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Third Post-Newtonian Spin-orbit Effect in the Gravitational Radiation Flux of Compact Binaries

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Abstract. Gravitational waves contain tail effects that are due to the backscattering of linear waves in the curved space-time geometry around the source. The knowledge as well as the accuracy of the two-body inspiraling post-Newtonian (PN) dynamics and of the gravitational-wave signal has been recently improved, notably by computing the spin-orbit (SO) terms induced by tail effects in the gravitational-wave energy flux at the 3PN order. Here we sketch this derivation, which yields the phasing formula including SO tail effects through the same 3PN order. Those results can be employed to improve the accuracy of analytical templates aimed at describing the whole process of inspiral, merger, and ringdown.

1. Introduction

By 2016 – 2018, the ground-based gravitational-wave detectors Virgo and LIGO will be upgraded to such a sensitivity that event rates for coalescing binary systems will increase by approximately a factor one thousand, making likely the first detection of gravitational waves from those systems. The search for gravitational waves from coalescing binary systems and the extraction of source parameters require a rather accurate knowledge of the waveform of the incoming signal. The post-Newtonian (PN) expansion is the most powerful approximation scheme in analytical relativity capable of describing the two-body dynamics and the gravitational-wave emission of inspiraling compact binary systems (Blanchet 2006).

The presence of spin effects adds substantial complexity to the gravitational waveforms, making it indispensable to include them into search templates.

We have recently improved (Blanchet et al. 2011) the knowledge and accuracy of the two-body inspiraling dynamics and gravitational-wave signal by computing the spin-orbit (SO) terms induced by tail contributions due to the back-scattering of linear waves in the curved space-time geometry around the source. Here we shall summarize this work, which is the continuation of Faye et al. (2006) and Blanchet et al. (2006) where we obtained the 2.5PN SO contributions in the equations of motion and gravitational-wave energy flux.

After briefly reviewing the post-Newtonian multipole moment formalism and discussing relevant properties of tails, we describe how spin effects are included to our scheme and derive the binary's evolution equations when black holes carry spins. Next, we investigate the time evolution of the moving triad and solve the precessing dynamics at the relevant PN order. We then compute the tails, which depend on the recent past history of the source, restricting ourselves to quasi-circular adiabatic inspiral. At last, we derive the 3PN SO tail effects in the energy flux and in the gravitational phasing.

2. Wave generation formalism

The gravitational waveform $h^{\rm TT}_{ij}$, generated by an isolated source described by a stress-energy tensor $T^{\mu\nu}$ with compact support, and propagating in the asymptotic regions of the source, is the transverse trace-free (TT) projection of the metric deviation at the leading-order $1/R$ in the distance to the source. It is parametrized by symmetric trace-free (STF) mass-type moments U_L and current-type ones V_L, referred to as radiative moments, which constitute observable quantities at infinity from the source. The general expression of the TT waveform, in a suitable radiative coordinate system $X^\mu = (c\,T, \mathbf{X})$, reads (Thorne 1980), when neglecting terms $O(1/R^2)$ with $R = |\mathbf{X}|$

$$h^{\rm TT}_{ij} = \frac{4G}{c^2R}\,\mathcal{P}^{\rm TT}_{ijkl}\sum_{\ell=2}^{+\infty}\frac{N_{L-2}}{c^\ell \ell!}\Big[U_{klL-2} - \frac{2\ell}{c(\ell+1)}\,N_m\,\varepsilon_{mn(k}\,V_{l)nL-2}\Big]. \qquad (1)$$

The moments U_L and V_L are functions of the retarded time $T_R \equiv T - R/c$. The integer ℓ refers to the multipolar order and $\mathbf{N} = \mathbf{X}/R$ is the unit vector pointing from the source to the far away detector. The tensor $\mathcal{P}^{\rm TT}_{ijkl}$ denotes the transverse-traceless (TT) projector $\mathcal{P}_{i(k}\mathcal{P}_{l)j} - \mathcal{P}_{ij}\mathcal{P}_{kl}/2$, where $\mathcal{P}_{ij} = \delta_{ij} - N_iN_j$ is the projector orthogonal to $\mathbf{N}$. The quantity ε_{ijk} is the Levi-Civita symbol such that $\varepsilon_{123} = 1$. Round parentheses indicate symmetrization. After plugging Eq. (1) into the standard expression for the gravitational-wave energy flux, we get

$$\mathcal{F} = \sum_{\ell=2}^{+\infty}\frac{G}{c^{2\ell+1}}\,\frac{(\ell+1)(\ell+2)}{(\ell-1)\ell\,\ell!(2\ell+1)!!}\Big[U^{(1)}_L U^{(1)}_L + \frac{4\ell^2}{c^2(\ell+1)^2}V^{(1)}_L V^{(1)}_L\Big], \qquad (2)$$

where the superscript (1) stands for the first time derivative (Thorne 1980).

In the multipolar-post-Minkowskian formalism, developed by Blanchet & Damour (1986, 1988, 1992), the radiative moments are linked to six sets of multipole moments characterizing the source, collectively called the "source" moments and denoted I_L, J_L, W_L, X_L, Y_L, Z_L. The relation between U_L, V_L and the former quantities encode all the non-linearities in the wave propagation between the source and the detector. After being re-expanded in a PN way, they are seen to contain the contribution of the so-called gravitational-wave tails, due to backscattering of linear waves onto the space-time curvature associated with the total mass of the source itself. The expression for $U_L(T_R)$ at the 1.5PN order, where the tail effects first appear, is

$$U_L(T_R) = I^{(\ell)}_L(T_R) + \frac{2GM}{c^3}\int_{-\infty}^{T_R}\mathrm{d}t\,I^{(\ell+2)}_L(t)\ln\Big(\frac{T_R-t}{2\tau'_0}\Big) + O\Big(\frac{1}{c^3}\Big)_{\rm non-tail}. \qquad (3)$$

Here τ'_0 is a freely specifiable time scale (Blanchet 1995). A similar equation links $V_L(T_R)$ to J_L. For the present application, W_L, X_L, Y_L and Z_L, associated to a possible gauge transformation performed at linear order, play no role. It will be sufficient to consider I_L at 1.5PN and J_L at 0.5PN order. The 1.5PN mass moments are given by an integral extending over the mass density $\sigma \equiv (T^{00}+T^{ii})/c^2$ and the current density $\sigma_i \equiv T^{0i}/c$ of the matter source (Blanchet et al. 2006):

$$I_L = \int \mathrm{d}^3\mathbf{x}\Big[\hat{x}_L\sigma + \frac{1}{2c^2(2\ell+3)}\,\hat{x}_L|\mathbf{x}|^2\,\sigma^{(2)} - \frac{4(2\ell+1)}{c^2(\ell+1)(2\ell+3)}\,\hat{x}_{iL}\,\sigma^{(1)}_i\Big] + O\Big(\frac{1}{c^4}\Big). \qquad (4)$$

The 0.5PN part of J_L is obtained by substituting $\varepsilon_{ab\langle i_\ell}x_{L-1\rangle a}\,\sigma_b$ to $\hat{x}_L\,\sigma$ in the first term.

To control the past behavior of the tail integral in Eq. (3), we assume that, at early times, the source was formed from a bunch of freely falling particles moving on some hyperbolic-like orbits, and forming at a later time a gravitationally bound system by emission of gravitational radiation. This ensures that the integral $\mathcal{U}_L(T_R) \equiv \int_{-\infty}^{T_R}\mathrm{d}t\,I^{(\ell+2)}_L(t)\,\ln[(T_R-t)/2\tau'_0]$ is convergent. To compute it, it is convenient to perform the Fourier decomposition $I_L(t) = \int_{-\infty}^{+\infty}\mathrm{d}\Omega/(2\pi)\,\widetilde{I}_L(\Omega)\,e^{-\mathrm{i}\Omega t}$, and commute the tail integral with the Fourier one. We obtain a closed-form expression by resorting to standard mathematical formulae for the integrals of $e^{-i\Omega t}\ln t$. The result reads (with $\mathrm{s}(\Omega) \equiv \frac{\Omega}{|\Omega|}$ and γ_E being the Euler constant)

$$\mathcal{U}_L(T_R) = \mathrm{i}\int_{-\infty}^{+\infty} \frac{\mathrm{d}\Omega}{2\pi}\,(-\mathrm{i}\Omega)^{\ell+1}\,\widetilde{I}_L(\Omega)\,e^{-\mathrm{i}\,\Omega\,T_R}\left[\frac{\pi}{2}\mathrm{s}(\Omega) + \mathrm{i}\Big(\ln(2|\Omega|\tau_0') + \gamma_{\mathrm{E}}\Big)\right]. \tag{5}$$

3. Applications to spinning binaries

3.1. Spin vectors for point-like objects

Following our previous works (Faye et al. 2006; Blanchet et al. 2006), we base our calculations on the model of point-particles with spins (Mathisson 1937; Papapetrou 1951; Tulczyjew 1959; Bailey & Israel 1980). The stress-energy tensor $T^{\mu\nu}$ of a system of spinning particles is the sum of a monopolar piece, made of Dirac delta-functions, plus the dipolar or spin piece, made of gradients of delta-functions:

$$T^{\mu\nu} = c^2\sum_A\int_{-\infty}^{+\infty}\mathrm{d}\tau_A\left\{p_A^\mu u_A^\nu\frac{\delta^{(4)}(x-y_A)}{\sqrt{-g_A}} - \frac{1}{c}\,\nabla_\rho\left[S_A^{\rho\mu}\,u_A^\nu\frac{\delta^{(4)}(x-y_A)}{\sqrt{-g_A}}\right]\right\}, \tag{6}$$

where $\delta^{(4)}$ is the four-dimensional Dirac function, x^μ is the field point, y_A^μ is the world-line of particle A, $u_A^\mu = \mathrm{d}y_A^\mu/(c\mathrm{d}\tau_A)$ is the four-velocity such that $g^A_{\mu\nu}u_A^\mu u_A^\nu = -1$ (with $g^A_{\mu\nu} \equiv g_{\mu\nu}(y_A)$ denoting the metric at the particle's location), p_A^μ is the linear momentum of the particle, and $S_A^{\mu\nu}$ denotes its antisymmetric spin angular momentum. Our spin variables has been rescaled by a factor c ($S^{\mu\nu} = c\,S^{\mu\nu}_{\mathrm{true}}$) so as to have a non-zero Newtonian limit for fast rotating compact objects. Imposing that the antisymmetric part of $T^{\mu\nu}$ should vanish yields the covariant equations of evolution for the spin, while the covariant equations of motion follow from the conservation relation $\nabla_\nu T^{\mu\nu} = 0$ (Papapetrou 1951).

In order to fix unphysical degrees of freedom associated with an arbitrariness in the definition of $S^{\mu\nu}$ in the case of point particles (due to the freedom in the choice of the location of the center-of-mass worldline within the extended bodies), we adopt the covariant supplementary spin condition $S_A^{\mu\nu}\,p_\nu^A = 0$ (Tulczyjew 1959). We restrict the computation to linear terms in the spins, neglecting $\mathcal{O}(S_A^2)$. Then the mass $m_A \equiv (-p_A^\mu p_\mu^A/c^2)^{1/2}$ is conserved and $S_A^{\mu\nu}$ is parallel transported along the worldline of body A. As is standard, we introduce a spin vector variable S_A^i with constant magnitude (see e.g., Kidder 1995), for instance by projecting $S_A^{\mu\nu}$ onto some orthonormal basis of the A-particle rest space and taking the Hodge dual with ε_{ijk}.

The post-Newtonian expressions of the densities σ, σ_i entering the source moments (4), as well as the stress density $\sigma_{ij} \equiv T^{ij}$, are obtained iteratively from the truncated components of the stress-energy tensor (6). They are used in turn to compute the metric at the next iteration step of the perturbative scheme.

Inserting the former matter densities into Eq. (4) (or its counterpart for V_L), we see that spins arise at 1.5PN order in I_L and 0.5PN order in J_L. By virtue of Eq. (2), leading SO terms in the gravitational-wave flux are thus seen to be of half-integer PN order. However, due to the $1/c^3$ factor in front of the tail integrals, as shown in Eq. (3), terms of integer orders may also appear: (i) when the source moments are contained in the tail integrands, or (ii) when they are multiplied by non-spin tail integrals. The leading 3PN terms coming from U_{ij} and V_{ij} are precisely those we want to compute.

3.2. 1.5PN dynamics with spin-orbit effects

To reduce the accelerations generated by the time differentiations of I_{ij} and J_{ij} and, most importantly, to find the time dependence of $I_{ij}^{(4)}$ and $J_{ij}^{(4)}$ required for calculating integrals such as the one in Eq (3), we shall rely on the 1.5PN dynamics. The corresponding PN equations of evolution at this order are derived by inserting the 1.5PN metric into the covariant equations and discarding $O(1/c^4)$ remainders. Going to the center-of-mass frame, we denote by $\mathbf{x} = \mathbf{y}_1 - \mathbf{y}_2$ the relative position of the particles (and $r = |\mathbf{x}|$, $\mathbf{v} = \mathrm{d}\mathbf{x}/\mathrm{d}t$). Following Kidder (1995) we introduce an orthonormal moving triad $\{\mathbf{n}, \boldsymbol{\lambda}, \boldsymbol{\ell}\}$ defined by $\mathbf{n} = \mathbf{x}/r$, $\boldsymbol{\ell} = \mathbf{L}_\mathrm{N}/|\mathbf{L}_\mathrm{N}|$, where $\mathbf{L}_\mathrm{N} \equiv m\nu\,\mathbf{x}\times\mathbf{v}$ denotes the Newtonian orbital angular momentum (with $m = m_1 + m_2$ and $\nu = m_1 m_2/m^2$), and $\boldsymbol{\lambda} = \boldsymbol{\ell}\times\mathbf{n}$. For general orbits, the time derivatives of those vectors may be written in the form

$$\frac{\mathrm{d}\mathbf{n}}{\mathrm{d}t} = \omega\,\boldsymbol{\lambda}\,, \qquad \frac{\mathrm{d}\boldsymbol{\lambda}}{\mathrm{d}t} = -\omega\,\mathbf{n} - \omega_\mathrm{prec}\,\boldsymbol{\ell}\,, \qquad \frac{\mathrm{d}\boldsymbol{\ell}}{\mathrm{d}t} = \omega_\mathrm{prec}\,\boldsymbol{\lambda}\,. \tag{7}$$

The angular frequency ω at which the separation $\mathbf{n}$ rotates in the instantaneous orbital plane is the so-called orbital frequency, while the third equation (7) defines the precessional frequency ω_prec, which gives thus the variation of $\boldsymbol{\ell}$ in the direction of $\boldsymbol{\lambda}$.

Instead of writing the equations of motion in the form of a link between $d\mathbf{v}/dt$ and the kinematic variables, we give the expressions of ω, ω_prec. Focusing from now on to the case of quasi-circular motion, for which r is constant apart from the 2.5PN secular radiation damping, we have (see e.g., Faye et al. 2006)

$$\omega^2 = \frac{G\,m}{r^3}\Big\{1 + \gamma\,(-3+\nu) + \gamma^{3/2}\,(-5s_\ell - 3\delta\sigma_\ell)\Big\} + O\Big(\frac{1}{c^4}\Big)\,, \tag{8}$$

$$\omega_\mathrm{prec} = -\omega\,\gamma^{3/2}\Big(7s_n + 3\delta\sigma_n\Big) + O\Big(\frac{1}{c^4}\Big)\,, \tag{9}$$

where the dimensionless spin variables are defined by $\mathbf{s} = (\mathbf{S}_1 + \mathbf{S}_2)/(Gm^2)$ and $\sigma = (\mathbf{S}_2/m_2 - \mathbf{S}_1/m_1)/(Gm)$, whereas $\delta \equiv X_1 - X_2$, $s_\ell \equiv \mathbf{s}\cdot\boldsymbol{\ell}$ and $s_n \equiv \mathbf{s}\cdot\mathbf{n}$; $\gamma \equiv Gm/(rc^2)$ is the PN parameter. The spin vectors $\mathbf{S}_A$ satisfy the usual-looking precession equations

$$\frac{\mathrm{d}\mathbf{S}_A}{\mathrm{d}t} = \boldsymbol{\Omega}_A \times \mathbf{S}_A\,, \qquad \text{with}\ \ \boldsymbol{\Omega}_1 = \omega\,\gamma\left[\frac{3}{4} + \frac{\nu}{2} - \frac{3}{4}\delta\right]\boldsymbol{\ell} + O\Big(\frac{1}{c^4}\Big)\,, \tag{10}$$

showing that the spins, at the 1.5PN order, precess around the direction of $\boldsymbol{\ell}$, and at the quasi-constant rate $\Omega_A = |\boldsymbol{\Omega}_A|$. To obtain $\boldsymbol{\Omega}_2$ we simply have to change δ into $-\delta$.

4. Computation of the tails

The only time dependence in the source moments composing the tail integrals occurs through the triad vectors and the spins at the 1.5PN order. We adopt the following strategy to integrate Eqs. (7): (i) we express the three moving triad vectors in terms of the Euler angles that link the latter triad to some orthonormal fixed triad whose third vectors points to the direction of the total angular momentum at some time t_0; (ii) we express the appropriate brick combinations of those angles as a function of S^A_n, S^A_λ, S^A_ℓ and $\phi \equiv \int \mathrm{d}t\,\omega(t)$; (iii) we solve for the precession equations (10) in the triad basis and insert the results into the formulae we got for $\mathbf{n}$, $\boldsymbol{\lambda}$, $\boldsymbol{\ell}$ in the previous step.

Although the binary continuous spectrum of frequencies of $I_L(t)$ should contain all orbital frequencies at any epoch in the past, say $\omega(t)$ with $t \leq T_R$, differing from the current orbital frequency $\omega(T_R)$ due to gravitational radiation damping, we can actually compute the tail integral by considering only the *current* frequency $\omega(T_R)$, modulo small error terms of negligible order $O(\ln c/c^5)$. Similarly, we can regard the precession frequencies $\boldsymbol{\Omega}_A(t)$ of the two spins as constant in the calculation. Thus, we may replace $\phi(t)$ by $\omega(t-t_0) + \phi(t_0)$ and $\Omega_A(t)$ by $\Omega_A(T_R)$ in the tail integrand. This takes the form of a sum of complex exponentials of the type $e^{\mathrm{i}\,\omega_{npq}t}$, with $\omega_{npq} = n\,\omega + p\,\Omega_1 + q\,\Omega_2$, times certain coefficients A_L^{npq}. It is then straightforward to take the Fourier transform of the source moments, e.g. $I_L(t)$, and apply formula (5), which leads to

$$\mathcal{U}_L(T_R) = \sum_{n,p,q} \mathrm{i}\,A_L^{npq}(-\mathrm{i}\,\omega_{npq})^{\ell+1} e^{-\mathrm{i}\,\omega_{npq}T_R}\left[\frac{\pi}{2}\mathrm{s}(\omega_{npq}) + \mathrm{i}\Big(\ln(2|\omega_{npq}|\tau_0') + \gamma_\mathrm{E}\Big)\right]\,. \tag{11}$$

5. Energy flux and orbital phasing

Inserting the explicit expressions for U_{ij} and V_{ij} into the energy flux (2) yields the net SO tail contribution at 3PN order, in terms of the gauge-invariant PN parameter $x \equiv (Gm\omega/c^3)^{2/3}$ and the spins. The SO terms have to be included in either the instantaneous moments in front of the tail integrals, or in the tail integrals themselves. This gives several "direct" SO contributions coming from tails at relative 1.5PN order (for the mass quadrupole tail) or 0.5PN order (for the current quadrupole tail) which are then added together. In addition there is the crucial contribution due to the reduction to circular orbits of the standard (non-spin) tail integral at 1.5PN order, for which the relation between the orbital separation r and the orbital frequency ω [as given by the inverse of Eq. (8)] provides a supplementary SO term at relative 1.5PN order, which contributes *in fine* at the same 3PN level as the "direct" SO tail terms. Finally we obtain

$$\delta\mathcal{F} = \frac{32}{5}\frac{c^5}{G}\, x^8\, \nu^2\Big[-16\pi\, s_\ell - \frac{31\pi}{6}\,\delta\,\sigma_\ell\Big], \tag{12}$$

Using an energy balance argument, we finally equate the averaged evolution of the binding energy of the binary reduced for quasi-circular orbits $\mathrm{d}E(x, s_\ell, \sigma_\ell)/\mathrm{d}t$ (see e.g., Blanchet et al. 2006) to $-\mathcal{F}(x, s_\ell, \sigma_\ell)$, where $\mathcal{F}$ is the sum of the 2PN SO flux obtained by Kidder (1995), the 3PN non-spin contribution investigated by Blanchet et al. (2002, 2004) and the correction (12) we just calculated. The time derivative applied to E acts only to the frequency variable since, as we have checked, s_ℓ and σ_ℓ are secularly conserved (neglecting SS contributions). We find the following secular variation of frequency, denoted $\dot\omega$ for simplicity, at the 3PN order including SO effects:

$$\begin{aligned}
\frac{\dot\omega}{\omega^2} = \frac{96}{5}\,\nu\, x^{5/2}\Big\{&1 + x\Big(-\frac{743}{336} - \frac{11}{4}\nu\Big) + x^{3/2}\Big(4\pi - \frac{47}{3}s_\ell - \frac{25}{4}\delta\sigma_\ell\Big)\\
&+x^2\Big(\frac{34103}{18144} + \frac{13661}{2016}\nu + \frac{59}{18}\nu^2\Big)\\
&+x^{5/2}\Big(-\frac{4159}{672}\pi - \frac{5861}{144}s_\ell - \frac{809}{84}\delta\sigma_\ell + \nu\Big[-\frac{189}{8}\pi + \frac{1001}{12}s_\ell + \frac{281}{8}\delta\sigma_\ell\Big]\Big)\\
&+x^3\Big(\frac{16447322263}{139708800} + \frac{16}{3}\pi^2 - \frac{1712}{105}\gamma_\mathrm{E} - \frac{856}{105}\ln(16\,x) - \frac{188\pi}{3}s_\ell - \frac{151\pi}{6}\delta\sigma_\ell\\
&\quad+\nu\Big[-\frac{56198689}{217728} + \frac{451}{48}\pi^2\Big] + \frac{541}{896}\nu^2 - \frac{5605}{2592}\nu^3\Big)\Big\}.
\end{aligned} \tag{13}$$

Integrating this, we get the "carrier" phase $\phi(t) = \int \omega\, dt$. The total phase $\Phi = \phi - \alpha + O(1/c^4)$, where α is the precessional phase, can be computed numerically or manually.

References

Bailey, I., & Israel, W. 1980, Ann. Phys., 130, 188

Blanchet, L. 1995, Phys. Rev. D, 51, 2559

— 2006, Living Rev. Rel., 9, 4. gr-qc/0202016

Blanchet, L., Buonanno, A., & Faye, G. 2006, Phys. Rev. D, 74, 104034. Erratum *Phys. Rev. D*, 75:049903(E), 2007

— 2011, Phys. Rev. D, 84, 064041

Blanchet, L., & Damour, T. 1986, Phil. Trans. Roy. Soc. Lond. A, 320, 379

— 1988, Phys. Rev. D, 37, 1410

— 1992, Phys. Rev. D, 46, 4304

Blanchet, L., Damour, T., Esposito-Farèse, G., & Iyer, B. R. 2004, Phys. Rev. Lett., 93, 091101

Blanchet, L., Faye, G., Iyer, B. R., & Joguet, B. 2002, Phys. Rev. D, 65, 061501(R). Erratum *Phys. Rev. D*, 71:129902(E), 2005

Faye, G., Blanchet, L., & Buonanno, A. 2006, Phys. Rev. D, 74, 104033

Kidder, L. E. 1995, Phys. Rev. D, 52, 821
Mathisson, M. 1937, Acta Phys. Polon., 6, 163
Papapetrou, A. 1951, Proc. R. Soc. London A, 209, 248
Thorne, K. 1980, Rev. Mod. Phys., 52, 299
Tulczyjew, W. 1959, Acta Phys. Polon., 18, 393