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Spectral approach to axisymmetric evolution of Einstein's equations

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Abstract. We present a new formulation of Einstein's equations for an axisymmetric spacetime with vanishing twist in vacuum. We propose a fully constrained scheme and use spherical polar coordinates. A general problem for this choice is the occurrence of coordinate singularities on the axis of symmetry and at the origin. Spherical harmonics are manifestly regular on the axis and hence take care of that issue automatically. In addition a spectral approach has computational advantages when the equations are implemented. Therefore we spectrally decompose all the variables in the appropriate harmonics. A central point in the formulation is the gauge choice. One of our results is that the commonly used maximal-isothermal gauge turns out to be incompatible with tensor harmonic expansions, and we introduce a new gauge that is better suited. We also address the regularisation of the coordinate singularity at the origin.

1. Introduction

In this paper we consider the vacuum Einstein equations under the assumption of axisymmetry, i.e. there is an everywhere spacelike Killing vector field with closed orbits, ∂_φ in spherical polar coordinates $(t, r, \vartheta, \varphi)$ adapted to the symmetry. As an initial step we further assume that ∂_φ is hypersurface-orthogonal (or twist-free), so there is no rotation.

The motivation for considering this situation comes from the obvious fact that due to the reduced dimensionality, it is much less computationally demanding than the case without symmetries. On the other hand we are not oversimplifying too much in the sense that our situation shares important features and properties of the full theory such as the existence of gravitational waves. In fact it was shown in [1] that it is not possible to assume any further reasonable symmetry in the given situation when demanding gravitational radiation. Due to the so-called Birkhoff theorem (for interesting historic remarks see [2]), vacuum spherical symmetry is non-dynamical and thus not of interest to us.

We focus on isolated systems here, i.e. we assume spacetime is asymptotically flat. This provides us with appropriate fall-off conditions that can be used at an artificial boundary far away from the situation of interest.

There are many interesting applications we have in mind. These include the collapse of gravitational waves, in particular critical phenomena. Up to date there are, to the best of our knowledge, only two successful implementations finding critical phenomena for the vacuum axisymmetric Einstein equations [3, 4]. These studies obtained rather different results for different initial data, and further investigations are needed. Another interesting question in mathematical relativity is the one of stability of solutions such as black holes. Our setup will allow us to study the evolution of perturbations compatible with our symmetry assumptions.

Our aim here is to derive a fully constrained formulation in spherical polar coordinates which may be implemented by making use of a spectral approach for the angular part. A well-known fact is that only six of the ten Einstein equations are of dynamical character, the other four are constraints. An approach that is often applied to numerical investigations of Einstein's equations is to solve the constraints only on an initial hypersurface and then to evolve the system according to the remaining evolution equations. This gives rise to the so-called free evolution. Analytically the constraints remain preserved, which justifies the approach. Numerically, on the other hand, there may exist constraint-violating modes, which cause instabilities. Therefore our approach consists in enforcing the constraints at each timestep. By construction there are no constraint-violating modes then. Such formulations are called fully constrained, for previous publications see e.g. [5, 6] for the case without symmetries and [7] for the case of axisymmetry. A disadvantage of solving the constraints is that they are of elliptic nature and hence computationally much more involved. We will introduce ideas how to save computational cost at other points in the formulation.

This brings us to the choice of a coordinate system. At least for the implementation it is very common to introduce a fixed coordinate system. In contrast to many previous formulations for similar situations which use cylindrical polar coordinates, we choose spherical polar coordinates $(t, r, \vartheta, \varphi)$. Besides the motivation from astrophysics, where many objects have an approximate spherical shape, the main reason is a mathematical one. If spacetime has a topology $\mathcal{M}^2 \times S^2$ then spherical harmonics $Y_{\ell m}(\vartheta, \varphi)$ form an appropriate system for the spectral expansion on the sphere S^2. Another nice property is that fall-off conditions to be imposed at the outer boundary are usually given as an expansion in inverse powers of r. For the choice of spherical coordinates and spectral expansion in general relativity see for example [8, 9, 10].

In these proceedings we mainly focus on conceptual issues of the formulation, in particular the gauge choice. In section 2 we introduce our spectral expansion. It will be applied in section 3, where we describe the derivation of the nonlinear equations, further their linearisation and regularisation. In particular we focus on the gauge choice and investigate two possibilities in detail. In the last chapter we briefly summarize and give an outlook on work in progress.

For us general relativity is given by the Einstein equations on a four-dimensional Lorentzian manifold $(\mathcal{M}^4, g)$ which is metric compatible, torsion-free and globally hyperbolic. Indices i, j run from 1 to 3.

2. Spectral approach

A general problem when using spherical polar coordinates is the occurrence of coordinate singularities at the origin $(r = 0)$ and the axis of symmetry $(\vartheta = 0, \pi)$. As an example consider the flat Laplacian in spherical coordinates,

$$\triangle = \partial_r^2 + \frac{2}{r}\partial_r + \frac{1}{r^2}\left(\partial_\vartheta^2 + \frac{\cos\vartheta}{\sin\vartheta}\partial_\vartheta\right). \tag{1}$$

In fact many equations contain operators that have some similarity with the Laplacian, which is why we will take it as a model to illustrate our ideas.

Since spherical harmonics are regular on the axis, they take care of that issue automatically. Because of the assumed twist-free axisymmetry, all quantities are φ-independent. This implies in particular that the spherical harmonics reduce to the $m = 0$ harmonics

$$Y := Y_\ell(\vartheta) = \sqrt{\frac{2\ell + 1}{4\pi}} P_\ell(\cos\vartheta) \tag{2}$$

with Legendre polynomials $P_\ell(\cos\vartheta)$. We shall omit the index ℓ if it is clear from the context. However general relativity is a tensor theory. We need to know the particular behaviour of each component of a scalar function, a vector and a symmetric two-tensor [11, 12]. Let $\hat{g}_{AB}$ be the round metric on the unit sphere and $\hat{\nabla}$ its covariant derivative $(A, B = \vartheta, \varphi)$. The general even parity quantities are $Y_A := \partial_A Y$ and $Y_{AB} := [\hat{\nabla}_A\hat{\nabla}_B Y]^{\mathrm{tf}}$, the odd parity quantities play no role because of hypersurface-orthogonality. In the case of twist-free axisymmetry the relevant basis harmonics are given by

$$Y = {}_0Y, \tag{3a}$$

$$Y_\vartheta = -\frac{1}{2}\sqrt{\ell(\ell+1)}\left({}_1Y - {}_{-1}Y\right), \tag{3b}$$

$$Y_{\vartheta\vartheta} = -\left[\frac{\cos\vartheta}{\sin\vartheta}Y_\vartheta + \frac{\ell(\ell+1)}{2}Y\right] = \frac{1}{4}\sqrt{(\ell-1)\ell(\ell+1)(\ell+2)}\left({}_2Y + {}_{-2}Y\right), \tag{3c}$$

where we have used the properties of the Legendre functions to eliminate the second ϑ-derivatives in (3c). For completeness we have also given the expressions in terms of the spin-weighted harmonics ${}_sY$ [12]. In the following we will refer to Y, Y_ϑ and $Y_{\vartheta\vartheta}$ as the scalar, vector and tensor harmonics, respectively. Fields expanding in those harmonics will be called scalar, vector and tensor quantities.

We explicitly give the expansion of some components of the spatial metric γ_{ij} needed in the following:

$$\gamma_{rr} = HY, \tag{4a}$$

$$\gamma_{\vartheta\vartheta} = r^2\left(K - \frac{\ell(\ell+1)}{2}G\right)Y - r^2\frac{\cos\vartheta}{\sin\vartheta}GY_\vartheta = r^2\left(KY + GY_{\vartheta\vartheta}\right), \tag{4b}$$

$$\gamma_{\varphi\varphi} = r^2\sin^2\vartheta\left(K + \frac{\ell(\ell+1)}{2}G\right)Y + r^2\cos\vartheta\sin\vartheta GY_\vartheta = r^2\sin^2\vartheta\left(KY - GY_{\vartheta\vartheta}\right), \tag{4c}$$

where H, K and G are functions of t and r only and a sum over ℓ is implied.

3. Formulation

Our starting point is the usual 3+1 decomposition of general relativity $(\mathcal{M}^4, g) \mapsto (\Sigma^3, \gamma, K)$ [13, 14]. Here Σ^3 is a level set of three dimensional spacelike hypersurfaces and γ and K are their first and second fundamental forms. The evolution takes place along the timelike vector field t (recall that $(\mathcal{M}^4, g)$ is globally hyperbolic by assumption). The line element is given by

$$\mathrm{d}s^2 = -\alpha^2\mathrm{d}t^2 + \gamma_{ij}\left(\mathrm{d}x^i + \beta^i\mathrm{d}t\right)\left(\mathrm{d}x^j + \beta^j\mathrm{d}t\right), \tag{5}$$

where α is the lapse function and β^i the components of the shift vector. We obtain six evolution equations each for γ_{ij} and K_{ij}, a Hamiltonian constraint and three momentum constraints.

Now, in this setting, twist-free axisymmetry means that all variables are φ-independent, $\gamma_{r\varphi} = \gamma_{\vartheta\varphi} = 0$ and $\beta^\varphi = 0$. These identities are preserved under time evolution. It follows that also $K_{r\varphi}$ and $K_{\vartheta\varphi}$ have to vanish and the φ-momentum constraint is identically satisfied.

The diffeomorphism invariance of general relativity is encoded in the lapse and shift. To fix the gauge we have to choose a slicing condition and two spatial gauge conditions.

3.1. Choice of a gauge

An evident choice is the so-called maximal-isothermal gauge, see [15] for a review. This is a combination of maximal slicing, $\mathrm{tr}K = 0 = \partial_t(\mathrm{tr}K)$, and the quasi-isotropic condition. The latter one consists of the diagonal gauge,

$$\gamma_{r\vartheta} = 0 = \partial_t\gamma_{r\vartheta}, \tag{6}$$

and a condition that separates the remaining φ-part of the spatial metric by relating the other components as

$$\gamma_{\vartheta\vartheta} = r^2 \gamma_{rr}, \tag{7}$$

which is also preserved in time, $\partial_t\left(\gamma_{\vartheta\vartheta} - r^2\gamma_{rr}\right) = 0$. This is a widely used gauge in axisymmetric simulations [3, 16, 17, 7] and also analytically well studied [15]. As we will see later in section 3.2 after the linearisation and expansion in spherical harmonics, the maximal-isothermal gauge is unfortunately *not* an appropriate gauge for our purposes. In order to find a well-suited condition we decide to keep maximal slicing and the diagonal gauge as before but to come up with a new condition for the diagonal components of metric, namely

$$\gamma_{\vartheta\vartheta} = r^4 \sin^2\vartheta\ \gamma^{\varphi\varphi}(\gamma_{rr})^2. \tag{8}$$

This gauge should also be preserved in time, $\partial_t\left(\gamma_{\vartheta\vartheta} - r^4\sin^2\vartheta\ \gamma^{\varphi\varphi}(\gamma_{rr})^2\right) = 0$. Note that the nonlinear condition (8) relates all the remaining components of the spatial metric. We will show in section 3.2 that this is indeed an appropriate gauge for our purposes. The three preservation equations in t for the gauge choices give us further elliptic equations to be solved at each time step in the fully constrained system.

Having the new gauge condition at hand we can now follow the usual procedure in the 3+1 formulation of general relativity to derive the nonlinear constraints and evolution equations. For obvious reasons one should choose variables in such a way that their linearisation has a convenient expansion in scalar, vector and tensor harmonics in the sense explained at the end of section 2. We note that it is sometimes useful to add appropriate multiples of the constraints (which vanish for a solution to Einstein's equations) to some of the equations such that the linearisation of the equations also expand in a definite way.

We have eight variables, namely the lapse function, two components of the shift vector, two components of the spatial metric and three components of the extrinsic curvature. On the other hand we have six elliptic equations: the Hamiltonian and two momentum constraints, the preservation of maximal slicing, the diagonal gauge condition (6) and the newly introduced gauge condition (8). Since we are looking for a fully constrained formulation, we will explicitly solve all of these elliptic equations in our scheme. Furthermore we have five evolution equations, two for the remaining components of the spatial metric γ_{ij} and three for the extrinsic curvature K_{ij}. Thus the system is overdetermined. In the following we will concentrate on those two evolution equations for the components of γ_{ij} and K_{ij} that expand in the linearisation in tensor harmonics. Besides being reasonable to choose two canonically conjugated variables as evolved fields, we expect the tensor quantities to carry the gravitational wave degrees of freedom (at least in linearised theory). The other evolution equations may be used for consistency checks but will not be considered in the remainder of this paper.

3.2. Linearisation

Having derived the equations, we next linearise them about a flat background spacetime. This means we expand all quantities in the form $f = f|_{\text{flat}} + \epsilon\tilde{f}$ and just keep terms of linear order in ϵ, ignoring higher-order terms. We obtain two evolution equations and six constraints, all dependent on (t, r, ϑ). As expected we are faced with singularities both on the axis and at the origin. One should think of the Laplacian in (1) as a model operator.

On the linearised level we expand all variables in the corresponding spherical harmonics as given at the end of section 2. E.g. for a variable $\tilde{f}$ that expands in scalar harmonics, we have

$$\tilde{f}(t, r, \vartheta) = \sum_\ell \hat{f}_\ell(t, r) Y_\ell(\vartheta). \tag{9}$$

Applying the expansion of the metric coefficients (4) to the quasi-isotropic condition (7), one finds

$$r^2\left(H - K + \frac{\ell(\ell+1)}{2}G\right)Y + r^2\frac{\cos\vartheta}{\sin\vartheta}GY_\vartheta = 0, \tag{10}$$

which implies $G = 0$ and hence $H = K$. Thus only one degree of freedom for the spatial metric remains. Therefore the only situation that is compatible with this choice is the one of spherical symmetry.

On the other hand, linearising (8) about flat space leads to the condition

$$\gamma_{\vartheta\vartheta} = 2r^2\gamma_{rr} - \frac{\gamma_{\varphi\varphi}}{\sin^2\vartheta} \tag{11}$$

and hence, again by using (4),

$$2r^2(K-H)Y = 0. \tag{12}$$

Therefore $H = K$ and G arbitrary are two remaining degrees of freedom, which shows that these conditions are indeed well suited.

The ϑ-dependence is completely absorbed in the spherical harmonics. Thus the interesting part is now contained in $\hat{f}_\ell$ in (9), which depends on (t, r) only. On the linear level one expects a decoupling of all the different ℓ-modes. This is indeed the case provided one considers the "correct" nonlinear equations as explained in the previous subsection 3.1, adding appropriate multiples of the constraints. Therefore we obtain, for each ℓ-mode, a $(1+1)$-dimensional system of equations in (t, r). The equations are still formally singular at the origin $r = 0$.

3.3. Regularisation

In order to regularise the just obtained equations at the origin $r = 0$, we follow a procedure proposed e.g. in [18, 19]. Let us again take the Laplace operator (1) as an example. After expansion in (scalar) spherical harmonics, the Laplace equation reads

$$\partial_r^2\hat{f}_\ell + \frac{2}{r}\partial_r\hat{f}_\ell - \frac{\ell(\ell+1)}{r^2}\hat{f}_\ell = 0. \tag{13}$$

This equation can be solved explicitly. Its solution is a superposition of a singular part proportional to $r^{-(\ell+1)}$ and a regular part proportional to r^ℓ. Since we only consider solutions that are smooth at $r = 0$, we require the integration constant of the part proportional to $r^{-(\ell+1)}$ to vanish. If we isolate the leading-order behaviour by setting $\hat{f}_\ell(t,r) =: r^\ell\bar{f}_\ell(t,r)$, we can continue to work with the barred quantities, which expand, close to the origin $r = 0$, in even power series in r. Therefore (13) is now manifestly regular,

$$\partial_r^2\bar{f}_\ell + \frac{2}{r}(\ell+1)\partial_r\bar{f}_\ell = 0. \tag{14}$$

For our set of equations we have to pull out factors of either r^ℓ, $r^{\ell+1}$ or $r^{\ell+2}$ to obtain similar results, but indeed, the system can be completely regularised in this way.

A further nice property is that, with a little bit of rearranging, one obtains a hierarchy of equations. For the implementation we evolve, from one time step to the next, the two tensor variables by using the evolution equations. Here we remark that, when taking the second time derivative of the tensor component of γ_{ij} and using the evolution equation for the corresponding component of K_{ij}, the principal part of the equations is just the ordinary wave equation. Then, on the new time level, we solve successively the constraints to obtain the remaining variables on that time slice before stepping to the next slice. Given the tensor part of γ_{ij}, the Hamiltonian constraint gives us the scalar component of γ_{ij}, while the preservation of maximal slicing determines the lapse function. Given the tensor contribution to K_{ij}, the two momentum constraints allow us to determine the remaining (scalar and vector) components of the extrinsic curvature. Finally the preservation of the spatial gauge conditions determines the components of the shift vector.

4. Conclusion

We have formulated Einstein's equations for an isolated system in twist-free axisymmetry. Key features of our formulation are that the scheme is fully constrained and uses a spherical polar coordinate system. In its linearisation about a flat background, a spectral expansion in spherical harmonics may be used and the resulting equations are fully regularisable at the origin. A central result presented in this paper is the fact that the well-understood and frequently applied maximal-isothermal gauge is not compatible with tensor spherical harmonic expansions. Instead we proposed another gauge condition which is well-suited. Currently we are in the process of coding the linearised system and finding a proper way to include the non-linearities. Possible directions for further studies include a deeper analysis of the properties of the system, the inclusion of a non-vanishing twist to allow for rotating spacetimes and, ultimately, the application of the code to physically and mathematically interesting situations such as those mentioned in section 1.

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