (4.1) we obtain

$$\begin{aligned} [\delta_{\Xi}, \delta_{\Pi}] e_{\mu}^A &= D_{\mu} (\delta_{\Xi} \Pi^A - \delta_{\Pi} \Xi^A) + f^A{}_{BC} (\delta_{\Xi} \omega_{\mu}^B \Pi^C - \delta_{\Pi} \omega_{\mu}^B \Xi^C) \\ &= \delta_{(\delta_{\Xi} \Pi - \delta_{\Pi} \Xi)} e_{\mu}^A + f^A{}_{BC} (\delta_{\Xi} \omega_{\mu}^B \Pi^C - \delta_{\Pi} \omega_{\mu}^B \Xi^C) . \end{aligned} \quad (4.7)$$

The first term is a local translation of the vielbein and therefore can again be interpreted as a gauge transformation in the metric-like formulation. For the second term it might in general not be possible to rewrite it as a gauge transformation on the vielbein. On the other hand, on-shell the last term is a generalised local Lorentz transformation of the vielbein as can be checked by using the Jacobi identity,

$$\begin{aligned} f^A{}_{BC} (\delta_{\Xi} \omega_{\mu}^B \Pi^C - \delta_{\Pi} \omega_{\mu}^B \Xi^C) &= -\frac{1}{l^2} f^A{}_{BC} (f^B{}_{DE} e_{\mu}^D \Xi^E \Pi^C - f^B{}_{DE} e_{\mu}^D \Pi^E \Xi^C) \\ &= \frac{1}{l^2} f^A{}_{BD} (f^B{}_{EC} \Xi^E \Pi^C) e_{\mu}^D . \end{aligned} \quad (4.8)$$

In the metric-like fields all frame indices are contracted with invariant tensors, and the local Lorentz transformations do not have any effect. Hence we find that on-shell the gauge algebra in the metric-like formulation is obtained by translating

$$[\delta_{\Xi}, \delta_{\Pi}] = \delta_{(\delta_{\Xi} \Pi - \delta_{\Pi} \Xi)} \quad (4.9)$$

into metric-like quantities.

4.2 Off-shell closure for spin-2 transformations

In the last subsection we have shown that after imposing the vielbein's equation of motion the second term in (4.7) can be written as a local Lorentz transformation. In this section we will show that in the special case in which at least one of the parameters describes a spin-2 transformation, i.e. $\Xi^A = e_{\mu}^A \xi^{\mu}$, the last term of (4.7) is a local Lorentz transformation even off-shell. Firstly, using (4.4) we calculate the variation of the spin connection in this special case.

$$\begin{aligned} f^A{}_{BC} \delta_{\Xi} \omega_{[\mu}^B e_{\nu]}^C &= -f^A{}_{BC} R_{\mu\nu}^B e_{\sigma}^C \xi^{\sigma} \\ &= -f^A{}_{BC} (R_{\mu\sigma}^B e_{\nu}^C + R_{\sigma\nu}^B e_{\mu}^C) \xi^{\sigma} \\ &= 2 f^A{}_{BC} R_{\sigma[\mu}^B e_{\nu]}^C \xi^{\sigma} , \end{aligned} \quad (4.10)$$

where we have used a Bianchi-like identity $f^A{}_{BC} R^B{}_{[\mu\nu]} e^C{}_{\sigma]} = 0$ (see appendix B). As the vielbein is non-degenerate we conclude from (4.10) that

$$\delta_{\Xi}\omega_{\mu}^{\mathcal{B}} = 2\xi^{\sigma} R_{\sigma\mu}^{\mathcal{B}} \quad (4.11)$$

is the induced transformation of the spin connection under a spin-2 transformation. Plugging this result into the second term of (4.7) and using (4.4) we obtain

$$\begin{aligned} f^A{}_{BC} (\delta_{\Xi}\omega_{\mu}^{\mathcal{B}} \Pi^C - \delta_{\Pi}\omega_{\mu}^{\mathcal{B}} \Xi^C) &= f^A{}_{BC} (2\xi^{\nu} R_{\nu\mu}^{\mathcal{B}} \Pi^C - \delta_{\Pi}\omega_{\mu}^{\mathcal{B}} e_{\nu}^C \xi^{\nu}) \\ &= f^A{}_{BC} \left(2\xi^{\nu} R_{\nu\mu}^{\mathcal{B}} \Pi^C - 2\delta_{\Pi}\omega_{[\mu}^{\mathcal{B}} e_{\nu]}^C \xi^{\nu} - \delta_{\Pi}\omega_{\nu}^{\mathcal{B}} e_{\mu}^C \xi^{\nu} \right) \\ &= f^A{}_{BC} (2\xi^{\nu} R_{\nu\mu}^{\mathcal{B}} \Pi^C + 2\xi^{\nu} R_{\mu\nu}^{\mathcal{B}} \Pi^C - \delta_{\Pi}\omega_{\nu}^{\mathcal{B}} e_{\mu}^C \xi^{\nu}) \\ &= -f^A{}_{BC} \delta_{\Pi}\omega_{\nu}^{\mathcal{B}} e_{\mu}^C \xi^{\nu} . \end{aligned} \quad (4.12)$$

But the final expression is just a generalised local Lorentz transformation and we have therefore shown that the commutator of a spin-2 transformation with any other transformation can be expressed as a gauge transformation also off-shell.

In the following we will compute the various commutators that arise in the algebra of metric-like gauge transformations explicitly.

4.3 Spin-2 spin-2 commutator

Here we will consider the case of both transformations being diffeomorphisms, i.e. $\Pi^A = e_{\mu}^A \pi^{\mu}$ and $\Xi^A = e_{\mu}^A \xi^{\mu}$. As shown in the previous section this commutator closes off-shell and using (4.7) we can calculate the resulting transformation

$$\begin{aligned} \delta_{\Pi} (e_{\mu}^A \xi^{\mu}) - \delta_{\Xi} (e_{\mu}^A \pi^{\mu}) &= D_{\mu} (e_{\nu}^A \pi^{\nu}) \xi^{\mu} - \xi \leftrightarrow \pi \\ &= -e_{\nu}^A \mathcal{L}_{\pi} \xi^{\nu} + 2\xi^{\mu} \pi^{\nu} D_{[\mu} e_{\nu]}^A \end{aligned} \quad (4.13)$$

where $\mathcal{L}_{\pi} \xi^{\nu} = \pi^{\mu} \partial_{\mu} \xi^{\nu} - \xi^{\mu} \partial_{\mu} \pi^{\nu}$ is the Lie derivative. But the last term in the last line vanishes as we impose the torsion constraint (1.5). By (2.5) the result of this commutator therefore induces a diffeomorphism with vector field $-\mathcal{L}_{\pi} \xi^{\nu}$.

4.4 Spin-3 spin-2 commutator

We now want to discuss the commutator of a spin-3 and a spin-2 transformation. The spin-3 transformation is parameterised by

$$\Xi^A = \mathcal{S}^A{}_{\mu_1\mu_2} \xi^{\mu_1\mu_2} , \quad (4.14)$$

where \mathcal{S} is given in (2.25). The result for the commutator will not depend on the precise form of \mathcal{S} , but only on the property that it is built from the vielbeins. In fact we can also consider the more general case of the commutator of a spin- $(s+1)$ and a spin-2 transformation without any additional complication, where the spin-2 and the spin- $(s+1)$ transformations are parameterised by

$$\Pi^A = e_{\sigma}^A \pi^{\sigma} \quad \text{and} \quad \Xi^A = \mathcal{S}^A{}_{\mu_1\dots\mu_s}(e) \xi^{\mu_1\dots\mu_s} . \quad (4.15)$$

Here, $\mathcal{S}^A{}_{\mu_1\dots\mu_s}(e)$ is built by contracting vielbeins and it is completely symmetric in all space-time indices. For a result that we need later we consider the following space-time tensor,

$$\mathcal{O}_{\nu\mu_1\dots\mu_s} = \kappa_{AB} e_{\nu}^A \mathcal{S}^B{}_{\mu_1\dots\mu_s}(e) . \quad (4.16)$$

Because it is constructed from the vielbeins, under the spin-2 transformation the tensor $\mathcal{O}_{\nu\mu_1\dots\mu_s}$ changes by the Lie derivative along π ,

$$\delta_{\Pi}\mathcal{O}_{\nu\mu_1\dots\mu_s} = \pi^\sigma \nabla_\sigma \mathcal{O}_{\nu\mu_1\dots\mu_s} + s \nabla_{(\mu_1} \pi^\sigma \mathcal{O}_{|\nu\sigma|\mu_2\dots\mu_s)} + \nabla_\nu \pi^\sigma \mathcal{O}_{\sigma\mu_1\dots\mu_s} . \quad (4.17)$$

The lhs of this equation can be calculated by explicitly evaluating the variation of the vielbein, i.e.

$$\begin{aligned} \delta_{\Pi}\mathcal{O}_{\nu\mu_1\dots\mu_s} &= \kappa_{\mathcal{A}\mathcal{B}} D_\nu (e_\sigma^{\mathcal{A}} \pi^\sigma) \mathcal{S}^{\mathcal{B}}_{\mu_1\dots\mu_s} + \kappa_{\mathcal{A}\mathcal{B}} e_\nu^{\mathcal{A}} (\delta_{\Pi} \mathcal{S}^{\mathcal{B}}_{\mu_1\dots\mu_s}) \\ &= \kappa_{\mathcal{A}\mathcal{B}} \pi^\sigma (D_\sigma e_\nu^{\mathcal{A}}) \mathcal{S}^{\mathcal{B}}_{\mu_1\dots\mu_s} + \kappa_{\mathcal{A}\mathcal{B}} e_\nu^{\mathcal{A}} (\delta_{\Pi} \mathcal{S}^{\mathcal{B}}_{\mu_1\dots\mu_s}) + \nabla_\nu \pi^\sigma \mathcal{O}_{\sigma\mu_1\dots\mu_s} , \end{aligned} \quad (4.18)$$

where we used (1.5) and suppressed the dependency of $\mathcal{S}^{\mathcal{B}}_{\mu_1\dots\mu_s}$ on the vielbeins to simplify notation. Combining (4.17) with (4.18) yields

$$\kappa_{\mathcal{A}\mathcal{B}} e_\nu^{\mathcal{A}} (\delta_{\Pi} \mathcal{S}^{\mathcal{B}}_{\mu_1\dots\mu_s}) = \kappa_{\mathcal{A}\mathcal{B}} e_\nu^{\mathcal{A}} \pi^\sigma D_\sigma \mathcal{S}^{\mathcal{B}}_{\mu_1\dots\mu_s} + s \kappa_{\mathcal{A}\mathcal{B}} e_\nu^{\mathcal{A}} \nabla_{(\mu_1} \pi^\sigma \mathcal{S}^{\mathcal{B}}_{|\sigma|\mu_2\dots\mu_s)} . \quad (4.19)$$

We therefore conclude that

$$\delta_{\Pi} \mathcal{S}^{\mathcal{B}}_{\mu_1\dots\mu_s} = \pi^\sigma D_\sigma \mathcal{S}^{\mathcal{B}}_{\mu_1\dots\mu_s} + s \nabla_{(\mu_1} \pi^\sigma \mathcal{S}^{\mathcal{B}}_{|\sigma|\mu_2\dots\mu_s)} . \quad (4.20)$$

We are now in the position to determine the commutator of the spin-2 transformation Π and the spin- $(s+1)$ transformation Ξ given in (4.15), and we find

$$\begin{aligned} \delta_{\Xi} \Pi^{\mathcal{A}} - \delta_{\Pi} \Xi^{\mathcal{A}} &= \pi^\sigma D_\sigma (\mathcal{S}^{\mathcal{A}}_{\mu_1\dots\mu_s} \xi^{\mu_1\dots\mu_s}) - \xi^{\mu_1\dots\mu_s} \delta_{\Pi} \mathcal{S}^{\mathcal{A}}_{\mu_1\dots\mu_s} \\ &= \mathcal{S}^{\mathcal{A}}_{\mu_1\dots\mu_s} \left(\pi^\sigma \nabla_\sigma \xi^{\mu_1\dots\mu_s} - s \xi^{\sigma(\mu_1\dots\mu_{s-1}} \nabla_\sigma \pi^{\mu_s)} \right) \\ &= \mathcal{S}^{\mathcal{A}}_{\mu_1\dots\mu_s} (\mathcal{L}_\pi \xi^{\mu_1\dots\mu_s}) . \end{aligned} \quad (4.21)$$

Thus the commutator is a spin- $(s+1)$ transformation whose parameter is given by the Lie derivative of the original spin- $(s+1)$ -parameter. In particular in our case we find

$$[\delta_\xi^{(3)}, \delta_\pi^{(2)}] = \delta_{\mathcal{L}_\pi \xi}^{(3)} . \quad (4.22)$$

4.5 Spin-3 spin-3 commutator

In contrast to the commutation relation involving at least one spin-2 transformation we currently do not have an all order result for the commutator of two spin-3 transformations. The commutator is specified by traceless parameters $\hat{\xi}^{\mu\nu}$ and $\hat{\pi}^{\mu\nu}$, and generically it will lead to a combination of a spin-2 transformation and a spin-3 transformation, i.e.

$$[\delta_{\Pi}, \delta_{\Xi}] e_\mu^{\mathcal{A}} = \delta_{\mathcal{S}(u,v)} e_\mu^{\mathcal{A}} , \quad (4.23)$$

where

$$\mathcal{S}^{\mathcal{A}}(u, v) = \mathcal{S}^{\mathcal{A}}_{\mu} v^\mu + \mathcal{S}^{\mathcal{A}}_{\rho\sigma} u^{\rho\sigma} \quad (4.24)$$

denotes the map defined in (2.11). In the following we will determine the parameters $u^{\mu\nu}$ and v^μ perturbatively in the spin-3 field. First we will calculate explicitly the spin-2 parameter v^μ by only considering zeroth order contributions. Then we will use the algorithm discussed in section 3.1 to determine these parameters at linear order.

4.5.1 Spin-2 parameter v^μ

This contribution was already calculated in [21] using a different method. We need to evaluate

$$\begin{aligned}
\delta_{\Pi}^{(3)} \Xi^{\mathcal{A}} &= 3 \delta_{\Pi}^{(3)} \{ \delta_{\mathcal{B}}^{\mathcal{A}} - P_{\mathcal{B}}^{\mathcal{A}} \} d^{\mathcal{B}}{}_{\mathcal{C}\mathcal{D}} e_{\mu}^{\mathcal{C}} e_{\nu}^{\mathcal{D}} \hat{\xi}^{\mu\nu} + 3 (\delta_{\mathcal{B}}^{\mathcal{A}} - P_{\mathcal{B}}^{\mathcal{A}}) \delta_{\Pi}^{(3)} \left\{ d^{\mathcal{B}}{}_{\mathcal{C}\mathcal{D}} e_{\mu}^{\mathcal{C}} e_{\nu}^{\mathcal{D}} \hat{\xi}^{\mu\nu} \right\} \\
&= -3 \left(\delta_{\Pi}^{(3)} P_{\mathcal{B}}^{\mathcal{A}} \right) d^{\mathcal{B}}{}_{\mathcal{C}\mathcal{D}} e_{\mu}^{\mathcal{C}} e_{\nu}^{\mathcal{D}} \hat{\xi}^{\mu\nu} \\
&\quad + 6 (\delta_{\mathcal{B}}^{\mathcal{A}} - P_{\mathcal{B}}^{\mathcal{A}}) d^{\mathcal{B}}{}_{\mathcal{C}\mathcal{D}} (D_{\mu} \Pi^{\mathcal{C}}) e_{\nu}^{\mathcal{D}} \hat{\xi}^{\mu\nu} \\
&\quad + 3 (\delta_{\mathcal{B}}^{\mathcal{A}} - P_{\mathcal{B}}^{\mathcal{A}}) d^{\mathcal{B}}{}_{\mathcal{C}\mathcal{D}} e_{\mu}^{\mathcal{C}} e_{\nu}^{\mathcal{D}} \left(\delta_{\Pi}^{(3)} \hat{\xi}^{\mu\nu} \right) .
\end{aligned} \tag{4.25}$$

The variation of the projector $P_{\mathcal{B}}^{\mathcal{A}}$ is given by

$$\begin{aligned}
\delta_{\Pi}^{(3)} P_{\mathcal{B}}^{\mathcal{A}} &= (D_{\mu} \Pi^{\mathcal{A}}) g^{\mu\nu} e_{\nu}^{\mathcal{C}} \kappa_{\mathcal{B}\mathcal{C}} + e_{\mu}^{\mathcal{A}} g^{\mu\nu} \kappa_{\mathcal{B}\mathcal{C}} (D_{\nu} \Pi^{\mathcal{C}}) \\
&\quad - 2 e_{\mu}^{\mathcal{A}} \kappa_{\mathcal{E}\mathcal{F}} e_{\nu}^{\mathcal{C}} e_{(\sigma}^{\mathcal{E}} D_{\rho)} \Pi^{\mathcal{F}} g^{\sigma\nu} g^{\rho\mu} \kappa_{\mathcal{B}\mathcal{C}} ,
\end{aligned} \tag{4.26}$$

where the last term arises due to the variation of the inverse metric in the projector. We will now evaluate (4.25) at leading order. Let us focus on the last term in (4.25) first. By using (3.31) it can be checked easily that this term is of higher order as

$$\delta_{\Pi}^{(3)} \hat{\xi}^{\mu\nu} = \mathcal{O}(E) . \tag{4.27}$$

Note that if we choose $\mathcal{A} = A$ all terms in (4.25) will be at least of linear order. For $\mathcal{A} = a$ we can easily deduce that the second term in (4.25) does not contribute as

$$(\delta_b^a - P_b^a) = 0 \text{ and } P_B^a = \mathcal{O}(E) . \tag{4.28}$$

So only the first term in (4.25) will contribute to leading order. From $\mathcal{A} = a$ it follows that to leading order we have to choose $\mathcal{B} = B$. The variation of the projector is then given by

$$\delta_{\Pi}^{(3)} P_B^a = 3 e_{\mu}^a g^{\mu\nu} \kappa_{BC} d^C{}_{ef} e_{\rho}^e e_{\tau}^f \nabla_{\nu} \hat{\pi}^{\rho\tau} + \mathcal{O}(E) . \tag{4.29}$$

Plugging this in the only non-vanishing term of (4.25) we find at leading order

$$\begin{aligned}
\delta_{\Pi}^{(3)} \Xi^a - \delta_{\Xi}^{(3)} \Pi^a &= -9 d_{Bef} d^B{}_{cd} e_{\sigma}^a g^{\sigma\tau} e_{\rho}^e e_{\gamma}^f e_{\mu}^c e_{\nu}^d \left(\hat{\xi}^{\mu\nu} \nabla_{\tau} \hat{\pi}^{\rho\gamma} - \hat{\xi} \leftrightarrow \hat{\pi} \right) \\
&= -18 e_{\sigma}^a g^{\sigma\tau} \left(\hat{\xi}^{\mu\nu} \nabla_{\tau} \hat{\pi}_{\mu\nu} - \hat{\xi} \leftrightarrow \hat{\pi} \right) ,
\end{aligned} \tag{4.30}$$

where we have used the identity (A.10b) in the last step.

But by (2.5) the result in (4.30) corresponds to a spin-2 transformation with the parameter

$$v^{\mu} = -18 g^{\mu\nu} \left(\xi^{\rho\sigma} \nabla_{\nu} \pi_{\rho\sigma} - \frac{1}{3} \xi^{\rho}{}_{\rho} \nabla_{\nu} \pi^{\sigma}{}_{\sigma} - \xi \leftrightarrow \pi \right) . \tag{4.31}$$

4.5.2 Spin-3 parameter $u^{\mu\nu}$

To determine the spin-3 parameter $u^{\mu\nu}$ we make the following ansatz

$$\delta_{\Pi}^{(3)} \Xi^{\mathcal{A}} - \delta_{\Xi}^{(3)} \Pi^{\mathcal{A}} = \mathcal{S}^{\mathcal{A}}(u^{\mu\nu}, v^{\mu}) , \tag{4.32}$$

where $\mathcal{S}^{\mathcal{A}}$ is defined as in (4.24). The parameter v^{μ} cannot be corrected by terms linear in the spin-3 field as we cannot build a vector by contracting a spin-3 field, a covariant derivative and the parameter

$u^{\mu\nu}$. In order to solve this equation we make an ansatz for the linear order of $u^{\mu\nu}$ by considering all possible contractions of

$$\xi^{\rho\sigma}, \pi^{\rho\sigma} \text{ and } \phi_{\sigma\tau\rho} \quad (4.33)$$

with two symmetric free indices, μ and ν , and antisymmetric with respect to the exchange of ξ and π . We use the algorithm described in section 3.3 to determine the coefficients of the ansatz. The result for $u^{\alpha\beta}$ contains three different contributions denoted by

$$u^{\alpha\beta} = u_1^{\alpha\beta} + u_2^{\alpha\beta} + u_3^{\alpha\beta}. \quad (4.34)$$

Firstly terms with a derivative acting on the spin-3 field

$$\begin{aligned} u_1^{\alpha\beta} = & 6(-5(\hat{\xi}\hat{\pi})^{\alpha\beta\chi\delta}\nabla_\delta\phi_\chi + (\hat{\xi}\hat{\pi})^{\chi\delta}{}_\chi{}^\epsilon(g^{\alpha\beta}\nabla_\epsilon\phi_\delta - \nabla_\epsilon\phi^{\alpha\beta}{}_\delta) + 3(\hat{\xi}\hat{\pi})^{\alpha\beta\chi\delta}\nabla_\epsilon\phi_\chi\delta^\epsilon \\ & - 6(\hat{\xi}\hat{\pi})^{\chi\delta\epsilon\xi}g^{\alpha\beta}\nabla_\xi\phi_\chi\delta_\epsilon + 3(\hat{\xi}\hat{\pi})^{(\alpha|\chi}{}_\chi{}^{|\delta|}\nabla^\beta)\phi_\delta + 2(\hat{\xi}\hat{\pi})^{(\alpha|\chi}{}_\chi{}^{|\delta|}\nabla_\delta\phi^\beta) \\ & - 3(\hat{\xi}\hat{\pi})^{(\alpha|\chi}{}_\chi{}^\delta\nabla^{\epsilon|}\phi^\beta)_{\delta\epsilon} - 5(\hat{\xi}\hat{\pi})^{(\alpha|\chi\delta\epsilon|}\nabla_\chi\phi^\beta)_{\delta\epsilon} + 13(\hat{\xi}\hat{\pi})^{(\alpha|\chi\delta\epsilon|}\nabla_\delta\phi^\beta)_{\chi\epsilon} + k_1D_1^{\alpha\beta}, \end{aligned} \quad (4.35)$$

where we have used the following notation

$$(\hat{\xi}\hat{\pi})^{\mu\nu\rho\sigma} = \hat{\xi}^{\mu\nu}\hat{\pi}^{\rho\sigma} - \hat{\xi} \leftrightarrow \hat{\pi}. \quad (4.36)$$

The hatted tensors again denote the traceless components of the parameters, see (A.3). The term $D_1^{\alpha\beta}$ is given in appendix C and vanishes due to DDIs.

Secondly there are contributions with a derivative acting on one of the parameters,

$$\begin{aligned} u_2^{\alpha\beta} = & 12\left((\nabla\hat{\xi}\hat{\pi})^{\delta\alpha\beta}{}_\chi\delta\phi^\chi + (\nabla\hat{\xi}\hat{\pi})^\delta{}_\chi{}^\epsilon\delta_\epsilon\phi^\chi g^{\alpha\beta} - (\nabla\hat{\xi}\hat{\pi})^{\chi\delta\epsilon}{}_\chi\delta\phi^{\alpha\beta}{}_\epsilon - \frac{3}{2}(\nabla\hat{\xi}\hat{\pi})^{\chi\alpha\beta\delta\epsilon}\phi_\chi\delta_\epsilon\right. \\ & + 12(\nabla\hat{\xi}\hat{\pi})^{\chi\delta\epsilon}{}_\chi{}^\xi g^{\alpha\beta}\phi_{\delta\epsilon\xi} - (\nabla\hat{\xi}\hat{\pi})^{\chi(\alpha|\delta|\beta)}{}_\chi\phi_\delta - (\nabla\hat{\xi}\hat{\pi})^{\chi(\alpha|\delta|}{}_\chi\delta\phi^\beta) \\ & \left. - 2(\nabla\hat{\xi}\hat{\pi})^{\chi(\alpha|\delta}{}_\chi{}^{\epsilon|}\phi^\beta)_{\delta\epsilon} - 34(\nabla\hat{\xi}\hat{\pi})^{\chi\delta\epsilon(\alpha}{}_\chi\phi^\beta)_{\delta\epsilon}\right) + k_2D_2^{\alpha\beta} + k_3D_3^{\alpha\beta} + k_4D_4^{\alpha\beta} + k_5D_5^{\alpha\beta}. \end{aligned} \quad (4.37)$$

Here we used the notation

$$(\nabla\hat{\xi}\hat{\pi})_\mu{}^{\nu\rho\sigma\epsilon} = \hat{\pi}^{\nu\rho}\nabla_\mu\hat{\xi}^{\sigma\epsilon} - \hat{\pi} \leftrightarrow \hat{\xi}. \quad (4.38)$$

The terms $D_i^{\alpha\beta}$, $i = 2 \dots 5$, are given in appendix C and are identically zero due to DDIs.

Finally there are contributions containing the trace of the parameters of the gauge transformations

$$\begin{aligned} u_3^{\alpha\beta} = & 4(\hat{\xi}\pi' - \hat{\pi}\xi')^{\lambda\rho}\phi^{\alpha\beta}{}_{\lambda;\rho} + 4(\hat{\xi}\pi' - \hat{\pi}\xi')^{\lambda\rho}g^{\alpha\beta}\phi_{\lambda;\rho} + 2(\hat{\xi}\pi' - \hat{\pi}\xi')^{\alpha\beta}\phi^{\rho}{}_{;\rho} \\ & - 8(\hat{\xi}\pi' - \hat{\pi}\xi')^{(\alpha|\lambda|}\nabla^\beta)\phi_\lambda - 8(\hat{\xi}\pi' - \hat{\pi}\xi')^{(\alpha|\lambda|}\nabla_\lambda\phi^\beta) + 8(\hat{\xi}\pi' - \hat{\pi}\xi')^{(\alpha|\lambda}\nabla^{\rho|}\phi^\beta)_{\lambda\rho} \\ & - 8(\hat{\xi}\pi' - \hat{\pi}\xi')^{\lambda\rho}\nabla^{(\alpha}\phi^\beta)_{\lambda\rho}, \end{aligned} \quad (4.39)$$

where we denoted

$$(\hat{\xi}\pi' - \hat{\pi}\xi')^{\mu\nu} = \hat{\xi}^{\mu\nu}\pi^\sigma{}_\sigma - \hat{\pi}^{\mu\nu}\xi^\sigma{}_\sigma. \quad (4.40)$$

It might at first seem surprising that the commutator contains traces of the gauge parameters, whereas in a single gauge transformation only their traceless part contributes. This is due to the fact that the notion of the trace is field-dependent (it depends on the metric), and that the field changes under the gauge transformation.

Let us briefly explain this phenomenon in a very simple example. Consider an infinitesimal rotation of a vector $\vec{x} \in \mathbb{R}^3$ parameterised by a vector $\vec{v} \in \mathbb{R}^3$,

$$\delta_v\vec{x} = \vec{v} \times \vec{x}. \quad (4.41)$$

Obviously the component of \vec{v} parallel to \vec{x} , i.e. $\vec{v}_{\parallel} = \frac{(\vec{v}\cdot\vec{x})}{\|\vec{x}\|^2}\vec{x}$, does not contribute to the rotation. However the commutator of two rotations is given by

$$[\delta_v, \delta_w]\vec{x} = (\vec{w} \times \vec{v}) \times \vec{x} = \vec{v}(\vec{w} \cdot \vec{x}) - \vec{w}(\vec{v} \cdot \vec{x}). \quad (4.42)$$

Therefore the components parallel to \vec{x} contribute in the commutator although an individual rotation only depends on the component orthogonal to \vec{x} . This is completely analogous to the observation above that the traces of the spin-3 parameter contribute to the commutator.

This concludes the computation of the commutator of two spin-3 transformations at linear order in the spin-3 field. Together with the expression derived for the commutator of a spin-2 with either a spin-2 or spin-3 transformation, which are exact results, we have therefore determined the gauge algebra to leading order.

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A Conventions

We denote symmetrisation by a pair of parentheses,

$$A_{(\mu} B_{\nu)} = \frac{1}{2} (A_{\mu} B_{\nu} + A_{\nu} B_{\mu}) . \quad (\text{A.1})$$

Likewise square brackets denote antisymmetrisation. We often omit contracted indices of a tensor to simplify notation, for example

$$\phi_{\mu} \equiv \phi_{\mu\lambda}{}^{\lambda} . \quad (\text{A.2})$$

Furthermore we will use hats to denote the traceless projection of a contravariant rank 2 tensor, i.e.

$$\hat{\xi}^{\mu\nu} = \left(\delta_{\sigma}^{\mu} \delta_{\kappa}^{\nu} - \frac{1}{3} g^{\mu\nu} g_{\sigma\kappa} \right) \xi^{\sigma\kappa} . \quad (\text{A.3})$$

The algebra $sl(3, \mathbb{R})$ can be given in terms of generators J_a and T_{ab} with the commutation relations

$$[J_a, J_b] = \epsilon_{abc} J^c , \quad (\text{A.4a})$$

$$[J_a, T_{bc}] = 2 \epsilon^d{}_{a(b} T_{c)d} , \quad (\text{A.4b})$$

$$[T_{ab}, T_{cd}] = -2 \left(\eta_{a(c\epsilon d)be} + \eta_{b(c\epsilon d)ae} \right) J^e , \quad (\text{A.4c})$$

and $T_{[ab]} = \eta^{ab} T_{ab} = 0$. Here the Levi-Civita symbol is given by

$$\epsilon^{012} = -\epsilon_{012} = 1 , \quad (\text{A.5})$$

and indices can be raised and lowered by $\eta_{ab} = \text{diag}(-1, 1, 1)$. A 3×3 matrix representation for the T_{ab} is given by

$$T_{ab} = \left(J_a J_b + J_b J_a - \frac{2}{3} \eta_{ab} J_c J^c \right) , \quad (\text{A.6})$$

where J_a is in the three-dimensional representation of $sl(2, \mathbb{R}) \hookrightarrow sl(3, \mathbb{R})$. Furthermore $\{J_A\}$ denote a set of five independent generators built from the matrix representation T_{ab} . We use the notation $\{J_A\}$ for the set of all generators $\{J_a, J_A\}$.

The Killing form is defined to be one half of the matrix trace in the fundamental representation of $sl(3, \mathbb{R})$,

$$\kappa_{AB} = \frac{1}{2} \text{tr} (J_A J_B) , \quad (\text{A.7})$$

therefore $\kappa_{ab} = \eta_{ab}$ and $\kappa_{aB} = 0$. The anti-symmetric and symmetric structure constants are given by

$$f_{ABC} = \frac{1}{2} \text{tr}([J_A, J_B]J_C) , \quad (\text{A.8})$$

$$d_{ABC} = \frac{1}{2} \text{tr}(\{J_A, J_B\}J_C) , \quad (\text{A.9})$$

such that $f_{Abc} = f_{ABC} = 0$, $f_{abc} = \epsilon_{abc}$ and $d_{abc} = d_{ABc} = 0$. The structure constants satisfy a number of identities of which we used

$$d_{Abc} \kappa^{bc} = 0 , \quad (\text{A.10a})$$

$$d_{Abc} d^A{}_{de} = -\frac{2}{3} \kappa_{bc} \kappa_{de} + 2 \kappa_{d(b} \kappa_{c)e} . \quad (\text{A.10b})$$

B Bianchi-like identity

For the curvature of the spin connection we have the following Bianchi-like identity,

$$f^A{}_{BC} R^B{}_{[\mu\nu]} e^C{}_{\rho]} = 0 . \quad (\text{B.1})$$

For convenience we display its proof here. We evaluate

$$\begin{aligned} f^A{}_{BC} R^B{}_{[\mu\nu]} e^C{}_{\rho]} &= f^A{}_{BC} \left(\partial_{[\mu} \omega_{\nu]}^B e^C{}_{\rho]} + \frac{1}{2} f^B{}_{\mathcal{EF}} \omega_{[\mu}^{\mathcal{E}} \omega_{\nu]}^{\mathcal{F}} e^C{}_{\rho]} \right) \\ &= f^A{}_{BC} \partial_{[\mu} \omega_{\nu]}^B e^C{}_{\rho]} - \frac{1}{2} (f^A{}_{\mathcal{BE}} f^B{}_{\mathcal{FC}} + f^A{}_{\mathcal{BF}} f^B{}_{\mathcal{CE}}) \omega_{[\mu}^{\mathcal{E}} \omega_{\nu]}^{\mathcal{F}} e^C{}_{\rho]} \\ &= f^A{}_{BC} \partial_{[\mu} \omega_{\nu]}^B e^C{}_{\rho]} - \frac{1}{2} f^A{}_{\mathcal{BE}} \omega_{[\mu}^{\mathcal{E}} \partial_{\nu]} e^B{}_{\rho]} - \frac{1}{2} f^A{}_{\mathcal{BF}} \omega_{[\nu]}^{\mathcal{F}} \partial_{\mu]} e^B{}_{\rho]} \\ &= f^A{}_{BC} \partial_{[\mu} \omega_{\nu]}^B e^C{}_{\rho]} + f^A{}_{\mathcal{BE}} \partial_{[\mu} \omega_{\nu]}^{\mathcal{E}} e^B{}_{\rho]} \\ &= f^A{}_{BC} \partial_{[\mu} \left(\omega_{\nu]}^B e^C{}_{\rho]} \right) . \end{aligned} \quad (\text{B.2})$$

Here we have used (1.5) to obtain the third line. By using the torsion constraint (1.5) again we yield

$$f^A{}_{BC} \partial_{[\mu} \left(\omega_{\nu]}^B e^C{}_{\rho]} \right) = -\partial_{[\mu} \partial_{\nu]} e^A{}_{\rho]} = 0 , \quad (\text{B.3})$$

which concludes the proof of (B.1).

C DDI contributions to gauge algebra

In the following we will summarise the contributions to the parameter $u^{\alpha\beta}$ of the gauge algebra, given in (4.34), which vanish due to dimensional dependent identities. These might be helpful in comparing with our results.

First we give the term with a derivative acting on the spin-3 field.

$$\begin{aligned} D_1^{\alpha\beta} &= \frac{1}{2} \left((\hat{\xi}\hat{\pi})^{\alpha\beta\chi\delta} \nabla_{\delta} \phi_{\chi} - (\hat{\xi}\hat{\pi})^{\chi\delta}{}_{\chi}{}^{\epsilon} g^{\alpha\beta} \nabla_{\epsilon} \phi_{\delta} + (\hat{\xi}\hat{\pi})^{\chi\delta}{}_{\chi}{}^{\epsilon} \nabla_{\epsilon} \phi^{\alpha\beta}{}_{\delta} - (\hat{\xi}\hat{\pi})^{\alpha\beta\chi\delta} \nabla_{\epsilon} \phi_{\chi\delta}{}^{\epsilon} \right. \\ &\quad + 2(\hat{\xi}\hat{\pi})^{\chi\delta\epsilon\xi} g^{\alpha\beta} \nabla_{\xi} \phi_{\chi\delta\epsilon} - (\hat{\xi}\hat{\pi})^{(\alpha|\chi}{}_{\chi}{}^{\delta|\nabla\beta)} \phi_{\delta} + 2(\hat{\xi}\hat{\pi})^{(\alpha|\chi\delta\epsilon|\nabla\beta)} \phi_{\chi\delta\epsilon} + (\hat{\xi}\hat{\pi})^{(\alpha|\chi}{}_{\chi}{}^{\delta} \nabla^{\epsilon|} \phi^{\beta)}{}_{\delta\epsilon} \\ &\quad \left. + (\hat{\xi}\hat{\pi})^{(\alpha|\chi\delta\epsilon|\nabla_{\chi} \phi^{\beta)}{}_{\delta\epsilon} - 3(\hat{\xi}\hat{\pi})^{(\alpha|\chi\delta\epsilon|\nabla_{\delta} \phi^{\beta)}{}_{\chi\epsilon}} \right) . \end{aligned} \quad (\text{C.1})$$

Furthermore there are four more quantities with a derivative acting on the parameters.

$$\begin{aligned} D_2^{\alpha\beta} &= (\nabla_{\hat{\xi}\hat{\pi}})_{\chi}{}^{\delta\epsilon}{}_{\delta\epsilon} \phi^{\chi} g^{\alpha\beta} - (\nabla_{\hat{\xi}\hat{\pi}}) \chi^{\delta\epsilon}{}_{\delta\epsilon} \phi^{\alpha\beta}{}_{\chi} - (\nabla_{\hat{\xi}\hat{\pi}}) \chi^{\alpha\beta\delta\epsilon} \phi_{\chi\delta\epsilon} - (\nabla_{\hat{\xi}\hat{\pi}}) \chi^{\delta\epsilon\alpha\beta} \phi_{\chi\delta\epsilon} - 2(\nabla_{\hat{\xi}\hat{\pi}}) \chi^{\delta\epsilon}{}_{\delta}{}^{\xi} g^{\alpha\beta} \phi_{\chi\epsilon\xi} \\ &\quad - 2(\nabla_{\hat{\xi}\hat{\pi}}) \chi^{(\alpha|\delta|\beta)}{}_{\delta} \phi_{\chi} + 2(\nabla_{\hat{\xi}\hat{\pi}}) \chi^{(\alpha|\delta|\beta)\epsilon} \phi_{\chi\delta\epsilon} + 2(\nabla_{\hat{\xi}\hat{\pi}}) \chi^{(\alpha|\delta}{}_{\delta}{}^{\epsilon|} \phi^{\beta)}{}_{\chi\epsilon} + 2(\nabla_{\hat{\xi}\hat{\pi}}) \chi^{\delta\epsilon(\alpha} \phi^{\beta)}{}_{\chi\epsilon} \end{aligned} \quad (\text{C.2})$$

$$\begin{aligned}
D_3^{\alpha\beta} = & (\nabla\hat{\xi}\hat{\pi})^\delta \chi^\delta \alpha^\beta \phi^\chi + (\nabla\hat{\xi}\hat{\pi})^\delta \chi^\epsilon \delta_\epsilon \phi^\chi g^{\alpha\beta} - (\nabla\hat{\xi}\hat{\pi})^{\chi\delta\epsilon} \chi^\delta \phi^{\alpha\beta} \epsilon - (\nabla\hat{\xi}\hat{\pi})^{\chi\delta\epsilon\alpha\beta} \phi_{\chi\delta\epsilon} - (\nabla\hat{\xi}\hat{\pi})^{\chi\delta\epsilon} \delta^\xi g^{\alpha\beta} \phi_{\chi\epsilon\xi} \\
& + (\nabla\hat{\xi}\hat{\pi})^{\chi\delta\epsilon\xi} g^{\alpha\beta} \phi_{\delta\epsilon\xi} - (\nabla\hat{\xi}\hat{\pi})^{\chi\delta\epsilon} \chi^\xi g^{\alpha\beta} \phi_{\delta\epsilon\xi} - (\nabla\hat{\xi}\hat{\pi})^{(\alpha|\chi\delta|\beta)} \chi \phi_\delta - (\nabla\hat{\xi}\hat{\pi})^{\chi(\alpha|\delta|\beta)} \chi \phi_\delta \\
& - (\nabla\hat{\xi}\hat{\pi})^{(\alpha\beta)\chi\delta\epsilon} \phi_{\chi\delta\epsilon} + (\nabla\hat{\xi}\hat{\pi})^{(\alpha|\chi\delta|\beta)\epsilon} \phi_{\chi\delta\epsilon} + (\nabla\hat{\xi}\hat{\pi})^{(\alpha|\chi\delta\epsilon|\phi^\beta)} \delta_\epsilon + (\nabla\hat{\xi}\hat{\pi})^{\chi(\alpha|\delta|\beta)\epsilon} \phi_{\chi\delta\epsilon} \\
& + (\nabla\hat{\xi}\hat{\pi})^{\chi(\alpha|\delta\epsilon|\phi^\beta)} \delta_\epsilon - 2(\nabla\hat{\xi}\hat{\pi})^{\chi\delta(\alpha\epsilon|\phi^\beta)} \delta_\epsilon + (\nabla\hat{\xi}\hat{\pi})^{\chi\delta\epsilon(\alpha\chi\phi^\beta)} \delta_\epsilon + (\nabla\hat{\xi}\hat{\pi})^{\chi\delta\epsilon(\alpha\delta\phi^\beta)} \chi_\epsilon
\end{aligned} \tag{C.3}$$

$$\begin{aligned}
D_4^{\alpha\beta} = & -(\nabla\hat{\xi}\hat{\pi})^{\delta\alpha\beta} \chi_\delta \phi^\chi - (\nabla\hat{\xi}\hat{\pi})^\delta \delta_\epsilon \chi^\epsilon \phi^\chi g^{\alpha\beta} + (\nabla\hat{\xi}\hat{\pi})^{\chi\delta} \delta_\epsilon \phi^{\alpha\beta} \epsilon + (\nabla\hat{\xi}\hat{\pi})^{\chi\alpha\beta\delta\epsilon} \phi_{\chi\delta\epsilon} + (\nabla\hat{\xi}\hat{\pi})^{\chi\delta\epsilon} \delta^\xi g^{\alpha\beta} \phi_{\chi\epsilon\xi} \\
& + (\nabla\hat{\xi}\hat{\pi})^{\chi\delta\epsilon\xi} g^{\alpha\beta} \phi_{\delta\epsilon\xi} - (\nabla\hat{\xi}\hat{\pi})^{\chi\delta\epsilon} \chi^\xi g^{\alpha\beta} \phi_{\delta\epsilon\xi} + (\nabla\hat{\xi}\hat{\pi})^{(\alpha\beta)\chi\delta} \chi \phi_\delta + (\nabla\hat{\xi}\hat{\pi})^{\chi(\alpha\beta)\delta} \phi_\delta \\
& - (\nabla\hat{\xi}\hat{\pi})^{(\alpha\beta)\chi\delta\epsilon} \phi_{\chi\delta\epsilon} + (\nabla\hat{\xi}\hat{\pi})^{(\alpha|\chi\delta|\beta)\epsilon} \phi_{\chi\delta\epsilon} - (\nabla\hat{\xi}\hat{\pi})^{(\alpha|\chi\delta\epsilon|\phi^\beta)} \delta_\epsilon - (\nabla\hat{\xi}\hat{\pi})^{\chi(\alpha|\delta|\beta)\epsilon} \phi_{\chi\delta\epsilon} \\
& - (\nabla\hat{\xi}\hat{\pi})^{\chi(\alpha\delta\epsilon|\phi^\beta)} \delta_\epsilon + 2(\nabla\hat{\xi}\hat{\pi})^{\chi(\alpha|\delta\epsilon|\phi^\beta)} \delta_\epsilon - (\nabla\hat{\xi}\hat{\pi})^{\chi(\alpha|\delta\epsilon|\phi^\beta)} \chi_\epsilon - (\nabla\hat{\xi}\hat{\pi})^{\chi\delta(\alpha\epsilon|\phi^\beta)} \delta_\epsilon
\end{aligned} \tag{C.4}$$

$$\begin{aligned}
D_5^{\alpha\beta} = & -(\nabla\hat{\xi}\hat{\pi})^{\delta\alpha\beta} \chi_\delta \phi^\chi - (\nabla\hat{\xi}\hat{\pi})^\delta \chi_\delta \alpha^\beta \phi^\chi + (\nabla\hat{\xi}\hat{\pi})_\chi \delta_\epsilon \phi^\chi g^{\alpha\beta} - (\nabla\hat{\xi}\hat{\pi})^\delta \chi^\epsilon \delta_\epsilon \phi^\chi g^{\alpha\beta} - (\nabla\hat{\xi}\hat{\pi})^\delta \delta_\epsilon \chi^\epsilon \phi^\chi g^{\alpha\beta} \\
& + (\nabla\hat{\xi}\hat{\pi})^{(\alpha\beta)\chi\delta} \chi \phi_\delta + (\nabla\hat{\xi}\hat{\pi})^{(\alpha|\chi\delta|\beta)} \chi \phi_\delta - (\nabla\hat{\xi}\hat{\pi})^{(\alpha|\chi\delta|\phi^\beta)} \chi \phi_\delta + (\nabla\hat{\xi}\hat{\pi})^{\chi(\alpha\beta)\delta} \phi_\delta + (\nabla\hat{\xi}\hat{\pi})^{\chi(\alpha|\delta|\beta)} \chi \phi_\delta \\
& - 2(\nabla\hat{\xi}\hat{\pi})^{\chi(\alpha|\delta|\beta)} \delta \phi_\chi + (\nabla\hat{\xi}\hat{\pi})^{\chi(\alpha|\delta|\phi^\beta)} \chi \phi_\delta + (\nabla\hat{\xi}\hat{\pi})^{\chi\delta(\alpha\delta\phi^\beta)} .
\end{aligned} \tag{C.5}$$

The quantities $(\nabla\hat{\xi}\hat{\pi})$ and $(\hat{\xi}\hat{\pi})$ are defined in (4.38) and (4.36).

D Higher order corrections to spin-3 transformations

In this section we list the higher order corrections to the gauge transformations given in section 3.4. By making a particular choice of the undetermined constants due to dimensional dependent identities we reduced the size of the expressions considerably. For the gauge transformation of the spin-3 field there are corrections with a derivative acting on the ϕ field,

$$\begin{aligned}
(\hat{\xi}\phi\nabla\phi)_{\alpha\beta\chi} = & 18\left(2\phi^\delta \hat{\xi}^\delta \nabla_\epsilon \phi_{\alpha\beta\chi} + \hat{\xi}^{\delta\epsilon} (3\phi_{\delta\epsilon}^\xi \nabla_\xi \phi_{\alpha\beta\chi} + \phi_{\alpha\beta\chi} (7\nabla_\epsilon \phi_\delta - 3\nabla_\xi \phi_{\delta\epsilon}^\xi)) - 4\phi_{(\alpha\hat{\xi}\beta\chi)} \nabla^\delta \phi_\delta \right. \\
& + 8\phi_{(\alpha\hat{\xi}\beta}^\delta \nabla_\chi) \phi_\delta + 9\phi_{(\alpha\hat{\xi}\beta}^\delta \nabla_{|\delta|\phi_\chi)} - 5\phi^\delta \hat{\xi}_{(\alpha\beta}^\delta \nabla_{|\delta|\phi_\chi)} - 2\phi^\delta \hat{\xi}_{(\alpha|\delta|\phi_\chi)} \nabla_\beta \phi_\chi - 8\phi_{(\alpha\hat{\xi}\beta}^\delta \nabla^\epsilon \phi_{\chi\delta\epsilon}) \\
& - 2\phi_{(\alpha\hat{\xi}\beta}^\delta \nabla_{|\delta|\phi_\beta\chi)\epsilon} + \phi^\delta \hat{\xi}_{(\alpha|\delta|\phi_\beta\chi)\epsilon} + 13\phi^\delta \hat{\xi}_{(\alpha^\epsilon \nabla_{|\delta|\phi_\beta\chi)\epsilon} - 3\phi^\delta \hat{\xi}_{(\alpha^\epsilon \nabla_{|\epsilon|\phi_\beta\chi)\delta} + 12\hat{\xi}_{(\alpha\beta}\phi_\chi)^\delta \nabla_\delta \phi_\epsilon \\
& + 5\hat{\xi}_{(\alpha^\delta \phi_\beta\chi)\delta} \nabla^\epsilon \phi_\epsilon - 12\hat{\xi}_{(\alpha^\delta \phi_\beta\chi)\epsilon} \nabla_\delta \phi_\epsilon - 7\hat{\xi}_{(\alpha^\delta \phi_\beta\chi)\epsilon} \nabla_\epsilon \phi_\delta - 6\hat{\xi}_{(\alpha^\delta \phi_\beta|\delta|\phi_\chi)} \nabla_\chi \phi_\epsilon \\
& + 8\hat{\xi}_{(\alpha^\delta \phi_\beta|\delta} \nabla_{\epsilon|\phi_\chi)} - 3\hat{\xi}^{\delta\epsilon} \phi_{(\alpha\beta|\delta|\phi_\chi)\epsilon} - 4\hat{\xi}^{\delta\epsilon} \phi_{(\alpha\beta|\delta} \nabla_{\epsilon|\phi_\chi)} - 8\hat{\xi}_{(\alpha\beta}\phi_\chi)^\delta \nabla^\epsilon \phi_{\delta\epsilon\xi} + \hat{\xi}_{(\alpha\beta}\phi^{\delta\epsilon\xi} \nabla_\chi) \phi_{\delta\epsilon\xi} \\
& + 5\hat{\xi}_{(\alpha\beta}\phi^{\delta\epsilon\xi} \nabla_{|\delta|\phi_\chi)\epsilon\xi} + 11\hat{\xi}_{(\alpha^\delta \phi_\beta\chi)\epsilon} \nabla^\xi \phi_{\delta\epsilon\xi} + \hat{\xi}_{(\alpha^\delta \phi_\beta}^{\epsilon\xi} \nabla_\chi) \phi_{\delta\epsilon\xi} + 6\hat{\xi}_{(\alpha^\delta \phi_\beta|\delta|\phi_\chi)} \nabla^\xi \phi_{\chi\epsilon\xi} \\
& + 3\hat{\xi}_{(\alpha^\delta \phi_\beta}^{\epsilon\xi} \nabla_{|\delta|\phi_\chi)\epsilon\xi} - 18\hat{\xi}_{(\alpha^\delta \phi_\beta}^{\epsilon\xi} \nabla_{|\epsilon|\phi_\chi)\delta\xi} - \hat{\xi}_{(\alpha^\delta \phi_\beta|\delta|\phi_\chi)\epsilon\xi} \nabla_\beta \phi_{\chi\epsilon\xi} - 11\hat{\xi}_{(\alpha^\delta \phi_\beta|\delta} \nabla_{\epsilon\xi} \phi_{\beta\chi)\xi} \\
& - 9\hat{\xi}^{\delta\epsilon} \phi_{(\alpha\beta}^\xi \nabla_\chi) \phi_{\delta\epsilon\xi} + 3\hat{\xi}^{\delta\epsilon} \phi_{(\alpha\beta|\delta|\phi_\chi)\epsilon\xi} + 7\hat{\xi}^{\delta\epsilon} \phi_{(\alpha\beta}^\xi \nabla_{|\delta|\phi_\chi)\epsilon\xi} + 3\hat{\xi}^{\delta\epsilon} \phi_{(\alpha\beta}^\xi \nabla_{|\xi|\phi_\chi)\delta\epsilon} \\
& - 7\hat{\xi}^{\delta\epsilon} \phi_{(\alpha|\delta}^\xi \nabla_{\epsilon|\phi_\beta\chi)\xi} - 9\hat{\xi}^{\delta\epsilon} \phi_{(\alpha|\delta}^\xi \nabla_{\xi|\phi_\beta\chi)\epsilon} - 3\phi_{(\alpha\hat{\xi}\beta\chi)} \nabla_\delta \phi_\epsilon + 2\phi^\delta \hat{\xi}_{(\alpha|\delta|\phi_\beta\chi)} \nabla^\epsilon \phi_\epsilon \\
& - 2\phi^\delta \hat{\xi}_{(\alpha^\epsilon g_{\beta\chi})} \nabla_\delta \phi_\epsilon - 3\phi^\delta \hat{\xi}_{(\alpha^\epsilon g_{\beta\chi})} \nabla_\epsilon \phi_\delta - 3\phi^\delta \hat{\xi}_{\delta^\epsilon} g_{(\alpha\beta} \nabla_\chi) \phi_\epsilon - 3\phi^\delta \hat{\xi}_{\delta^\epsilon} g_{(\alpha\beta} \nabla_{|\epsilon|\phi_\chi)} \\
& + \phi_{(\alpha\hat{\xi}\beta}^\delta g_{\beta\chi}) \nabla^\xi \phi_{\delta\epsilon\xi} + 3\phi^\delta \hat{\xi}_{(\alpha^\epsilon g_{\beta\chi})} \nabla^\xi \phi_{\delta\epsilon\xi} + 3\phi^\delta \hat{\xi}_{\delta^\epsilon} g_{(\alpha\beta} \nabla^\xi \phi_{\chi\epsilon\xi} - 7\phi^\delta \hat{\xi}_{\delta^\epsilon} g_{(\alpha\beta} \nabla_{|\delta|\phi_\chi)\epsilon\xi} \\
& + 3\phi^\delta \hat{\xi}_{\delta^\epsilon} g_{(\alpha\beta} \nabla_{|\epsilon|\phi_\chi)\delta\xi} - 3\hat{\xi}_{(\alpha^\delta g_{\beta\chi})} \phi_{\delta\epsilon\xi} \nabla_\epsilon \phi_\xi - 2\hat{\xi}^{\delta\epsilon} g_{(\alpha\beta}\phi_\chi)_{\delta\epsilon} \nabla^\xi \phi_\xi + 9\hat{\xi}^{\delta\epsilon} g_{(\alpha\beta}\phi_\chi)_{\delta^\xi} \nabla_\epsilon \phi_\xi \\
& + 3\hat{\xi}^{\delta\epsilon} g_{(\alpha\beta}\phi_\chi)_{\delta^\xi} \nabla_\xi \phi_\epsilon - 4\hat{\xi}^{\delta\epsilon} g_{(\alpha\beta}\phi_{|\delta\epsilon}^\xi \nabla_{\xi|\phi_\chi)} + \hat{\xi}_{(\alpha^\delta g_{\beta\chi})} \phi_{\delta\epsilon\xi} \nabla^\gamma \phi_{\epsilon\xi\gamma} + \hat{\xi}_{(\alpha^\delta g_{\beta\chi})} \phi_{\delta\epsilon\xi\gamma} \nabla_\delta \phi_{\epsilon\xi\gamma} \\
& - \hat{\xi}_{(\alpha^\delta g_{\beta\chi})} \phi_{\delta\epsilon\xi\gamma} \nabla_\epsilon \phi_{\delta\xi\gamma} - 8\hat{\xi}^{\delta\epsilon} g_{(\alpha\beta}\phi_\chi)_{\delta^\xi} \nabla^\gamma \phi_{\epsilon\xi\gamma} - 6\hat{\xi}^{\delta\epsilon} g_{(\alpha\beta}\phi_\chi)_{\delta^\xi\gamma} \nabla_\delta \phi_{\epsilon\xi\gamma} + 6\hat{\xi}^{\delta\epsilon} g_{(\alpha\beta}\phi_\chi)_{\delta^\xi\gamma} \nabla_\xi \phi_{\delta\epsilon\gamma} \\
& \left. + 6\hat{\xi}^{\delta\epsilon} g_{(\alpha\beta}\phi_{|\delta|\phi_\chi)\epsilon\xi\gamma} + 8\hat{\xi}^{\delta\epsilon} g_{(\alpha\beta}\phi_{|\delta}^{\xi\gamma} \nabla_{\xi|\phi_\chi)\epsilon\gamma} \right) .
\end{aligned} \tag{D.1}$$

Then there are contributions with a derivative acting on the parameter.

$$\begin{aligned}
(\nabla \hat{\xi} \phi \phi)_{\alpha\beta\chi} = & -9 \left(12\phi^\delta \phi_{\alpha\beta\chi} \nabla_\epsilon \hat{\xi} \delta^\epsilon + 4\phi_{\alpha\beta\chi} \phi^{\delta\epsilon\xi} \nabla_\xi \hat{\xi} \delta_\epsilon + 14\phi_{(\alpha\phi\beta} \nabla^\delta \hat{\xi}_{\chi)\delta} + 6\phi_{(\alpha\phi^\delta} \nabla_\beta \hat{\xi}_{\chi)\delta} \right. \\
& + \phi^\delta \phi_\delta \nabla_{(\alpha} \hat{\xi}_{\beta\chi)} - 16\phi_{(\alpha\phi\beta\chi)}^\delta \nabla^\epsilon \hat{\xi} \delta_\epsilon + 16\phi_{(\alpha\phi\beta} \delta^\epsilon \nabla_\chi) \hat{\xi} \delta_\epsilon - 8\phi_{(\alpha\phi\beta} \delta^\epsilon \nabla_{|\delta|} \hat{\xi}_{\chi)\epsilon} - 12\phi^\delta \phi_{(\alpha\beta} \delta^\epsilon \nabla_\chi) \hat{\xi} \delta_\epsilon \\
& - 8\phi^\delta \phi_{(\alpha\beta|\delta|} \nabla^\epsilon \hat{\xi}_{\chi)\epsilon} + 32\phi^\delta \phi_{(\alpha\beta} \delta^\epsilon \nabla_{|\delta|} \hat{\xi}_{\chi)\epsilon} - 12\phi^\delta \phi_{(\alpha\beta} \delta^\epsilon \nabla_{|\epsilon|} \hat{\xi}_{\chi)\delta} + 8\phi^\delta \phi_{(\alpha|\delta} \delta^\epsilon \nabla_{\epsilon|} \hat{\xi}_{\beta\chi)} \\
& + 4\phi_{(\alpha\beta} \delta^\epsilon \phi_\chi) \delta^\epsilon \nabla^\xi \hat{\xi}_{\epsilon\xi} - 4\phi_{(\alpha\beta} \delta^\epsilon \phi_\chi) \delta^\epsilon \nabla_\epsilon \hat{\xi} \delta_\xi + 14\phi_{(\alpha\beta} \delta^\epsilon \phi_{|\delta|} \delta^\epsilon \nabla_\chi) \hat{\xi}_{\epsilon\xi} - 12\phi_{(\alpha\beta} \delta^\epsilon \phi_{|\delta} \delta^\epsilon \nabla_{\epsilon|} \hat{\xi}_{\chi)\xi} \\
& - 6\phi_{(\alpha} \delta^\epsilon \phi_{\beta|\delta\epsilon|} \nabla^\xi \hat{\xi}_{\chi)\xi} + 8\phi_{(\alpha} \delta^\epsilon \phi_{|\delta\epsilon} \delta^\xi \nabla_\xi \hat{\xi}_{\beta\chi)} - 12\phi_{(\alpha\phi^\delta} g_{\beta\chi)} \nabla^\epsilon \hat{\xi} \delta_\epsilon + 3\phi^\delta \phi^\epsilon g_{(\alpha\beta} \nabla_\chi) \hat{\xi} \delta_\epsilon \\
& - \phi^\delta \phi_\delta g_{(\alpha\beta} \nabla^\epsilon \hat{\xi}_{\chi)\epsilon} - 4\phi_{(\alpha} g_{\beta\chi)} \phi^{\delta\epsilon\xi} \nabla_\delta \hat{\xi}_{\epsilon\xi} + 10\phi^\delta g_{(\alpha\beta} \phi_\chi) \delta^\epsilon \nabla^\xi \hat{\xi}_{\epsilon\xi} - 16\phi^\delta g_{(\alpha\beta} \phi_\chi) \delta^\epsilon \nabla_\delta \hat{\xi}_{\epsilon\xi} \\
& + 12\phi^\delta g_{(\alpha\beta} \phi_\chi) \delta^\epsilon \nabla_\epsilon \hat{\xi} \delta_\xi - 6\phi^\delta g_{(\alpha\beta} \phi_{|\delta|} \delta^\epsilon \nabla_\chi) \hat{\xi}_{\epsilon\xi} + 2\phi^\delta g_{(\alpha\beta} \phi_{|\delta} \delta^\epsilon \nabla_{\epsilon|} \hat{\xi}_{\chi)\xi} + 2g_{(\alpha\beta} \phi_\chi) \delta^\epsilon \phi_{\delta\epsilon} \delta^\xi \nabla^\gamma \hat{\xi}_{\xi\gamma} \\
& - 2g_{(\alpha\beta} \phi_\chi) \delta^\epsilon \phi_{\delta} \delta^{\xi\gamma} \nabla_\epsilon \hat{\xi}_{\xi\gamma} + 10g_{(\alpha\beta} \phi_\chi) \delta^\epsilon \phi_{\delta} \delta^{\xi\gamma} \nabla_\xi \hat{\xi}_{\epsilon\gamma} + g_{(\alpha\beta} \phi^{\delta\epsilon\xi} \phi_{|\delta\epsilon\xi|} \nabla^\gamma \hat{\xi}_{\chi)\gamma} \\
& \left. - 2g_{(\alpha\beta} \phi^{\delta\epsilon\xi} \phi_{|\delta\epsilon} \delta^\gamma \nabla_{\xi|} \hat{\xi}_{\chi)\gamma} - 16\phi_{(\alpha\phi^\delta} \nabla_{|\delta|} \hat{\xi}_{\beta\chi)} \right) . \tag{D.2}
\end{aligned}$$

Finally the transformation of the metric to cubic order is given by

$$\begin{aligned}
(\hat{\xi}\phi\phi\nabla\phi)_{\alpha\beta} = & 18(16\phi^X\hat{\xi}^{\delta\epsilon}\phi_{\alpha\beta\delta}\nabla_X\phi_\epsilon - 8\phi^X\hat{\xi}^{\delta\epsilon}g_{\alpha\beta}\phi_{\delta\epsilon}\xi\nabla_X\phi_\xi + 16\phi^X\hat{\xi}^{\delta\epsilon}\phi_{\delta\epsilon}\xi\nabla_X\phi_{\alpha\beta\xi} \\
& + 16\phi^X\hat{\xi}^{\delta\epsilon}g_{\alpha\beta}\phi_{\delta\epsilon}\xi^\gamma\nabla_X\phi_{\epsilon\xi\gamma} + 5\phi^X\phi^\delta\hat{\xi}_{\alpha\beta}\nabla_\delta\phi_\chi - 14\phi_\chi\phi^X\hat{\xi}_{\alpha\beta}\nabla_\delta\phi^\delta + 4\phi^X\phi^\delta\hat{\xi}_X^\epsilon g_{\alpha\beta}\nabla_\delta\phi_\epsilon \\
& + 12\hat{\xi}^{\chi\delta}\phi_{\alpha\beta}^\epsilon\phi_{\chi\epsilon}\xi\nabla_\delta\phi_\xi + 8\hat{\xi}^{\chi\delta}g_{\alpha\beta}\phi_{\chi\epsilon}\xi\phi_{\epsilon\xi}\gamma\nabla_\delta\phi_\gamma - 10\phi^X\phi^\delta\hat{\xi}^{\epsilon\xi}g_{\alpha\beta}\nabla_\delta\phi_{\chi\epsilon\xi} - 8\phi^X\hat{\xi}_X^\delta g_{\alpha\beta}\phi^{\epsilon\xi\gamma}\nabla_\delta\phi_{\epsilon\xi\gamma} \\
& + 8\hat{\xi}^{\chi\delta}g_{\alpha\beta}\phi_{\chi\epsilon}\xi\phi_\epsilon^{\gamma\eta}\nabla_\delta\phi_{\xi\gamma\eta} + 12\phi^X\phi^\delta\hat{\xi}_X^\epsilon g_{\alpha\beta}\nabla_\epsilon\phi_\delta - 20\phi_\chi\phi^X\hat{\xi}^{\delta\epsilon}g_{\alpha\beta}\nabla_\epsilon\phi_\delta + 14\phi^X\hat{\xi}_{\alpha\beta}\phi_{\chi\delta}\nabla_\epsilon\phi_\delta \\
& - \phi^X\phi^\delta\hat{\xi}_{\chi\delta}g_{\alpha\beta}\nabla_\epsilon\phi^\epsilon - 9\phi^X\phi^\delta\hat{\xi}_{\alpha\beta}\nabla_\epsilon\phi_{\chi\delta}^\epsilon + 20\phi^X\hat{\xi}^{\delta\epsilon}g_{\alpha\beta}\phi_{\chi\delta}\xi^\gamma\nabla_\epsilon\phi_{\delta\xi\gamma} + 8\hat{\xi}^{\chi\delta}\phi_{\alpha\beta}^\epsilon\phi_{\chi\epsilon}\xi\nabla_\xi\phi_\delta \\
& - 8\hat{\xi}^{\chi\delta}\phi_{\alpha\beta}^\epsilon\phi_{\chi\delta}\xi\nabla_\xi\phi_\epsilon + 18\hat{\xi}_{\alpha\beta}\phi_{\chi\delta}\xi\phi_{\chi\delta\epsilon}\nabla_\xi\phi_\epsilon - 10\phi^X\hat{\xi}^{\delta\epsilon}g_{\alpha\beta}\phi_{\chi\delta\epsilon}\nabla_\xi\phi_\xi + 4\hat{\xi}_{\alpha\beta}\phi_{\chi\delta\epsilon}\phi^{\chi\delta\epsilon}\nabla_\xi\phi_\xi \\
& - 8\phi^X\hat{\xi}_X^\delta\phi_{\delta\epsilon}\xi\nabla_\xi\phi_{\alpha\beta\epsilon} - 12\phi^X\phi^\delta\hat{\xi}_X^\epsilon g_{\alpha\beta}\nabla_\xi\phi_{\delta\epsilon}^\xi + 10\phi_\chi\phi^X\hat{\xi}^{\delta\epsilon}g_{\alpha\beta}\nabla_\xi\phi_{\delta\epsilon}^\xi + 18\phi^X\hat{\xi}_{\alpha\beta}\phi_{\chi\delta}\nabla_\xi\phi_{\delta\epsilon}^\xi \\
& + 16\hat{\xi}^{\chi\delta}g_{\alpha\beta}\phi_{\chi\epsilon}\xi\phi_{\epsilon\xi}\gamma\nabla_\gamma\phi_\delta + 20\hat{\xi}^{\chi\delta}g_{\alpha\beta}\phi_{\chi\epsilon}\phi_{\delta\epsilon}\gamma\nabla_\gamma\phi_\xi + 6\hat{\xi}^{\chi\delta}g_{\alpha\beta}\phi_{\chi\epsilon}\phi_{\delta\epsilon\xi}\nabla_\gamma\phi_\gamma \\
& + 4\hat{\xi}^{\chi\delta}\phi_{\chi\epsilon}\xi\phi_{\delta\epsilon}\gamma\nabla_\gamma\phi_{\alpha\beta\xi} - 24\hat{\xi}^{\chi\delta}\phi_{\chi\delta}\phi_\epsilon^{\xi\gamma}\nabla_\gamma\phi_{\alpha\beta\xi} - 8\hat{\xi}^{\chi\delta}\phi_{\alpha\beta}^\epsilon\phi_\epsilon^{\xi\gamma}\nabla_\gamma\phi_{\chi\delta\xi} + 16\phi^X\hat{\xi}^{\delta\epsilon}g_{\alpha\beta}\phi_{\chi\delta}\xi^\gamma\nabla_\gamma\phi_{\delta\epsilon\xi} \\
& + 24\hat{\xi}^{\chi\delta}\phi_{\alpha\beta}^\epsilon\phi_{\chi\delta}\xi^\gamma\nabla_\gamma\phi_{\delta\epsilon\xi} - 12\hat{\xi}_{\alpha\beta}\phi_{\chi\delta}\xi^\gamma\phi_{\chi\delta\epsilon}\nabla_\gamma\phi_{\delta\epsilon\xi} - 16\hat{\xi}^{\chi\delta}\phi_{\alpha\beta\chi}\phi^{\epsilon\xi\gamma}\nabla_\gamma\phi_{\delta\epsilon\xi} - 12\hat{\xi}^{\chi\delta}\phi_{\alpha\beta}^\epsilon\phi_{\chi\epsilon}\xi\nabla_\gamma\phi_{\delta\xi}^\gamma \\
& - 14\hat{\xi}_{\alpha\beta}\phi_{\chi\delta}\xi\phi_{\chi\delta\epsilon}\nabla_\gamma\phi_{\epsilon\xi}^\gamma + 8\phi^X\hat{\xi}_X^\delta g_{\alpha\beta}\phi_{\delta\epsilon}\xi^\gamma\nabla_\gamma\phi_{\epsilon\xi}^\gamma - 4\hat{\xi}^{\chi\delta}g_{\alpha\beta}\phi_{\epsilon\xi}^\gamma\nabla_\gamma\phi_{\chi\delta\gamma} \\
& - 40\hat{\xi}^{\chi\delta}g_{\alpha\beta}\phi_{\chi\epsilon}\xi\phi_\epsilon^{\gamma\eta}\nabla_\eta\phi_{\delta\xi\gamma} - 8\hat{\xi}^{\chi\delta}g_{\alpha\beta}\phi_{\chi\epsilon}\xi\phi_{\epsilon\xi}\gamma\nabla_\eta\phi_{\delta\gamma}^\eta - 16\hat{\xi}^{\chi\delta}g_{\alpha\beta}\phi_{\chi\epsilon}\phi_{\delta\epsilon}\gamma\nabla_\eta\phi_{\xi\gamma}^\eta \\
& + 8\hat{\xi}^{\chi\delta}g_{\alpha\beta}\phi_{\chi\delta}\phi_\epsilon^{\xi\gamma}\nabla_\eta\phi_{\xi\gamma}^\eta + 2\phi_{(\alpha}\phi^X\hat{\xi}_{\beta)\chi}\nabla^\delta\phi_\delta - 20\phi_{(\alpha}\phi^X\hat{\xi}_{\beta)}^\delta\nabla_\chi\phi_\delta - 12\phi_{(\alpha}\phi^X\hat{\xi}_{\beta)}^\delta\nabla_\delta\phi_\chi \\
& + 20\phi^X\phi_{\chi(\alpha}\hat{\xi}_{\beta)}^\delta\nabla_\delta\phi_\delta - 12\phi^X\phi^\delta\hat{\xi}_{(\alpha|\chi|}\nabla_{\beta)}\phi_\delta + 20\phi^X\phi_{\chi(\alpha}\hat{\xi}_{\beta)}^\delta\nabla_{|\delta|}\phi_{\beta)} - 4\phi^X\phi^\delta\hat{\xi}_{(\alpha|\chi}\nabla_{\delta|}\phi_{\beta)} \\
& + 12\phi_{(\alpha}\phi^X\hat{\xi}_{\beta)}^\delta\nabla^\epsilon\phi_{\chi\delta\epsilon} + 16\phi_{(\alpha}\phi^X\hat{\xi}^{\delta\epsilon}\nabla_{|\chi|}\phi_{\beta)\delta\epsilon} - 20\phi^X\phi_{\chi(\alpha}\hat{\xi}_{\beta)}^\delta\nabla^\epsilon\phi_{\beta)\delta\epsilon} + 12\phi^X\phi^\delta\hat{\xi}_{(\alpha|\chi|}\nabla^\epsilon\phi_{\beta)\delta\epsilon} \\
& + 20\phi^X\phi^\delta\hat{\xi}_{(\alpha}\nabla_{|\chi|}\phi_{\beta)\delta\epsilon} + 16\phi_{(\alpha}\hat{\xi}^{\chi\delta}\phi_{\beta)\chi}\nabla^\epsilon\phi_\delta + 20\phi^X\hat{\xi}_{(\alpha}\phi_{\beta)\chi}\nabla^\epsilon\phi_\epsilon + 8\phi^X\hat{\xi}_{(\alpha}\phi_{\beta)\delta}\nabla_\chi\phi_\epsilon \\
& - 8\phi_{(\alpha}\hat{\xi}_{\beta)}^\chi\phi_{\chi\delta}\nabla^\epsilon\phi_{\delta\epsilon\xi} + 8\phi_{(\alpha}\hat{\xi}_{\beta)}^\chi\phi_{\delta\epsilon\xi}\nabla_\chi\phi_{\delta\epsilon\xi} + 16\phi_{(\alpha}\hat{\xi}_{\beta)}^\chi\phi_{\delta\epsilon\xi}\nabla_\delta\phi_{\chi\epsilon\xi} - 24\phi_{(\alpha}\hat{\xi}^{\chi\delta}\phi_{\beta)}^\epsilon\nabla_\epsilon\phi_{\chi\delta\xi} \\
& - 8\phi_{(\alpha}\hat{\xi}^{\chi\delta}\phi_{|\chi}\nabla_{\epsilon|\phi_{\beta)\delta\xi} - 8\phi^X\hat{\xi}_{(\alpha|\chi|}\phi_{\beta)}^\delta\nabla^\epsilon\phi_{\delta\epsilon\xi} + 8\phi^X\hat{\xi}_{(\alpha}\phi_{\beta)}^\delta\nabla_\chi\phi_{\delta\epsilon\xi} + 8\phi^X\hat{\xi}_{(\alpha|\chi|}\phi_{\beta)}^{\delta\epsilon\xi}\nabla_{\beta)}\phi_{\delta\epsilon\xi} \\
& - 20\phi^X\hat{\xi}_{(\alpha}\phi_{|\chi|}\nabla_{\beta)}\phi_{\delta\epsilon\xi} - 20\phi^X\hat{\xi}_{(\alpha}\phi_{|\chi}\nabla_{\beta)}\phi_{\delta\epsilon\xi} - 32\phi^X\hat{\xi}_{(\alpha}\phi_{|\chi}\nabla_{\beta)}\phi_{\delta\epsilon\xi} - 16\phi^X\hat{\xi}_{(\alpha}\phi_{|\chi}\nabla_{\beta)}\phi_{\delta\epsilon\xi} \\
& + 8\phi^X\hat{\xi}_X^\delta\phi_{(\alpha}\nabla_{|\epsilon|\phi_{\beta)\delta\xi} - 40\phi^X\hat{\xi}^{\delta\epsilon}\phi_{(\alpha|\delta}\nabla_{\chi|}\phi_{\beta)\epsilon\xi} - 12\hat{\xi}_{(\alpha}^X\phi_{\beta)}^\delta\phi_{\chi\delta\epsilon}\nabla^\xi\phi_\xi - 20\hat{\xi}_{(\alpha}^X\phi_{\beta)}^\delta\phi_{\chi\delta}\nabla^\epsilon\phi_\xi \\
& - 12\hat{\xi}_{(\alpha}^X\phi_{\beta)}^\delta\phi_{\chi\delta}\nabla_\xi\phi_\epsilon - 8\hat{\xi}_{(\alpha}^X\phi_{\beta)}^\delta\phi_{\delta\epsilon}\xi\nabla_\chi\phi_\xi - 8\hat{\xi}_{(\alpha}^X\phi_{|\chi}\nabla_{\delta\epsilon|\phi_{\beta)}^\xi\nabla_{\beta)}\phi_\xi \\
& - 16\hat{\xi}_{(\alpha}^X\phi_{|\chi}\nabla_{\delta\epsilon|\phi_{\beta)}^\xi\nabla_{\xi|\phi_{\beta)} + 8\hat{\xi}^{\chi\delta}\phi_{(\alpha|\chi|}\nabla_{\beta)}\phi_\delta\nabla_\epsilon\phi_\xi - 12\hat{\xi}^{\chi\delta}\phi_{(\alpha|\chi|}\nabla_{\beta)}\phi_\epsilon\nabla_\delta\phi_\xi \\
& - 40\hat{\xi}^{\chi\delta}\phi_{(\alpha|\chi|}\nabla_{\beta)}\phi_\epsilon\nabla_\xi\phi_\delta - 12\hat{\xi}^{\chi\delta}\phi_{(\alpha|\chi}\nabla_{\delta\epsilon|\phi_{\beta)}^\xi\nabla_{\beta)}\phi_\xi + 12\hat{\xi}^{\chi\delta}\phi_{(\alpha}\nabla_{|\chi\delta\epsilon|\phi_{\beta)}^\xi\nabla_{\beta)}\phi_\xi - 24\hat{\xi}^{\chi\delta}\phi_{(\alpha|\chi}\nabla_{\delta\epsilon|\phi_{\beta)}^\xi\nabla_{\xi|\phi_{\beta)} \\
& + 24\hat{\xi}^{\chi\delta}\phi_{(\alpha}\nabla_{|\chi\delta\epsilon|\phi_{\beta)}^\xi\nabla_{\xi|\phi_{\beta)} - 16\hat{\xi}_{(\alpha}^X\phi_{\beta)\chi}\nabla_{\delta\epsilon}\phi_{\epsilon\xi\gamma} + 8\hat{\xi}_{(\alpha}^X\phi_{\beta)\chi}\nabla_{\delta\epsilon}\phi_{\epsilon\xi\gamma} + 32\hat{\xi}_{(\alpha}^X\phi_{\beta)}^\delta\phi_{\chi\delta}\nabla^\gamma\phi_{\epsilon\xi\gamma} \\
& - 8\hat{\xi}_{(\alpha}^X\phi_{\beta)}^\delta\phi_{\chi\delta}\xi^\gamma\nabla_\xi\phi_{\delta\epsilon\gamma} + 8\hat{\xi}_{(\alpha}^X\phi_{\beta)}^\delta\phi_{\delta\epsilon}\xi\nabla_\gamma\phi_{\chi\xi\gamma} - 8\hat{\xi}_{(\alpha}^X\phi_{\beta)}^\delta\phi_{\delta\epsilon}\xi^\gamma\nabla_\chi\phi_{\epsilon\xi\gamma} \\
& - 8\hat{\xi}_{(\alpha}^X\phi_{|\chi}\nabla_{\delta\epsilon|\phi_{\beta)}^\xi\nabla_{\beta)}\phi_{\epsilon\xi\gamma} + 8\hat{\xi}_{(\alpha}^X\phi_{|\chi}\nabla_{\delta\epsilon|\phi_{\beta)}^\xi\nabla_{\beta)}\phi_{\xi\gamma} + 40\hat{\xi}_{(\alpha}^X\phi_{|\chi}\nabla_{\delta\epsilon|\phi_{\beta)}^\xi\nabla_{\xi|\phi_{\beta)}\epsilon\gamma \\
& + 8\hat{\xi}_{(\alpha}^X\phi_{\delta\epsilon|\phi_{\beta)}^\xi\nabla_{\xi|\phi_{\beta)\chi\gamma} + 12\hat{\xi}^{\chi\delta}\phi_{(\alpha|\chi|}\nabla_{\beta)}\phi_\delta\nabla_\xi\phi_\xi - 24\hat{\xi}^{\chi\delta}\phi_{(\alpha|\chi|}\nabla_{\beta)}\phi_\delta\nabla_\xi\phi_\xi \\
& + 16\hat{\xi}^{\chi\delta}\phi_{(\alpha|\chi|}\nabla_{\beta)}\phi_\delta\nabla_\xi\phi_\xi + 28\hat{\xi}^{\chi\delta}\phi_{(\alpha}\nabla_{\xi|\phi_{\beta)\delta\gamma} + 12\hat{\xi}^{\chi\delta}\phi_{(\alpha|\chi}\nabla_{\delta\epsilon|\phi_{\beta)}^\xi\nabla_{\beta)}\phi_{\xi\gamma} \\
& + 64\hat{\xi}^{\chi\delta}\phi_{(\alpha|\chi}\nabla_{\delta\epsilon|\phi_{\beta)}^\xi\nabla_{\xi|\phi_{\beta)\delta\gamma} - 12\hat{\xi}^{\chi\delta}\phi_{(\alpha}\nabla_{|\chi\delta\epsilon|\phi_{\beta)}^\xi\nabla_{\beta)}\phi_{\xi\gamma} - 8\hat{\xi}^{\chi\delta}\phi_{(\alpha}\nabla_{|\chi\delta}\nabla_{\epsilon|\phi_{\beta)\xi\gamma} \\
& + 8\hat{\xi}^{\chi\delta}\phi_{(\alpha}\nabla_{|\chi\delta}\nabla_{\gamma|\phi_{\beta)\epsilon\xi} - 4\hat{\xi}^{\chi\delta}\phi_{(\alpha}\nabla_{|\chi\epsilon}\nabla_{\xi|\phi_{\beta)\delta\gamma} + 4\hat{\xi}^{\chi\delta}\phi_{(\alpha}\nabla_{|\chi\epsilon}\nabla_{\gamma|\phi_{\beta)\delta\xi} \\
& - 16\hat{\xi}^{\chi\delta}\phi_{(\alpha}\nabla_{|\epsilon\xi}\nabla_{\gamma|\phi_{\beta)\chi\delta} - 8\phi^X\hat{\xi}^{\delta\epsilon}\phi_{\alpha\beta}\nabla_X\phi_{\delta\epsilon\xi}) . \tag{D.3}
\end{aligned}$$

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