

Polynomial Kernels for λ -extendible Properties Parameterized Above the Poljak-Turzík Bound

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Abstract

Poljak and Turzík (*Discrete Mathematics* 1986) introduced the notion of λ -extendible properties of graphs as a generalization of the property of being bipartite. They showed that for any $0 < \lambda < 1$ and λ -extendible property Π , any connected graph G on n vertices and m edges contains a spanning subgraph $H \in \Pi$ with at least $\lambda m + \frac{1-\lambda}{2}(n-1)$ edges. The property of being bipartite is λ -extendible for $\lambda = 1/2$, and so the Poljak-Turzík bound generalizes the well-known Edwards-Erdős bound for MAX-CUT. Other examples of λ -extendible properties include: being an acyclic oriented graph, a balanced signed graph, or a q -colorable graph for some $q \in \mathbb{N}$.

Mnich et. al. (*FSTTCS* 2012) defined the closely related notion of *strong* λ -extendibility. They showed that the problem of finding a subgraph satisfying a given strongly λ -extendible property Π is fixed-parameter tractable (FPT) when parameterized above the Poljak-Turzík bound—*does there exist a spanning subgraph H of a connected graph G such that $H \in \Pi$ and H has at least $\lambda m + \frac{1-\lambda}{2}(n-1) + k$ edges?*—subject to the condition that the problem is FPT on a certain simple class of graphs called *almost-forests of cliques*. This generalized an earlier result of Crowston et al. (*ICALP* 2012) for MAX-CUT, to all strongly λ -extendible properties which satisfy the additional criterion.

In this paper we settle the kernelization complexity of nearly all problems parameterized above Poljak-Turzík bounds, in the affirmative. We show that these problems admit quadratic kernels (cubic when $\lambda = 1/2$), *without using* the assumption that the problem is FPT on almost-forests of cliques. Thus our results not only remove the technical condition of being FPT on almost-forests of cliques from previous results, but also unify and extend previously known kernelization results in this direction. Our results add to the select list of *generic* kernelization results known in the literature.

Keywords and phrases Kernelization, Lambda Extension, Above-Guarantee Parameterization, MaxCut

1 Introduction

In parameterized complexity each problem instance I comes with a parameter k , and a parameterized problem is said to be *fixed parameter tractable* (FPT) if for each instance (I, k) the problem can be solved in time $f(k)|I|^{\mathcal{O}(1)}$ where f is some computable function. The parameterized problem is said to admit a *polynomial kernel* if there is a polynomial time algorithm, called a *kernelization* algorithm, that reduces the input instance down to an instance with size bounded by a polynomial $p(k)$ in k , while preserving the answer. This reduced instance is called a $p(k)$ *kernel* for the problem. The study of kernelization is a major research frontier of Parameterized Complexity; many important recent advances in



the area pertain to kernelization. These include general results showing that certain classes of parameterized problems have polynomial kernels [1, 4, 12, 13] and randomized kernelization based on matroid tools [17, 18]. The recent development of a framework for ruling out polynomial kernels under certain complexity-theoretic assumptions [3, 8, 14] has added a new dimension to the field and strengthened its connections to classical complexity. For overviews of kernelization we refer to surveys [2, 15] and to the corresponding chapters in books on Parameterized Complexity [16, 22]. In this paper we give a generic kernelization result for a class of problems parameterized above guaranteed lower bounds.

Context and Related Work. Many interesting graph problems are about finding a largest subgraph H of the input graph G , where graph H satisfies some specified property and its size is defined as the number of its edges. For many properties this problem is NP-hard, and for some of these we know nontrivial lower bounds for the size of H . In these latter cases, the apposite parameterization “by problem size” is: Given graph G and parameter $k \in \mathbb{N}$, does G have a subgraph H which has (i) the specified property and (ii) at least k more edges than the best known lower bound? MAX-CUT is a sterling example of such a problem. The problem asks for a largest *bipartite* subgraph H of the input graph G ; it is NP-complete [24], and the well-known *Edwards-Erdős bound* [10, 11] tells us that any connected loop-less graph on n vertices and m edges has a bipartite subgraph with at least $\frac{m}{2} + \frac{n-1}{4}$ edges. This lower bound is also the best possible, in the sense that it is tight for an infinite family of graphs—for example, for the set of all cliques with an odd number of vertices.

Poljak and Turzík investigated the *reason* why bipartite subgraphs satisfy the Edwards-Erdős bound, and they abstracted out a sufficient condition for *any* graph property to have such a lower bound. They defined the notion of a λ -extendible property for $0 < \lambda < 1$, and showed that for any λ -extendible property Π , any connected graph $G = (V, E)$ contains a spanning subgraph $H = (V, F) \in \Pi$ with at least $\lambda|E| + \frac{1-\lambda}{2}(|V|-1)$ edges [23]. The property of being bipartite is λ -extendible for $\lambda = 1/2$, and so the Poljak and Turzík result implies the Edwards-Erdős bound. Other examples of λ -extendible properties—with different values of λ —include q -colorability and acyclicity in oriented graphs.

In their pioneering paper which introduced the notion of “above-guarantee” parameterization, Mahajan and Raman [19] posed the parameterized tractability of MAX-CUT above its tight lower bound (MAX-CUT ATLB)—*Given a connected graph G with n vertices and m edges and a parameter $k \in \mathbb{N}$, does G have a bipartite subgraph with at least $\frac{m}{2} + \frac{n-1}{4} + k$ edges?*—as an open problem. This was recently resolved by Crowston et al. who showed that MAX-CUT ATLB can be solved in $2^{\mathcal{O}(k)} \cdot n^4$ time and has a kernel with $\mathcal{O}(k^5)$ vertices [7]. Following this, Mnich et al. [20] generalized the FPT result of Crowston et al. to *all* graph properties which (i) satisfy a (potentially) stronger notion which they dubbed *strong λ -extendibility*, and (ii) are FPT on a certain simple class of graphs called *almost-forests of cliques*. That is, they showed that for any strongly λ -extendible graph property Π which satisfies the simplicity criterion, the following problem—called ABOVE POLJAK-TURZÍK (Π), or APT(Π) for short—is FPT: *Given a connected graph G with n vertices and m edges and a parameter $k \in \mathbb{N}$, does G have a spanning subgraph $H \in \Pi$ with at least $\lambda m + \frac{1-\lambda}{2}(n-1) + k$ edges?* Problems which satisfy these conditions include MAX-CUT, ORIENTED MAX ACYCLIC DIGRAPH, MAX q -COLORABLE SUBGRAPH and, more generally, any graph property which is equivalent to having a homomorphism to a fixed vertex-transitive graph [20].

Our Results and their Implications. Our main result is that for almost all strongly λ -extendible properties Π of (possibly oriented or edge-labelled) graphs, the ABOVE POLJAK-TURZÍK (Π) problem has kernels with $\mathcal{O}(k^2)$ or $\mathcal{O}(k^3)$ vertices. Here “almost all” includes

the following: (i) *all* strongly λ -extendible properties for $\lambda \neq \frac{1}{2}$, (ii) *all* strongly λ -extendible properties which contain all orientations and labels (if applicable) of the graph K_3 (triangle), and (iii) all *hereditary* strongly λ -extendible properties for simple or oriented graphs. In particular, our result implies kernels with $\mathcal{O}(k^2)$ vertices for MAX q -COLORABLE SUBGRAPH and other problems defined by homomorphisms to vertex-transitive graphs.

We address both the questions left open by Mnich et al. [20], albeit in different ways. Firstly, we resolve the kernelization question for strongly λ -extendible properties, except for the special cases of non-hereditary $\frac{1}{2}$ -extendible properties which do not contain some orientation or labelling of the triangle, or hereditary $\frac{1}{2}$ -extendible properties which do not contain some labelling of the triangle. Note that for non-hereditary properties, we may expect to find kernelization very difficult, as a large subgraph with the property can disappear entirely if we delete even a small part of the graph. For the cases when the membership of the triangle depends on its labelling, we may expect the rules of kernelization to depend greatly on the family of labellings, and so it is difficult to produce a general result.

Secondly, we get rid of the simplicity criterion required by Mnich et al. Showing that a specific problem is FPT on almost-forests of cliques takes—in general—a non-trivial amount of work, as can be seen from the corresponding proofs for MAX-CUT [6, Lemma 9], ORIENTED MAX ACYCLIC DIGRAPH, and having a homomorphism to a vertex transitive graph [21, Lemmas 27, 31]. Mnich et al. had proposed that a way to get around this problem was to find a logic which captures all problems which are FPT on almost-forests of cliques, and had left open the problem of finding the right logic. The proof of our main result shows that *all* strongly λ -extendible properties—save for the special cases—are FPT on almost-forests of cliques: in fact, that they have polynomial size kernels on this class of graphs. No special logic is required to capture these problems, and this answers their second open problem.

Formally, our main result is as follows:

► **Theorem 1.** *Let $0 < \lambda < 1$, and let Π be a strongly λ -extendible property of (possibly oriented and/or labelled) graphs. Then the ABOVE POLJAK-TURZÍK (Π) problem has a kernel on $\mathcal{O}(k^2)$ vertices if conditions 1 or 2 holds, and a kernel on $\mathcal{O}(k^3)$ vertices if only 3 holds:*

1. $\lambda \neq \frac{1}{2}$;
2. All orientations and labels (if applicable) of the graph K_3 belong to Π ;
3. Π is a hereditary property of simple or oriented graphs.

As a corollary, we get that a number of specific problems have polynomial kernels when parameterized above their respective Poljak-Turzík bounds:

► **Corollary 2.** *The ABOVE POLJAK-TURZÍK (Π) parameterization of MAX q -COLORABLE SUBGRAPH, $q > 2$, has a kernel on $\mathcal{O}(k^2)$ vertices, and the ABOVE POLJAK-TURZÍK (Π) parameterization of ORIENTED MAX ACYCLIC DIGRAPH has a kernel on $\mathcal{O}(k^3)$ vertices. Furthermore, the ABOVE POLJAK-TURZÍK (Π) parameterization of any problem which is defined by homomorphism to a vertex-transitive graph with at least 3 vertices has a kernel on $\mathcal{O}(k^2)$ vertices.*

The corollary follows from Theorem 1 using the fact that each of these problems is λ -extendible for different values of λ [20].

An outline of the proof. We now give an intuitive outline of our proof of Theorem 1. Our proof starts from a key result of Mnich et al.

► **Proposition 1** ([20]). Let Π be a strongly λ -extendible property and let (G, k) be an instance of $\text{APT}(\Pi)$. Then in polynomial time, we can either decide that (G, k) is a YES-instance or find a set $S \subseteq V(G)$ such that $|S| < \frac{6k}{1-\lambda}$ and $G - S$ is a forest of cliques.

Proposition 1 is a classical WIN/WIN result, and either outputs that the given instance is a YES instance or outputs a set $S \subseteq V(G); |S| < \frac{6k}{1-\lambda}$. In the former case we return a trivial YES instance. In the latter case we know that $G - S$ is a forest of cliques and $|S| < \frac{6k}{1-\lambda}$; thus $G - S$ has a very special structure. For $\lambda \neq \frac{1}{2}$, or when all orientations or labels of the graph K_3 have the property, we show combinatorially that if the combined sizes of the cliques are too big then either we can get some “extra edges”, or we can apply a reduction rule. We then show that the reduced instance has size polynomial in k . For $\lambda = \frac{1}{2}$, we need the extra technical condition that the property is hereditary, and defined only for simple or oriented graphs. In this case we can show that the problem either contains (all orientations of) K_3 , or is exactly MAX-CUT, or that we can bound the number and sizes of the cliques. In any of these cases the problem admits a polynomial kernel.

A block of a graph G is a maximal 2-connected subgraph of G . Note that a block \mathcal{B} of G may consist of a single vertex and no edges, if that vertex is isolated in G .

Let G, S be as in Proposition 1, and let Q be the set of cut vertices of $G - S$. For any block B of $G - S$, let $B_{\text{int}} = V(B) \setminus Q$ be the *interior* of B . Let \mathcal{B} be the set of blocks of $G - S$. A *block neighbor* of a block B is a block B' such that $|V(B) \cap V(B')| = 1$. Given a sequence of blocks $B_0, B_1, \dots, B_l, B_{l+1}$ in $G - S$, the subgraph induced by $V(B_1) \cup \dots \cup V(B_l)$ is a *block path* if, for every $1 \leq i \leq l$, $V(B_i)$ contains exactly two vertices from Q , and B_i has exactly two block neighbors B_{i-1} and B_{i+1} . A block B in $G - S$ is a *leaf block* if $V(B)$ contains exactly one vertex from Q . A block in $G - S$ is an *isolated block* if it contains no vertex from Q . Observe that an isolated block has no block neighbour, while a leaf block has *at least* one block neighbour.

Let \mathcal{B}_0 and \mathcal{B}_1 be the set of isolated blocks and leaf blocks, respectively, contained in \mathcal{B} . Let \mathcal{B}_2 be the set of blocks $B \in \mathcal{B}$ such that B is a block in some block path of $G - S$. Finally, let $\mathcal{B}_{\geq 3} = \mathcal{B} \setminus (\mathcal{B}_0 \cup \mathcal{B}_1 \cup \mathcal{B}_2)$. Thus:

- \mathcal{B}_0 is the set of all blocks of $G - S$ which contain no cut vertex of $G - S$, and therefore have no block neighbour;
- \mathcal{B}_1 is the set of all blocks of $G - S$ which contain exactly one cut vertex of $G - S$, and therefore have at least one block neighbour;
- \mathcal{B}_2 is the set of all blocks of $G - S$ which (i) contain exactly two cut vertices of $G - S$, and (ii) have exactly two block neighbours; and,
- \mathcal{B}_3 is the set of all the remaining blocks of $G - S$. A block of $G - S$ is in \mathcal{B}_3 if and only if it (i) contains at least two cut vertices of $G - S$, and (ii) has at least three block neighbours.

In order to bound the number of vertices in $G - S$ it is enough to bound (i) the number of blocks, and (ii) the size of each block. When $\lambda \neq \frac{1}{2}$ or the property includes all orientations and labellings of K_3 , we show (Lemma 26) that all blocks with two or more vertices have positive excess. Using this fact, we can bound the number of vertices in blocks of \mathcal{B}_1 or \mathcal{B}_2 directly, and it remains only to bound $|\mathcal{B}_0|$. In the remaining case, we have to bound each of $|\mathcal{B}_0|, |\mathcal{B}_1|, |\mathcal{B}_2|, |\mathcal{B}_{\geq 3}|$ and the size of each block separately. We bound these numbers over a number of lemmas.

2 Definitions

We use \uplus to denote the disjoint union of sets. We use “graph” to denote simple graphs without self-loops, directions, or labels, and use standard graph terminology used by Diestel [9] for the terms which we do not explicitly define. Each edge in an *oriented* graph has one of two directions $\{<, >\}$, while each edge in a *labelled* graph has an associated label $\ell \in L$ chosen from a finite set L . A *graph property* is a subclass of the class of all (possibly labelled and/or oriented) graphs. For a labelled and/or oriented graph G , we use $U(G)$ to denote the underlying simple graph; for any graph property of simple graphs, we say that G has the property if $U(G)$ does: for instance, G is connected if $U(G)$ is. For a (possibly labelled and/or oriented) graph $G = (V, E)$ and weight function $w : E(G) \rightarrow \mathbb{R}^+$, we use $w(F)$ to denote the sum of the weights of all the edges in $F \subseteq E$. We use K_j to denote the complete simple graph on j vertices for $j \in \mathbb{N}$, and K to denote an arbitrary complete simple graph. For a graph property Π , we say that $K_j \in \Pi$ if $G \in \Pi$ for every (oriented, labelled) graph G such that $U(G) = K_j$. A connected (possibly labelled and/or oriented) graph is a *tree of cliques* if the vertex set of each block of the graph forms a clique. We use $\mathcal{C}(G)$ to denote the set of connected components of graph G . A *forest of cliques* is a graph whose connected components are trees of cliques. A graph G is 2-connected if and only if it does not contain cut vertices.

Mnich et al. [20] defined the following variant of Poljak and Turzík’s notion of λ -extendibility [23].

► **Definition 3.** Let \mathcal{G} be a class of (possibly labelled and/or oriented) graphs and let $0 < \lambda < 1$. A graph property Π is *strongly λ -extendible* on \mathcal{G} if it satisfies the following properties:

- INCLUSIVENESS $\{G \in \mathcal{G} : U(G) \in \{K_1, K_2\}\} \subseteq \Pi$. That is, $K_1 \in \Pi$, and every possible orientation and labelling of the graph K_2 is in Π ;
- BLOCK ADDITIVITY $G \in \mathcal{G}$ belongs to Π if and only if every block of G belongs to Π ;
- STRONG λ -SUBGRAPH EXTENSION Let $G \in \mathcal{G}$ and let (U, W) be a partition of $V(G)$, such that $G[U] \in \Pi$ and $G[W] \in \Pi$. For any weight function $w : E(G) \rightarrow \mathbb{R}^+$ there exists an $F \subseteq E(U, W)$ with $w(F) \geq \lambda w(E(U, W))$, such that $G - (E(U, W) \setminus F) \in \Pi$.

In the rest of the paper we use \mathcal{G} to denote a class of (possibly labelled and/or oriented) graphs, and Π to denote an arbitrary—but fixed—strongly λ -extendible property defined on \mathcal{G} for some $0 < \lambda < 1$. The focus of our work is the following “above-guarantee” parameterized problem:

ABOVE POLJAK-TURZÍK (Π) (APT(Π))	
<i>Input:</i>	A connected graph $G = (V, E)$ and an integer k .
<i>Parameter:</i>	k
<i>Question:</i>	Is there a spanning subgraph $H = (V, F) \in \Pi$ of G such that $ F \geq \lambda E + \frac{1-\lambda}{2}(V - 1) + k$?

Let $G \in \mathcal{G}$. A Π -*subgraph* of G is a **spanning** subgraph of G which is in Π . Let $\beta_\Pi(G)$ denote the maximum number of edges in any Π -subgraph of G , and let $\gamma_\Pi(G)$ denote the Poljak-Turzík bound on G ; that is, $\gamma_\Pi(G) = \lambda|E(G)| + \frac{1-\lambda}{2}(|V(G)| - |\mathcal{C}(G)|)$. The *excess of Π on G* , denoted $ex_\Pi(G)$, is equal to $\beta_\Pi(G) - \gamma_\Pi(G)$. Thus, given a connected graph G and $k \in \mathbb{N}$ as inputs, the APT(Π) problem asks whether $ex_\Pi(G) \geq k$. We omit the subscript Π when it is clear from the context. We use $ex(K_j)$ to denote the minimum value of $ex(G)$ for any (oriented, labelled) graph G such that $K_j = U(G)$. Thus, for example, if $ex(K_3) = t$ then any graph G with underlying graph K_3 has a Π -subgraph with at least $\gamma(G) + t$

edges, regardless of orientations or labellings on the edges of G . We say that a strongly λ -extendible property *diverges on cliques* if there exists $j \in \mathbb{N}$ such that $ex(K_j) > \frac{1-\lambda}{2}$. We say that a simple connected graph \tilde{K} is an *almost-clique* if there exists $V' \subseteq V(\tilde{K})$ with $|V'| \leq 1$ (possibly V' is empty) such that $\tilde{K} - V'$ is a clique. For an almost-clique \tilde{K} , we use $ex(\tilde{K})$ to denote the minimum value of $ex(G)$ for any (oriented, labelled) graph G such that $\tilde{K} = U(G)$, and we say that $\tilde{K} \in \Pi$ if and only if $G \in \Pi$ for every (oriented, labelled) graph G with underlying graph \tilde{K} .

► **Definition 4.** We use AK_{Π}^+ to denote the class of all graphs $G \in \mathcal{G}$ such that $U(G)$ is an almost-clique and $ex_{\Pi}(G) > 0$. For any strongly λ -extendible property which diverges on cliques, we use \inf_{AK} to denote the value $\inf_{(G \in AK^+)} ex(G)$.

Note that the class AK_{Π}^+ contains an infinite number of graphs. Hence, it could be the case that $\inf_{AK} = 0$. In the next section, we will show that for any strongly λ -extendible property which diverges on cliques, it holds that $\inf_{AK} > 0$.

3 Preliminary Results

We begin with some preliminary results. The first two lemmas state how, in two special cases, the excess of a graph G can be bounded in terms of the excesses of its subgraphs.

► **Lemma 5.** *Let G be a connected (possibly labelled and/or oriented) graph and let v be a cut vertex of G . Then $ex(G) = \sum_{X \in \mathcal{C}(G-\{v\})} ex(G[V(X) \cup \{v\}])$.*

Proof. Recall that by definition, $\gamma(G) = \lambda|E(G)| + \frac{1-\lambda}{2}(|V(G)| - 1)$. Observe first that

$$|E(G)| = \sum_{X \in \mathcal{C}(G-\{v\})} |E(G[V(X) \cup \{v\}])|,$$

and

$$|V(G)| - 1 = \sum_{X \in \mathcal{C}(G-\{v\})} |V(X)| = \sum_{X \in \mathcal{C}(G-\{v\})} (|V(X) \cup \{v\}| - 1).$$

Thus

$$\begin{aligned} \gamma(G) &= \lambda \sum_{X \in \mathcal{C}(G-\{v\})} |E(G[V(X) \cup \{v\}])| + \frac{1-\lambda}{2} \sum_{X \in \mathcal{C}(G-\{v\})} (|V(X) \cup \{v\}| - 1) \\ &= \sum_{X \in \mathcal{C}(G-\{v\})} \left(\lambda |E(G[V(X) \cup \{v\}])| + \frac{1-\lambda}{2} (|V(X) \cup \{v\}| - 1) \right) \\ &= \sum_{X \in \mathcal{C}(G-\{v\})} \gamma(G[V(X) \cup \{v\}]). \end{aligned}$$

We now derive a similar expression for $\beta(G)$. For each $X \in \mathcal{C}(G-v)$, let H_X be a largest Π -subgraph of $G[V(X) \cup \{v\}]$, and let $H = \bigcup_{X \in \mathcal{C}(G-v)} H_X$. Since v is a cutvertex of graph G , we get that every block of H is a block of some such subgraph H_X . Hence we get—from the block additivity property of Π —that H is a Π -subgraph of G . Since no edge of G appears in two distinct subgraphs H_X , we get that $\beta(G) \geq \sum_{X \in \mathcal{C}(G-\{v\})} \beta(G[V(X) \cup \{v\}])$.

Now consider a largest Π -subgraph H of G , and let $H_X = H[V(X) \cup \{v\}]$ for each $X \in \mathcal{C}(G-\{v\})$. Since v is a cutvertex of graph G , we get that every block of each subgraph

H_X is a block of H . Hence we get—again, from the block additivity property of Π —that each H_X is a Π -subgraph of the corresponding subgraph $G[V(X) \cup \{v\}]$. Since each edge of the subgraph H lies in at least one such H_X , we get that $\beta(G) \leq \sum_{X \in \mathcal{C}(G-\{v\})} \beta(G[V(X) \cup \{v\}])$.

Thus $\beta(G) = \sum_{X \in \mathcal{C}(G-\{v\})} \beta(G[V(X) \cup \{v\}])$, and so

$$\begin{aligned} ex(G) &= \beta(G) - \gamma(G) \\ &= \sum_{X \in \mathcal{C}(G-\{v\})} \beta(G[V(X) \cup \{v\}]) - \sum_{X \in \mathcal{C}(G-\{v\})} \gamma(G[V(X) \cup \{v\}]) \\ &= \sum_{X \in \mathcal{C}(G-\{v\})} \beta(G[V(X) \cup \{v\}]) - \gamma(G[V(X) \cup \{v\}]) \\ &= \sum_{X \in \mathcal{C}(G-\{v\})} ex(G[V(X) \cup \{v\}]). \end{aligned} \quad \blacktriangleleft$$

► **Lemma 6.** *Let $G \in \mathcal{G}$ be a connected graph, and let $V(G) = V_1 \uplus V_2$. Let c_1 be the number of components of $G[V_1]$ and c_2 the number of components of $G[V_2]$. If $ex(G[V_1]) \geq k_1$ and $ex(G[V_2]) \geq k_2$, then $ex(G) \geq k_1 + k_2 - \frac{1-\lambda}{2}(c_1 + c_2 - 1)$.*

Proof. Let $E_i = E(G[V_i])$ for $i \in \{1, 2\}$. Then $E(G) = E_1 \uplus E_2 \uplus E(V_1, V_2)$. By definition, $\gamma(G_i) = \lambda|E_i| + \frac{1-\lambda}{2}(|V_i| - c_i)$ for $i \in \{1, 2\}$, and

$$\begin{aligned} \gamma(G) &= \lambda|E(G)| + \frac{1-\lambda}{2}(|V(G)| - 1) \\ &= \lambda(|E_1| + |E_2| + |E(V_1, V_2)|) + \frac{1-\lambda}{2}(|V_1| + |V_2| - 1) \\ &= [\lambda|E_1| + \frac{1-\lambda}{2}(|V_1| - c_1)] + [\lambda|E_2| + \frac{1-\lambda}{2}(|V_2| - c_2)] \\ &\quad + \lambda|E(V_1, V_2)| + \frac{1-\lambda}{2}(c_1 + c_2 - 1) \\ &= \gamma(G[V_1]) + \gamma(G[V_2]) + \lambda|E(V_1, V_2)| + \frac{1-\lambda}{2}(c_1 + c_2 - 1). \end{aligned}$$

Let H_i be a largest Π -subgraph of $G[V_i]$ for $i \in \{1, 2\}$. We apply the strong λ -subgraph extension property to the graph $(V, E(H_1) \cup E(H_2) \cup E(V_1, V_2))$, its vertex partition (V_1, V_2) , and a weight function which assigns unit weights to all its edges. We get that there exists a Π -subgraph H of G such that $H = (V, E(H_1) \uplus E(H_2) \uplus F)$, where $F \subseteq E(V_1, V_2)$ is such that $|F| \geq \lambda|E(V_1, V_2)|$. Therefore $\beta(G) \geq \beta(G[V_1]) + \beta(G[V_2]) + \lambda|E(V_1, V_2)|$. So we get that

$$\begin{aligned} ex(G) &= \beta(G) - \gamma(G) \\ &\geq [\beta(G[V_1]) + \beta(G[V_2]) + \lambda|E(V_1, V_2)|] \\ &\quad - [\gamma(G[V_1]) + \gamma(G[V_2]) + \lambda|E(V_1, V_2)| + \frac{1-\lambda}{2}(c_1 + c_2 - 1)] \\ &= ex(G[V_1]) + ex(G[V_2]) - \frac{1-\lambda}{2}(c_1 + c_2 - 1) \\ &\geq k_1 + k_2 - \frac{1-\lambda}{2}(c_1 + c_2 - 1). \end{aligned} \quad \blacktriangleleft$$

We now prove some useful facts about strongly λ -extendible properties which diverge on cliques. In particular, we show that for a property Π which diverges on cliques, $ex(K_j)$ increases as j increases; this motivated our choice of the name. We also show that $\inf_{AK} is necessarily a constant greater than 0.$

► **Lemma 7.** *Let $ex(K_j) = a \geq \frac{1-\lambda}{2}$ for some $j \in \mathbb{N}$. Then, for every almost-clique \tilde{K} with at least $j+1$ vertices, $ex(\tilde{K}) \geq a - \frac{1-\lambda}{2}$.*

Proof. Let $G \in \mathcal{G}$ be a graph such that $U(G) = \tilde{K}$, where \tilde{K} is an almost-clique with at least $j+1$ vertices. Let V' be a minimum-sized subset of $V(\tilde{K})$ such that $\tilde{K} - V'$ is a clique. Set V_1 to be any subset of exactly $|V(G)| - j$ vertices of G such that (i) $V' \subseteq V_1$, and (ii) $G[V_1]$ is connected. Set $V_2 = V(G) \setminus V_1$. Observe that G is connected, $V(G) = V_1 \uplus V_2$, $G[V_1]$ is connected, and $U(G[V_2]) = K_j$. Further, $ex(G[V_1])$ is—trivially—at least 0, and $ex(G[V_2])$ is—by assumption—at least a . So by Lemma 6, we get that $ex(G) \geq a - \frac{1-\lambda}{2}$. ◀

► **Lemma 8.** *Let Π be a strongly λ -extendible property which diverges on cliques, and let j, a be such that $ex(K_j) = \frac{1-\lambda}{2} + a$, $a > 0$. Then $ex(K_{rj}) \geq \frac{1-\lambda}{2} + ra$ for each $r \in \mathbb{N}^+$. Furthermore, $\lim_{s \rightarrow +\infty} ex(K_s) = +\infty$.*

Proof. We prove the first part of the lemma by induction on r . The claim holds for $r = 1$ by assumption. Suppose that the claim holds for some $r \geq 1$. We show that it holds for $r+1$ as well. Let $G = K_{(r+1)j}$, and consider a partition of $V(G)$ into two parts U, W with $|U| = j, |W| = rj$. Note that $G[U] = K_j, G[W] = K_{rj}$. By assumption we have that $ex(G[U]) = \frac{1-\lambda}{2} + a$, and from the induction hypothesis we get that $ex(G[W]) \geq \frac{1-\lambda}{2} + ra$. Lemma 6 now tells us that $ex(G) \geq \frac{1-\lambda}{2} + (r+1)a$, and this completes the induction step.

Now consider the function $f : \mathbb{N}^+ \rightarrow \mathbb{R}^+$ defined as $f(r) = ex(K_{rj})$. Our arguments above show also that f is an *unbounded* function. Indeed, $f(r+1) = ex(K_{(r+1)j}) \geq ex(K_{rj}) + ex(K_j) - \frac{1-\lambda}{2} = ex(K_{rj}) + \frac{1-\lambda}{2} + a - \frac{1-\lambda}{2} = ex(K_{rj}) + a = f(r) + a$. We use this to argue that given any $x \in \mathbb{R}^+$, there is an $r_x \in \mathbb{N}^+$ such that $\forall r \geq r_x . ex(K_r) > x$; this would prove the second part of the lemma. So let $x \in \mathbb{R}^+$. We choose $y \in \mathbb{N}^+$ such that $f(y) = ex(K_{yj}) = a > x + \frac{1-\lambda}{2}$. Since f is unbounded, such a choice of y exists. We set $r_x = yj$, and from Lemma 7 we get that $\forall r > r_x . ex(K_r) \geq a - \frac{1-\lambda}{2} > x$. ◀

► **Lemma 9.** *Let Π be a strongly λ -extendible property which diverges on cliques. Then $\inf_{AK} > 0$.*

Proof. Since Π diverges on cliques, there exist $j \in \mathbb{N}^+, a \in \mathbb{R}^+$ such that $ex(K_j) = \frac{1-\lambda}{2} + a$. Then, by Lemma 7, for every graph $G \in AK^+$ with at least $j+1$ vertices, $ex(G) \geq a$. Now observe that $\{G \in AK^+ : |V(G)| \leq j\}$ is a finite set, hence the minimum of $ex(G)$ over this set is defined and is positive. So we have that $\inf_{AK} \geq \min(a, \min_{\{G \in AK^+ : |V(G)| \leq j\}} ex(G)) > 0$. ◀

4 Polynomial kernel for divergence

In this section we show that $\text{APT}(\Pi)$ has a polynomial kernel, as long as Π diverges on cliques and all cliques with at least two vertices have positive excess.

Recall the partition $\mathcal{B}_0, \mathcal{B}_1, \mathcal{B}_2, \mathcal{B}_{\geq 3}$ of the blocks of $G - S$. Since $|S| < \frac{6k}{1-\lambda}$, and the number of cut vertices in $G - S$ is bounded by the number of blocks in $G - S$, it is enough to prove upper bounds on $|\mathcal{B}_0|, |\mathcal{B}_1|, |\mathcal{B}_2|, |\mathcal{B}_{\geq 3}|$, and $|B_{\text{int}}|$ for every block B in $G - S$.

In order to prove the main result of this section, Theorem 22, it is enough to bound $|\mathcal{B}_0|$, together with the number and size of all cliques with positive excess. This is because only the blocks in \mathcal{B}_0 may have fewer than two vertices.

We will prove bounds on $|\mathcal{B}_0|, |\mathcal{B}_1|$ and $|\mathcal{B}_{\geq 3}|$ (subject to a reduction rule), and a bound on $|B_{\text{int}}|$ for all blocks B in $G - S$. We do not give a bound on $|\mathcal{B}_2|$ directly, but we do give a bound on the number of cliques with positive excess, which is enough. The bound on $|\mathcal{B}_1|$

is used to bound the number of components in $G - S$, which is required to prove Theorem 22. The bounds on $|\mathcal{B}_{\geq 3}|$ and $|B_{\text{int}}|$ are not used by the proof of Theorem 22, but they will be useful in the following sections.

4.1 Bounding $|\mathcal{B}_0|$ and $|\mathcal{B}_1|$

► **Definition 10.** Let v be a cut vertex of G and let $X \subseteq V(G) \setminus \{v\}$ be such that $G[X]$ is a component of $G - \{v\}$ and the underlying graph of $G[X \cup \{v\}]$ is a 2-connected almost-clique. Then we say that $G[X \cup \{v\}]$ is a *dangling component* with root v .

To bound the number of isolated and leaf blocks in $G - S$, we require the following reduction rule.

► **Reduction Rule 1.** Let $G \in \mathcal{G}$ be a connected graph with at least two 2-connected components and let G' be a dangling component. Then if $ex(G') = 0$, delete $G' - \{v\}$ (where v is root of G') and leave k the same.

► **Lemma 11.** *Reduction Rule 1 is valid.*

Proof. Let G'' be the graph obtained after an application of the rule. By Lemma 5, $\beta(G) = \beta(G') + \beta(G'')$ and $\gamma(G) = \gamma(G') + \gamma(G'')$, which is enough to ensure that $ex(G) = ex(G'')$. ◀

► **Lemma 12.** *Reduction Rule 1 applies in polynomial time if Π diverges on cliques.*

Proof. In polynomial time it is possible to find all 2-connected components of G and to check which ones have an underlying graph which is an almost-clique. Thus, in polynomial time we can find all dangling components. Now, we claim that in constant time it is possible to evaluate whether their excess is zero. In fact, by the definition of divergence on cliques, it holds that $ex(K_j) > \frac{1-\lambda}{2}$ for some j . Given a subgraph G' whose underlying graph is an almost-clique, if G' has at least $j + 1$ vertices Lemma 7 ensures that $ex(G') > 0$. On the other hand, if G' has at most j vertices, a brute force algorithm which finds the largest Π -subgraph of G' runs in time $O(2^{j^2})$, where j is a constant which only depends on Π . ◀

► **Lemma 13.** *Let Π be a strongly λ -extendible property which diverges on cliques and let G be a connected graph reduced under Reduction Rule 1. Then the number of dangling components is less than $\frac{k}{\inf_{AK}}$, or the instance is a YES-instance.*

Proof. Let B_1, \dots, B_l be the dangling components of G . Since the graph is reduced under Reduction Rule 1, $ex(B_i) > 0$ for every $1 \leq i \leq l$. Since Π diverges on cliques, Lemma 9 ensures that $\inf_{AK} > 0$. Let $G' = G - (\cup_{i=1}^l (B_i)_{\text{int}})$.

By Lemma 5, $ex(G) = ex(G') + \sum_{i=1}^l ex(B_i) \geq 0 + \inf_{AK} l$. Then if $l \geq \frac{k}{\inf_{AK}}$ the instance is a YES-instance. ◀

► **Theorem 14.** *Let Π be a strongly λ -extendible property which diverges on cliques and let G be a connected graph reduced under Reduction Rule 1. If there exists $s \in S$ such that $\sum_{B \in \mathcal{B}} |N_G(B_{\text{int}}) \cap \{s\}|$ is at least $(\frac{16}{1-\lambda} + \frac{2}{\inf_{AK}})k - 2$, then the instance is a YES-instance.*

Proof. Let $U = \{s\}$. For every block B of $G - S$ such that $|N_G(B_{\text{int}}) \cap \{s\}| = 1$, pick a vertex in $N(s) \cap B_{\text{int}}$ and add it to U . Since the vertices are chosen in the interior of different blocks of $G - S$, $G[U]$ is a star and therefore it is in Π by block additivity. By Lemma 5, $ex(G[U]) = \frac{1-\lambda}{2}d$, where d is the degree of s in $G[U]$. Let c be the number of components of $G - U$, and assume that U is constructed such that d is maximal and c is minimal. By Lemma 6, $ex(G) \geq \frac{1-\lambda}{2}(d - c)$.

We will now show that c is bounded. The number of components of $G - U$ which contain a vertex of $S \setminus \{s\}$ is bounded by $(|S| - 1) < \frac{6k}{1-\lambda} - 1$. In addition, the number of components of $G - S$ which contain at least two blocks from which a vertex has been added to U is at most $\frac{d}{2}$.

Now, let T be a component of $G - S$ such that in the graph G , no vertex in $T - U$ has a neighbor in $S \setminus \{s\}$ and $|U \cap V(T)| = 1$. Firstly, suppose that T contains only one block B of $G - S$. Let $\{v\} = U \cap V(T)$. Note that, by the current assumptions, $N(S \setminus \{s\}) \cap V(T) \subseteq \{v\}$. If v is the only neighbor of s in T , then it is a cut vertex in G , hence B is a dangling component of G . If s has another neighbor v' in T and v has no neighbor in S different from s , then s is a cut vertex, therefore $G[V(B) \cup \{s\}]$ is a dangling component. Finally, if v has at least two neighbors in S and s has at least another neighbor v' in T , let U' be the star obtained by replacing v with v' in U , and observe that T is connected to $S \setminus \{s\}$ in $G - U'$, contradicting the minimality of c .

Now, suppose that T contains at least two blocks of $G - S$. In this case, every block except B does not contain neighbors of S . In particular, this holds for at least one leaf block B' in T . Hence, B' is a dangling component.

This ensures that carefully choosing the vertices of U we may assume that any component of $G - U$ still contains a vertex of $S \setminus \{s\}$, or contains at least two blocks from which a vertex of U has been chosen, or contains part of a dangling component. Hence, the number of components of $G - U$ is at most $\frac{6k}{1-\lambda} - 1 + \frac{d}{2} + \frac{k}{\inf_{AK}}$.

Therefore, if $d \geq (\frac{16}{1-\lambda} + \frac{2}{\inf_{AK}})k - 2$, then $ex(G) \geq k$. \blacktriangleleft

► **Corollary 15.** *Let Π be a strongly λ -extendible property which diverges on cliques and let G be a connected graph reduced under Reduction Rule 1. If $\sum_{s \in S} \sum_{B \in \mathcal{B}} |N_G(B_{int}) \cap \{s\}|$ is at least $((\frac{16}{1-\lambda} + \frac{2}{\inf_{AK}})k - 2)\frac{6k}{1-\lambda}$, then the instance is a YES-instance.*

► **Corollary 16.** *Let Π be a strongly λ -extendible property which diverges on cliques and let G be a connected graph reduced under Reduction Rule 1. Then $|\mathcal{B}_0| + |\mathcal{B}_1| \leq ((\frac{16}{1-\lambda} + \frac{2}{\inf_{AK}})k - 2)\frac{6k}{1-\lambda} + \frac{k}{\inf_{AK}}$, or the instance is a YES-instance.*

Proof. Note that every isolated or leaf block either has a neighbor of S in its interior or is a dangling block. The claim follows from Lemma 13 and Corollary 15. \blacktriangleleft

► **Corollary 17.** *Let Π be a strongly λ -extendible property which diverges on cliques and let G be a connected graph reduced under Reduction Rule 1. Then either $G - S$ has at most $((\frac{16}{1-\lambda} + \frac{2}{\inf_{AK}})k - 2)\frac{6k}{1-\lambda} + \frac{k}{\inf_{AK}}$ components, or the instance is a YES-instance.*

Proof. Since a component of $G - S$ contains at least one block from $\mathcal{B}_0 \cup \mathcal{B}_1$, the result follows applying Corollary 16. \blacktriangleleft

4.2 Bounding blocks with positive excess

► **Lemma 18.** *Let Π be a strongly λ -extendible property which diverges on cliques. If $G - S$ contains at least $((\frac{16}{1-\lambda} + \frac{2}{\inf_{AK}})k - 1)\frac{6k}{\inf_{AK}(1-\lambda)} + \frac{k}{(\inf_{AK})^2} + \frac{k-1}{\inf_{AK}}$ blocks with positive excess, then the instance is a YES-instance.*

Proof. Let d be the number of blocks in $G - S$ with positive excess, and let G' be the union of all components of $G - S$ which contain a block with positive excess. Observe that by Corollary 17, we may assume G' has at most $((\frac{16}{1-\lambda} + \frac{2}{\inf_{AK}})k - 2)\frac{6k}{1-\lambda} + \frac{k}{\inf_{AK}}$ components. Observe that by repeated use of Lemma 5, $ex(G') \geq d \inf_{AK}$. Now let $G'' = G - G'$, and observe that G'' has at most $|S| \leq \frac{6k}{1-\lambda}$ components. Then by Lemma 6,

$ex(G) \geq d \inf_{AK} - \frac{1-\lambda}{2} \left(\left(\frac{16}{1-\lambda} + \frac{2}{\inf_{AK}} \right) k - 2 \right) \frac{6k}{1-\lambda} + \frac{k}{\inf_{AK}} + \frac{6k}{1-\lambda} - 1$. Therefore if $d \geq \left(\left(\frac{16}{1-\lambda} + \frac{2}{\inf_{AK}} \right) k - 1 \right) \frac{6k}{\inf_{AK}(1-\lambda)} + \frac{k-1}{(\inf_{AK})^2} + \frac{k-1}{\inf_{AK}}$, the instance is a YES-instance. \blacktriangleleft

4.3 Bounding $|\mathcal{B}_{\geq 3}|$

The following lemma is not required to prove the last theorem in this section, but it will be used in Section 6.

► Lemma 19. *Let $G \in \mathcal{G}$ be a connected graph and $S \subseteq V(G)$ be such that $G - S$ is a forest of cliques. Then $|\mathcal{B}_{\geq 3}| \leq 3|\mathcal{B}_1|$.*

Proof. The proof is by induction on $|\mathcal{B}|$. We may assume that $G - S$ is connected, otherwise we can prove the bound separately for every component. If $|\mathcal{B}| = 1$, then $|\mathcal{B}_{\geq 3}| = 0$ and the bound trivially holds. Suppose now that $|\mathcal{B}| = l + 1 \geq 2$ and that the bound holds for every tree of cliques with at most l blocks. Let $B \in \mathcal{B}$ be a leaf block and let v be its root. Let $G' = G - (V(B) \setminus \{v\})$. $G' - S$ is a tree of cliques with at most l blocks, so by induction hypothesis $|\mathcal{B}'_{\geq 3}| \leq 3|\mathcal{B}'_1|$. Now, note that only block neighbors of B can be influenced by the removal of B : in other words, if a block B' is not a block neighbor of B and $B' \in \mathcal{B}_i$, then $B' \in \mathcal{B}'_i$ for every $i \in \{1, 2, \geq 3\}$.

We distinguish three cases. Recall that Q is the set of cutvertices of $G - S$. Let Q' be the set of cutvertices of $G' - S$.

Case 1 (*B has at least three block neighbors*): In this case it holds that $Q = Q'$, which ensures that the removal of B does not increase the number of leaf blocks, that is, $|\mathcal{B}'_1| = |\mathcal{B}_1| - 1$. In addition, if a block neighbor B' of B is in $\mathcal{B}_{\geq 3}$, then it is in $\mathcal{B}'_{\geq 3}$, which means that $|\mathcal{B}'_{\geq 3}| = |\mathcal{B}_{\geq 3}|$. Therefore in this case, using the induction hypothesis it follows that $|\mathcal{B}_{\geq 3}| = |\mathcal{B}'_{\geq 3}| \leq 3|\mathcal{B}'_1| \leq 3|\mathcal{B}_1| - 3$.

Case 2 (*B has two block neighbors*): As in the previous case $Q = Q'$, hence $|\mathcal{B}'_1| = |\mathcal{B}_1| - 1$. On the other hand, if a block neighbor B' of B is in $\mathcal{B}_{\geq 3}$, it could be the case that B' is in \mathcal{B}'_2 . Therefore, $|\mathcal{B}'_{\geq 3}| \geq |\mathcal{B}_{\geq 3}| - 2$. Using the induction hypothesis it follows that $|\mathcal{B}_{\geq 3}| \leq |\mathcal{B}'_{\geq 3}| + 2 \leq 3|\mathcal{B}'_1| + 2 \leq 3|\mathcal{B}_1|$.

Case 3 (*B has exactly one block neighbor*): Let B' be the only block neighbor of B . Again, we distinguish three cases. If $B' \in \mathcal{B}_1$, then B and B' are the only blocks of $G - S$ and $|\mathcal{B}_{\geq 3}| = 0$, therefore the bound trivially holds. If $B' \in \mathcal{B}_2$, then B' is a leaf block in $G' - S$, hence $|\mathcal{B}_1| = |\mathcal{B}'_1|$ and $|\mathcal{B}_{\geq 3}| = |\mathcal{B}'_{\geq 3}|$: the bound follows using the induction hypothesis.

Lastly, let $B' \in \mathcal{B}_{\geq 3}$. If $|V(B') \cap Q| \geq 3$, then $|\mathcal{B}'_{\geq 3}| \geq |\mathcal{B}_{\geq 3}| - 1$ and $|\mathcal{B}'_1| = |\mathcal{B}_1| - 1$: therefore, by induction hypothesis, $|\mathcal{B}_{\geq 3}| \leq |\mathcal{B}'_{\geq 3}| + 1 \leq 3|\mathcal{B}'_1| + 1 \leq 3|\mathcal{B}_1|$. Otherwise, $|V(B') \cap Q| = 2$ and B' is a leaf block in $G' - S$. In this case, $|\mathcal{B}'_1| = |\mathcal{B}_1|$ and $|\mathcal{B}'_{\geq 3}| = |\mathcal{B}_{\geq 3}| - 1$. Now, consider the graph $G'' = G' - (V(B') \setminus \{v'\})$, where v' is the root of B' in $G' - S$. Removing B' from G' corresponds either to case 1 or 2, hence it holds that $|\mathcal{B}''_1| = |\mathcal{B}'_1| - 1$ and $|\mathcal{B}''_{\geq 3}| \geq |\mathcal{B}'_{\geq 3}| - 2$. Therefore, using the induction hypothesis on $G'' - S$ (which is a tree of cliques with $l - 1$ blocks) it follows that $|\mathcal{B}_{\geq 3}| = |\mathcal{B}'_{\geq 3}| + 1 \leq |\mathcal{B}''_{\geq 3}| + 3 \leq 3|\mathcal{B}''_1| + 3 = 3|\mathcal{B}'_1| = 3|\mathcal{B}_1|$, which concludes the proof. \blacktriangleleft

► Corollary 20. *Let Π be a strongly λ -extendible property which diverges on cliques and let G be a connected graph reduced under Reduction Rule 1. Then $|\mathcal{B}_0| + |\mathcal{B}_1| + |\mathcal{B}_{\geq 3}| \leq 4 \left(\left(\frac{16}{1-\lambda} + \frac{2}{\inf_{AK}} \right) k - 2 \right) \frac{6k}{1-\lambda} + \frac{k}{\inf_{AK}}$, or the instance is a YES-instance.*

Proof. The bound follows from Corollary 16 and Lemma 19. \blacktriangleleft

4.4 Bounding $|B_{\text{int}}|$

► **Lemma 21.** *Let Π be a strongly λ -extendible property which diverges on cliques, and let j, a be such that $ex(K_j) = \frac{1-\lambda}{2} + a$, $a > 0$. If $|B_{\text{int}}| \geq \lceil \frac{4k}{a} + \frac{1-\lambda}{2a} \rceil j$ for any block B of $G - S$, then the instance is a YES-instance.*

Proof. Note that $G - B_{\text{int}}$ has at most $\frac{6k}{1-\lambda} + 1$ components, since every component of $G - S$ which does not contain B still has an edge to a vertex of S , while the only component of $G - S$ that could be disconnected from S is the one containing B . Therefore, if $ex(B_{\text{int}}) \geq \frac{1-\lambda}{2}(\frac{6k}{1-\lambda} + 1) + k = 4k + \frac{1-\lambda}{2}$, Lemma 6 ensures that we have a YES-instance.

By Lemma 8, if r is an integer such that $r \geq \frac{4k}{a} + \frac{1-\lambda}{2a}$, then $ex(K_{rj}) \geq 4k + (1-\lambda)$. This shows that if $|B_{\text{int}}| = rj$ then $ex(B_{\text{int}}) \geq 4k + (1-\lambda)$. Furthermore, by Lemma 7, if $|B_{\text{int}}| \geq rj$ then $ex(B_{\text{int}}) \geq 4k + \frac{1-\lambda}{2}$. Thus, if $|B_{\text{int}}| \geq \lceil \frac{4k}{a} + \frac{1-\lambda}{2a} \rceil j$ we have a YES-instance. ◀

4.5 Combined kernel

► **Theorem 22.** *Let Π be a strongly λ -extendible property which diverges on cliques, and suppose $ex(K_i) > 0$ for all $i \geq 2$. Then $\text{APT}(\Pi)$ has a kernel with at most $\mathcal{O}(k^2)$ vertices.*

Proof. Let $j \in \mathbb{N}$ be such that $ex(K_j) = \frac{1-\lambda}{2} + a$, where $a > 0$. Let $V(G - S) = V' \cup V'' \cup V'''$, where V' is the set of vertices contained in blocks with exactly one vertex, V'' is the set of vertices contained in blocks with between 2 and $j - 1$ vertices and V''' is the set of vertices contained in blocks with at least j vertices (note that in general $V'' \cap V''' \neq \emptyset$). Observe that the blocks containing V' are isolated blocks, therefore by Corollary 16 there exists a constant c_1 depending only on Π such that $|V'| \leq c_1 k^2$, or the instance is a YES-instance. By Lemma 18, there exists a constant c_2 depending only on Π such that $|V''| \leq c_2(j - 1)k^2$, or the instance is a YES-instance.

Now, let \mathcal{B}'' be the set of blocks of $G - S$ which contain at least j vertices. For every block $B \in \mathcal{B}''$, let $rj + l$ be the number of its vertices, where $0 \leq l < j$. Note that, by Lemma 8 and Lemma 6, $ex(B) \geq ra$. Let $d = \sum_{B \in \mathcal{B}''} r$ and let G'' be the union of all components of $G - S$ which contain a block in \mathcal{B}'' . By Corollary 17, we may assume that G'' has at most $((\frac{16}{1-\lambda} + \frac{2}{\inf_{AK}})k - 2)\frac{6k}{1-\lambda} + \frac{k}{\inf_{AK}}$ components. Furthermore, by repeated use of Lemma 5, we get that $ex(G'') \geq da$. Observe that $G - G''$ has at most $|S| \leq \frac{6k}{1-\lambda}$ components: then, by Lemma 6, $ex(G) \geq da - \frac{1-\lambda}{2}(((\frac{16}{1-\lambda} + \frac{2}{\inf_{AK}})k - 2)\frac{6k}{1-\lambda} + \frac{k}{\inf_{AK}} + \frac{6k}{1-\lambda} - 1)$. Therefore if $d \geq ((\frac{16}{1-\lambda} + \frac{2}{\inf_{AK}})k - 1)\frac{6k}{a(1-\lambda)} + \frac{k}{(a\inf_{AK})} + \frac{k-1}{a}$, the instance is a YES-instance. Otherwise, $|V''| \leq 2dj \leq c_3 j k^2$, where c_3 is a constant which depends only on Π , which means that $|V(G)| \leq \frac{6k}{1-\lambda} + (c_1 + c_2(j - 1) + c_3 j)k^2$. ◀

5 Kernel when $\lambda \neq \frac{1}{2}$ or $K_3 \in \Pi$

► **Lemma 23.** *Let Π be a strongly λ -extendible property with $\lambda \neq \frac{1}{2}$. Then $ex(K_3) \geq 1 - 2\lambda$ and, if $\lambda > \frac{1}{2}$, $ex(K_3) = 2 - 2\lambda$. In particular, $ex(K_3) > 0$ in every case.*

Proof. Note that $\beta(G) \geq 2$ for any connected graph $G \in \mathcal{G}$ with at least two edges, because any graph whose underlying graph is a path on three vertices is in Π by inclusiveness and block additivity. Therefore, $\beta(K_3) \geq 2$, which ensures that $ex(K_3) \geq 2 - (3\lambda + \frac{1-\lambda}{2} \cdot 2) = 1 - 2\lambda$, which is strictly greater than zero if $\lambda < \frac{1}{2}$.

Now, assume $\lambda > \frac{1}{2}$, let $G \in \mathcal{G}$ be such that $U(G) = K_3$ and let $V(G) = \{v_1, v_2, v_3\}$. Consider $U = \{v_1, v_2\}$ and $W = \{v_3\}$ and note that $G[U], G[W] \in \Pi$ by inclusiveness. Then, by the strong λ -subgraph extension property, it holds that $G \in \Pi$, which ensures that $\beta(K_3) = 3$. This means that $ex(K_3) = 3 - (3\lambda + \frac{1-\lambda}{2} \cdot 2) = 2 - 2\lambda > 0$. ◀

► **Lemma 24.** *Let Π be a strongly λ -extendible property. If $\lambda \neq \frac{1}{2}$, then $ex(K_3) > \frac{1-\lambda}{2}$ or $ex(K_4) > \frac{1-\lambda}{2}$. In particular, Π diverges on cliques.*

Proof. If $\lambda > \frac{1}{2}$, then by Lemma 23 $ex(K_3) = 2 - 2\lambda > \frac{1-\lambda}{2}$. If $\lambda < \frac{1}{3}$, then by Lemma 23 $ex(K_3) \geq 1 - 2\lambda$, which is greater than $\frac{1-\lambda}{2}$. Lastly, if $\frac{1}{3} \leq \lambda < \frac{1}{2}$, let $G \in \mathcal{G}$ be such that $U(G) = K_4$ and let $V(G) = \{v_1, v_2, v_3, v_4\}$. Consider $U = \{v_1, v_2\}$ and $W = \{v_3, v_4\}$ and note that $G[U], G[W] \in \Pi$ by inclusiveness. By the strong λ -subgraph extension property, it holds that $\beta(G) \geq 4$, since $\lambda > \frac{1}{4}$. Therefore, $ex(K_4) \geq 4 - 6\lambda - \frac{1-\lambda}{2}3 = \frac{5}{2} - \frac{9}{2}\lambda$ which is greater than $\frac{1-\lambda}{2}$. ◀

► **Lemma 25.** *Let Π be a strongly λ -extendible property. If $K_3 \in \Pi$, then $ex(K_3) > \frac{1-\lambda}{2}$. In particular, Π diverges on cliques.*

Proof. If $K_3 \in \Pi$, then $\beta(K_3) = 3$, which means that $ex(K_3) = 2 - 2\lambda > \frac{1-\lambda}{2}$. ◀

► **Lemma 26.** *Let Π be a strongly λ -extendible property. If $\lambda \neq \frac{1}{2}$ or $K_3 \in \Pi$, then $ex(K_i) > 0$ for all $i \geq 2$.*

Proof. By Lemma 24 and Lemma 25, $ex(K_3) > \frac{1-\lambda}{2}$ or $ex(K_4) > \frac{1-\lambda}{2}$. In the first case, by Lemma 7, it holds that $ex(K_j) > 0$ for all $j \geq 4$, while in the second case, using the same Lemma, $ex(K_j) > 0$ for all $j \geq 5$. In addition, by Lemma 23, $ex(K_3) > 0$. Finally, $ex(K_2) = 1 - (\lambda + \frac{1-\lambda}{2}) = \frac{1-\lambda}{2} > 0$. ◀

► **Theorem 27.** *Let Π be a strongly λ -extendible property. If $\lambda \neq \frac{1}{2}$ or $K_3 \in \Pi$, then $\text{APT}(\Pi)$ has a kernel with $\mathcal{O}(k^2)$ vertices.*

Proof. By Lemma 24 or Lemma 25, Π diverges on cliques. Furthermore, by Lemma 26, $ex(K_i) > 0$ for all $i \geq 2$. Then, by Theorem 22, $\text{APT}(\Pi)$ has a kernel with at most $\mathcal{O}(k^2)$ vertices. ◀

By Theorem 27, the only remaining cases to consider are those for which $\lambda = 1/2$ and Π does not contain all triangles. We do this in the following section.

6 Kernel when $\lambda = \frac{1}{2}$

► **Definition 28.** A graph property Π is *hereditary* if, for any graph G and vertex-induced subgraph G' of G , if $G \in \Pi$ then $G' \in \Pi$.

► **Theorem 29.** *Let Π be a strongly λ -extendible property with $\lambda = \frac{1}{2}$. Suppose Π is hereditary and $G \notin \Pi$ for any $G \in \mathcal{G}$ such that $U(G) = K_3$. Then $\Pi = \{G \in \mathcal{G} : G \text{ is bipartite}\}$.*

Proof. First, assume for the sake of contradiction that Π contains a non-bipartite graph H . Then H contains an odd cycle C_l . By choosing l as small as possible we may assume that C_l is a vertex-induced subgraph of H . Then, since Π is hereditary, C_l is in Π . Note that if $l = 3$, then $U(C_3) = K_3$, so this is not the case. Consider the graph H' obtained from C_l adding a new vertex v and an edge from v to every vertex of C_l . Since both C_l and $K_1 = \{v\}$ are in Π , by the strong λ -subgraph extension property we can find a subgraph of H' which contains C_l , v and at least half of the edges between v and C_l . Since l is odd, for any choice of $\frac{l+1}{2}$ edges there are two of them, say $e_1 = vx$ and $e_2 = vy$, such that the edge xy is in C_l . Therefore, since Π is hereditary, $H'[v, x, y] \in \Pi$, which leads to a contradiction, as $U(H'[v, x, y]) = K_3$.

Now, we will show that all connected bipartite graphs are in Π , independently from any possible labelling and/or orientation. We will proceed by induction. The claim is trivially

true for $j = 1, 2$. Assume $j \geq 3$ and that every bipartite graph with $l < j$ vertices is in Π . Consider any connected bipartite graph H with j vertices. Consider a vertex v such that $H' = H - \{v\}$ is connected. By induction hypothesis, $H' \in \Pi$. Consider the graph H'' obtained from H' and G_2 , where G_2 is any graph in \mathcal{G} with $U(G_2) = K_2$ (let $V(G_2) = \{v_1, v_2\}$), adding an edge from v_i to $w \in V(H')$ if and only if in H there is an edge from v to w . Colour red the edges from v_1 and blue the edges from v_2 .

Since $G_2 \in \Pi$ by inclusiveness and $H' \in \Pi$, by the strong λ -subgraph extension property there must exist a subgraph \tilde{H} of H'' which contains G_2 , H' and at least half of the edges between them. Note that the red edges are exactly half of the edges and that if \tilde{H} contains all of them and no blue edges, then we can conclude that H is in Π by block additivity. The same holds if \tilde{H} contains every blue edge and no red edge.

If, on the contrary, \tilde{H} contains one red and one blue edge, we will show that it contains a vertex-induced cycle of odd length, which leads to a contradiction according to the first part of the proof. First, suppose that both these edges contain $w \in V(H')$: if this happens, \tilde{H} contains a cycle of length 3 as a vertex-induced subgraph.

Now, suppose \tilde{H} contains a red edge v_1w_1 and a blue edge v_2w_2 . Note that w_1 and w_2 are in the same partition and, since H' is connected, there is a path from w_1 to w_2 which has even length. Together with v_1w_1 , v_2w_2 and v_1v_2 , this gives a cycle of odd length. Choosing the shortest path between w_1 and w_2 , we may assume that the cycle is vertex-induced.

Thus, we conclude that the only possible choices for \tilde{H} are either picking the red edges or picking the blue edges, which concludes the proof. \blacktriangleleft

The above theorem is of interest due to the following theorem:

► **Theorem 30.** [5] MAX-CUT ATLB has a kernel with $\mathcal{O}(k^3)$ vertices.

6.1 Simple graphs

In this part, we assume that \mathcal{G} is the class of simple graphs, that is, without any labelling or orientation. Note that, in this case, there is only one graph, up to isomorphism, whose underlying graph is K_3 (namely, K_3 itself).

► **Theorem 31.** Let Π be a strongly λ -extendible property on simple graphs, with $\lambda = \frac{1}{2}$, and suppose Π is hereditary. Then $APT(\Pi)$ has a kernel with $\mathcal{O}(k^2)$ or $\mathcal{O}(k^3)$ vertices.

Proof. If $K_3 \notin \Pi$, by Theorem 29 Π is equal to Max Cut and therefore by Theorem 30 it admits a kernel with $\mathcal{O}(k^3)$. On the other hand, if $K_3 \in \Pi$, then by Theorem 27 Π admits a kernel with $\mathcal{O}(k^2)$ vertices. \blacktriangleleft

6.2 Oriented graphs

In this part, we assume that \mathcal{G} is the class of oriented graphs, without any labelling.

► **Definition 32.** Let $\vec{K}_3 \in \mathcal{G}$ be such that $U(\vec{K}_3) = K_3$, $V(\vec{K}_3) = \{v_1, v_2, v_3\}$ and $o((v_i, v_{i+1})) = >$ for $1 \leq i \leq 2$ and $o((v_1, v_3)) = <$. We will call \vec{K}_3 the *oriented* triangle.

Similarly, let $\overleftrightarrow{K}_3 \in \mathcal{G}$ be such that $U(\overleftrightarrow{K}_3) = K_3$, $V(\overleftrightarrow{K}_3) = \{u_1, u_2, u_3\}$ and $o((u_i, u_j)) = >$ for every $i < j$, $1 \leq i, j \leq 3$. We will call \overleftrightarrow{K}_3 the *non-oriented* triangle.

It is not difficult to see that, up to isomorphism, \vec{K}_3 and \overleftrightarrow{K}_3 are the only graphs in \mathcal{G} with K_3 as underlying graph.

► **Lemma 33.** *Let Π be a strongly λ -extendible property on oriented graphs, with $\lambda = \frac{1}{2}$, and suppose Π is hereditary. If $\vec{K}_3 \in \Pi$, then $\overleftrightarrow{K}_3 \in \Pi$.*

Proof. Consider the graph H obtained by adding a vertex v to \vec{K}_3 and an edge from v to every vertex of \vec{K}_3 , such that $o((v, v_i)) = >$ for every $1 \leq i \leq 3$. Since $\vec{K}_3 \in \Pi$, by the strong λ -subgraph extension property there exists a Π -subgraph H' of H which contains \vec{K}_3 , v and at least two edges between \vec{K}_3 and v : without loss of generality, assume these edges are vv_1 and vv_2 . Then since Π is hereditary $H'[v, v_1, v_2] \in \Pi$ and note that $H'[v, v_1, v_2]$ is isomorphic to \overleftrightarrow{K}_3 . ◀

► **Lemma 34.** *Let Π be a strongly λ -extendible property on oriented graphs, with $\lambda = \frac{1}{2}$, and suppose Π is hereditary. If $\vec{K}_3 \in \Pi$, then $ex(K_4) > \frac{1}{4}$. In particular, Π diverges on cliques.*

Proof. Let $H \in \mathcal{G}$ be such that $U(H) = K_4$ and let $V(H) = \{w_1, w_2, w_3, w_4\}$. If $H[w_1, w_2, w_3] = \vec{K}_3$ and $H[w_2, w_3, w_4] = \vec{K}_3$, then $H[w_1, w_2, w_4] = \vec{K}_3$. Hence, for any orientation on the edges of H , the graph contains \vec{K}_3 as a vertex-induced subgraph. Now, since $\vec{K}_3 \in \Pi$, by the strong λ -subgraph extension property there exists a Π -subgraph of H which contains at least 5 edges, which means that $\beta(H) \geq 5$. This ensures that $ex(K_4) \geq 5 - (3 + \frac{3}{4}) = \frac{5}{4}$, which concludes the proof. ◀

► **Lemma 35.** *Let Π be a strongly λ -extendible property on oriented graphs, with $\lambda = \frac{1}{2}$, and suppose Π is hereditary, $\vec{K}_3 \notin \Pi$ and $\overleftrightarrow{K}_3 \in \Pi$. Then $ex(K_j) > 0$ for every $j \neq 3$.*

Proof. Note that $ex(K_4) > \frac{1}{4}$, then by Lemma 7 $ex(K_j) > 0$ for every $j \geq 4$. In addition, $ex(K_2) = \frac{1}{4}$. ◀

Let \mathcal{B}_2^0 be the subset of \mathcal{B}_2 which contains all the blocks with excess zero, and have no internal vertices in $N(S)$. Let Q_0 denote the set of cut vertices of $G - S$ which only appear in blocks in \mathcal{B}_2^0 . Note that every vertex in Q_0 appears in exactly two blocks in \mathcal{B}_2^0 .

► **Lemma 36.** *Let Π be a strongly λ -extendible property on oriented graphs, with $\lambda = \frac{1}{2}$, and suppose Π is hereditary, $\vec{K}_3 \notin \Pi$ and $\overleftrightarrow{K}_3 \in \Pi$. Let (G, k) be an instance of $APT(\Pi)$ reduced by Reduction Rule 1. For any $s \in S$, if $|Q_0 \cap N(s)| \geq ((32 + \frac{2}{\inf_{AK}})k - 2)48k - \frac{4k}{\inf_{AK}} + 4k$, then the instance is a YES-instance.*

Proof. First, note that all the blocks in \mathcal{B}_2^0 are isomorphic to \vec{K}_3 by Lemma 35. Observe that every vertex in Q_0 has at most two neighbours in Q_0 . Since all vertices in Q_0 are cut vertices of $G - S$, it follows that $G[Q_0 \cap N(s)]$ is a disjoint union of paths. It follows that we can find a set $Q'_0 \subseteq Q_0 \cap N(s)$ such that $|Q'_0| \geq \frac{|Q_0 \cap N(s)|}{2}$ and Q'_0 is an independent set.

For each $v \in Q'_0$, let B_1, B_2 be the two blocks in \mathcal{B}_2^0 that contain v , and let v_i be the unique vertex in $(B_i)_{\text{int}}$, for $i \in \{1, 2\}$. Then let $U = \{s\} \cup Q'_0 \cup \{v_i : v \in Q'_0, i \in \{1, 2\}\}$, and observe that $G[U]$ is a tree with $3|Q'_0|$ edges. It follows that $G[U] \in \Pi$ and $ex(G[U]) = \frac{3|Q'_0|}{4}$. By Lemma 6, $ex(G) \geq \frac{3|Q'_0| - c}{4}$, where c is the number of components of $G - U$.

Consider the components of $G - U$. Each component either contains a block in $\mathcal{B}_1 \cup \mathcal{B}_{\geq 3}$ or it is part of a block path of $G - S$ containing two vertices from Q'_0 : by Corollary 20 there are at most $4((\frac{16}{1-\lambda} + \frac{2}{\inf_{AK}})k - 2)\frac{6k}{1-\lambda} + \frac{k}{\inf_{AK}} = ((32 + \frac{2}{\inf_{AK}})k - 2)48k + \frac{4k}{\inf_{AK}}$ components of the first kind, while there are at most $|Q'_0|$ of the second kind.

Thus, if $2|Q'_0| - ((32 + \frac{2}{\inf_{AK}})k - 2)48k - \frac{4k}{\inf_{AK}} \geq 4k$ then we have a YES-instance; otherwise $|Q_0 \cap N(s)| \leq 2|Q'_0| \leq ((32 + \frac{2}{\inf_{AK}})k - 2)48k - \frac{4k}{\inf_{AK}} + 4k$. ◀

► **Reduction Rule 2.** Let $B_1, B_2 \in \mathcal{B}_2$ be such that $V(B_1) \cap V(B_2) = \{v\}$, $B_1 = \vec{K}_3 = B_2$, $\{v\} \cap N(S) = \emptyset$ and $(B_i)_{\text{int}} \cap N(S) = \emptyset$ for $i = 1, 2$. Let $\{w_i\} = (B_i)_{\text{int}}$ and $\{u_i\} = V(B_i) \setminus \{v, w_i\}$ for $i = 1, 2$. If $G - \{v\}$ is disconnected, delete v, w_1, w_2 , identify u_1 and u_2 and set $k' = k$. Otherwise, delete v, w_1, w_2 and set $k' = k - \frac{1}{4}$.

► **Lemma 37.** Let Π be a strongly λ -extendible property on oriented graphs, with $\lambda = \frac{1}{2}$, and suppose Π is hereditary. If $\vec{K}_3 \notin \Pi$, then Rule 2 is valid.

Proof. Let G' be the graph which is obtained after an application of the rule. If $G - \{v\}$ is disconnected, let G'' be the graph obtained from G deleting v, w_1 and w_2 and without identifying any vertices. Then, note that G'' has two connected components, one containing u_1 and the other containing u_2 : hence, G' is connected. Additionally, $ex(G') = ex(G'')$.

Let $\tilde{G} = G''$ if $G - \{v\}$ is disconnected and $\tilde{G} = G'$ otherwise. We have to show that $ex(\tilde{G}) = ex(G)$ if $G - \{v\}$ is disconnected and $ex(\tilde{G}) = ex(G) - \frac{1}{4}$ otherwise. In order to do this, we will show that for any maximal Π -subgraph \tilde{H} of \tilde{G} there exists a Π -subgraph H of G such that $H[V(\tilde{G})] = \tilde{H}$ and $|E(H[V(B_1) \cup V(B_2)])| = \gamma(G[V(B_1) \cup V(B_2)])$. Then the result follows because if $G - \{v\}$ is disconnected, then $\gamma(G) = \gamma(G - \{v, w_1, w_2\}) + \gamma(G[V(B_1) \cup V(B_2)])$, and if $G - \{v\}$ is connected, then $\gamma(G) = \gamma(G - \{v, w_1, w_2\}) + \gamma(G[V(B_1) \cup V(B_2)]) - \frac{1}{4}$.

Let \tilde{H} be any maximal Π -subgraph of \tilde{G} . Note that by block additivity and inclusiveness, $G[v, w_1, w_2] \in \Pi$. Then, by the strong λ -subgraph extension property there exists a Π -subgraph H of G which contains \tilde{H} , $G[v, w_1, w_2]$ and at least half of the edges between them. Note that these edges are exactly four: vu_1, w_1u_1, vu_2 and w_2u_2 . If vu_1 and w_1u_1 are in $E(H)$, then since Π is hereditary it holds that $\vec{K}_3 \in \Pi$, which is a contradiction. Similarly if vu_2 and w_2u_2 are in $E(H)$. This means that exactly two edges among them are in $E(H)$, that is that $|E(H[V(B_1) \cup V(B_2)])| = 4 = \gamma(G[V(B_1) \cup V(B_2)])$. ◀

► **Theorem 38.** Let Π be a strongly λ -extendible property on oriented graphs, with $\lambda = \frac{1}{2}$, and suppose Π is hereditary. Then $\text{APT}(\Pi)$ has a kernel with $\mathcal{O}(k^2)$ if $K_3 \in \Pi$ and has a kernel with $\mathcal{O}(k^3)$ vertices otherwise.

Proof. If $\vec{K}_3 \in \Pi$, by Lemma 33 $\vec{K}_3 \in \Pi$. This means that $K_3 \in \Pi$ and, by Theorem 27, $\text{APT}(\Pi)$ has a kernel with $\mathcal{O}(k^2)$ vertices. On the other hand, if $\vec{K}_3 \notin \Pi$ and $\vec{K}_3 \notin \Pi$, then by Theorem 29 Π is equal to Max Cut and by Theorem 30 it admits a kernel with $\mathcal{O}(k^3)$ vertices.

Lastly, suppose $\vec{K}_3 \in \Pi$ and $\vec{K}_3 \notin \Pi$. By Lemma 34, Π diverges on cliques. Let (G, k) be an instance of $\text{APT}(\Pi)$ reduced by Reduction Rule 1 and 2 (note that in this case Rule 2 is valid by Lemma 37).

By Corollary 20, we may assume $|\mathcal{B}_0| + |\mathcal{B}_1| + |\mathcal{B}_{\geq 3}| \leq 4 \left(\left(\frac{16}{1-\lambda} + \frac{2}{\text{inf}_{AK}} \right) k - 2 \right) \frac{6k}{1-\lambda} + \frac{k}{\text{inf}_{AK}}$.

We now need to consider different types of blocks in \mathcal{B}_2 separately. Let \mathcal{B}_2^+ be the blocks in \mathcal{B}_2 with positive excess. By Lemma 18, we may assume the number of such blocks is at most $(32 + \frac{2}{\text{inf}_{AK}})k - 1) \frac{12k}{\text{inf}_{AK}} + \frac{k}{(\text{inf}_{AK})^2} + \frac{k-1}{\text{inf}_{AK}}$.

Let \mathcal{B}_2' be the blocks in $\mathcal{B}_2 \setminus \mathcal{B}_2^+$ which have an interior vertex in $N(S)$. By Corollary 15, we may assume the number of such blocks is at most $((32 + \frac{2}{\text{inf}_{AK}})k - 2)12k$.

Let \mathcal{B}_2'' be the blocks in $\mathcal{B}_2 \setminus (\mathcal{B}_2^+ \cup \mathcal{B}_2')$ which contain a vertex in $Q \cap N(S)$. Observe that these blocks must either contain a vertex of $Q_0 \cap N(S)$ or be adjacent to a block in $\mathcal{B}_1, \mathcal{B}_{\geq 3}, \mathcal{B}_2^+$ or \mathcal{B}_2' . Furthermore they must be in block paths between such blocks, from which it follows that $|\mathcal{B}_2''| \leq 2(|\mathcal{B}_1| + |\mathcal{B}_{\geq 3}| + |\mathcal{B}_2^+| + |\mathcal{B}_2'| + |Q_0 \cap N(S)|)$.

Finally let $\mathcal{B}_2''' = \mathcal{B}_2 \setminus (\mathcal{B}_2^+ \cup \mathcal{B}_2' \cup \mathcal{B}_2'')$. These are just the blocks in \mathcal{B}_2 with excess 0 which contain no neighbors of S . By Reduction Rule 2, no two such blocks can be adjacent. Therefore every block in \mathcal{B}_2''' is adjacent to two blocks from $\mathcal{B}_1, \mathcal{B}_{\geq 3}, \mathcal{B}_2^+, \mathcal{B}_2'$ or \mathcal{B}_2'' . It follows that $|\mathcal{B}_2'''| \leq |\mathcal{B}_1| + |\mathcal{B}_{\geq 3}| + |\mathcal{B}_2^+| + |\mathcal{B}_2'| + |\mathcal{B}_2''|$.

Hence, we may conclude that $|\mathcal{B}_0| + |\mathcal{B}_1| + |\mathcal{B}_{\geq 3}| + |\mathcal{B}_2^+| \leq c_1 k^2$ for some constant c_1 depending only on Π . Furthermore, note that by Lemma 36 and the fact that $|S| \leq 12k$, we may assume that $|Q_0 \cap N(S)| \leq ((32 + \frac{2}{\inf_{AK}})k - 2)48k - \frac{4k}{\inf_{AK}} + 4k)12k$. Then we may conclude from the above that $|\mathcal{B}_2'| + |\mathcal{B}_2''| + |\mathcal{B}_2'''| \leq c_2 k^3$ for some constant c_2 depending only on Π .

Therefore the total number of blocks in $G - S$ is at most $c_1 k^2 + c_2 k^3$. It follows that $|Q|$, the number of cut vertices of $G - S$, is at most $c_1 k^2 + c_2 k^3$.

By Lemma 21, we may assume that the number of internal vertices for any block is at most $c_3 k$, for some constant c_3 depending only on Π . It follows that the number of vertices in blocks from $\mathcal{B}_0, \mathcal{B}_1, \mathcal{B}_{\geq 3}$ or \mathcal{B}_2^+ is at most $c_1 c_3 k^3 + c_1 k^2 + c_2 k^3$. To bound the number of vertices in blocks from $\mathcal{B}_2' \cup \mathcal{B}_2'' \cup \mathcal{B}_2'''$, note that each of these blocks contains at most 3 vertices, by Lemma 35 and the fact that these blocks have excess 0 by definition. Therefore the number of vertices in blocks from $\mathcal{B}_2' \cup \mathcal{B}_2'' \cup \mathcal{B}_2'''$ is at most $3c_2 k^3$. Finally, recalling that $|S| \leq 12k$, we have that the number of vertices in G is $\mathcal{O}(k^3)$. ◀

Putting together Theorem 27, Theorem 31, and Theorem 38, we get our main result, Theorem 1.

7 Conclusion

We have succeeded in showing that $\text{APT}(\Pi)$ has a polynomial kernel for nearly all strongly λ -extendible Π . The only cases in which the polynomial kernel question remains open are those in which $\lambda = \frac{1}{2}$ and either Π is not hereditary, or membership in Π depends on the labellings on edges. For the cases when $\lambda \neq \frac{1}{2}$ or Π contains all triangles, we could show the existence of a kernel with $\mathcal{O}(k^2)$ vertices. It would be desirable to show a $\mathcal{O}(k^2)$ kernel in all cases.

The bound of Poljak and Turzík extends to edge-weighted graphs - for any strongly λ -extendible property Π and any connected graph G with weight function $w : E(G) \rightarrow \mathbb{R}^+$, there exists a subgraph H of G such that $H \in \Pi$ and $w(H) \geq \lambda w(G) + \frac{(1-\lambda)w(T)}{2}$, where T is a minimum weight spanning tree of G . The natural question following from our results is whether the weighted version of $\text{APT}(\Pi)$ affords a polynomial kernel.

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