PHYSICAL REVIEW D **87,** 044009 (2013)

Spin effects on gravitational waves from inspiraling compact binaries at second post-Newtonian order

Alessandra Buonanno,[1] Guillaume Faye,[2] and Tanja Hinderer[1]

[1]*Department of Physics, Maryland Center for Fundamental Physics and Joint Space-Science Institute, University of Maryland, College Park, Maryland 20742, USA*

[2]*Institut d'Astrophysique de Paris, UMR 7095 CNRS, Université Pierre and Marie Curie, 98$^{\mathrm{bis}}$ boulevard Arago, 75014 Paris, France*

(Received 27 September 2012; published 5 February 2013)

We calculate the gravitational waveform for spinning, precessing compact binary inspirals through second post-Newtonian order in the amplitude. When spins are collinear with the orbital angular momentum and the orbits are quasicircular, we further provide explicit expressions for the gravitational-wave polarizations and the decomposition into spin-weighted spherical-harmonic modes. Knowledge of the second post-Newtonian spin terms in the waveform could be used to improve the physical content of analytical templates for data analysis of compact binary inspirals and for more accurate comparisons with numerical-relativity simulations.

DOI: 10.1103/PhysRevD.87.044009 PACS numbers: 04.30.−w, 04.25.−g

I. INTRODUCTION

Coalescing compact binary systems are a key source of gravitational radiation for ground-based gravitational-wave detectors such as the advanced Laser Interferometer Gravitational Wave Observatory (LIGO) [1], the advanced Virgo [2], the GEO-HF [3], the Large Cryogenic Gravitational Telescope (LCGT) (or KAGRA) [4], coming into operation within the next few years, and future space-based detectors [5,6]. For this class of gravitational-wave sources, the signal detection and interpretation will be based on the method of matched filtering [7,8], where the noisy detector output is cross correlated with a bank of theoretical templates. The accuracy requirement on the templates is that they remain as much as possible phase coherent with the signal over the hundreds to thousands of cycles of inspiral that are within the detector's sensitive bandwidth.

Constructing such accurate templates has motivated a significant research effort during the past 30 years. In the regime where the separation between the two bodies is large, gravitational waveforms can be computed using the post-Newtonian (PN) approximation method [9–11]. In the post-Newtonian scheme, the results are written as an asymptotic expansion in powers of v_A/c, with v_A being the magnitude of the orbital coordinate velocity $\boldsymbol{v}_A$ of body A at a given time. This approximation is physically relevant for $v_A/c \ll 1$, i.e., in the so-called inspiraling regime where the radiation reaction forces, of order $\sim(v_A/c)^5$, are negligible over an orbital period and act adiabatically on a quasiconservative system. In the domain of validity of the post-Newtonian scheme, the separation $r \sim (Gm_A/v^2) \sim (c/v)^2$, with $m = m_1 + m_2$ and $v = |\boldsymbol{v}| \equiv |\boldsymbol{v}_1 - \boldsymbol{v}_2|$, remains large with respect to the radii of both compact objects $\sim Gm_A/c^2$ or, in other words, the bodies can be regarded effectively as point particles.

Post-Newtonian waveforms cease to be reliable near the end of the inspiral and the coalescence phase, where numerical-relativity simulations should be used to predict the gravitational-wave signal [12–14]. By combining the information from post-Newtonian predictions and the numerical-relativity simulations it is possible to accurately and analytically describe the gravitational-wave signal during the entire inspiral, plunge, merger and ringdown process [15–23].

For nonspinning binaries, the post-Newtonian expansion has been iterated to 3.5PN order beyond the leading Newtonian order in the gravitational-wave phasing [24–26]. The gravitational-wave amplitude has been computed through 3PN order [27–30] and the quadrupole mode through 3.5PN order [31]. However, black hole binaries could potentially have large spins [32], which may be misaligned with the orbital angular momentum, in which case the precession effects add significant complexity to the emitted gravitational waves [33]. Ignoring the effects of black hole spins could lead to a reduction in the signal-to-noise ratio and decrease the detection efficiency [34,35] although this should be overcome with phenomenological and physical models [21,36–43]. To maximize the payoffs for astrophysics will require extracting the source parameters from the gravitational-wave signal using template models computed from the most accurate physical prediction available [44–47]. Spin effects in the waveform are currently known through much lower post-Newtonian order than for nonspinning binaries. More specifically, spin effects are known through 2.5PN order in the phase [48–50], 1.5PN order in the polarizations for spin-orbit effects [51,52], 2PN order for the spin$_1$-spin$_2$ effects [51,53] and partially 3PN order in the polarizations for the tail-induced spin-orbit effects [54].

In this paper, we compute all spin effects in the gravitational-wave strain tensor through 2PN order. This requires knowledge of the influence of the spins on the system's orbital dynamics as well as on the radiative multipole moments. At this PN order, nonlinear spin effects attributable to the spin-induced quadrupole moments of the compact objects first appear. Using results from Refs. [55–58], we derive the stress-energy tensor with self-spin terms and compute the self-induced quadrupole terms in the equations of motion and in the source multipole moments at 2PN order. Our results are in agreement with previous calculations [59–62].

The two main inputs entering our calculation of the gravitational-wave strain tensor through 2PN order are (i) the results of Refs. [50,51,59] for the influence of the spins on the system's orbital dynamics, which have also been derived by effective field theory and canonical methods [56,63–68], and (ii) the spin effects in the system's radiative multipole moments [50]. Recently, the necessary knowledge to compute the waveform at 2.5PN order was obtained using the effective field theory approach [62,64]. Here we use (i) and (ii) in the multipolar wave generation formalism [69–71] to obtain the waveform for spinning, precessing binaries through 2PN order. To compute the gravitational polarizations from this result, one must specify an appropriate source frame and project the strain tensor onto a polarization triad. For precessing systems, there are several frames that could be employed [8,35,51,72–76]. For nonprecessing binaries with the spins collinear to the orbital angular momentum, the most natural frame is the one used for nonspinning binaries. Therefore, instead of choosing one frame, for simplicity, we specialize to the nonprecessing case and quasicircular orbits and provide the explicit expressions for the gravitational polarizations. Lengthy calculations are performed with the help of the scientific software MATHEMATICA®, supplemented by the package xTensor [77] dedicated to tensor calculus. Our generic, precessing result is available in MATHEMATICA format upon request and can be used to compute the polarizations for specific choices of frame. We notice that the 2PN terms in the polarizations, for circular orbits, linear in the spins were also computed in Ref. [78]. However, these results contain errors in the multipole moments, which were corrected in Ref. [50].

For future work at the interface of analytical and numerical relativity, we also explicitly compute the decomposition of the strain tensor into spin-weighted spherical-harmonic modes for nonprecessing spinning binaries on circular orbits. The test-particle limit of these results can also be directly compared with the black-hole perturbation calculations of Refs. [79,80], and we verify that the relevant terms agree.

The organization of the paper is as follows. In Sec. II, we review the Lagrangian for compact objects with self-induced spin effects [55–57,61], compute the stress-energy tensor and derive the self-induced spin couplings in the two-body acceleration and source multipole moments [59–62]. In Sec. III we summarize the necessary information about spin effects in the equations of motion and the wave generation necessary for our calculation. In Sec. IV B we calculate the spin-orbit effects at 2PN order in the strain tensor for generic precessing binaries. In Sec. IV C we complete the knowledge of 2PN spin-spin terms by including the spin self-induced quadrupole terms in addition to the spin_1-spin_2 terms obtained in Ref. [51]. In Sec. IV E we specialize to quasicircular orbits and explicitly give the polarization tensors for nonprecessing systems. Then, in Sec. IV F we decompose the polarizations into spin-weighted spherical-harmonic modes. Finally, Sec. V summarizes our main findings.

We use lowercase Latin letters $a, b, \ldots, i, j, \ldots$, for indices of spatial tensors. Spatial indices are contracted with the Euclidean metric, with up or down placement of the indices having no meaning and repeated indices summed over. We use angular brackets to denote the symmetric, trace-free (STF) projection of tensors, e.g., $T_{\langle ij\rangle} = \mathrm{STF}[T_{ij}] = T_{(ij)} - \frac{1}{3}\delta_{ij}T_{kk}$, where the round parentheses indicate the symmetrization operation. Square parentheses indicate antisymmetrized indices, e.g., $T_{[ij]} = \frac{1}{2}(T_{ij} - T_{ji})$. The letter $L = i_1 \ldots i_\ell$ signifies a multi-index composed of ℓ STF indices. The transverse-traceless (TT) projection operator is denoted $\mathcal{P}^{\mathrm{TT}}_{ijab} = \mathcal{P}_{a(i}\mathcal{P}_{j)b} - \frac{1}{2}\mathcal{P}_{ij}\mathcal{P}_{ab}$, where $\mathcal{P}_{ij} = \delta_{ij} - N_iN_j$ is the projector orthogonal to the unit direction $\boldsymbol{N} = \boldsymbol{X}/R$ of a radiative coordinate system $X^\mu = (cT, \boldsymbol{X})$, where the boldface denotes a spatial three-vector. As usual, $g_{\mu\nu}$ represents the space-time metric and g its determinant. The quantity ε_{ijk} is the antisymmetric Levi-Cività symbol, with $\varepsilon_{123} = 1$, and $\epsilon_{\mu\nu\rho\sigma}$ stands for the Levi-Cività four-volume form, with $\epsilon_{0123} = +\sqrt{-g}$. Henceforth, we shall indicate the spin_1-spin_2 terms with S_1S_2, the spin_1^2, spin_2^2 terms with S^2 and the total spin-spin terms with SS. Throughout the paper, we retain only the terms relevant to our calculations and omit all other terms, which either are already known or appear at a higher post-Newtonian order than required for our purposes.

II. MODELING SPINNING COMPACT OBJECTS WITH SELF-INDUCED QUADRUPOLES

In this section we review the construction of a Lagrangian for compact objects with self-induced quadrupole spin effects [55–57,61,81], compute the stress-energy tensor and derive the self-induced spin couplings in the two-body acceleration and source multipole moments. Our findings are in agreement with previous results [59–62].

A. Lagrangian for compact objects with self-induced spin effects

A Lagrangian for a system of spinning compact objects with nondynamical[1] self-induced quadrupole moments can be obtained by augmenting the Lagrangian for point

[1]We shall not include kinetic terms in the Lagrangian for the quadrupole moment that can describe resonance effects in neutron stars.

particles with $L_A^{S^2}$ describing the quadrupole-curvature coupling for each body A. Since the action for body A must admit a covariant representation, the corresponding Lagrangian $L_A^{S^2}$ should be a function of the four-velocity u_A^μ, the metric $g_{\mu\nu}$, the Riemann tensor $R^\lambda{}_{\rho\mu\nu}$ and its covariant derivatives, evaluated at the worldline point y_A^μ, and the spin variables entering via the antisymmetric spin tensor $S_A^{\mu\nu}$.

The spin tensor $S_A^{\mu\nu}$ contains six degrees of freedom. It is well known that in order to reduce them to the three physical degrees of freedom a spin supplementary condition (SSC) should be imposed [82]. This is equivalent to performing a shift of the worldline y_A^μ. In this paper we specialize to the SSC $S_A^{\mu\nu} p_\nu^A = 0$, which is equivalent to $S_A^{\mu\nu} u_\nu^A = 0$ since $p_A^\mu \approx m_A c u_A^\mu$ through 2.5PN order. To ensure the preservation of the SSC under the evolution, we follow Ref. [57] and introduce the spin tensor $\mathcal{S}_A^{\mu\nu} = S_A^{\mu\nu} + 2u_A^{[\mu} S_A^{\nu]\lambda} u_\lambda^A$. The spin tensor $\mathcal{S}_A^{\mu\nu}$ automatically satisfies the algebraic identity $\mathcal{S}_A^{\mu\nu} u_\nu^A = 0$, which provides three constraints that can be used to reduce the spin degrees of freedom from six to three.

From the above discussion and Refs. [56,83], we assume that the Lagrangian of particle A is of the form $L_A^{S^2} = L_{A\mu\nu\lambda\rho} \mathcal{S}_A^{\mu\nu} \mathcal{S}_A^{\lambda\rho}$, where $L_{A\mu\nu\lambda\rho}$ is a polynomial in the Riemann tensor and its derivatives, as well as the 4-velocity u_A^μ. As noticed in Ref. [84], any term proportional to $\nabla_{\dots} R_{\alpha\beta}$ evaluated at point y_A^μ can be recast into a redefinition of the gravitational field. As a result, the Riemann tensor may be replaced for each of its occurrences by the Weyl tensor $C^\lambda{}_{\rho\mu\nu}$, which can be decomposed into a combination of the gravitoelectric- and gravitomagnetic-type STF tidal quadrupole moments $G_{\mu\nu}^A \equiv G_{\mu\nu}(y_A^\alpha) \equiv -c^2 R_{\mu\alpha\nu\beta} u_A^\alpha u_A^\beta$ and $H_{\mu\nu}^A \equiv H_{\mu\nu}(y_A^\alpha) \equiv 2c^3 R^{A*}_{\mu\alpha\nu\beta} u_A^\alpha u_A^\beta$ with $R^*_{\mu\nu\alpha\beta} \equiv \frac{1}{2}\epsilon_{\mu\nu\rho\sigma} R^{\rho\sigma}{}_{\alpha\beta}$. More generally, the multiple space derivatives of $C^\lambda{}_{\rho\mu\nu}$ at point y_A^μ may be expressed in terms of some STF tidal multipole moments $G^A_{\mu_1\dots\mu_\ell}$ and $H^A_{\mu_1\dots\mu_\ell}$, of parity 1 and -1, respectively. However, those higher-order moments will play no role in this paper.

Taking into account that the contraction of the velocity vector u_A^ν with both $G_{\mu\nu}^A$ and $\mathcal{S}_A^{\mu\nu}$ vanishes, that the spin and tidal multipole tensors are traceless, and that the Lagrangian must obey parity and time-reversal symmetries, we obtain [55–57,83]

$$L_A^{S^2} = -\frac{\kappa_A}{2m_A c^2} G_{\mu\nu} S^\mu_{A\lambda} S_A^{\lambda\nu}. \tag{2.1}$$

Here, we have also assumed that the rotating body is axially symmetric, and we have replaced $\mathcal{S}_A^{\mu\nu}$ with $S_A^{\mu\nu}$ since the difference between these spin variables contributes to the equations of motion at $\mathcal{O}(S_A^3)$, where $S_A = \sqrt{|S_A^\mu S_\mu^A|}$ with $S_\mu^A = \epsilon_{\rho\sigma\nu\mu} S_A^{\rho\sigma} p_A^\nu/(2m_A c)$.

For a neutron star the numerical constant κ_A in Eq. (2.1) depends on the equation of state of the fluid [85]. For an isolated black hole $\kappa_A = 1$ [59,60], but for a black hole in a compact binary κ_A can deviate from 1. However, these deviations occur at PN orders that are much higher than the ones considered here. We notice that the leading contribution $\kappa_A = 1$ can be obtained by computing the acceleration of body A from Eq. (2.1) in a compact binary for $m_A \ll m$ and match it with the acceleration of a test particle in the gravitational field of a Kerr black hole of mass m [83].

B. Effective stress-energy tensor with self-induced quadrupoles

The piece of the stress-energy tensor encoding the self-induced quadrupole dynamics of body A reads by definition

$$T^{\mu\nu}_{\text{quad},A} = \frac{2}{\sqrt{-g}} \frac{\delta}{\delta g_{\mu\nu}(x)} \int d\tau_A L_A^{S^2}[y_A^\alpha(\tau_A), S_A^{\alpha\beta}(\tau_A)], \tag{2.2}$$

where $L_A^{S^2}$ is the Lagrangian (2.1). To determine the action of the operator $\delta/\delta g_{\mu\nu}$, which stands for the usual "functional derivative" with respect to the field $g_{\mu\nu}$, we need to adopt a specific model for the spin. The rotational state of the extended object A is usually represented by a tetrad of orthonormal vectors $e^\mu_{A\bar\alpha}(\tau_A)$ with $\bar\alpha \in \{0,1,2,3\}$ along the worldline y_A^μ with affine parameter τ_A. The corresponding angular rotation tensor is then defined as $\Omega_A^{\mu\nu} = \eta^{\bar\alpha\bar\beta} e^\mu_{A\bar\alpha} De^\nu_{A\bar\beta}/d\tau_A$. We now make the reasonable physical hypothesis that the rotation of the axially symmetric object takes place about the symmetry axis. The moment of inertia I_A along that direction is a 2PN-order quantity $\sim G^2 m_A^3/c^4$ for compactness parameters of order 1, whereas $\Omega_A^{\mu\nu} \sim V_A/R_A$, R_A being the radius of body A and V_A its typical internal velocity, is roughly equal to $c^3/(Gm_A)$. In the weak field limit where G goes formally to zero, the spin must satisfy the relation $S_A^{\mu\nu} = I_A \Omega_A^{\mu\nu}$, as in special relativity [86]. In the presence of a nonnegligible gravitational field, this relation is expected to be modified by nonminimal coupled terms proportional to positive powers of $R^A_{\mu\nu\alpha\beta}$ times positive powers of I_A and $S_A^{\mu\nu}$ [83]:

$$\hat S_A^{\mu\nu} = I_A\left[\Omega_A^{\mu\nu} + \mathcal{O}\left(\frac{\hat S_A}{c^2}\right)\right]. \tag{2.3}$$

Here we use a hat to distinguish the generic spin variable from the one related to our specific spin model. The corrections $I_A \times \mathcal{O}(\hat S_A/c^2)$ are not relevant for the two-body dynamics in this paper because they correspond to the 4.5PN order when taking into account the factor $\mathcal{O}(1/c)$ contained in the spin variable.

Using the definition (2.3) for the spin variables, we compute in a covariant manner the variation of the action

$$\mathcal{A}^{S^2} = \int d\tau_A L_A^{S^2}(\tau_A) = \int \frac{d^4x}{c}\sqrt{-g}\int d\lambda_A L_A^{S^2}(\lambda_A)\frac{\delta^4(x^\alpha - y_A^\alpha(\lambda_A))}{\sqrt{-g}}, \tag{2.4}$$

when the metric varies by $\delta g_{\mu\nu}(x)$, and find the following quadrupolar piece of the stress-energy tensor:

$$T^{\mu\nu}_{\text{quad},A} = \frac{\kappa_A}{m_A c^2}\Big[\frac{n_A^*}{2}\Big(-3u_A^\mu u_A^\nu G^A_{\lambda\rho}\hat{S}_A^{\lambda\sigma}\hat{S}_\sigma^{A\rho} - c^2 R^{(\mu}_{A\lambda\rho\tau}u_A^{\nu)}\hat{S}^\lambda_{A\sigma}\hat{S}_A^{\sigma\rho}u_A^\tau + G^{(\mu}_{A\lambda}\hat{S}^{\nu)}_{A\rho}\hat{S}_A^{\rho\lambda}\Big) + \nabla_\rho\Big(I_A c n_A^*\Big(G_\lambda^{A(\mu}u_A^{\nu)}\hat{S}_A^{\lambda\rho} - G_\lambda^{A\rho}\hat{S}_A^{\lambda(\mu}u_A^{\nu)}\Big)\Big)\Big] - 2\nabla_\lambda\nabla_\rho\Big[n_A^*\hat{S}_A^{\sigma[\lambda}u_A^{(\mu]}\hat{S}_\sigma^{A[\nu)}u_A^{\rho]}\Big], \tag{2.5}$$

where we have indicated with n_A^* the Dirac-type scalar density $n_A^*(x^\mu) = \int d\lambda_A \delta^4(x^\mu - y_A^\mu(\lambda_A))/\sqrt{-g(x^\nu)}$ and, in the last term, we have adopted the convention that symmetrization of indices applies after antisymmetrization. As derived in Ref. [81], the most general form of the effective stress-energy tensor is

$$T^{\mu\nu}_{\text{skel},A}(x^\mu) = \sum_{\ell=0}^{+\infty}\nabla_{\lambda_1}\nabla_{\lambda_2}\ldots\nabla_{\lambda_\ell}[t_A^{\mu\nu|\lambda_1\lambda_2\ldots\lambda_\ell}(\tau_A)n_A^*(x^\mu)], \tag{2.6}$$

where τ_A is the proper time of the Ath worldline at event y_A^μ with $y_A^0 = x^0$ and the coefficients $t_A^{\mu\nu|\lambda_1\lambda_2\ldots\lambda_\ell}(\tau_A)$ are the "skeleton" multipole moments. The latter are not arbitrary but satisfy algebraic constraints imposed by the equation of conservation $\nabla_\nu T^{\mu\nu}_{\text{skel}} = 0$. Let us check that we can indeed recast the total stress-energy tensor, including the monopolar, dipolar and quadrupolar pieces, in the form of (2.6). If we add $T^{\mu\nu}_{\text{quad}}$ to the monopolar and dipolar contributions [49,81,87–89]

$$T^{\mu\nu}_{\text{mon+dipole}} = \sum_A[n_A^*\tilde{p}_A^{(\mu}u_A^{\nu)}c + \nabla_\lambda(n_A^* c u_A^{(\mu}\tilde{S}_A^{\nu)\lambda})], \tag{2.7}$$

and redefine the spin variable entering the quadrupolar piece as

$$S_A^{\mu\nu} = \tilde{S}_A^{\mu\nu} - \frac{2\kappa_A}{m_A c^2}I_A\hat{S}_A^{\lambda[\mu}G^{\nu]}_{A\lambda}, \tag{2.8}$$

we obtain the total stress-energy tensor in the form

$$T^{\mu\nu} = \sum_A\Big[n_A^*\Big(p_A^{(\mu}u_A^{\nu)}c + \frac{1}{3}R^{(\mu}_{A\tau\lambda\rho}J_A^{\nu)\tau\lambda\rho}c^2\Big) + \nabla_\lambda(n_A^* c u_A^{(\mu}S_A^{\nu)\lambda}) - \frac{2}{3}\nabla_\lambda\nabla_\rho(n_A^* c^2 J_A^{\lambda(\mu\nu)\rho})\Big], \tag{2.9a}$$

where the four-rank tensor $J_A^{\lambda\rho\mu\nu}$ takes the following expression in our effective description:

$$J_A^{\lambda\rho\mu\nu} = \frac{3\kappa_A}{m_A c^2}S_A^{\sigma[\lambda}u_A^{\rho]}S_\sigma^{A[\mu}u_A^{\nu]}. \tag{2.9b}$$

Consistently with the approximation already made in the spin model (2.3), we have neglected here the difference of order $I_A \times \mathcal{O}(\hat{S}_A/c^2)$ between the spins $\hat{S}_A^{\mu\nu}$ and $S_A^{\mu\nu}$ in the above formula. The net result is that Eq. (2.9a) matches Eq. (2.6) for $\ell = 0, 1, 2$ as expected. Moreover, Eq. (2.9b) agrees with Refs. [58,61].

Lastly, the conservation of the stress-energy tensor (2.9a) is equivalent to the equation of motion for the particle worldline, supplemented by the spin precession equation [58]. They read

$$\frac{Dp_A^\mu}{d\tau_A} = -\frac{c}{2}R^\mu_{A\rho\nu\lambda}u_A^\rho S_A^{\nu\lambda} - \frac{c^2}{3}\nabla_\tau R^\mu_{A\rho\nu\lambda}J_A^{\tau\rho\nu\lambda}, \tag{2.10a}$$

$$\frac{DS^{\mu\nu}}{d\tau_A} = 2cp_A^{[\mu}u_A^{\nu]} + \frac{4c^2}{3}R^{[\mu}_{A\tau\lambda\rho}J_A^{\nu]\tau\lambda\rho}. \tag{2.10b}$$

Those equations are in full agreement with the equations of evolution derived from the Dixon formalism truncated at the quadrupolar order [90].

C. Self-induced quadrupole terms in the 2PN binary dynamics and source multipole moments

Once the stress-energy tensor has been derived, the post-Newtonian equations of motion and the source multipole moments parametrizing the linearized gravitational field outside the system can be computed by means of the usual standard techniques [10]. At 2PN order, the accelerations including the self-spin interactions were obtained in Refs. [59,60], but the self-induced quadrupole effects in the source multipole moments were never explicitly included in the standard version of the post-Newtonian scheme, although recently they were calculated at 3PN order using effective-field-theory techniques [91]. Here we can use the results of the previous section, which constitutes a natural extension of the standard post-Newtonian approximation for spinning compact bodies [49], and explicitly derive the self-induced quadrupole couplings in the 2PN dynamics and source multipole moments.

Henceforth, we define the spin vectors S_A^i by the relation $S_i^A/c = g_{ij}^A S_A^j$, where S_i^A is the three-form induced on the hypersurface $t = \text{const}$ by S_μ^A. Note that it is S_A^i/c that has the dimension of a spin, while S_A^i has been rescaled in order to have a nonzero Newtonian limit for compact objects.

In the post-Newtonian formalism for point particles in the harmonic gauge, it is convenient to represent effectively the source by the mass density $\sigma = (T^{00} + T^{ii})/c^2$, the current density $\sigma_i = T^{0i}/c$, and the stress density $\sigma_{ij} = T^{ij}$. They are essentially the components of the stress-energy tensor rescaled so as not to vanish in the formal limit $c \to 0$ for weakly stressed, standard matter. At 2PN order, the second term in the right-hand side of Eq. (2.9a) does not contribute. From the last term, we obtain the following self-spin contributions:

$$\sigma^{S^2} = \frac{\kappa_1}{2m_1c^2}\partial_{ij}[\delta_1 S_1^{ki}S_1^{kj}] + 1 \leftrightarrow 2 + \mathcal{O}\left(\frac{S_A^2}{c^4}\right), \quad (2.11a)$$

$$\sigma_i^{S^2} = \mathcal{O}\left(\frac{S_A^2}{c^2}\right), \quad (2.11b)$$

$$\sigma_{ij}^{S^2} = \mathcal{O}\left(\frac{S_A^2}{c^2}\right), \quad (2.11c)$$

where $1 \leftrightarrow 2$ represents the counterpart of the preceding term with particles 1 and 2 exchanged, and $\delta_1 \equiv \delta^3(\boldsymbol{x} - \boldsymbol{y}_1)$.

At 2PN order, the spin^2 part of the equations of motion (2.10a) for, say, the first particle, reduce to

$$\frac{D(u_i^1 c)}{d\tau_1} = \text{non-}S_1^2 \text{ terms} - \frac{\kappa_1}{2m_1^2}\partial_k R^1_{i0j0}S_1^{lk}S_1^{lj} + \mathcal{O}\left(\frac{S_1^2}{c^4}\right). \quad (2.12)$$

The only occurrence of self-spin interactions at 2PN order on the left-hand side of the above equation comes from the gradient of the time component of the metric, $g_{00} = -1 + 2V/c^2 + \mathcal{O}(1/c^4)$, where the Newton-like potential V satisfies $\Box V = -4\pi G\sigma$. Although V coincides with the Newtonian potential U in the leading approximation, it contains higher order corrections, including quadratic-in-spin terms coming from the mass density (2.11a), which are smaller than U by a factor $\mathcal{O}(1/c^4)$. They read

$$V_{S^2} = -\frac{2\pi G\kappa_1}{m_1c^2}\partial_{ij}\Delta^{-1}[\delta_1 S_1^{ki}S_1^{kj}] + 1 \leftrightarrow 2 + \mathcal{O}\left(\frac{S_A^2}{c^4}\right)$$
$$= \frac{G\kappa_1}{2m_1c^2}\partial_{ij}\frac{1}{r_1}S_1^{ki}S_1^{kj} + 1 \leftrightarrow 2 + \mathcal{O}\left(\frac{S_A^2}{c^4}\right), \quad (2.13)$$

with $\partial_i = \partial/\partial x^i$ and $r_1 \equiv |\boldsymbol{x} - \boldsymbol{y}_1|$, the symbol Δ^{-1} holding for the retarded integral operator. Other potentials appear at the 1PN approximation or beyond, but their sources cannot contain a self-induced quadrupole below $\mathcal{O}(1/c^4)$; thus they are negligible here. The self-induced spin part of the acceleration $\boldsymbol{a}_1$ of the first particle is therefore given by

$$(a_1^i)_{S^2} = -c^2(\Gamma^0{}_{0i})_{S^2} - \frac{\kappa_1}{2m_1^2}\partial_k R^1_{i0j0}S_1^{lk}S_1^{lj} + \mathcal{O}\left(\frac{S_A^2}{c^4}\right). \quad (2.14)$$

Replacement of the Christoffel symbols $\Gamma^\lambda{}_{\mu\nu}$ and the Riemann tensor by the leading order values

$$\Gamma^0{}_{0i} = -\frac{\partial_i V}{c^2} + \mathcal{O}\left(\frac{1}{c^4}\right), \qquad R_{i0j0} = -\frac{\partial_{ij}U}{c^2} + \mathcal{O}\left(\frac{1}{c^4}\right), \quad (2.15)$$

with $U = Gm_1/r_1 + Gm_2/r_2 + \mathcal{O}(1/c^2)$ yields the more explicit result (posing $\partial_{1i} \equiv \partial/\partial y_1^i$):

$$(a_1^i)_{S^2} = -\frac{G}{2c^2}\partial_{1ijk}\frac{1}{r}\left[\frac{\kappa_2}{m_2}S_2^jS_2^k + \frac{m_2\kappa_1}{m_1^2}S_1^jS_1^k\right] + \mathcal{O}\left(\frac{1}{c^4}\right), \quad (2.16)$$

which agrees with Refs. [59,60] in the center-of-mass frame, for $S_A^i/c = \varepsilon_{ijk}S^{jk} + \mathcal{O}(1/c^3)$.

Self-induced quadrupolar deformations of the bodies also produce 2PN-order terms in the source multipole moments I_L and J_L. Those are defined as volume integrals whose integrands are certain polynomials in the densities σ, σ_i and σ_{ij} as well as some gravitational potentials, such as V, that parametrize the metric. Now, since those potentials are multiplied by prefactors of order $\mathcal{O}(1/c^2)$ and cannot contain themselves spin^2 interactions below the 2PN order, monomials involving one potential or more may be ignored for the calculation. The remaining sources are linear in the σ variables. With the help of the general formula (5.15) of Ref. [92], it is then immediate to get the self-spin contribution to I_L:

$$I_L^{S^2} = \int d^3\boldsymbol{x}\, x^{\langle i_1}\dots x^{i_\ell\rangle}\sigma_{S^2} + \mathcal{O}\left(\frac{S_A^2}{c^4}\right). \quad (2.17)$$

Inserting expression (2.11a) for σ_{S^2} and performing a straightforward integration, we arrive at

$$I_L^{S^2} = \frac{\kappa_1}{2m_1c^2}\partial_{1ij}(y_1^{\langle i_1}\dots y_1^{i_\ell\rangle})S_1^{ki}S_1^{kj} + 1 \leftrightarrow 2 + \mathcal{O}\left(\frac{S_A^2}{c^4}\right). \quad (2.18)$$

We can show similarly that J_L is of order $\mathcal{O}(S_A^2/c^2)$. As a result, at the accuracy level required for the 2PN waveform, the only terms quadratic in one of the spins that originate from the source moments come from the quadrupole $\ell = 2$, for which we have

$$I_{ij}^{S^2} = -\frac{\kappa_1}{m_1c^4}S_1^{\langle i}S_1^{j\rangle} + 1 \leftrightarrow 2 + \mathcal{O}\left(\frac{1}{c^6}\right), \quad (2.19)$$

whereas similar terms in $(I_L)_{\ell\geq 3}$ or $(J_L)_{\ell\geq 2}$ lie beyond our approximation. The above correction to the mass quadrupole agrees with that of Porto *et al.* [91] truncated at 2PN order. It is formally of order $\mathcal{O}(1/c^4)$ but, because $\dot{\boldsymbol{S}}_A = \mathcal{O}(1/c^2)$, it is cast to the 3PN order in the waveform expansion given below [see Eq. (4.1)] after the second time derivative is applied. This result was already argued in Ref. [93].

III. TWO-BODY DYNAMICS WITH SPIN EFFECTS THROUGH 2PN ORDER

The equations of motion in harmonic coordinates for the relative orbital separation $\boldsymbol{x} = r\boldsymbol{n}$ in the center of mass frame are [10]

$$\frac{d^2x^i}{dt^2} = a^i_{\rm Newt} + \frac{1}{c^2}a^i_{\rm 1PN} + \frac{1}{c^3}a^i_{\rm SO} + \frac{1}{c^4}[a^i_{\rm S_1S_2} + a^i_{\rm S^2} + a^i_{\rm 2PN}], \tag{3.1a}$$

where

$$\boldsymbol{a}_{\rm Newt} = -\frac{Gm}{r^2}\boldsymbol{n}, \tag{3.1b}$$

$$\boldsymbol{a}_{\rm 1PN} = -\frac{Gm}{r^2}\left\{\left[(1+3\nu)v^2 - \frac{3}{2}\nu\dot{r}^2 - 2(2+\nu)\frac{Gm}{r}\right]\boldsymbol{n} - 2\dot{r}(2-\nu)\boldsymbol{v}\right\}, \tag{3.1c}$$

with $m \equiv m_1 + m_2$, $\nu \equiv m_1 m_2/m^2$, $\boldsymbol{n} = \boldsymbol{x}/r$ and $\boldsymbol{v} = d\boldsymbol{x}/dt$. The 2PN acceleration given, e.g., in Ref. [51] will not be needed for our calculation. The spin-orbit terms are [51]

$$\boldsymbol{a}_{\rm SO} = \frac{G}{r^3}\{6[(\boldsymbol{n}\times\boldsymbol{v})\cdot(2\boldsymbol{S}+\delta\boldsymbol{\Sigma})]\boldsymbol{n} - [\boldsymbol{v}\times(7\boldsymbol{S}+3\delta\boldsymbol{\Sigma})] + 3\dot{r}[\boldsymbol{n}\times(3\boldsymbol{S}+\delta\boldsymbol{\Sigma})]\}, \tag{3.1d}$$

where we denote with $\delta = (m_1 - m_2)/m$ and

$$\boldsymbol{S} \equiv \boldsymbol{S}_1 + \boldsymbol{S}_2, \tag{3.2a}$$

$$\boldsymbol{\Sigma} \equiv m\left[\frac{\boldsymbol{S}_2}{m_2} - \frac{\boldsymbol{S}_1}{m_1}\right]. \tag{3.2b}$$

The spin$_1$-spin$_2$ interaction terms are [51]

$$\boldsymbol{a}_{\rm S_1S_2} = -\frac{3G}{m\nu r^4}\{[(\boldsymbol{S}_1\cdot\boldsymbol{S}_2) - 5(\boldsymbol{n}\cdot\boldsymbol{S}_1)(\boldsymbol{n}\cdot\boldsymbol{S}_2)]\boldsymbol{n} + (\boldsymbol{n}\cdot\boldsymbol{S}_1)\boldsymbol{S}_2 + (\boldsymbol{n}\cdot\boldsymbol{S}_2)\boldsymbol{S}_1\}. \tag{3.3a}$$

As originally computed in Ref. [59] [see Eq. (2.16) above], an additional term due to the influence of the spin-induced mass quadrupole moment on the motion arises at 2PN order:

$$\boldsymbol{a}_{\rm S^2} = -\frac{3G}{2m\nu r^4}\left\{\boldsymbol{n}\left[\frac{\kappa_1}{q}S_1^2 + q\kappa_2 S_2^2\right] + 2\left[\frac{\kappa_1}{q}(\boldsymbol{n}\cdot\boldsymbol{S}_1)\boldsymbol{S}_1 + q\kappa_2(\boldsymbol{n}\cdot\boldsymbol{S}_2)\boldsymbol{S}_2\right] - \boldsymbol{n}\left[\frac{5\kappa_1}{q}(\boldsymbol{n}\cdot\boldsymbol{S}_1)^2 + 5q\kappa_2(\boldsymbol{n}\cdot\boldsymbol{S}_2)^2\right]\right\}. \tag{3.3b}$$

Here, $q = m_1/m_2$ is the mass ratio, and we recall that the parameters κ_A characterize the mass quadrupole moments of the bodies.

We find that the quadratic spin contribution to the acceleration can be rewritten in a simpler way by introducing the spin variables

$$\boldsymbol{S}_0^+ = \frac{m}{m_1}\left(\frac{\kappa_1}{\kappa_2}\right)^{1/4}(1+\sqrt{1-\kappa_1\kappa_2})^{1/2}\boldsymbol{S}_1 + \frac{m}{m_2}\left(\frac{\kappa_2}{\kappa_1}\right)^{1/4}(1-\sqrt{1-\kappa_1\kappa_2})^{1/2}\boldsymbol{S}_2, \tag{3.4}$$

and $\boldsymbol{S}_0^-$, which is obtained by exchanging the labels 1 and 2 in the above equation.[2] Those variables generalize the quantity $\boldsymbol{S}_0$ of Ref. [60] in the case where the two bodies are not black holes. In terms of these spin variables the spin-spin part of the acceleration reads

$$\boldsymbol{a}_{\rm S_1S_2} + \boldsymbol{a}_{\rm S^2} = -\frac{3G}{2mr^4}[\boldsymbol{n}(\boldsymbol{S}_0^+\cdot\boldsymbol{S}_0^-) + (\boldsymbol{n}\cdot\boldsymbol{S}_0^+)\boldsymbol{S}_0^- + (\boldsymbol{n}\cdot\boldsymbol{S}_0^-)\boldsymbol{S}_0^+ - 5\boldsymbol{n}(\boldsymbol{n}\cdot\boldsymbol{S}_0^+)(\boldsymbol{n}\cdot\boldsymbol{S}_0^-)]. \tag{3.5}$$

The spin precession equations through 2PN order are [51,94]

$$\frac{d\boldsymbol{S}}{dt} = \frac{Gm\nu}{c^2r^2}\{[-4(\boldsymbol{v}\cdot\boldsymbol{S}) - 2\delta(\boldsymbol{v}\cdot\boldsymbol{\Sigma})]\boldsymbol{n} + [3(\boldsymbol{n}\cdot\boldsymbol{S}) + \delta(\boldsymbol{n}\cdot\boldsymbol{\Sigma})]\boldsymbol{v} + \dot{r}[2\boldsymbol{S}+\delta\boldsymbol{\Sigma}]\}, \tag{3.6a}$$

$$\frac{d\boldsymbol{\Sigma}}{dt} = \frac{Gm}{c^2r^2}\{[-2\delta(\boldsymbol{v}\cdot\boldsymbol{S}) - 2(1-2\nu)(\boldsymbol{v}\cdot\boldsymbol{\Sigma})]\boldsymbol{n} + [\delta(\boldsymbol{n}\cdot\boldsymbol{S}) + (1-\nu)(\boldsymbol{n}\cdot\boldsymbol{\Sigma})]\boldsymbol{v} + \dot{r}[\delta\boldsymbol{S} + (1-2\nu)\boldsymbol{\Sigma}]\}. \tag{3.6b}$$

It is often convenient to use a different set of spin variables $S^{\rm c}_{Ai}$ whose magnitude remains constant and that obey precession equations of the form $d\boldsymbol{S}^{\rm c}_A/dt = \boldsymbol{\Omega}_A \times \boldsymbol{S}^{\rm c}_A$. The relationship between the spin variables appearing in the equations of motion above and the constant spin variables is [50]

$$\boldsymbol{S}_{\rm c} = \boldsymbol{S} + \frac{Gm\nu}{rc^2}[2\boldsymbol{S}+\delta\boldsymbol{\Sigma}] - \frac{\nu}{2c^2}[(\boldsymbol{v}\cdot\boldsymbol{S}) + \delta(\boldsymbol{v}\cdot\boldsymbol{\Sigma})]\boldsymbol{v}, \tag{3.7a}$$

$$\boldsymbol{\Sigma}_{\rm c} = \boldsymbol{\Sigma} + \frac{Gm}{rc^2}[\delta\boldsymbol{S} + (1-2\nu)\boldsymbol{\Sigma}] - \frac{1}{2c^2}[\delta(\boldsymbol{v}\cdot\boldsymbol{S}) + (1-3\nu)(\boldsymbol{v}\cdot\boldsymbol{\Sigma})]\boldsymbol{v}. \tag{3.7b}$$

IV. WAVEFORMS WITH SPIN EFFECTS AT 2PN ORDER

A. General formalism

The gravitational radiation from the two-body system is calculated from symmetric trace-free radiative multipole moments I_L and J_L using the general formula from Ref. [69] truncated at 2PN order

[2] In the formal limit where the induced quadrupole of at least one body vanishes, so that e.g., $\kappa_2 \to 0$, we may define the effective spins as $\boldsymbol{S}_0^+ = \frac{m}{m_1}\sqrt{2}\boldsymbol{S}_1$, $\boldsymbol{S}_0^- = \frac{m}{m_1}\frac{\kappa_1}{\sqrt{2}}\boldsymbol{S}_1 + \frac{m}{m_2}\sqrt{2}\boldsymbol{S}_2$.

$$h_{ij}^{\mathrm{TT}} = \frac{2G}{Rc^4}\bigg\{I_{ab}^{(2)} + \frac{1}{3c}I_{abc}^{(3)}N^c + \frac{1}{12c^2}I_{abcd}^{(4)}N^cN^d + \frac{1}{60c^3}I_{abcde}^{(5)}N^cN^dN^e + \frac{1}{360c^4}I_{abcdef}^{(6)}N^cN^dN^eN^f + N^k\varepsilon_{cka}\bigg[\frac{4}{3c}J_{bc}^{(2)} + \frac{1}{2c^2}J_{bcd}^{(3)}N^d + \frac{2}{15c^3}J_{bcde}^{(4)}N^dN^e + \frac{1}{36c^4}J_{bcdef}^{(5)}N^dN^eN^f\bigg]\bigg\}\mathcal{P}_{ijab}^{\mathrm{TT}}, \tag{4.1}$$

where $\boldsymbol{N}$ is the unit vector pointing from the center of mass of the source to the observer's location and R is the distance between the source and the observer. Here, the superscript (n) signifies the nth time derivative, and the transverse-traceless projection operator is

$$\mathcal{P}_{ijab}^{\mathrm{TT}} = \mathcal{P}_{a(i}\mathcal{P}_{j)b} - \frac{1}{2}\mathcal{P}_{ij}\mathcal{P}_{ab}, \tag{4.2}$$

with $\mathcal{P}_{ij} = \delta_{ij} - N_iN_j$.

The gravitational radiation (4.1) can be rewritten in a post-Newtonian expansion as

$$h_{ij}^{\mathrm{TT}} = \frac{1}{c^4}\bigg(h_{ij\,\mathrm{TT}}^{\mathrm{Newt}} + \frac{1}{c^2}h_{ij\,\mathrm{TT}}^{\mathrm{1PN}} + \frac{1}{c^2}h_{ij\,\mathrm{TT}}^{\mathrm{1PNSO}} + \frac{1}{c^3}h_{ij\,\mathrm{TT}}^{\mathrm{1.5PNSO}} + \frac{1}{c^4}h_{ij\,\mathrm{TT}}^{\mathrm{2PN}} + \frac{1}{c^4}h_{ij\,\mathrm{TT}}^{\mathrm{2PNSO}} + \frac{1}{c^4}h_{ij\,\mathrm{TT}}^{\mathrm{2PNSS}} + \cdots\bigg). \tag{4.3}$$

The 1PN and 1.5PN spin terms are given explicitly in Refs. [51,52]. The terms in the source multipole moments that are *a priori* needed to compute the spin-orbit waveform exactly at 2PN order are identified by considering their schematic structure,

$$I_L = I_L^{\mathrm{Newt}} + \frac{1}{c^2}I_L^{\mathrm{1PN}} + \frac{1}{c^3}I_L^{\mathrm{SO}} + \frac{1}{c^4}(I_L^{\mathrm{2PN}} + I_L^{\mathrm{SS}}), \tag{4.4a}$$

$$J_L = J_L^{\mathrm{Newt}} + \frac{1}{c}J_L^{\mathrm{SO}} + \frac{1}{c^2}J_L^{\mathrm{1PN}} + \frac{1}{c^3}J_L^{\mathrm{1.5PNSO}}, \tag{4.4b}$$

together with the scalings of Eqs. (4.1) and (3.1a). Specifically, the following pieces are required: $(I_{abc}^{\mathrm{Newt}})^{(3)}$ using the 1.5PN motion and $(I_{abc}^{\mathrm{SO}})^{(3)}$ with $\boldsymbol{a}^{\mathrm{Newt}}$, $(J_{ab}^{\mathrm{SO}})^{(2)}$ with the 1PN motion and the spin evolution, $(J_{ab}^{\mathrm{1.5PNSO}})^{(2)}$ with $\boldsymbol{a}^{\mathrm{Newt}}$, $(J_{ab}^{\mathrm{Newt}})^{(2)}$ with the 1.5PN accurate motion, and $(J_{abcd}^{\mathrm{SO}})^{(4)}$ with $\boldsymbol{a}^{\mathrm{Newt}}$. For the SS part, we need $(I_{ab}^{\mathrm{Newt}})^{(2)}$ with a^{SS}, as the time derivative of I_{ab}^{SS} does not contribute at 2PN order. When we write the waveform in terms of the constant magnitude spin variables, there is an additional contribution to the 2PN spin piece of the waveform coming from J_{ab}^{SO} with $\boldsymbol{a}^{\mathrm{Newt}}$ and the 1PN conversion factor in Σ^{c}. The relevant spin contributions to the multipole moments are [50]

$$J_{ij}^{\mathrm{spin}} = \frac{\nu}{c}\bigg\{-\frac{3}{2}rn^{\langle i}\Sigma^{j\rangle}\bigg\} + \frac{\nu}{c^3}\bigg\{\bigg(\frac{3}{7}-\frac{16}{7}\nu\bigg)r\dot{r}v^{\langle i}\Sigma^{j\rangle} + \frac{3}{7}\delta r\dot{r}v^{\langle i}S^{j\rangle} + \bigg[\bigg(\frac{27}{14}-\frac{109}{14}\nu\bigg)(\boldsymbol{v}\cdot\boldsymbol{\Sigma}) + \frac{27}{14}\delta(\boldsymbol{v}\cdot\boldsymbol{S})\bigg]rn^{\langle i}v^{j\rangle} + \bigg[\bigg(-\frac{11}{14}+\frac{47}{14}\nu\bigg)(\boldsymbol{n}\cdot\boldsymbol{\Sigma}) - \frac{11}{14}\delta(\boldsymbol{n}\cdot\boldsymbol{S})\bigg]rv^{\langle i}v^{j\rangle} + \bigg[\bigg(\frac{19}{28}+\frac{13}{28}\nu\bigg)\frac{Gm}{r} + \bigg(-\frac{29}{28}+\frac{143}{28}\nu\bigg)v^2\bigg]rn^{\langle i}\Sigma^{j\rangle} + \bigg[\bigg(-\frac{4}{7}+\frac{31}{14}\nu\bigg)(\boldsymbol{n}\cdot\boldsymbol{\Sigma}) - \frac{29}{14}\delta(\boldsymbol{n}\cdot\boldsymbol{S})\bigg]Gmn^{\langle i}n^{j\rangle} + \bigg[-\frac{1}{14}\frac{Gm}{r} - \frac{2}{7}v^2\bigg]\delta rn^{\langle i}S^{j\rangle}\bigg\}, \tag{4.5a}$$

$$I_{ijk}^{\mathrm{spin}} = \frac{\nu}{c^3}r^2\bigg\{-\frac{9}{2}\delta n^{\langle i}n^j(\boldsymbol{v}\times\boldsymbol{S})^{k\rangle} - \frac{3}{2}(3-11\nu)n^{\langle i}n^j(\boldsymbol{v}\times\boldsymbol{\Sigma})^{k\rangle} + 3\delta n^{\langle i}v^j(\boldsymbol{n}\times\boldsymbol{S})^{k\rangle} + 3(1-3\nu)n^{\langle i}v^j(\boldsymbol{n}\times\boldsymbol{\Sigma})^{k\rangle}\bigg\}, \tag{4.5b}$$

$$J_{ijkl}^{\mathrm{spin}} = -\frac{5\nu}{2c}r^3\{\delta n^{\langle i}n^jn^kS^{l\rangle} + (1-3\nu)n^{\langle i}n^jn^k\Sigma^{l\rangle}\}. \tag{4.5c}$$

The nonspinning contributions to the multipole moments that we employed in our calculation are

$$I_{ij} = m\nu r^2n^{\langle i}n^{j\rangle}, \tag{4.6a}$$

$$I_{ijk} = -m\nu r^3\delta n^{\langle i}n^jn^{k\rangle}, \tag{4.6b}$$

$$J_{ij} = -m\nu r^2\delta\varepsilon_{ab\langle i}n^{j\rangle}n^av^b. \tag{4.6c}$$

B. Spin-orbit effects

Using the multipole moments of Eqs. (4.5) and (4.6) in Eq. (4.1) and substituting the equations of motion (3.1) and (3.3b), we find the following 2PN spin-orbit piece:

$$
\begin{aligned}
h_{ij\,\mathrm{TT}}^{\mathrm{2PNSO}} = \frac{2G^2 m\nu}{r^2 R}\mathcal{P}_{ijab}^{\mathrm{TT}}\bigg\{ & n^a n^b\bigg[\frac{5}{2}(3-13\nu)\dot{r}^2(\boldsymbol{n}\times\boldsymbol{\Sigma}_{\mathrm{c}})\cdot\boldsymbol{N} + 30(1-4\nu)(\boldsymbol{n}\cdot\boldsymbol{N})\dot{r}(\boldsymbol{n}\times\boldsymbol{v})\cdot\boldsymbol{\Sigma}_{\mathrm{c}} \\
& - (7-29\nu)\dot{r}(\boldsymbol{v}\times\boldsymbol{\Sigma}_{\mathrm{c}})\cdot\boldsymbol{N} - 6(1-4\nu)(\boldsymbol{v}\cdot\boldsymbol{N})(\boldsymbol{n}\times\boldsymbol{v})\cdot\boldsymbol{\Sigma}_{\mathrm{c}} - \frac{1}{2}(3-13\nu)v^2(\boldsymbol{n}\times\boldsymbol{\Sigma}_{\mathrm{c}})\cdot\boldsymbol{N} \\
& - \frac{2Gm}{3r}(1-5\nu)(\boldsymbol{n}\times\boldsymbol{\Sigma}_{\mathrm{c}})\cdot\boldsymbol{N} + \delta\bigg(\frac{35}{2}\dot{r}^2(\boldsymbol{n}\times\boldsymbol{S}_{\mathrm{c}})\cdot\boldsymbol{N} - \frac{7}{2}v^2(\boldsymbol{n}\times\boldsymbol{S}_{\mathrm{c}})\cdot\boldsymbol{N} + 60(\boldsymbol{n}\cdot\boldsymbol{N})\dot{r}(\boldsymbol{n}\times\boldsymbol{v})\cdot\boldsymbol{S}_{\mathrm{c}} \\
& - 12(\boldsymbol{v}\cdot\boldsymbol{N})(\boldsymbol{n}\times\boldsymbol{v})\cdot\boldsymbol{S}_{\mathrm{c}} - 13\dot{r}(\boldsymbol{v}\times\boldsymbol{S}_{\mathrm{c}})\cdot\boldsymbol{N}\bigg)\bigg] + n^a(\boldsymbol{n}\times\boldsymbol{S}_{\mathrm{c}})^b\delta\Big[35(\boldsymbol{n}\cdot\boldsymbol{N})\dot{r}^2 - 14(\boldsymbol{v}\cdot\boldsymbol{N})\dot{r} - 7(\boldsymbol{n}\cdot\boldsymbol{N})v^2\Big] \\
& + n^a(\boldsymbol{n}\times\boldsymbol{N})^b\bigg[\frac{5}{2}(3-13\nu)\dot{r}^2(\boldsymbol{n}\cdot\boldsymbol{\Sigma}_{\mathrm{c}}) - \frac{1}{2}(3-13\nu)v^2(\boldsymbol{n}\cdot\boldsymbol{\Sigma}_{\mathrm{c}}) + \frac{15}{2}(1-3\nu)\dot{r}^2(\boldsymbol{n}\cdot\boldsymbol{N})(\boldsymbol{N}\cdot\boldsymbol{\Sigma}_{\mathrm{c}}) \\
& - 5(1-3\nu)\dot{r}(\boldsymbol{v}\cdot\boldsymbol{N})(\boldsymbol{N}\cdot\boldsymbol{\Sigma}_{\mathrm{c}}) - \frac{3}{2}(1-3\nu)v^2(\boldsymbol{n}\cdot\boldsymbol{N})(\boldsymbol{N}\cdot\boldsymbol{\Sigma}_{\mathrm{c}}) - \frac{2Gm}{r}(1-3\nu)(\boldsymbol{n}\cdot\boldsymbol{N})(\boldsymbol{N}\cdot\boldsymbol{\Sigma}_{\mathrm{c}}) \\
& + \frac{4Gm}{3r}(1-5\nu)(\boldsymbol{n}\cdot\boldsymbol{\Sigma}_{\mathrm{c}}) - (3+11\nu)\dot{r}(\boldsymbol{v}\cdot\boldsymbol{\Sigma}_{\mathrm{c}}) + \delta\bigg(\frac{4Gm}{r}(\boldsymbol{n}\cdot\boldsymbol{S}_{\mathrm{c}}) + \frac{35}{2}\dot{r}^2(\boldsymbol{n}\cdot\boldsymbol{S}_{\mathrm{c}}) - \frac{7}{2}v^2(\boldsymbol{n}\cdot\boldsymbol{S}_{\mathrm{c}}) \\
& + \frac{15}{2}\dot{r}^2(\boldsymbol{n}\cdot\boldsymbol{N})(\boldsymbol{N}\cdot\boldsymbol{S}_{\mathrm{c}}) - \frac{2Gm}{r}(\boldsymbol{n}\cdot\boldsymbol{N})(\boldsymbol{N}\cdot\boldsymbol{S}_{\mathrm{c}}) - \frac{3}{2}v^2(\boldsymbol{n}\cdot\boldsymbol{N})(\boldsymbol{N}\cdot\boldsymbol{S}_{\mathrm{c}}) - 5\dot{r}(\boldsymbol{v}\cdot\boldsymbol{N})(\boldsymbol{N}\cdot\boldsymbol{S}_{\mathrm{c}}) + \dot{r}(\boldsymbol{v}\cdot\boldsymbol{S}_{\mathrm{c}})\bigg)\bigg] \\
& + n^a(\boldsymbol{n}\times\boldsymbol{\Sigma}_{\mathrm{c}})^b\bigg[5(3-13\nu)(\boldsymbol{n}\cdot\boldsymbol{N})\dot{r}^2 - (3-13\nu)(\boldsymbol{n}\cdot\boldsymbol{N})v^2 - 2(3-14\nu)(\boldsymbol{v}\cdot\boldsymbol{N})\dot{r} - \frac{4Gm}{3r}(1-5\nu)(\boldsymbol{n}\cdot\boldsymbol{N})\bigg] \\
& + n^a(\boldsymbol{n}\times\boldsymbol{v})^b\dot{r}[2(1-4\nu)(\boldsymbol{N}\cdot\boldsymbol{\Sigma}_{\mathrm{c}}) + 6\delta(\boldsymbol{N}\cdot\boldsymbol{S}_{\mathrm{c}})] + (\boldsymbol{n}\times\boldsymbol{N})^a\Sigma_c^b\bigg[\frac{5}{4}(1+7\nu)\dot{r}^2 + \frac{15}{4}(1-3\nu)(\boldsymbol{n}\cdot\boldsymbol{N})^2\dot{r}^2 \\
& - 5(1-3\nu)(\boldsymbol{n}\cdot\boldsymbol{N})(\boldsymbol{v}\cdot\boldsymbol{N})\dot{r} + \frac{5}{3}(1-3\nu)(\boldsymbol{v}\cdot\boldsymbol{N})^2 + \frac{1}{12}(11-25\nu)v^2 - \frac{3}{4}(1-3\nu)(\boldsymbol{n}\cdot\boldsymbol{N})^2v^2 - \frac{Gm}{3r}(11+2\nu) \\
& - \frac{Gm}{r}(1-3\nu)(\boldsymbol{n}\cdot\boldsymbol{N})^2\bigg] + (\boldsymbol{n}\times\boldsymbol{N})^a S_c^b\delta\bigg[-\frac{5}{4}\dot{r}^2 + \frac{15}{4}(\boldsymbol{n}\cdot\boldsymbol{N})^2\dot{r}^2 - 5(\boldsymbol{n}\cdot\boldsymbol{N})(\boldsymbol{v}\cdot\boldsymbol{N})\dot{r} \\
& + \frac{5}{3}(\boldsymbol{v}\cdot\boldsymbol{N})^2 + \frac{1}{4}v^2 - \frac{3}{4}(\boldsymbol{n}\cdot\boldsymbol{N})^2v^2 - \frac{Gm}{r}(\boldsymbol{n}\cdot\boldsymbol{N})^2\bigg] + (\boldsymbol{n}\times\boldsymbol{v})^a\Sigma_c^b(1-4\nu)[2(\boldsymbol{v}\cdot\boldsymbol{N}) - 2(\boldsymbol{n}\cdot\boldsymbol{N})\dot{r}] \\
& + n^a v^b\bigg[36(-1+4\nu)(\boldsymbol{n}\cdot\boldsymbol{N})(\boldsymbol{n}\times\boldsymbol{v})\cdot\boldsymbol{\Sigma}_{\mathrm{c}} - 4(2-9\nu)\dot{r}(\boldsymbol{n}\times\boldsymbol{\Sigma}_{\mathrm{c}})\cdot\boldsymbol{N} + \frac{2}{3}(13-55\nu)(\boldsymbol{v}\times\boldsymbol{\Sigma}_{\mathrm{c}})\cdot\boldsymbol{N} \\
& + \delta\bigg(-72(\boldsymbol{n}\cdot\boldsymbol{N})(\boldsymbol{n}\times\boldsymbol{v})\cdot\boldsymbol{S}_{\mathrm{c}} - 20\dot{r}(\boldsymbol{n}\times\boldsymbol{S}_{\mathrm{c}})\cdot\boldsymbol{N} + \frac{50}{3}(\boldsymbol{v}\times\boldsymbol{S}_{\mathrm{c}})\cdot\boldsymbol{N}\bigg)\bigg] + (\boldsymbol{n}\times\boldsymbol{v})^a S_c^b\delta\bigg[-6(\boldsymbol{n}\cdot\boldsymbol{N})\dot{r} + \frac{14}{3}(\boldsymbol{v}\cdot\boldsymbol{N})\bigg] \\
& + n^a(\boldsymbol{v}\times\boldsymbol{S}_{\mathrm{c}})^b\delta[-26\dot{r}(\boldsymbol{n}\cdot\boldsymbol{N}) + 12(\boldsymbol{v}\cdot\boldsymbol{N})] + n^a(\boldsymbol{v}\times\boldsymbol{\Sigma}_{\mathrm{c}})^b\bigg[2(-7+29\nu)\dot{r}(\boldsymbol{n}\cdot\boldsymbol{N}) + \frac{2}{3}(10-43\nu)(\boldsymbol{v}\cdot\boldsymbol{N})\bigg] \\
& + v^a(\boldsymbol{v}\times\boldsymbol{S}_{\mathrm{c}})^b\delta\frac{64}{3}(\boldsymbol{n}\cdot\boldsymbol{N}) + v^a(\boldsymbol{n}\times\boldsymbol{\Sigma}_{\mathrm{c}})^b\bigg[-2(5-22\nu)\dot{r}(\boldsymbol{n}\cdot\boldsymbol{N}) + \frac{4}{3}(1-6\nu)(\boldsymbol{v}\cdot\boldsymbol{N})\bigg] \\
& + v^a(\boldsymbol{v}\times\boldsymbol{\Sigma}_{\mathrm{c}})^b\frac{2}{3}(16-67\nu)(\boldsymbol{n}\cdot\boldsymbol{N}) + v^a(\boldsymbol{n}\times\boldsymbol{S}_{\mathrm{c}})^b\delta\bigg[-26\dot{r}(\boldsymbol{n}\cdot\boldsymbol{N}) + \frac{4}{3}(\boldsymbol{v}\cdot\boldsymbol{N})\bigg] \\
& + v^a(\boldsymbol{n}\times\boldsymbol{v})^b\bigg[2(-1+4\nu)(\boldsymbol{N}\cdot\boldsymbol{\Sigma}_{\mathrm{c}}) - \frac{14}{3}\delta(\boldsymbol{N}\cdot\boldsymbol{S}_{\mathrm{c}})\bigg] + v^a(\boldsymbol{n}\times\boldsymbol{N})^b\bigg[-(3-23\nu)\dot{r}(\boldsymbol{n}\cdot\boldsymbol{\Sigma}_{\mathrm{c}}) \\
& - 5(1-3\nu)\dot{r}(\boldsymbol{n}\cdot\boldsymbol{N})(\boldsymbol{N}\cdot\boldsymbol{\Sigma}_{\mathrm{c}}) + \frac{2}{3}(1+8\nu)(\boldsymbol{v}\cdot\boldsymbol{\Sigma}_{\mathrm{c}}) + \frac{10}{3}(1-3\nu)(\boldsymbol{v}\cdot\boldsymbol{N})(\boldsymbol{N}\cdot\boldsymbol{\Sigma}_{\mathrm{c}}) + \delta\bigg(\frac{10}{3}(\boldsymbol{v}\cdot\boldsymbol{N})(\boldsymbol{N}\cdot\boldsymbol{S}_{\mathrm{c}}) \\
& - 11\dot{r}(\boldsymbol{n}\cdot\boldsymbol{S}_{\mathrm{c}}) - 5\dot{r}(\boldsymbol{n}\cdot\boldsymbol{N})(\boldsymbol{N}\cdot\boldsymbol{S}_{\mathrm{c}}) - \frac{2}{3}(\boldsymbol{v}\cdot\boldsymbol{S}_{\mathrm{c}})\bigg)\bigg] + S_{\mathrm{c}}^a(\boldsymbol{v}\times\boldsymbol{N})^b\delta\bigg[\frac{5}{6}\dot{r} - \frac{5}{2}\dot{r}(\boldsymbol{n}\cdot\boldsymbol{N})^2 + \frac{10}{3}(\boldsymbol{v}\cdot\boldsymbol{N})(\boldsymbol{n}\cdot\boldsymbol{N})\bigg] \\
& + \Sigma_{\mathrm{c}}^a(\boldsymbol{v}\times\boldsymbol{N})^b\bigg[-\frac{29}{6}(1+\nu)\dot{r} - \frac{5}{2}(1-3\nu)\dot{r}(\boldsymbol{n}\cdot\boldsymbol{N})^2 + \frac{10}{3}(1-3\nu)(\boldsymbol{v}\cdot\boldsymbol{N})(\boldsymbol{n}\cdot\boldsymbol{N})\bigg] \\
& + v^a(\boldsymbol{v}\times\boldsymbol{N})^b\bigg[-\frac{40\nu}{3}(\boldsymbol{n}\cdot\boldsymbol{\Sigma}_{\mathrm{c}}) + \frac{10}{3}(1-3\nu)(\boldsymbol{n}\cdot\boldsymbol{N})(\boldsymbol{N}\cdot\boldsymbol{\Sigma}_{\mathrm{c}}) + \delta\bigg(\frac{20}{3}(\boldsymbol{n}\cdot\boldsymbol{S}_{\mathrm{c}}) + \frac{10}{3}(\boldsymbol{n}\cdot\boldsymbol{N})(\boldsymbol{N}\cdot\boldsymbol{S}_{\mathrm{c}})\bigg)\bigg] \\
& + v^a v^b\bigg[\bigg(\frac{2}{3}-4\nu\bigg)(\boldsymbol{n}\times\boldsymbol{\Sigma}_{\mathrm{c}})\cdot\boldsymbol{N} + \frac{2}{3}\delta(\boldsymbol{n}\times\boldsymbol{S}_{\mathrm{c}})\cdot\boldsymbol{N}\bigg] + (\boldsymbol{\Sigma}_{\mathrm{c}}\times\boldsymbol{N})^a n^b\bigg[\frac{5}{4}(1+7\nu)\dot{r}^2 + \frac{15}{4}(1-3\nu)\dot{r}^2(\boldsymbol{n}\cdot\boldsymbol{N})^2
\end{aligned}
$$

$$+ 5(-1 + 3\nu)\dot{r}(\boldsymbol{n}\cdot\boldsymbol{N})(\boldsymbol{v}\cdot\boldsymbol{N}) + \frac{5}{3}(1-3\nu)(\boldsymbol{v}\cdot\boldsymbol{N})^2 + \frac{1}{12}(11-25\nu)v^2 + \frac{3}{4}(-1+3\nu)(\boldsymbol{n}\cdot\boldsymbol{N})^2 v^2$$
$$+ \frac{Gm}{3r}(-17+10\nu) + \frac{Gm}{r}(-1+3\nu)(\boldsymbol{n}\cdot\boldsymbol{N})^2\Bigg] + (\boldsymbol{S}_\mathrm{c}\times\boldsymbol{N})^a n^b \delta\Bigg[-\frac{5}{4}\dot{r}^2 + \frac{15}{4}\dot{r}^2(\boldsymbol{n}\cdot\boldsymbol{N})^2 - 5\dot{r}(\boldsymbol{n}\cdot\boldsymbol{N})(\boldsymbol{v}\cdot\boldsymbol{N})$$
$$+ \frac{5}{3}(\boldsymbol{v}\cdot\boldsymbol{N})^2 + \frac{1}{4}v^2 - \frac{3}{4}v^2(\boldsymbol{n}\cdot\boldsymbol{N})^2 - \frac{2Gm}{r} - \frac{Gm}{r}(\boldsymbol{n}\cdot\boldsymbol{N})^2\Bigg] + (\boldsymbol{\Sigma}_\mathrm{c}\times\boldsymbol{N})^a v^b\Bigg[-\frac{29}{6}(1+\nu)\dot{r}$$
$$+ \frac{5}{2}(-1+3\nu)\dot{r}(\boldsymbol{n}\cdot\boldsymbol{N})^2 + \frac{10}{3}(1-3\nu)(\boldsymbol{n}\cdot\boldsymbol{N})(\boldsymbol{v}\cdot\boldsymbol{N})\Bigg] + (\boldsymbol{S}_\mathrm{c}\times\boldsymbol{N})^a v^b \delta\Bigg[\frac{5}{6}\dot{r} - \frac{5}{2}\dot{r}(\boldsymbol{n}\cdot\boldsymbol{N})^2 + \frac{10}{3}(\boldsymbol{n}\cdot\boldsymbol{N})(\boldsymbol{v}\cdot\boldsymbol{N})\Bigg]$$
$$+ (\boldsymbol{v}\times\boldsymbol{N})^a n^b\Bigg[(-3+23\nu)\dot{r}(\boldsymbol{n}\cdot\boldsymbol{\Sigma}_\mathrm{c}) + 5(-1+3\nu)\dot{r}(\boldsymbol{n}\cdot\boldsymbol{N})(\boldsymbol{\Sigma}_\mathrm{c}\cdot\boldsymbol{N}) + \frac{1}{3}(5+7\nu)(\boldsymbol{v}\cdot\boldsymbol{\Sigma}_\mathrm{c})$$
$$+ \frac{10}{3}(1-3\nu)(\boldsymbol{v}\cdot\boldsymbol{N})(\boldsymbol{\Sigma}_\mathrm{c}\cdot\boldsymbol{N}) + \delta\left(-11\dot{r}(\boldsymbol{n}\cdot\boldsymbol{S}_\mathrm{c}) - 5\dot{r}(\boldsymbol{n}\cdot\boldsymbol{N})(\boldsymbol{S}_\mathrm{c}\cdot\boldsymbol{N}) + \frac{1}{3}(\boldsymbol{v}\cdot\boldsymbol{S}_\mathrm{c}) + \frac{10}{3}(\boldsymbol{v}\cdot\boldsymbol{N})(\boldsymbol{S}_\mathrm{c}\cdot\boldsymbol{N})\right)\Bigg]\Bigg]. \tag{4.7}$$

These contributions add linearly to the other known terms in the waveform. Note that in Eq. (4.7) we have already anticipated the transverse-traceless projection and simplified the expression (4.7) using $\delta^{ij}_\mathrm{TT} = N^i_\mathrm{TT} = N^j_\mathrm{TT} = 0$ and the interchange identity [51]

$$\mathcal{P}^\mathrm{TT}_{ijab} A^a (\boldsymbol{B}\times\boldsymbol{N})^b = \mathcal{P}^\mathrm{TT}_{ijab} B^a (\boldsymbol{A}\times\boldsymbol{N})^b, \tag{4.8}$$

for any vectors $\boldsymbol{A}$ and $\boldsymbol{B}$.

C. Spin-spin effects

Spin-spin terms in the waveform at 2PN order are entirely attributable to the equations of motion; they arise when substituting $\boldsymbol{a}^\mathrm{SS}$ in the time derivatives of I^Newt_{ab}. The second time derivative of the contribution $I^{\mathrm{S}^2}_{ab}$ given in Eq. (2.19) is at least of 3PN order (because of the fact that spins are constant at leading approximation) and therefore vanishes for our calculation. We derive

$$h^{2\mathrm{PNSS}}_{ij\,\mathrm{TT}} = \frac{6G^2\nu}{r^3 R}\mathcal{P}^\mathrm{TT}_{ijab}\{n^a n^b[5(\boldsymbol{n}\cdot\boldsymbol{S}_0^+)(\boldsymbol{n}\cdot\boldsymbol{S}_0^-) - (\boldsymbol{S}_0^+\cdot\boldsymbol{S}_0^-)] - n^a S_0^{+b}(\boldsymbol{n}\cdot\boldsymbol{S}_0^-) - n^a S_0^{-b}(\boldsymbol{n}\cdot\boldsymbol{S}_0^+)\}. \tag{4.9}$$

We notice that the spin-orbit contributions at 2PN order are zero for an equal-mass, equal-spin black-hole binary. This is a consequence of the multipoles (4.5) being zero for this highly symmetric binary configuration.

The general results (4.7) and (4.9) are available as a MATHEMATICA notebook upon request to be used to compute the gravitational polarizations and spherical harmonic modes for precessing binaries for any choice of the source frame and the polarization triad [8,35,51,72–76]. Below, we shall derive the polarizations and spin-weighted spherical-harmonic modes for the case of nonprecessing compact binaries on circular orbits.

D. Reduction to quasicircular orbits

We now specialize Eqs. (4.7) and (4.9) to the case of orbits that have a constant separation r in the absence of radiation reaction and for which the precession time scale is much longer than an orbital period. The details of the derivation of the modified Kepler law relating the orbit-averaged orbital angular frequency ω and the orbit-averaged orbital separation are discussed in Ref. [93]. The instantaneous accelerations (3.1) and (3.5) are projected onto a triad consisting of the following unit vectors: $\boldsymbol{n} = \boldsymbol{x}/r$, the vector $\boldsymbol{\ell} = \boldsymbol{L}_\mathrm{N}/|\boldsymbol{L}_\mathrm{N}|$ orthogonal to the instantaneous orbital plane, where $\boldsymbol{L}_\mathrm{N} = m\nu\boldsymbol{x}\times\boldsymbol{v}$ denotes the Newtonian orbital angular momentum, and $\boldsymbol{\lambda} = \boldsymbol{\ell}\times\boldsymbol{n}$. The orbital separation r and angular frequency ω are decomposed into their orbit averaged piece, indicated by an overbar, and remaining fluctuating pieces, $r = \bar{r} + \delta r$ and $\omega = \bar{\omega} + \delta\omega$. Projecting the equations of motion along $\boldsymbol{\lambda}$ yields the equality $2\omega\dot{r} + \dot{\omega}r$ or, equivalently [93],

$$\frac{d}{dt}(\omega r^2) = -\frac{3G}{2m\omega r^3 c^4}\frac{d}{dt}(\boldsymbol{n}\cdot\boldsymbol{S}_0^+)(\boldsymbol{n}\cdot\boldsymbol{S}_0^-). \tag{4.10}$$

At the 2PN order, r and ω can be replaced by the constants $\bar{r}$ and $\bar{\omega}$, respectively, on the right-hand side. The expression for ωr^2 follows from (i) dropping the time derivatives in the above equation, and (ii) adding an integration constant determined by averaging ωr^2 over an orbit. Inserting the result in the projection along $\boldsymbol{n}$ of the equations of motion,

$$\ddot{r} - \omega^2 r = (\boldsymbol{n}\cdot\boldsymbol{a}), \tag{4.11}$$

and linearizing in δr, we find an explicit solution to the differential equation given by

$$\dot{r} = \frac{d\delta r}{dt} = -\frac{\omega}{2m^2 rc^4}[(\boldsymbol{n}\cdot\boldsymbol{S}_0^+)(\boldsymbol{\lambda}\cdot\boldsymbol{S}_0^-) + (\boldsymbol{\lambda}\cdot\boldsymbol{S}_0^+)(\boldsymbol{n}\cdot\boldsymbol{S}_0^-)], \tag{4.12a}$$

$$\omega^2 = \frac{\ddot{r} - (\boldsymbol{n}\cdot\boldsymbol{a})}{r} = \frac{Gm}{r^3}\bigg[1 - (3-\nu)\frac{Gm}{rc^2} - \left(\frac{Gm}{rc^2}\right)^{\frac{1}{2}}\frac{5(\boldsymbol{\ell}\cdot\boldsymbol{S}_{\rm c}) + 3\delta(\boldsymbol{\ell}\cdot\boldsymbol{\Sigma}_{\rm c})}{mrc^2} + \frac{1}{2m^2r^2c^4}((\boldsymbol{S}_0^+\cdot\boldsymbol{S}_0^-) + 2(\boldsymbol{\ell}\cdot\boldsymbol{S}_0^+)(\boldsymbol{\ell}\cdot\boldsymbol{S}_0^-) - 5(\boldsymbol{n}\cdot\boldsymbol{S}_0^+)(\boldsymbol{n}\cdot\boldsymbol{S}_0^-))\bigg]. \tag{4.12b}$$

Inverting Eq. (4.12b) to write r as a function of ω in Eq. (4.7) and inserting there the expression (4.12a) of $\dot{r}$, we obtain the following spin-orbit terms in the waveform:

$$\begin{aligned} h^{\rm 2PNSO}_{ij\,\rm TT} = \frac{G^2\nu m\omega^2}{3R}\mathcal{P}^{\rm TT}_{ijab}\{&n^a n^b[4(1-7\nu)(\boldsymbol{\ell}\cdot\boldsymbol{\Sigma}_{\rm c})(\boldsymbol{\lambda}\cdot\boldsymbol{N}) - (13-59\nu)(\boldsymbol{n}\times\boldsymbol{\Sigma}_{\rm c})\cdot\boldsymbol{N} - 21\delta(\boldsymbol{n}\times\boldsymbol{S}_{\rm c})\cdot\boldsymbol{N}] \\ &+ \lambda^a\lambda^b[4(7-24\nu)(\boldsymbol{\ell}\cdot\boldsymbol{\Sigma}^{\rm c})(\boldsymbol{\lambda}\cdot\boldsymbol{N}) + 4(1-6\nu)(\boldsymbol{n}\times\boldsymbol{\Sigma}^{\rm c})\cdot\boldsymbol{N} + \delta(4(\boldsymbol{n}\times\boldsymbol{S}^{\rm c})\cdot\boldsymbol{N} + 52(\boldsymbol{\ell}\cdot\boldsymbol{S}^{\rm c})(\boldsymbol{\lambda}\cdot\boldsymbol{N}))] \\ &+ \lambda^a n^b[4(13-55\nu)(\boldsymbol{\lambda}\times\boldsymbol{\Sigma}_{\rm c})\cdot\boldsymbol{N} + 2(-63+239\nu)(\boldsymbol{n}\cdot\boldsymbol{N})(\boldsymbol{\ell}\cdot\boldsymbol{\Sigma}_{\rm c}) + \delta(100(\boldsymbol{\lambda}\times\boldsymbol{S}_{\rm c})\cdot\boldsymbol{N} \\ &- 262(\boldsymbol{n}\cdot\boldsymbol{N})(\boldsymbol{\ell}\cdot\boldsymbol{S}_{\rm c}))] + \Sigma_{\rm c}^a\ell^b 12(1-4\nu)(\boldsymbol{\lambda}\cdot\boldsymbol{N}) + \lambda^a\ell^b[12(-1+4\nu)(\boldsymbol{N}\cdot\boldsymbol{\Sigma}_{\rm c}) + 8(1-6\nu)(\boldsymbol{\lambda}\cdot\boldsymbol{\Sigma}_{\rm c})(\boldsymbol{\lambda}\cdot\boldsymbol{N}) \\ &+ 4(-16+67\nu)(\boldsymbol{n}\cdot\boldsymbol{\Sigma}_{\rm c})(\boldsymbol{n}\cdot\boldsymbol{N}) + \delta(-28(\boldsymbol{N}\cdot\boldsymbol{S}_{\rm c}) + 8(\boldsymbol{\lambda}\cdot\boldsymbol{S}_{\rm c})(\boldsymbol{\lambda}\cdot\boldsymbol{N}) - 128(\boldsymbol{n}\cdot\boldsymbol{S}_{\rm c})(\boldsymbol{n}\cdot\boldsymbol{N}))] \\ &+ n^a\ell^b[2(-13+59\nu)(\boldsymbol{\lambda}\cdot\boldsymbol{\Sigma}_{\rm c})(\boldsymbol{n}\cdot\boldsymbol{N}) + 4(-10+43\nu)(\boldsymbol{n}\cdot\boldsymbol{\Sigma}_{\rm c})(\boldsymbol{\lambda}\cdot\boldsymbol{N}) + \delta(-42(\boldsymbol{\lambda}\cdot\boldsymbol{S}_{\rm c})(\boldsymbol{n}\cdot\boldsymbol{N}) \\ &- 72(\boldsymbol{n}\cdot\boldsymbol{S}_{\rm c})(\boldsymbol{\lambda}\cdot\boldsymbol{N}))] + S_{\rm c}^a\ell^b 28\delta(\boldsymbol{\lambda}\cdot\boldsymbol{N}) + n^a(\boldsymbol{n}\times\boldsymbol{N})^b[-(1+\nu)(\boldsymbol{n}\cdot\boldsymbol{\Sigma}_{\rm c}) - 21(1-3\nu)(\boldsymbol{n}\cdot\boldsymbol{N})(\boldsymbol{N}\cdot\boldsymbol{\Sigma}_{\rm c}) \\ &+ \delta(3(\boldsymbol{n}\cdot\boldsymbol{S}_{\rm c}) - 21(\boldsymbol{n}\cdot\boldsymbol{N})(\boldsymbol{N}\cdot\boldsymbol{S}_{\rm c}))] + \lambda^a(\boldsymbol{n}\times\boldsymbol{N})^b[2(7+23\nu)(\boldsymbol{\lambda}\cdot\boldsymbol{\Sigma}_{\rm c}) + 40(1-3\nu)(\boldsymbol{N}\cdot\boldsymbol{\Sigma}_{\rm c})(\boldsymbol{\lambda}\cdot\boldsymbol{N}) \\ &+ \delta(40(\boldsymbol{N}\cdot\boldsymbol{S}_{\rm c})(\boldsymbol{\lambda}\cdot\boldsymbol{N}) - 2(\boldsymbol{\lambda}\cdot\boldsymbol{S}_{\rm c}))] + \Sigma_{\rm c}^a(\boldsymbol{n}\times\boldsymbol{N})^b[-(21+17\nu) + 20(1-3\nu)(\boldsymbol{\lambda}\cdot\boldsymbol{N})^2 \\ &+ 21(-1+3\nu)(\boldsymbol{n}\cdot\boldsymbol{N})^2] + S_{\rm c}^a(\boldsymbol{n}\times\boldsymbol{N})^b\delta[-9 + 20(\boldsymbol{\lambda}\cdot\boldsymbol{N})^2 - 21(\boldsymbol{n}\cdot\boldsymbol{N})^2] + S_{\rm c}^a(\boldsymbol{\lambda}\times\boldsymbol{N})^b 40\delta(\boldsymbol{\lambda}\cdot\boldsymbol{N})(\boldsymbol{n}\cdot\boldsymbol{N}) \\ &+ \lambda^a(\boldsymbol{\lambda}\times\boldsymbol{N})^b[-80\nu(\boldsymbol{n}\cdot\boldsymbol{\Sigma}_{\rm c}) + 20(1-3\nu)(\boldsymbol{n}\cdot\boldsymbol{N})(\boldsymbol{N}\cdot\boldsymbol{\Sigma}_{\rm c}) + \delta(40(\boldsymbol{n}\cdot\boldsymbol{S}_{\rm c}) + 20(\boldsymbol{N}\cdot\boldsymbol{S}_{\rm c})(\boldsymbol{n}\cdot\boldsymbol{N}))] \\ &+ \Sigma_{\rm c}^a(\boldsymbol{\lambda}\times\boldsymbol{N})^b 40(1-3\nu)(\boldsymbol{\lambda}\cdot\boldsymbol{N})(\boldsymbol{n}\cdot\boldsymbol{N})\}. \end{aligned} \tag{4.13}$$

Here, we have used that

$$(\boldsymbol{n}\times\boldsymbol{S}_{\rm c})^i = -\lambda^i(\boldsymbol{\ell}\cdot\boldsymbol{S}_{\rm c}) + \ell^i(\boldsymbol{\lambda}\cdot\boldsymbol{S}_{\rm c}), \tag{4.14}$$

and similarly for $\boldsymbol{\Sigma}_{\rm c}$.

Finally, we derive the 2PN spin-spin terms for circular orbits. They read

$$\begin{aligned} h^{\rm 2PNSS}_{ij\,\rm TT} = \frac{2G\nu\omega^2}{mR}\mathcal{P}^{\rm TT}_{ijab}\bigg\{&n^a n^b\bigg[-\frac{8}{3}(\boldsymbol{S}_0^+\cdot\boldsymbol{S}_0^-) + \frac{2}{3}(\boldsymbol{\ell}\cdot\boldsymbol{S}_0^+)(\boldsymbol{\ell}\cdot\boldsymbol{S}_0^-) + \frac{40}{3}(\boldsymbol{n}\cdot\boldsymbol{S}_0^+)(\boldsymbol{n}\cdot\boldsymbol{S}_0^-)\bigg] \\ &+ \lambda^a\lambda^b\bigg[\frac{2}{3}(\boldsymbol{S}_0^+\cdot\boldsymbol{S}_0^-) + \frac{4}{3}(\boldsymbol{\ell}\cdot\boldsymbol{S}_0^+)(\boldsymbol{\ell}\cdot\boldsymbol{S}_0^-) - \frac{10}{3}(\boldsymbol{n}\cdot\boldsymbol{S}_0^+)(\boldsymbol{n}\cdot\boldsymbol{S}_0^-)\bigg] - 2n^a\lambda^b[(\boldsymbol{n}\cdot\boldsymbol{S}_0^+)(\boldsymbol{\lambda}\cdot\boldsymbol{S}_0^-) \\ &+ (\boldsymbol{n}\cdot\boldsymbol{S}_0^-)(\boldsymbol{\lambda}\cdot\boldsymbol{S}_0^+)] - 3(\boldsymbol{n}\cdot\boldsymbol{S}_0^+)n^{(a}S_0^{-b)} - 3(\boldsymbol{n}\cdot\boldsymbol{S}_0^-)n^{(a}S_0^{+b)}\bigg\}. \end{aligned} \tag{4.15}$$

E. Polarizations for nonprecessing, spinning compact bodies

The two polarization states h_+ and $h_\times$ are obtained by choosing a coordinate system and taking linear combinations of the components of $h^{\rm TT}_{ij}$. Using an orthonormal triad consisting of $\boldsymbol{N}$ and two polarization vectors $\boldsymbol{P}$ and $\boldsymbol{Q}$, the polarizations are

$$h_+ = \frac{1}{2}(P^iP^j - Q^iQ^j)h^{\rm TT}_{ij}, \tag{4.16a}$$

$$h_\times = \frac{1}{2}(P^iQ^j + Q^iP^j)h^{\rm TT}_{ij}. \tag{4.16b}$$

Although different choices of $\boldsymbol{P}$ and $\boldsymbol{Q}$ give different polarizations, the particular linear combination of h_+ and $h_\times$ corresponding to the physical strain measured in a detector is independent of the convention used. For nonspinning binaries, one usually chooses a coordinate system such that the orbital plane lies in the x-y plane, and the direction of gravitational-wave propagation $\boldsymbol{N}$ is in the x-z plane.

When the spins of the bodies are aligned or antialigned with the orbital angular momentum, the system's evolution is qualitatively similar to the case of nonspinning bodies. This case is characterized by the absence of precession of the spins and orbital angular momentum, and thus the orbital plane remains fixed in space. However, the effect of the spins gives a contribution to the phase and a correction to the amplitude of the waveform, which we explicitly provide in this subsection. We use the conventions that the z axis coincides with $\boldsymbol{\ell}$ and the vectors $\boldsymbol{n}$, $\boldsymbol{\lambda}$ and $\boldsymbol{N}$ have the following (x, y, z) components:

$$\boldsymbol{\ell} = (0, 0, 1), \qquad \boldsymbol{N} = (\sin\theta, 0, \cos\theta), \tag{4.17a}$$
$$\boldsymbol{n} = (\sin\Phi, -\cos\Phi, 0), \qquad \boldsymbol{\lambda} = (\cos\Phi, \sin\Phi, 0), \tag{4.17b}$$

where Φ is the orbital phase defined such that at the initial time, $\boldsymbol{n}$ points in the x direction. We use the following polarization vectors:

$$\boldsymbol{P} = \boldsymbol{N} \times \boldsymbol{\ell}, \qquad \boldsymbol{Q} = \boldsymbol{N} \times \boldsymbol{P}. \tag{4.18}$$

The vector $\boldsymbol{P}$ is the ascending node where the orbital separation vector crosses the plane of the sky from below. With these conventions, Eqs. (4.16) with Eqs. (4.13), specialized to the case that the only nonvanishing spin components are $(\boldsymbol{\Sigma}^{\rm c}\cdot\boldsymbol{\ell})$ and $(\boldsymbol{S}^{\rm c}\cdot\boldsymbol{\ell})$, become

$$\begin{aligned}
h_+^{\rm 2PN\,spin} = &-\frac{G^2\nu m\omega^2}{12R}\cos\Phi\sin\theta\{3\delta(\boldsymbol{\ell}\cdot\boldsymbol{S}_{\rm c})(-33+\cos^2\theta)+[(-93+167\nu)+9(1-3\nu)\cos^2\theta](\boldsymbol{\ell}\cdot\boldsymbol{\Sigma}_{\rm c})\}\\
&-\frac{9G^2\nu m\omega^2}{4R}\cos(3\Phi)\sin\theta\{\delta(5-\cos^2\theta)(\boldsymbol{\ell}\cdot\boldsymbol{S}_{\rm c})+3(1-3\nu)\sin^2\theta(\boldsymbol{\ell}\cdot\boldsymbol{\Sigma}_{\rm c})\}\\
&-\frac{2G\nu\omega^2}{mR}\cos(2\Phi)(1+\cos^2\theta)(\boldsymbol{\ell}\cdot\boldsymbol{S}_0^+)(\boldsymbol{\ell}\cdot\boldsymbol{S}_0^-),
\end{aligned}\tag{4.19}$$

$$\begin{aligned}
h_\times^{\rm 2PN\,spin} = &-\frac{G^2\nu m\omega^2}{48R}\sin\Phi\sin(2\theta)\{6\delta(\boldsymbol{\ell}\cdot\boldsymbol{S}_{\rm c})(-33+\cos^2\theta)+[(-171+289\nu)+3(1-3\nu)\cos(2\theta)](\boldsymbol{\ell}\cdot\boldsymbol{\Sigma}_{\rm c})\}\\
&-\frac{9G^2\nu m\omega^2}{8R}\sin(3\Phi)\sin(2\theta)\{\delta(\boldsymbol{\ell}\cdot\boldsymbol{S}_{\rm c})(7-3\cos^2\theta)+3(1-3\nu)\sin^2\theta(\boldsymbol{\ell}\cdot\boldsymbol{\Sigma}_{\rm c})\}\\
&-\frac{4G\nu\omega^2}{mR}\sin(2\Phi)\cos\theta(\boldsymbol{\ell}\cdot\boldsymbol{S}_0^+)(\boldsymbol{\ell}\cdot\boldsymbol{S}_0^-).
\end{aligned}\tag{4.20}$$

Here, the convention for the 2PN spin pieces of the polarizations is analogous to that adopted for the PN expansion of the waveform (4.3), with the expansion coefficients related by Eqs. (4.16) at each PN order.

F. Gravitational modes for nonprecessing, spinning compact bodies

The gravitational wave modes are obtained by expanding the complex polarization

$$h = h_+ - ih_\times, \tag{4.21}$$

into spin-weighted $s=-2$ spherical harmonics as

$$h(\theta,\phi) = \sum_{\ell=2}^{+\infty}\sum_{m=-\ell}^{\ell} h_{\ell m\,-2}Y^{\ell m}(\theta,\phi), \tag{4.22}$$

where

$$_{-s}Y^{\ell m}(\theta,\phi) = (-1)^s\sqrt{\frac{2\ell+1}{4\pi}}d^\ell_{sm}(\theta)e^{im\phi}, \tag{4.23}$$

with

$$\begin{aligned}
d^\ell_{sm}(\theta) = &\sum_{k=\max(0,m-s)}^{\min(\ell+m,\ell-s)}\frac{(-1)^k}{k!}\\
&\times\frac{\sqrt{(\ell+m)!(\ell-m)!(\ell+s)!(\ell-s)!}}{(k-m+s)!(\ell+m-k)!(\ell-k-s)!}\\
&\times(\cos(\theta/2))^{2\ell+m-2k-s}(\sin(\theta/2))^{2k-m+s}.
\end{aligned}\tag{4.24}$$

The modes $h_{\ell m}$ can be extracted by computing

$$h_{\ell m} = \int d\Omega\, h(\theta,\phi)\,_{-2}Y^{\ell m*}(\theta,\phi), \tag{4.25}$$

where the integration is over the solid angle $\int d\Omega = \int_0^\pi \sin\theta d\theta\int_0^{2\pi}d\phi$, and using the orthogonality property $\int d\Omega\,_{-s}Y^{\ell m}(\theta,\phi)\,_{-s}Y^{\ell' m'*}(\theta,\phi) = \delta^{\ell\ell'}\delta^{mm'}$, where $\delta^{\ell\ell'}$ is the Kronecker symbol and the star denotes complex conjugation. Using Eqs. (4.19) and (4.20) in Eq. (4.25) we find the following nonvanishing modes:

$$(h_{\ell m})^{\rm 2PN\,spin} = -\frac{2G^2m\nu\omega^2}{R}\sqrt{\frac{16\pi}{5}}e^{-im\Phi}\hat{h}_{\ell m}, \tag{4.26}$$

$$\hat{h}_{21} = -\frac{43}{21}\delta(\boldsymbol{\ell}\cdot\boldsymbol{S}_{\rm c})+\frac{1}{42}(-79+139\nu)(\boldsymbol{\ell}\cdot\boldsymbol{\Sigma}_{\rm c}), \tag{4.27a}$$
$$\hat{h}_{22} = \frac{(\boldsymbol{\ell}\cdot\boldsymbol{S}_0^+)(\boldsymbol{\ell}\cdot\boldsymbol{S}_0^-)}{Gm^2}, \tag{4.27b}$$
$$\hat{h}_{31} = \frac{1}{24\sqrt{14}}\delta(\boldsymbol{\ell}\cdot\boldsymbol{S}_{\rm c})+\frac{5}{24\sqrt{14}}(1-3\nu)(\boldsymbol{\ell}\cdot\boldsymbol{\Sigma}_{\rm c}), \tag{4.27c}$$
$$\hat{h}_{33} = -\frac{3\sqrt{105}}{8\sqrt{2}}\delta(\boldsymbol{\ell}\cdot\boldsymbol{S}_{\rm c})-\frac{9}{8}\sqrt{\frac{15}{14}}(1-3\nu)(\boldsymbol{\ell}\cdot\boldsymbol{\Sigma}_{\rm c}), \tag{4.27d}$$
$$\hat{h}_{41} = \frac{\sqrt{5}}{168\sqrt{2}}\delta(\boldsymbol{\ell}\cdot\boldsymbol{S}_{\rm c})+\frac{\sqrt{5}}{168\sqrt{2}}(1-3\nu)(\boldsymbol{\ell}\cdot\boldsymbol{\Sigma}_{\rm c}), \tag{4.27e}$$
$$\hat{h}_{43} = \frac{9\sqrt{5}}{8\sqrt{14}}\delta(\boldsymbol{\ell}\cdot\boldsymbol{S}_{\rm c})+\frac{9\sqrt{5}}{8\sqrt{14}}(1-3\nu)(\boldsymbol{\ell}\cdot\boldsymbol{\Sigma}_{\rm c}). \tag{4.27f}$$

We have explicitly checked that in the test-mass limit $\nu\to 0$, Eqs. (4.27) reduce to the 2PN $\mathcal{O}(q)$ and $\mathcal{O}(q^2)$ terms given in Eqs. (22) of Ref. [80] (see also Ref. [79]), after accounting for the factor of $(-i)^m$ attributable to the different conventions for the phase origin, as explained in Ref. [52].

It is interesting to note from Eq. (4.27b) that in the nonprecessing case, the dominant h_{22} mode contains only terms that are quadratic in the spin at 2PN order. By contrast, for precessing binaries, the 2PN spin-orbit terms will give a nonvanishing contribution to the 22-mode.

V. CONCLUSIONS

We have extended the knowledge of the spin terms in the gravitational-wave strain tensor to 2PN accuracy for precessing binaries. Our result includes the spin-orbit as well as the spin$_1$-spin$_2$ and spin$_1^2$, spin$_2^2$ effects. The quadratic-in-spin terms are entirely from the equations of motion, whereas the 2PN spin-orbit terms come from both the corrections to the orbital dynamics and the radiation field.

For a given choice of an orthonormal polarization triad and a source frame, the gravitational-wave polarizations can be obtained by projecting our result for the gravitational-wave strain tensor given in Secs. IV B and IV C orthogonal to the propagation direction. For precessing binaries, there is no preferred unique choice of the source frame [8,35,51,72–76], but in the case that the spins are collinear with the orbital angular momentum, the procedure to obtain the polarizations can be carried out in a similar fashion as for nonspinning binaries. For the nonprecessing case and circular orbits, we provided ready-to-use expressions for the gravitational polarizations in Sec. IV E, which could be directly employed in time-domain post-Newtonian, phenomenological and effective-one-body–based template models [19–23,51,52].

In view of the current interest in interfacing analytical and numerical relativity, we also provided the decomposition of the waveform into spin-weighted spherical harmonic modes for nonprecessing binaries and quasicircular orbits. We verified that the test-particle limit of our result reduces to the expressions obtained from black-hole perturbation theory [79,80]. We noted that for spins collinear with the orbital angular momentum, the dominant h_{22} mode of the waveform contains only quadratic-in-spin effects since the spin-orbit contributions vanish in this case, although they are nonzero for generic, precessing configurations.

ACKNOWLEDGMENTS

A. B. acknowledges partial support from NSF Grants No. PHY-0903631 and No. PHY-1208881, and NASA Grant No. NNX09AI81G. A. B. also thanks the Kavli Institute for Theoretical Physics (supported by the NSF Grant No. PHY11-25915) for hospitality during the preparation of this manuscript. T. H. acknowledges support from NSF Grants No. PHY-0903631 and No. PHY-1208881, and the Maryland Center for Fundamental Physics. We thank Gilles Esposito-Farèse, Larry Kidder, and Etienne Racine for useful interactions, as well as David Delavaquerie for help in finalizing one of our MATHEMATICA codes.

APPENDIX: USEFUL IDENTITIES

According to the way the waveform is computed, the result may take various forms, which are not immediately seen to be equivalent. Their difference vanishes because of some dimensional identities valid in three dimensions. They all amount to expressing the fact that a tensor with four antisymmetrized indices must vanish. We shall present here two of such identities, which turned out to be particularly useful for our checks, together with Eqs. (5.2) of Ref. [49].

Let $\boldsymbol{U}_A = U_A^i$, for $A \in \{1, 2, 3\}$, be three vectors of $\mathbb{R}^3$. The first identity tells us that for any vector $\boldsymbol{U}$, we must have

$$(\boldsymbol{U}_1 \times \boldsymbol{U}_2)^{(i}[U_3^{j)}(\boldsymbol{U}_4 \cdot \boldsymbol{U}) - U_4^{j)}(\boldsymbol{U}_3 \cdot \boldsymbol{U})] = U_4^{(i}[(\boldsymbol{U} \times \boldsymbol{U}_1)^{j)}(\boldsymbol{U}_2 \cdot \boldsymbol{U}_3) - (\boldsymbol{U} \times \boldsymbol{U}_2)^{j)}(\boldsymbol{U}_1 \cdot \boldsymbol{U}_3)] + U_3^{(i}[(\boldsymbol{U} \times \boldsymbol{U}_2)^{j)}(\boldsymbol{U}_1 \cdot \boldsymbol{U}_4) - (\boldsymbol{U} \times \boldsymbol{U}_1)^{j)}(\boldsymbol{U}_2 \cdot \boldsymbol{U}_4)]. \tag{A1}$$

To show this, we compute $\varepsilon^i{}_{ab}\varepsilon^{mjk}\varepsilon_{mpq}U_1^a U_2^b U_3^p U_4^q$ in two different manners: (i) we group the first two epsilons, which are next expanded in terms of the identity tensor $\delta^i{}_j$ using the standard formula $\varepsilon_{iab}\varepsilon^{mjk} = 3!\delta^m{}_{[i}\delta^j{}_a\delta^k{}_{b]}$; (ii) we group the last two epsilons and apply the contracted version of the previous equation: $\varepsilon^{mjk}\varepsilon_{mpq} = 2\delta^j{}_{[p}\delta^k{}_{q]}$. One of the remaining free indices, say k, is finally contracted with U_k.

The second identity reads

$$\delta^{ij}[U_1^2U_2^2U_3^2 - U_1^2(\boldsymbol{U}_2 \cdot \boldsymbol{U}_3)^2 - U_2^2(\boldsymbol{U}_3 \cdot \boldsymbol{U}_1)^2 - U_3^2(\boldsymbol{U}_1 \cdot \boldsymbol{U}_2)^2 + 2(\boldsymbol{U}_1 \cdot \boldsymbol{U}_2)(\boldsymbol{U}_2 \cdot \boldsymbol{U}_3)(\boldsymbol{U}_3 \cdot \boldsymbol{U}_1)] + 2U_1^{(i}U_3^{j)}[U_2^2(\boldsymbol{U}_3 \cdot \boldsymbol{U}_1) - (\boldsymbol{U}_1 \cdot \boldsymbol{U}_2)(\boldsymbol{U}_2 \cdot \boldsymbol{U}_3)] + 2U_1^{(i}U_2^{j)}[U_3^2(\boldsymbol{U}_1 \cdot \boldsymbol{U}_2) - (\boldsymbol{U}_2 \cdot \boldsymbol{U}_3)(\boldsymbol{U}_3 \cdot \boldsymbol{U}_1)] + 2U_2^{(i}U_3^{j)}[U_1^2(\boldsymbol{U}_2 \cdot \boldsymbol{U}_3) - (\boldsymbol{U}_1 \cdot \boldsymbol{U}_2)(\boldsymbol{U}_1 \cdot \boldsymbol{U}_3)] + U_1^iU_1^j[(\boldsymbol{U}_2 \cdot \boldsymbol{U}_3)^2 - U_2^2U_3^2] + U_2^iU_2^j[(\boldsymbol{U}_1 \cdot \boldsymbol{U}_3)^2 - U_1^2U_3^2] + U_3^iU_3^j[(\boldsymbol{U}_1 \cdot \boldsymbol{U}_2)^2 - U_1^2U_2^2] = 0. \tag{A2}$$

It is proved by contracting the equality $U_1^{[a}U_2^bU_3^c\delta^{i]j} = 0$ with $U_{1a}U_{2b}U_{3c}$ and expanding. As the trace of the left-hand side of Eq. (A2) is identically zero, the nontrivial content of this identity consists of its STF part.

[1] B. Abbott *et al.* (LIGO Scientific Collaboration), Rep. Prog. Phys. **72**, 076901 (2009).
[2] F. Acernese *et al.*, Classical Quantum Gravity **25**, 184001 (2008).
[3] H. Grote (LIGO Scientific Collaboration), Classical Quantum Gravity **25**, 114043 (2008).
[4] K. Kuroda and the LCGT Collaboration, Classical Quantum Gravity **27**, 084004 (2010).
[5] T. A. Prince, P. Binetruy, J. Centrella, L. S. Finn, C. Hogan, G. Nelemans, E. S. Phinney, and B. Schutz (LISA International Science Team), Technical Report, LISA science case document, 2007, http://list.caltech .edu/mission_documents.
[6] http://sci.esa.int/lisa.
[7] L. S. Finn, Phys. Rev. D **46**, 5236 (1992).
[8] L. S. Finn and D. F. Chernoff, Phys. Rev. D **47**, 2198 (1993).
[9] M. Sasaki and H. Tagoshi, Living Rev. Relativity **6**, 6 (2003), http://www.livingreviews.org/lrr-2003-6.
[10] L. Blanchet, Living Rev. Relativity **9**, 4 (2006), http:// www.livingreviews.org/lrr-2006-4.
[11] T. Futamase and Y. Itoh, Living Rev. Relativity **10**, 2 (2007), http://www.livingreviews.org/lrr-2007-2.
[12] F. Pretorius, Phys. Rev. Lett. **95**, 121101 (2005).
[13] M. Campanelli, C. O. Lousto, P. Marronetti, and Y. Zlochower, Phys. Rev. Lett. **96**, 111101 (2006).
[14] J. G. Baker, J. Centrella, D.-I. Choi, M. Koppitz, and J. van Meter, Phys. Rev. Lett. **96**, 111102 (2006).
[15] A. Buonanno and T. Damour, Phys. Rev. D **59**, 084006 (1999).
[16] A. Buonanno and T. Damour, Phys. Rev. D **62**, 064015 (2000).
[17] T. Damour, P. Jaranowski, and G. Schafer, Phys. Rev. D **62**, 084011 (2000).
[18] A. Buonanno, G. B. Cook, and F. Pretorius, Phys. Rev. D **75**, 124018 (2007).
[19] P. Ajith *et al.*, Phys. Rev. D **77**, 104017 (2008).
[20] T. Damour and A. Nagar, Phys. Rev. D **79**, 081503 (2009).
[21] Y. Pan, A. Buonanno, L. T. Buchman, T. Chu, L. E. Kidder, H. P. Pfeiffer, and M. A. Scheel, Phys. Rev. D **81**, 084041 (2010).
[22] L. Santamaría *et al.*, Phys. Rev. D **82**, 064016 (2010).
[23] Y. Pan, A. Buonanno, M. Boyle, L. T. Buchman, L. E. Kidder, H. P. Pfeiffer, and M. A. Scheel, Phys. Rev. D **84**, 124052 (2011).
[24] L. Blanchet, T. Damour, and B. R. Iyer, Phys. Rev. D **51**, 5360 (1995).
[25] L. Blanchet, T. Damour, and G. Esposito-Farèse, Phys. Rev. D **69**, 124007 (2004).
[26] K. G. Arun, L. Blanchet, B. R. Iyer, and M. S. S. Qusailah, Classical Quantum Gravity **21**, 3771 (2004).
[27] L. Blanchet, B. R. Iyer, C. M. Will, and A. G. Wiseman, Classical Quantum Gravity **13**, 575 (1996).
[28] L. E. Kidder, L. Blanchet, and B. R. Iyer, Classical Quantum Gravity **24**, 5307 (2007).
[29] L. E. Kidder, Phys. Rev. D **77**, 044016 (2008).
[30] L. Blanchet, G. Faye, B. R. Iyer, and S. Sinha, Classical Quantum Gravity **25**, 165003 (2008).
[31] G. Faye, S. Marsat, L. Blanchet, and B. R. Iyer, Classical Quantum Gravity **29**, 175004 (2012).
[32] J. M. Miller, C. S. Reynolds, A. C. Fabian, G. Miniutti, and L. C. Gallo, Astrophys. J. **697**, 900 (2009).
[33] T. A. Apostolatos, C. Cutler, G. J. Sussman, and K. S. Thorne, Phys. Rev. D **49**, 6274 (1994).
[34] T. A. Apostolatos, Phys. Rev. D **54**, 2421 (1996).
[35] A. Buonanno, Y. Chen, and M. Vallisneri, Phys. Rev. D **67**, 104025 (2003).
[36] Y. Pan, A. Buonanno, Y. Chen, and M. Vallisneri, Phys. Rev. D **69**, 104017 (2004).
[37] A. Buonanno, Y. Chen, Y. Pan, and M. Vallisneri, Phys. Rev. D **70**, 104003 (2004).
[38] A. Buonanno, Y. Chen, Y. Pan, H. Tagoshi, and M. Vallisneri, Phys. Rev. D **72**, 084027 (2005).
[39] A. Buonanno, Y. Chen, and T. Damour, Phys. Rev. D **74**, 104005 (2006).
[40] P. Ajith, M. Hannam, S. Husa, Y. Chen, B. Bruegmann, N. Dorband, D. Mueller, F. Ohme, D. Pollney, C. Reisswig, L. Santamaría, and J. Seiler, Phys. Rev. Lett. **106**, 241101 (2011).
[41] P. Ajith, Phys. Rev. D **84**, 084037 (2011).
[42] D. A. Brown, A. Lundgren, and R. O'Shaughnessy, Phys. Rev. D **86**, 064020 (2012).
[43] A. Taracchini, Y. Pan, A. Buonanno, E. Barausse, M. Boyle, T. Chu, G. Lovelace, H. P. Pfeiffer, and M. A. Scheel, Phys. Rev. D **86**, 024011 (2012).
[44] C. Cutler and E. E. Flanagan, Phys. Rev. D **49**, 2658 (1994).
[45] E. Poisson and C. M. Will, Phys. Rev. D **52**, 848 (1995).
[46] M. V. van der Sluys, C. Röver, A. Stroeer, V. Raymond, I. Mandel, N. Christensen, V. Kalogera, R. Meyer, and A. Vecchio, Astrophys. J. Lett. **688**, L61 (2008).
[47] P. Ajith and S. Bose, Phys. Rev. D **79**, 084032 (2009).
[48] B. Mikoczi, M. Vasuth, and L. A. Gergely, Phys. Rev. D **71**, 124043 (2005).
[49] G. Faye, L. Blanchet, and A. Buonanno, Phys. Rev. D **74**, 104033 (2006).
[50] L. Blanchet, A. Buonanno, and G. Faye, Phys. Rev. D **74**, 104034 (2006).
[51] L. E. Kidder, Phys. Rev. D **52**, 821 (1995).
[52] K. G. Arun, A. Buonanno, G. Faye, and E. Ochsner, Phys. Rev. D **79**, 104023 (2009).
[53] C. M. Will and A. G. Wiseman, Phys. Rev. D **54**, 4813 (1996).
[54] L. Blanchet, A. Buonanno, and G. Faye, Phys. Rev. D **84**, 064041 (2011).
[55] I. Bailey and W. Israel, Ann. Phys. (N.Y.) **130**, 188 (1980).
[56] R. A. Porto and I. Z. Rothstein, Phys. Rev. Lett. **97**, 021101 (2006).
[57] R. A. Porto and I. Z. Rothstein, Phys. Rev. D **78**, 044013 (2008).
[58] J. Steinhoff and D. Puetzfeld, Phys. Rev. D **81**, 044019 (2010).
[59] E. Poisson, Phys. Rev. D **57**, 5287 (1998).
[60] T. Damour, Phys. Rev. D **64**, 124013 (2001).
[61] J. Steinhoff, Ann. Phys. (Berlin) **523**, 296 (2011).
[62] R. A. Porto, A. Ross, and I. Z. Rothstein, J. Cosmol. Astropart. Phys. 09 (2012) 028.
[63] R. A. Porto and I. Z. Rothstein, Phys. Rev. D **78**, 044012 (2008).
[64] R. A. Porto, Classical Quantum Gravity **27**, 205001 (2010).
[65] T. Damour, P. Jaranowski, and G. Schafer, Phys. Rev. D **77**, 064032 (2008).
[66] J. Steinhoff, G. Schäfer, and S. Hergt, Phys. Rev. D **77**, 104018 (2008).

[67] J. Steinhoff, S. Hergt, and G. Schäfer, Phys. Rev. D **77**, 081501 (2008).
[68] J. Steinhoff, S. Hergt, and G. Schäfer, Phys. Rev. D **78**, 101503 (2008).
[69] K. S. Thorne, Rev. Mod. Phys. **52**, 299 (1980).
[70] L. Blanchet and T. Damour, Ann. Inst. Henri Poincaré **50**, 377 (1989).
[71] L. Blanchet, Phys. Rev. D **51**, 2559 (1995).
[72] P. Schmidt, M. Hannam, S. Husa, and P. Ajith, Phys. Rev. D **84**, 024046 (2011).
[73] R. O'Shaughnessy, B. Vaishnav, J. Healy, Z. Meeks, and D. Shoemaker, Phys. Rev. D **84**, 124002 (2011).
[74] E. Ochsner and R. O'Shaughnessy, Phys. Rev. D **86**, 104037 (2012).
[75] M. Boyle, R. Owen, and H. P. Pfeiffer, Phys. Rev. D **84**, 124011 (2011).
[76] P. Schmidt, M. Hannam, and S. Husa, Phys. Rev. D **86**, 104063 (2012).
[77] J. M. Martín-García *et al.* (GPL 2002–2012), http://www.xact.es/.
[78] B. J. Owen, H. Tagoshi, and A. Ohashi, Phys. Rev. D **57**, 6168 (1998).
[79] H. Tagoshi, M. Shibata, T. Tanaka, and M. Sasaki, Phys. Rev. D **54**, 1439 (1996).
[80] Y. Pan, A. Buonanno, R. Fujita, E. Racine, and H. Tagoshi, Phys. Rev. D **83**, 064003 (2011).
[81] W. Tulczyjew, Acta Phys. Pol. **18**, 393 (1959).
[82] B. M. Barker and R. F. O'Connell, Gen. Relativ. Gravit. **11**, 149 (1979).
[83] R. A. Porto, Phys. Rev. D **73**, 104031 (2006).
[84] T. Damour and G. Esposito-Farèse, Phys. Rev. D **58**, 042001 (1998).
[85] W. G. Laarakkers and E. Poisson, Astrophys. J. **512**, 282 (1999).
[86] A. J. Hanson and T. Regge, Ann. Phys. (N.Y.) **87**, 498 (1974).
[87] A. Papapetrou, Proc. R. Soc. A **209**, 248 (1951).
[88] W. G. Dixon, Nuovo Cimento **34**, 317 (1964).
[89] H. Tagoshi, A. Ohashi, and B. J. Owen, Phys. Rev. D **63**, 044006 (2001).
[90] W. G. Dixon, Phil. Trans. R. Soc. A **277**, 59 (1974).
[91] R. A. Porto, A. Ross, and I. Z. Rothstein, J. Cosmol. Astropart. Phys. 03 (2011) 009.
[92] L. Blanchet, Classical Quantum Gravity **15**, 1971 (1998).
[93] E. Racine, A. Buonanno, and L. E. Kidder, Phys. Rev. D **80**, 044010 (2009).
[94] E. Racine, Phys. Rev. D **78**, 044021 (2008).