

# Near-optimal asymmetric binary matrix partitions\*

Fidaa Abed<sup>†</sup>Ioannis Caragiannis<sup>‡</sup>Alexandros A. Voudouris<sup>§</sup>

## Abstract

We study the asymmetric binary matrix partition problem that was recently introduced by Alon et al. (WINE 2013) to model the impact of asymmetric information on the revenue of the seller in take-it-or-leave-it sales. Instances of the problem consist of an  $n \times m$  binary matrix  $A$  and a probability distribution over its columns. A partition scheme  $B = (B_1, \dots, B_n)$  consists of a partition  $B_i$  for each row  $i$  of  $A$ . The partition  $B_i$  acts as a smoothing operator on row  $i$  that distributes the expected value of each partition subset proportionally to all its entries. Given a scheme  $B$  that induces a smooth matrix  $A^B$ , the partition value is the expected maximum column entry of  $A^B$ . The objective is to find a partition scheme such that the resulting partition value is maximized. We present a  $9/10$ -approximation algorithm for the case where the probability distribution is uniform and a  $(1 - 1/e)$ -approximation algorithm for non-uniform distributions, significantly improving results of Alon et al. Although our first algorithm is combinatorial (and very simple), the analysis is based on linear programming and duality arguments. In our second result we exploit a nice relation of the problem to submodular welfare maximization.

## 1 Introduction

We study the *asymmetric binary matrix partition problem*, recently proposed by Alon et al. [2]. Consider a matrix  $A \in \{0, 1\}^{n \times m}$  and a probability distribution  $p$  over its columns;  $p_j$  denotes the probability associated with column  $j$ . We distinguish between two cases for the probability distribution over the columns of the given matrix, depending on whether it is uniform or non-uniform. A partition scheme  $B = (B_1, \dots, B_n)$  for matrix  $A$  consists of a partition  $B_i$  of  $[m]$  for each row  $i$  of  $A$ . More specifically,  $B_i$  is a collection of  $k_i$  pairwise disjoint subsets  $B_{ik} \subseteq [m]$  (with  $1 \leq k \leq k_i$ ) such that  $\bigcup_{k=1}^{k_i} B_{ik} = [m]$ . We can think of each partition  $B_i$  as a smoothing operator, which acts on the entries of row  $i$  and changes their value to the expected value of the partition subset they belong to. Formally, the smooth value of an entry  $(i, j)$  such that  $j \in B_{ik}$  is defined as

$$A_{ij}^B = \frac{\sum_{\ell \in B_{ik}} p_\ell \cdot A_{i\ell}}{\sum_{\ell \in B_{ik}} p_\ell}.$$

Given a partition scheme  $B$  that induces the smooth matrix  $A^B$ , the resulting partition value is the expected maximum column entry of  $A^B$ , namely,

$$v^B(A, p) = \sum_{j \in [m]} p_j \cdot \max_i A_{ij}^B.$$

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\*This work was partially supported by the European Social Fund and Greek national funds through the research funding program Thales on “Algorithmic Game Theory” and by a Caratheodory research grant from the University of Patras.

<sup>†</sup>Max-Planck-Institut für Informatik, Saarbrücken, Germany. Email: fabled@mpi-inf.mpg.de

<sup>‡</sup>Computer Technology Institute and Press “Diophantus” & Department of Computer Engineering and Informatics, University of Patras, 26504 Rion, Greece. Email: caragian@ceid.upatras.gr

<sup>§</sup>Department of Computer Engineering and Informatics, University of Patras, 26504 Rion, Greece. Email: voudouris@ceid.upatras.gr

The objective of the asymmetric binary matrix partition problem is to find a partition scheme  $B$  such that the resulting partition value  $v^B(A, p)$  is maximized.

Alon et al. [2] were the first to consider the asymmetric matrix partition problem. They proved that the problem is APX-hard and provided a 0.563- and a  $1/13$ -approximation for uniform and non-uniform probability distributions, respectively. They also considered input matrices with non-negative non-binary entries and presented a  $1/2$ - and an  $\Omega(1/\log m)$ -approximation algorithm for uniform and non-uniform distributions, respectively. This interesting combinatorial optimization problem has apparent relations to revenue maximization in *take-it-or-leave-it sales*. For example, consider the following setting. There are  $m$  items and  $n$  potential buyers. Each buyer has a value for each item. Nature selects at random (according to some probability distribution) an item for sale and, then, the seller approaches the highest valuation buyer and offers the item to her at a price equal to her valuation. Can the seller exploit the fact that she has much more accurate information about the items for sale compared to the potential buyers? In particular, information asymmetry arises since the seller knows the realization of the randomly selected item whereas the buyers do not. The approach that is discussed in [2] is to let the seller define a buyer-specific signaling scheme. That is, for each buyer, the seller can partition the set of items into disjoint subsets (bundles) and report this partition to the buyer. After nature's random choice, the seller can reveal to each buyer the bundle that contains the realization, thus enabling her to update her valuation beliefs. The relation of this problem to asymmetric matrix partition should be apparent. Interestingly, the seller can achieve revenue from items for which no buyer has any value.

This scenario falls within the line of research that studies the impact of information asymmetry to the quality of markets. Akerlof [1] was the first to introduce a formal analysis of “markets of lemons”, where the seller has more information than the buyers regarding the quality of the products. Crawford and Sobel [7] studied how, in such markets, the seller can exploit her advantage in order to maximize revenue. In [17], Milgrom and Weber provided the “Linkage Principle” which states that the expected revenue is enhanced when bidders are provided with more information. This principle seems to suggest full transparency but, in [15] and [16] the authors suggest that careful bundling of the items is the best way to exploit information asymmetry. Many different frameworks that reveal information to the bidders have been proposed in the literature.

More recently, Ghosh et al. [12] considered full information and proposed a clustering scheme according to which, the items are partitioned into bundles and then, for each such bundle, a separate second-price auction is performed. In this way, the potential buyers cannot bid only for the items that they actually want; they also have to compete for items that they do not care. Hence, the demand for each item is increased and the revenue generated is more. Emek et al. [10] present complexity results in similar settings and Miltersen and Sheffet [19] considered fractional bundling schemes for signaling.

In this work we focus on the simplest binary case of asymmetric matrix partition which has been proved to be APX-hard. We present a  $9/10$ -approximation algorithm for the uniform case and a  $(1 - 1/e)$ -approximation algorithm for non-uniform distributions. Both results significantly improve previous bounds of Alon et al. [2]. The analysis of our first algorithm is quite interesting because, despite its purely combinatorial nature, it exploits linear programming techniques. Similar techniques have been used in a series of papers on variants of set cover (e.g. [3, 4, 5, 6]) by the second author; however, the application of the technique in the current context requires a quite involved reasoning about the structure of the solutions computed by the algorithm.

In our second result, we exploit a nice relation of the problem to submodular welfare maximization and use well-known algorithms from the literature. First, we discuss the application of a simple greedy  $1/2$ -approximation algorithm that has been studied by Lehmann et al. [14] and then apply Vondrák's smooth greedy algorithm [20] to achieve a  $(1 - 1/e)$ -approximation. Vondrák's algorithm is optimal in the value query model as Khot et al. [13] have proved. In a more powerful model where it is assumed that demand queries can be answered efficiently, Feige and Vondrák [11] have proved that  $(1 - 1/e + \epsilon)$ -approximation algorithms — where  $\epsilon$  is a small positive constant — are possible. We briefly discuss the

possibility/difficulty of applying such algorithms to asymmetric binary matrix partition and observe that the corresponding demand query problems are, in general, NP-hard.

The rest of the paper is structured as follows. We begin with preliminary definitions in section 2. Then, we present our  $9/10$ -approximation algorithm for the uniform case in section 3 and our  $(1 - 1/e)$ -approximation algorithm for the non-uniform case in section 4.

## 2 Preliminaries

Let  $A^+ = \{j \in [m] : \text{there exists a row } i \text{ such that } A_{ij} = 1\}$  denote the set of columns of  $A$  that contain at least one 1-value entry, and  $A^0 = [m] \setminus A^+$  denote the set of columns of  $A$  that contain only 0-value entries. In the next sections, we usually refer to the sets  $A^+$  and  $A^0$  as the sets of one-columns and zero-columns, respectively. Furthermore, let  $A_i^+ = \{j \in [m] : A_{ij} = 1\}$  and  $A_i^0 = \{j \in [m] : A_{ij} = 0\}$  denote the set of all 1- and 0-value entries of row  $i$ , respectively. Also, denote by  $r = \sum_{j \in A^+} p_j$  the total probability of the one-columns. We say that a one-column  $j$  is covered by a partition scheme  $B$  if there is at least one row  $i$  such that  $A_{ij} = 1$  and  $\{j\} \in B_i$ ; such a singleton  $\{j\}$  is called column-covering bundle. Subsequently, we say that a scheme  $B$  covers the columns of  $A^+$  if every one-column is covered. Finally, a partition subset is called mixed if it contains both 1- and 0-value entries.

The following structural properties were first observed in [2].

**Lemma 2.1.** *Given a uniform instance of the asymmetric binary matrix partition problem with a matrix  $A$ , there is an optimal partition scheme with the following properties:*

- *For each one-column  $j$ , there exists exactly one row  $i$  with  $A_{ij} = 1$  such that  $\{j\}$  is a column-covering bundle of  $B_i$ .*
- *For each zero-column  $j$ , there exists exactly one row  $i$  such that  $j$  is contained in the mixed bundle of  $B_i$  (and  $n - 1$  rows in which  $j$  is contained in the all-zero bundle).*
- *For each row  $i$ ,  $B_i$  contains at most one bundle containing all the one-columns (if any) that are not contained in column-covering bundles of  $B_i$ . Such a bundle is a mixed one if it contains zero-columns.*
- *For each row  $i$ , the zero-columns that are not contained in the mixed bundle of  $B_i$  form an all-zero bundle.*

The first two properties imply that we can think of the partition value as the sum of the contributions of the column-covering bundles and the contributions of the zero-columns in mixed bundles. The third property should be apparent; the 1-value entries that do not form column-covering bundles are bundled together with zeros in order to increase the contribution of the latter to the partition value. The fourth property makes  $B$  consistent to the definition of a partition scheme where the disjoint union of all the partition subsets in a row should give  $[m]$ . Clearly, the contribution of the all-zero bundles to the partition value is 0.

As we will see later in Section 4, we can consider the problem of computing an optimal partition scheme as a welfare maximization problem. In welfare maximization, there are  $m$  items and  $n$  agents; agent  $i$  has a valuation function  $v_i : 2^{[m]} \rightarrow \mathbb{R}^+$  that specifies her value for each subset of the items. I.e., for a set  $S$  of items,  $v_i(S)$  represents the value of agent  $i$  for  $S$ . Given a disjoint partition (or allocation)  $S = (S_1, S_2, \dots, S_n)$  of the items to the agents, where  $S_i$  denotes the set of items allocated to agent  $i$ , the social welfare is the sum of values of the agents for the sets of items allocated to them, i.e.,  $\text{SW}(S) = \sum_i v_i(S_i)$ . The term welfare maximization refers to the problem of computing an allocation of maximum social welfare. We will discuss only the variant of the problem where the valuations are monotone and submodular; following the literature, we use the term submodular welfare maximization to refer to it.

**Definition 2.1.** A valuation function  $v$  is monotone if  $v(S) \leq v(T)$  for any pair of sets  $S, T$  such that  $S \subseteq T$ . A valuation function  $v$  is submodular if  $v(S \cup \{x\}) - v(S) \geq v(T \cup \{x\}) - v(T)$  for any pair of sets  $S, T$  such that  $S \subseteq T$  and for any item  $x \notin T$ .

An important issue in (submodular) welfare maximization arises with the representation of valuation functions. A valuation function can be described in detail by listing explicitly the values for each of the  $2^m$  possible subsets of items. Unfortunately, this is clearly inefficient due to the necessity for exponential input size. A solution that has been proposed in the literature is to assume access to these functions by queries of a particular form. The simplest such form of queries reads as “what is the value of agent  $i$  for the set of items  $S$ ?” These are known as value queries. Another type of queries, known as demand queries, are phrased as follows: “Given a non-negative price for each item, compute a set  $S$  of items for which the difference of the valuation of agent  $i$  minus the sum of prices for the items in  $S$  is maximized.” Approximation algorithms that use a polynomial number of valuation or demand queries and obtain solutions to submodular welfare maximization with a constant approximation ratio are well-known in the literature. Our improved approximation algorithm for the non-uniform case of asymmetric binary matrix partition exploits such algorithms.

### 3 The uniform case

In this section, we present the analysis of a greedy approximation algorithm in the case where the probability distribution  $p$  over the columns of the given matrix is uniform.

Our algorithm uses a greedy completion procedure which, given a cover of the matrix, can complete the partition scheme  $B$  by including the zero-columns into mixed bundles such that the additional partition value is as high as possible. Formally, the greedy completion procedure goes over the zero-columns, one by one, and adds a zero-column to the mixed bundle of that row which maximizes the marginal contribution of the zero-column. The marginal contribution (or marginal partition value) of a zero-column when it is added to a mixed bundle consisting of  $x$  zero-columns and  $y$  1-value entries is given by the quantity

$$\Delta(x, y) = (x + 1) \frac{y}{x + y + 1} - x \frac{y}{x + y} = \frac{y^2}{(x + y)(x + y - 1)}.$$

Alon et al. [2] have shown that, in the uniform case, this greedy completion procedure yields the optimal contribution from the zero-columns to the partition value. We extensively use this property as well as the fact that  $\Delta(x, y)$  is non-decreasing with respect to  $y$ .

So, our algorithm consists of two phases. In the first phase, called the cover phase, our algorithm computes an arbitrary cover of  $A^+$ . In the second phase, called the greedy phase, it runs the greedy completion procedure mentioned above. In the rest of this section, we will show that this simple algorithm obtains an approximation ratio of 9/10; we will also show that our analysis is tight. Even though our algorithm is purely combinatorial, our analysis exploits linear programming duality.

The partition value obtained by the algorithm can be thought of as the sum of the partition value from the one-columns in the cover (this is exactly  $r$ ) plus the additional partition value obtained from the mixed bundles created during the greedy phase. In our analysis, we distinguish between two main cases. Denote by  $\rho$  the ratio between the number of 1-value entries in the mixed bundles of the optimal solution and the number of zero-columns. The first case is when  $\rho < 1$ ; in this case, the additional partition value obtained during the greedy phase of the algorithm is lower-bounded by the partition value we would have by creating bundles containing exactly one 1-value entry and either  $\lceil 1/\rho \rceil$  or  $\lfloor 1/\rho \rfloor$  zero-columns.

**Lemma 3.1.** *If  $\rho < 1$ , then the partition value obtained by the algorithm is at least 0.97 of the optimal one.*

*Proof.* In order to upper-bound the contribution of the zero-columns to the optimal solution, it suffices to assume that all mixed bundles have a ratio between 1-value entries and zero-columns equal to  $\rho$ . Then, an upper-bound for the optimal partition value is given by

$$\text{OPT} \leq r + (1 - r) \frac{\rho}{\rho + 1}.$$

On the other hand, we will lower-bound the partition value returned by the algorithm by considering the following (not necessarily optimal) formation of mixed bundles as an alternative to the greedy completion procedure used in the greedy phase. Among the  $m(1 - r)$  zero-columns, we pick  $\rho m(1 - r)$  1-value entries that were not used by the algorithm in the cover phase and assign each of them with either  $\lceil 1/\rho \rceil$  or  $\lfloor 1/\rho \rfloor$  zero-columns in order to create mixed bundles that contain all zero-columns. Clearly, this process yields an optimal partition value if  $1/\rho$  is an integer. Otherwise, denote by  $x = m(1 - r)(1 - \rho \lfloor 1/\rho \rfloor)$  the number of mixed bundles containing  $\lceil 1/\rho \rceil$  zero-columns. Then, the number of mixed bundles containing  $\lfloor 1/\rho \rfloor$  zero-columns will be  $\rho m(1 - r) - x = m(1 - r)(\rho \lceil 1/\rho \rceil - 1)$ . Observe that the smooth value of a zero-column is  $\frac{1}{1 + \lceil 1/\rho \rceil}$  in the first case and  $\frac{1}{1 + \lfloor 1/\rho \rfloor}$  in the second case. Hence, we can bound the partition value obtained by the algorithm as follows.

$$\text{ALG} \geq r + (1 - r)(1 - \rho \lfloor 1/\rho \rfloor) \frac{\lceil 1/\rho \rceil}{1 + \lceil 1/\rho \rceil} + (1 - r)(\rho \lceil 1/\rho \rceil - 1) \frac{\lfloor 1/\rho \rfloor}{1 + \lfloor 1/\rho \rfloor}.$$

Now, assuming that  $\rho \in (\frac{1}{k+1}, \frac{1}{k})$  for some integer  $k \geq 1$ , we have that  $\lfloor 1/\rho \rfloor = k$  and  $\lceil 1/\rho \rceil = k + 1$  and, hence,

$$\text{ALG} \geq r + (1 - r) \frac{1 + \rho k(k + 1)}{(k + 1)(k + 2)}.$$

We have

$$\begin{aligned} \frac{\text{ALG}}{\text{OPT}} &\geq \frac{r + (1 - r) \frac{1 + \rho k(k + 1)}{(k + 1)(k + 2)}}{r + (1 - r) \frac{\rho}{\rho + 1}} \geq \frac{\frac{1 + \rho k(k + 1)}{(k + 1)(k + 2)}}{\frac{\rho}{\rho + 1}} \\ &= \frac{(1 + 1/\rho)(1 + \rho k(k + 1))}{(k + 1)(k + 2)}. \end{aligned}$$

This last expression is minimized (with respect to  $\rho$ ) for  $1/\rho = \sqrt{k(k + 1)}$ . Hence,

$$\frac{\text{ALG}}{\text{OPT}} \geq \frac{\left(1 + \sqrt{k(k + 1)}\right)^2}{(k + 1)(k + 2)},$$

which is minimized for  $k = 1$  to approximately 0.97.  $\square$

For the case  $\rho \geq 1$ , we use completely different arguments. The first idea is to reason about the solution produced by the algorithm by reasoning about a set of mixed bundles that are obtained by decomposing the set of mixed bundles computed in the greedy phase. Then, the contribution of the zero-columns to the partition value in the solution computed by the algorithm is lower-bounded by their contribution to the partition value in the set of mixed bundles obtained after the decomposition.

The decomposition is defined as follows. It takes as input a bundle with  $y$  zero-columns and  $x$  1-value entries and decomposes it into  $y$  bundles containing exactly one zero-column and either  $\lfloor x/y \rfloor$  or  $\lceil x/y \rceil$  1-value entries. Note that if  $x/y$  is not an integer, there will be  $(x - y) \lceil x/y \rceil$  bundles with  $\lceil x/y \rceil$  1-values entries. The solution obtained after the decomposition to the solution returned by the algorithm has a very special structure. Due to the greedy phase of the algorithm, it turns out that the set of mixed bundles after the decomposition has a particular structure.

**Lemma 3.2.** *There exists an integer  $s \geq 1$  such that each bundle in the decomposition has at least  $s$  and at most  $3s$  1-value entries.*

*Proof.* Consider the application of the decomposition step to the mixed bundles that are computed by the algorithm and let  $s$  be the minimum number of 1-value entries among the decomposed mixed bundles. This implies that one of the mixed bundles computed by the algorithm has  $\mu$  zero-columns and at most  $(s+1)\mu - 1$  1-value entries. Denoting by  $\nu$  the number of 1-value entries in this bundle, we have that the marginal partition value when the last zero-column  $Z$  is included in the first mixed bundle is exactly

$$\Delta(\mu, \nu) = \frac{\nu^2}{(\nu + \mu)(\nu + \mu - 1)} \leq \frac{((s+1)\mu - 1)^2}{((s+2)\mu - 1)((s+2)\mu - 2)}$$

since  $\Delta(\mu, \nu)$  is increasing in  $\nu$  and  $\nu \leq (s+1)\mu - 1$ . The rightmost expression is decreasing in  $\mu \geq 1$  and, hence, the marginal partition value of  $Z$  is at most  $\frac{s}{s+1}$ .

Now assume for the sake of contradiction that one of the mixed bundles obtained after the decomposition has at least  $3s + 1$  1-value entries. Clearly, this must have been obtained by the decomposition of a mixed bundle (returned by the algorithm) with  $\lambda$  zero-columns and at least  $(3s + 1)\lambda$  1-value entries. Denote by  $\nu'$  the number of 1-value entries in this bundle and let us compute the marginal partition value if the zero-column  $Z$  would be included in this bundle. This would be

$$\begin{aligned} \Delta(\lambda + 1, \nu') &= \frac{\nu'^2}{(\nu' + \lambda + 1)(\nu' + \lambda)} \\ &\geq \frac{(3s + 1)^2 \lambda}{((3s + 2)\lambda + 1)(3s + 2)} \\ &\geq \frac{(3s + 1)^2}{(3s + 3)(3s + 2)}. \end{aligned}$$

The first inequality follows since the marginal partition value function is increasing in  $\nu' \geq (3s + 1)\lambda$  and the second one follows since  $\lambda \geq 1$ . Now, the last quantity can be easily be verified to be strictly higher than  $\frac{s}{s+1}$  and the algorithm should have put  $Z$  in this mixed bundle instead. We have reached the desired contradiction that proves the lemma.  $\square$

Now, our analysis proceeds as follows. For every triplet,  $r \in [0, 1]$ ,  $\rho \geq 1$  and integer  $s > 0$ , we will prove that any solution consisting of an arbitrary cover of the  $rm$  one-columns and the decomposed set of bundles containing at least  $s$  and at most  $3s$  1-value entries yields a 9/10-approximation of the optimal partition value. By the discussion above, this will also be the case for the solution returned by the algorithm. In order to account for the worst-case contribution of zero-columns to the partition value for a given triplet of parameters, we will use the following linear program, which we denote by  $LP(r, \rho, s)$ :

$$\begin{aligned} \text{minimize} \quad & \sum_{k=s}^{3s} \frac{k}{k+1} \theta_k \\ \text{subject to:} \quad & \sum_{k=s}^{3s} \theta_k = 1 - r \\ & \sum_{k=s}^{3s} k \theta_k \geq \rho(1 - r) - r \\ & \theta_k \geq 0, k = s, \dots, 3s \end{aligned}$$

The variable  $\theta_k$  denotes the total probability of the zero-columns that participate in bundles with  $k$  1-value entries. The objective is to minimize the contribution of the zero-columns to the partition value

obtained from the mixed bundles. The equality constraint means that all zero-columns have to participate in bundles. The inequality constraint requires that the number of 1-value entries in bundles used by the algorithm is at least the number of 1-value entries in mixed bundles of the optimal solution minus the 1-value entries in the cover since for every selection of the cover, the algorithm will have the same number of 1-value entries available to form mixed bundles. Due to the structure of the solution obtained after the decomposition and the way the linear program is formulated, we can make the following observation.

Informally, the linear program answers (rather pessimistically) the question: how inefficient can the algorithm be? In particular, given an instance with parameters  $r$  and  $\rho$ , the quantity  $\min_{\text{ints}} \text{LP}(r, \rho, s)$  yields a lower bound on the contribution of the zero-columns to the partition value and  $r + \min_{\text{ints}} \text{LP}(r, \rho, s)$  is a lower bound on the partition value. The next lemma completes the proof for the case  $\rho \geq 1$ .

**Lemma 3.3.** *For every  $r \in [0, 1]$  and  $\rho \geq 1$ ,*

$$r + \min_{\text{int } s} \text{LP}(r, \rho, s) \geq \frac{9}{10} \text{OPT}.$$

*Proof.* We will prove this lemma using LP-duality. The dual of  $\text{LP}(r, \rho, s)$  is:

$$\begin{aligned} & \text{maximize} && (1-r)\alpha + ((1-r)\rho - r)\beta \\ & \text{subject to:} && k\beta + \alpha \leq \frac{k}{k+1}, k = s, \dots, 3s \\ & && \beta \geq 0 \end{aligned}$$

As we did in Lemma 3.1, we will bound the optimal partition value as

$$\text{OPT} \leq r + (1-r) \frac{\rho}{\rho+1} = \frac{\rho+r}{\rho+1}.$$

Hence, it suffices to show that, for every triplet of parameters  $(r, \rho, s)$ , there is a feasible dual solution  $D$  that satisfies

$$r + D \geq \frac{9}{10} \frac{\rho+r}{\rho+1}. \quad (1)$$

The feasible region of the dual is defined by the lines  $\beta = 0$ ,  $\alpha = \frac{s}{s+1} - s\beta$  and  $\alpha = \frac{3s}{3s+1} - 3s\beta$ ; the remaining constraints can be easily seen to be redundant. The two important intersections of those lines are the points  $(\alpha, \beta) = \left(\frac{s}{s+1}, 0\right)$  and  $(\alpha, \beta) = \left(\frac{3s^2}{(s+1)(3s+1)}, \frac{1}{(s+1)(3s+1)}\right)$  with objective values  $D_1 = \frac{s}{s+1}(1-r)$  and  $D_2 = \frac{3s^2}{(s+1)(3s+1)}(1-r) + \frac{\rho(1-r)-r}{(s+1)(3s+1)}$ , respectively. We will show that one of these two points (depending on which has the highest dual objective value) can always be used as a feasible dual solution in order to prove inequality (1). We distinguish between several cases.

**Case I:**  $\rho - \frac{r}{1-r} \leq s$ . In this case observe that  $D_1 \geq D_2$  and, thus, it suffices to prove that

$$r + \frac{s}{s+1}(1-r) \geq \frac{9}{10} \frac{\rho+r}{\rho+1}. \quad (2)$$

**Subcase I.1:**  $s = 1$ . We have that the difference between the left hand side and the right hand side of inequality (2) yields:

$$\frac{1+r}{2} - \frac{9}{10} \frac{\rho+r}{\rho+1} = \frac{1}{2} - \frac{9\rho}{10(\rho+1)} + r \left( \frac{1}{2} - \frac{9}{10(\rho+1)} \right)$$

Since  $\rho \geq 1$ , we have that  $\frac{1}{2} - \frac{9}{10(\rho+1)} \geq 0$ , and we can lower bound the above quantity using the constraint  $r \geq \frac{\rho-1}{\rho}$  (which is obtained from the fact that  $\rho - \frac{r}{1-r} \leq 1$ ), as follows:

$$\frac{1+r}{2} - \frac{9}{10} \frac{\rho+r}{\rho+1} \geq \frac{1}{2} - \frac{9\rho}{10(\rho+1)} + \frac{\rho-1}{\rho} \left( \frac{1}{2} - \frac{9}{10(\rho+1)} \right) = \frac{(\rho-2)^2}{10\rho(\rho+1)} \geq 0$$

**Subcase I.2:**  $\rho - \frac{r}{1-r} > 1$  and  $\rho - \frac{r}{1-r} \leq s$  with  $s > 1$ . We have that the difference between the left hand side and the right hand side of inequality (2) can be lower-bounded by replacing  $s$  by  $\rho - \frac{r}{1-r}$ ; note that the function  $\frac{s}{s+1}$  is non-decreasing with respect to  $s$ . Hence, we have that

$$\begin{aligned} & r + \frac{s}{s+1}(1-r) - \frac{9}{10} \frac{\rho+r}{\rho+1} \\ & \geq r + \frac{\rho - \frac{r}{1-r}}{\rho - \frac{r}{1-r} + 1}(1-r) - \frac{9}{10} \frac{\rho+r}{\rho+1} \\ & \geq \frac{10(\rho+1)(\rho(1-r) - 2r+1)r}{10(\rho+1)(\rho - \frac{r}{1-r} + 1)(1-r)} + \frac{10(\rho(1-r) - r)(\rho+1)(1-r)}{10(\rho+1)(\rho - \frac{r}{1-r} + 1)(1-r)} \\ & \quad - \frac{9(\rho+1)(\rho(1-r) - 2r+1)}{10(\rho+1)(\rho - \frac{r}{1-r} + 1)(1-r)} \\ & = \frac{(8-\rho)r^2 - (\rho^2 + \rho + 9)r + \rho^2 + \rho}{10(\rho+1)(\rho - \frac{r}{1-r} + 1)(1-r)}. \end{aligned}$$

Now, observe that the quadratic function (with respect to  $r$ ) in the numerator of the above fraction is positive for  $r = 0$  and negative for  $r = 1$ . This means that for  $r = 1$  the value of this function is between its two roots. Hence, since  $r \leq \frac{\rho-1}{\rho}$ , it suffices to show that, for  $r = \frac{\rho-1}{\rho}$ , the quadratic function has a positive value. We have:

$$\begin{aligned} & (8-\rho) \left( \frac{\rho-1}{\rho} \right)^2 - (\rho^2 + \rho + 9) \frac{\rho-1}{\rho} + \rho^2 + \rho \\ & = \frac{1}{\rho^2} \left( (8-\rho)(\rho^2 - 2\rho + 1) - (\rho^2 - \rho)(\rho^2 + \rho + 9) + \rho^4 + \rho^3 \right) \\ & = \frac{2(\rho-2)^2}{\rho^2} \\ & \geq 0 \end{aligned}$$

**Case II:**  $\rho - \frac{r}{1-r} \geq s$ . Here, it holds that  $D_2 \geq D_1$  and, thus, it suffices to prove that

$$r + \frac{3s^2}{(s+1)(3s+1)}(1-r) + \frac{\rho(1-r) - r}{(s+1)(3s+1)} \geq \frac{9}{10} \frac{\rho+r}{\rho+1}. \quad (3)$$

Let  $\hat{s}$  denote the (not necessarily integer) value of  $s$  that minimizes  $D_2$ . Then, by nullifying the partial derivative of  $D_2$  with respect to  $s$  we have that  $\hat{s}$  has to satisfy the equality

$$\rho - \frac{r}{1-r} = \frac{6\hat{s}^2 + 3\hat{s}}{3\hat{s} + 2}.$$

The value of  $D_2$  at  $\hat{s}$  equals

$$D_2 = (1-r) \frac{3\hat{s}^2 + \rho - \frac{r}{1-r}}{(\hat{s}+1)(3\hat{s}+1)} = (1-r) \frac{3\hat{s}}{3\hat{s}+2}$$



and, thus, it suffices to prove that

$$r + (1-r) \frac{3\hat{s}}{3\hat{s}+2} \geq \frac{9}{10} \frac{\rho+r}{\rho+1}.$$

Now, observe that the function  $f(x) = \frac{3x}{3x+2}$  is non-decreasing, and that in the case where  $\hat{s} \geq 6$ , the above inequality holds trivially, since

$$r + (1-r) \frac{3\hat{s}}{3\hat{s}+2} \geq r + \frac{9}{10}(1-r) \geq \frac{9}{10} \frac{\rho+r}{\rho+1}.$$

Otherwise, in the case where  $s < 6$  we will use different arguments and we will prove directly the desired inequality

$$\frac{3s^2}{(s+1)(3s+1)}(1-r) + \frac{\rho(1-r)-r}{(s+1)(3s+1)} + r - \frac{9}{10} \frac{\rho+r}{\rho+1} \geq 0.$$

The left hand side of this inequality can be lower-bounded as follows:

$$\begin{aligned} & \frac{3s^2}{(s+1)(3s+1)}(1-r) + \frac{\rho(1-r)-r}{(s+1)(3s+1)} + r - \frac{9}{10} \frac{\rho+r}{\rho+1} \\ &= \frac{3s^2 + \rho}{(3s+1)(s+1)} - \frac{9\rho}{10(\rho+1)} - r \left( \frac{3s^2 + \rho + 1}{(3s+1)(s+1)} + \frac{9}{10(\rho+1)} - 1 \right) \\ &= \frac{10\rho^2 - (-3s^2 + 36s - 1)\rho + 30s^2}{10(3s+1)(s+1)(\rho+1)} - r \left( \frac{10\rho^2 - (40s-10)\rho + 27s^2 - 4s + 9}{10(3s+1)(s+1)(\rho+1)} \right) \\ &\geq \frac{10\rho^2 - (-3s^2 + 36s - 1)\rho + 30s^2}{10(3s+1)(s+1)(\rho+1)} \\ &\quad - r \left( \frac{10\rho^2 - (40s-10)\rho + 27s^2 - 4s + 9}{10(3s+1)(s+1)(\rho+1)} + \frac{13s^2 - 16s - 13/2}{10(3s+1)(s+1)(\rho+1)} \right) \\ &= \frac{10\rho^2 - (-3s^2 + 36s - 1)\rho + 30s^2}{10(3s+1)(s+1)(\rho+1)} - r \frac{10\rho^2 - (40s-10)\rho + 40s^2 - 20s + 5/2}{10(3s+1)(s+1)(\rho+1)} \end{aligned}$$

Now observe that the quadratic function  $10\rho^2 - (40s-10)\rho + 40s^2 - 20s + 5/2$  is positive for any value of  $\rho \geq 1$ . This means that we can use the inequalities  $r \leq \frac{\rho-s}{\rho-s+1}$  and  $\rho \geq s$  (which are implied by the constraint  $\rho - \frac{r}{1-r} \geq s$ ) and have that

$$\begin{aligned} & \frac{3s^2}{(s+1)(3s+1)}(1-r) + \frac{\rho(1-r)-r}{(s+1)(3s+1)} + r - \frac{9}{10} \frac{\rho+r}{\rho+1} \\ &\geq \frac{10\rho^2 - (-3s^2 + 36s - 1)\rho + 30s^2}{10(3s+1)(s+1)(\rho+1)} \\ &\quad - \frac{\rho-s}{\rho-s+1} \left( \frac{10\rho^2 - (40s-10)\rho + 40s^2 - 20s + 5/2}{10(3s+1)(s+1)(\rho+1)} \right). \end{aligned}$$

In order to simplify some of the terms, notice that it suffices to show that

$$\begin{aligned} & (10\rho^2 - (-3s^2 + 36s - 1)\rho + 30s^2) (\rho - s + 1) - (\rho - s) \left( 10\rho^2 - (40s-10)\rho + 40s^2 - 20s + \frac{5}{2} \right) \\ &= (3s^2 + 4s + 1)\rho^2 - \left( 3s^3 + 11s^2 + 7s + \frac{3}{2} \right) \rho + 10s^3 + 10s^2 + \frac{5}{2}s \\ &\geq 0, \end{aligned}$$

which is true since  $3s^2 + 4s + 1 > 0$  and the discriminant of this quadratic function with respect to  $\rho$  equals  $9s^6 - 54s^5 - 117s^4 - 67s^3 + 2s^2 + 11s + 9/4$  which is negative for  $s \in \{1, 2, \dots, 5\}$ .  $\square$

The next statement summarizes the discussion in this section.

**Theorem 3.4.** *The algorithm always yields a 9/10-approximation of the optimal partition value.*

Our analysis is tight as our next counter-example suggests.

**Theorem 3.5.** *There exists an instance of the uniform asymmetric binary matrix partition problem for which the greedy algorithm computes a partition scheme with value (at most) 9/10 of the optimal one.*

*Proof.* Consider the instance of the asymmetric binary matrix partition problem that consists of the matrix

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{pmatrix}$$

with  $p_i = 1/4$  for  $i = 1, 2, 3, 4$ . The optimal partition value is obtained by covering the one-columns using the 1-value entries in the first two rows and then bundling each of the two zero-columns with a pair of 1-value entries in the third and fourth row, respectively. This yields a partition value of  $5/6$ . The greedy algorithm may select to cover the one-columns using the 1-value entries  $A_{31}$  and  $A_{42}$ . This is possible since the greedy algorithm has no particular criterion for breaking ties when selecting the full cover. Given this full cover, the greedy completion procedure will assign each of the two zero-columns with one 1-value entry. The partition value is then  $3/4$ , i.e.,  $9/10$  times the optimal partition value.  $\square$

## 4 Asymmetric binary matrix partition as submodular welfare maximization

In general, the partition value can be thought of as the sum of a contribution to the partition value from each row. In particular, given a set  $S$  consisting of a set of 0-value entries  $S \cap A_i^0$  and a set of column-covering 1-value entries  $S \cap A_i^+$ , the contribution to the partition value of row  $i$  can be described by the function

$$R_i(S) = \sum_{j \in S \cap A_i^+} p_j + \frac{\sum_{j \in S \cap A_i^0} p_j \sum_{j \in A_i^+ \setminus S} p_j}{\sum_{j \in S \cap A_i^0} p_j + \sum_{j \in A_i^+ \setminus S} p_j}.$$

Then, the partition scheme can be thought of as a collection of disjoint sets  $B_i$  (with one set per row) such that  $B_i$  contains those columns whose entries achieve their maximum smooth value in row  $i$ . Hence, the partition value of the partition scheme  $B$  is  $v^B(A, p) = \sum_{i \in [n]} R_i(B_i)$ . Hence, the problem is essentially equivalent to welfare maximization where the rows act as the agents who will be allocated bundles of items. The set of items consists of two kinds of items: one-items corresponding to one-columns and zero-items corresponding to zero-columns.

**Lemma 4.1.** *For every row  $i$ , the function  $R_i$  is non-decreasing and submodular.*

*Proof.* We will show that the function  $R_i$  is non-decreasing and has decreasing marginal utilities, i.e.,

- (monotonicity) for every set  $S$  and item  $x \notin S$ , it holds that  $R_i(S) \leq R_i(S \cup \{x\})$ ;
- (decreasing marginal utilities) for every pair of sets  $S, T$  such that  $S \subseteq T$  and every item  $x \notin T$ , it holds that  $R_i(S \cup \{x\}) - R_i(S) \geq R_i(T \cup \{x\}) - R_i(T)$ .

In order to simplify notation, let us define the functions  $\alpha(S) = \sum_{j \in S \cap A_i^+} p_j$ ,  $\beta(S) = \sum_{j \in S \cap A_i^0} p_j$  and  $\gamma(S) = \sum_{j \in A_i^+ \setminus S} p_j$ . We can rewrite the function  $R_i$  as

$$R_i(S) = \alpha(S) + \frac{\beta(S) \cdot \gamma(S)}{\beta(S) + \gamma(S)}.$$

Let  $S, T \subseteq [m]$  be two sets of columns such that  $S \subseteq T$  and let  $x$  be a column that does not belong to set  $T$ . We distinguish between two cases depending on  $x$ . If  $x \in A_i^+$ , observe that

- $\alpha(S \cup \{x\}) = \alpha(S) + p_x$  and  $\alpha(T \cup \{x\}) = \alpha(T) + p_x$ ;
- $\beta(S \cup \{x\}) = \beta(S)$  and  $\beta(T \cup \{x\}) = \beta(T)$ ;
- $\gamma(S \cup \{x\}) = \gamma(S) - p_x$  and  $\gamma(T \cup \{x\}) = \gamma(T) - p_x$ .

Using the definition of function  $R_i$ , we have

$$\begin{aligned} R_i(S \cup \{x\}) - R_i(S) &= p_x + \beta(S) \left( \frac{\gamma(S) - p_x}{\beta(S) + \gamma(S) - p_x} - \frac{\gamma(S)}{\beta(S) + \gamma(S)} \right) \\ &= p_x - \frac{p_x \beta(S)^2}{(\beta(S) + \gamma(S)) \cdot (\beta(S) + \gamma(S) - p_x)} \\ &\geq p_x - \frac{p_x \beta(S)^2}{(\beta(S) + \gamma(T))(\beta(S) + \gamma(T) - p_x)} \\ &\geq p_x - \frac{p_x \beta(T)^2}{(\beta(T) + \gamma(T))(\beta(T) + \gamma(T) - p_x)} \\ &= R_i(T \cup \{x\}) - R_i(T). \end{aligned}$$

The first inequality follows since  $\gamma$  is clearly non-increasing and  $S \subseteq T$  and the second inequality follows by applying twice (with  $a = \gamma(T)$  and  $a = \gamma(T) - p_x$ , respectively) the fact that the function  $f(z) = \frac{z}{z+a}$  for  $a \geq 0$  is non-decreasing.

If instead  $x \in A_i^0$ , observe that

- $\alpha(S \cup \{x\}) = \alpha(S)$  and  $\alpha(T \cup \{x\}) = \alpha(T)$ ;
- $\beta(S \cup \{x\}) = \beta(S) + p_x$  and  $\beta(T \cup \{x\}) = \beta(T) + p_x$ ;
- $\gamma(S \cup \{x\}) = \gamma(S)$  and  $\gamma(T \cup \{x\}) = \gamma(T)$ .

Hence, we have

$$\begin{aligned} R_i(S \cup \{x\}) - R_i(S) &= \gamma(S) \left( \frac{\beta(S) + p_x}{\beta(S) + \gamma(S) + p_x} - \frac{\beta(S)}{\beta(S) + \gamma(S)} \right) \\ &= \frac{p_x \gamma(S)^2}{(\beta(S) + \gamma(S))(\beta(S) + \gamma(S) + p_x)} \\ &\geq \frac{p_x \gamma(S)^2}{(\beta(T) + \gamma(S))(\beta(T) + \gamma(S) + p_x)} \\ &\geq \frac{p_x \gamma(T)^2}{(\beta(T) + \gamma(T))(\beta(T) + \gamma(T) + p_x)} \\ &= R_i(T \cup \{x\}) - R_i(T). \end{aligned}$$

Again, the inequality follows since  $\beta$  is clearly non-decreasing and  $S \subseteq T$  and the second inequality following by applying twice (with  $a = \beta(T)$  and  $a = \beta(T) + p_x$ , respectively) the fact that the function  $f(z) = \frac{z}{z+a}$  with  $a \geq 0$  is non-decreasing.

We have completed the proof that  $R_i$  has decreasing marginal utilities. In order to establish monotonicity, it suffices to observe that the quantity at the right-hand side of the second equality in each of the above two derivations starting with  $R_i(S \cup \{x\}) - R_i(S)$  is non-negative.  $\square$

Lehmann, Lehmann and Nisan [14] studied the submodular welfare maximization problem and provided a simple algorithm that yields a  $1/2$ -approximation of the optimal welfare. Their algorithm considers the items one by one and assigns item  $j$  to that agent so that the marginal valuation (the additional value from the allocation of item  $j$ ) is maximized. In our setting, this algorithm can be implemented as follows. It considers the one-columns first and the zero-columns afterwards. Whenever considering a one-column, one of its 1-value entries forms a column-covering bundle (such a decision definitely maximizes the increase in the partition value). Whenever considering a zero-column, it includes it to a mixed bundle so that the increase in the partition value is maximized. Using the terminology of Alon et al. [2], the algorithm essentially starts with an arbitrary cover of the one-columns and then it runs the greedy completion procedure. Again, we will use the term greedy for this algorithm.

**Theorem 4.2.** *The greedy algorithm for the asymmetric binary matrix partition problem has approximation ratio at least  $1/2$ . This bound is tight.*

*Proof.* The lower bound holds by the equivalence of the greedy algorithm with the algorithm studied by Lehmann et al. [14]. Below, we prove the upper bound. In particular, we show that for every  $\epsilon > 0$ , there exists an instance of the problem in which the greedy algorithm obtains a partition scheme whose value is at most  $1/2 + \epsilon$  of the optimal one.

Let  $k > 0$  be a positive integer and  $\alpha$  significantly higher than  $k$ . Consider the instance of the asymmetric binary matrix partition that consists of the following  $(k + 1) \times (k + 1)$  matrix

$$A = \begin{pmatrix} 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \\ 1 & 1 & \cdots & 1 & 0 \end{pmatrix}$$

where  $p_i = \frac{1}{k+\alpha}$  for  $i \in [k]$  and  $p_{k+1} = \frac{\alpha}{k+\alpha}$ . So, the first  $k$  columns and rows of  $A$  form an identity matrix, the last column has only 0-value entries and the last row consists of  $k$  1-value entries in the first  $k$  columns. In order to lower-bound the optimal partition value, consider the partition scheme consisting of a full cover of the one-columns by the 1-value entries in the first  $k$  rows and a bundle containing the whole  $(k + 1)$ -th row. The optimal partition value is lower-bounded by the value of this partition scheme. By simple calculations, we obtain

$$\text{OPT} \geq \frac{k^2 + 2\alpha k}{(k + \alpha)^2}.$$

On the other hand, the greedy algorithm may select first to cover the  $k$  one-columns using the 1-value entries in the last row and, then, bundle the zero-column together with only one 1-value entry in some of the first  $k$  rows. The partition value of the greedy algorithm is then

$$\text{GREEDY} = \frac{k + (k + 1)\alpha}{(k + \alpha)(\alpha + 1)}.$$

Hence, the ratio between the two partition values is

$$\frac{\text{GREEDY}}{\text{OPT}} \leq \frac{(k + \alpha)(k + (k + 1)\alpha)}{(k^2 + 2\alpha k)(\alpha + 1)}.$$

Pick an arbitrarily small  $\delta > 0$ ; then, there exist a value for  $\alpha$  (significantly higher than  $k$ ) so that the above ratio satisfies  $\frac{\text{GREEDY}}{\text{OPT}} \leq \frac{k+1}{2k} + \delta$ . The theorem follows by picking  $k$  sufficiently large and  $\delta$  sufficiently small.  $\square$

Can we hope for a better approximation guarantee by starting with a particular full cover? Recall that, in principle, this might be possible in the uniform case as Lemma 2.1 indicates. Interestingly, this is not the case in general as the following statement suggests.

**Lemma 4.3.** *For every  $\epsilon > 0$ , there exists an instance of the asymmetric binary matrix partition problem in which any fully covered solution yields a partition value that is at most  $8/9 + \epsilon$  of the optimal one.*

*Proof.* Consider the instance of the asymmetric binary matrix partition problem consisting of the matrix

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \end{pmatrix}$$

with column probabilities  $p_i = \frac{1}{\beta+3}$  for  $i = 1, 2, 3$  and  $p_4 = \frac{\beta}{\beta+3}$  for  $\beta > 0$ . Observe that there are four fully covered solutions (depending on the selection of the column-covering bundle in the first two columns) and, in each of them, the zero-column is bundled together with a 1-value entry. Hence, by making calculation, we obtain that the partition value in these cases is  $\frac{4\beta+3}{(\beta+1)(\beta+3)}$ . In contrast, consider the solution in which the 1-value entries  $A_{11}$  and  $A_{22}$  form column-covering bundles in rows 1 and 2, the entries  $A_{32}$  and  $A_{33}$  are bundled together in row 3 and the entries  $A_{41}$ ,  $A_{43}$ , and  $A_{44}$  are bundled together in row 4. It can be verified that the partition value now becomes  $\frac{4.5\beta+5}{(\beta+2)(\beta+3)}$ . Clearly, the ratio of the two partition values approaches  $8/9$  from above as  $\beta$  tends to infinity. Hence, the theorem follows by selecting  $\beta$  sufficiently large for any given  $\epsilon > 0$ .  $\square$

We can use the more sophisticated smooth greedy algorithm by Vondrák [20], which uses value queries to obtain the following.

**Corollary 4.4.** *There exists a  $(1 - 1/e)$ -approximation algorithm for the asymmetric binary matrix partition problem.*

One might hope that due to the particular form of functions  $R_i$ , better approximation guarantees might be possible using the  $(1 - 1/e + \epsilon)$ -approximation algorithm of Feige and Vondrák [11] which requires that demand queries of the form

$$\text{given a price } q_j \text{ for every item } j \in [m], \text{ select the bundle } S \text{ that maximizes the difference} \\ R_i(S) - \sum_{j \in S} q_j$$

can be answered in polynomial time. Unfortunately, in our setting, this is not the case in spite of the very specific form of the function  $R_i$ .

**Lemma 4.5.** *Answering demand queries associated with the asymmetric binary matrix partition problem are NP-hard.*

*Proof.* We use reduction from PARTITION to show that the following (very restricted) decision version DQ of a demand query is NP-hard.

$$\text{DQ: Given a } 1 \times m \text{ binary matrix } A, \text{ probabilities } p_j \text{ and prices } q_j \text{ for column } j \in [m], \text{ is} \\ \text{there a set } S \subseteq [m] \text{ such that } R_i(S) - \sum_{j \in S} q_j \geq 5/18?$$

We start from an instance of PARTITION consisting of a collection  $C$  of  $t$  items of integer size  $w_1, w_2, \dots, w_t$  and the question of whether there exists a subset  $Y \subseteq C$  of items such that

$$\sum_{j \in Y} w_j = \sum_{j \in C \setminus Y} w_j = \frac{1}{2} \sum_{j \in C} w_j.$$

Define  $W = \sum_{j \in C} w_j$ . Given this instance, we construct an instance of DQ with  $m = t + 1$  as follows. The binary matrix  $A$  consists of a single row that contains  $t$  1-value entries with associated probabilities  $\frac{w_1}{2W}, \frac{w_2}{2W}, \dots, \frac{w_t}{2W}$  and a 0-value entry with associated probability  $1/2$ . Set the prices as  $q_j = \frac{5w_j}{18W}$  for  $j = 1, \dots, t$  and  $q_{t+1} = 0$ .

By the definition of the function  $R_i$ , given a set  $S \subseteq [t + 1]$ , we have

$$\begin{aligned} R_i(S) - \sum_{j \in S} q_j &= \frac{1}{2W} \sum_{j \in S \setminus \{t+1\}} w_j + \frac{\frac{1}{4W} \sum_{j \in [t] \setminus S} w_j}{\frac{1}{2} + \frac{1}{2W} \sum_{j \in [t] \setminus S} w_j} - \frac{5}{18W} \sum_{j \in S \setminus \{t+1\}} w_j \\ &= \frac{2}{9} - \frac{2}{9W} \sum_{j \in [t] \setminus S} w_j + \frac{\sum_{j \in [t] \setminus S} w_j}{2W + 2 \sum_{j \in [t] \setminus S} w_j}. \end{aligned}$$

Now, consider the function  $f(z) = \frac{2}{9} - \frac{2z}{9W} + \frac{z}{2W + 2z}$ ; the equality above implies that

$$R_i(S) - \sum_{j \in S} q_j = f\left(\sum_{j \in [t] \setminus S} w_j\right).$$

By nullifying the derivative of function  $f$ , we obtain that it has a unique maximum at  $z = W/2$ . Since  $f(W/2) = 5/18$ , the instance of DQ is equivalent to asking whether there exists a set  $S$  such that  $\sum_{j \in [t] \setminus S} w_j = W/2$ , which is equivalent to asking whether there exists a set of items of total size  $W/2$  in the instance of PARTITION.  $\square$

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