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Graßmannian integrals as matrix models for non-compact Yangian invariants

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Abstract

In the past years, there have been tremendous advances in the field of planar $\mathcal{N} = 4$ super Yang–Mills scattering amplitudes. At tree-level they were formulated as Graßmannian integrals and were shown to be invariant under the Yangian of the superconformal algebra $\mathfrak{psu}(2,2|4)$. Recently, Yangian invariant deformations of these integrals were introduced as a step towards regulated loop-amplitudes. However, in most cases it is still unclear how to evaluate these deformed integrals. In this work, we propose that changing variables to oscillator representations of $\mathfrak{psu}(2,2|4)$ turns the deformed Graßmannian integrals into certain matrix models. We exemplify our proposal by formulating Yangian invariants with oscillator representations of the non-compact algebra $\mathfrak{u}(p,q)$ as Graßmannian integrals. These generalize the Brezin–Gross–Witten and Leutwyler–Smilga matrix models. This approach might make elaborate matrix model technology available for the evaluation of Graßmannian integrals. Our invariants also include a matrix model formulation of the $\mathfrak{u}(p,q)$ R-matrix, which generates non-compact integrable spin chains.

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1. Introduction

The maximally supersymmetric Yang–Mills theory in four-dimensions, for short $\mathcal{N} = 4$ SYM, is a remarkably rich mathematical model. Even more so in the planar limit where the theory is conjectured to be integrable. By now this integrability is well established for the spectral problem of anomalous dimensions, see the comprehensive review series [1]. Less is known about integrability for scattering amplitudes. However, at tree-level the amplitudes can be encoded as surprisingly simple formulas, so-called Graßmannian integrals [2,3], see also [4]. The mere existence of such formulas already hints at an underlying integrable structure. Furthermore, it was shown that tree-level amplitudes are invariant under the Yangian of the superconformal algebra $\mathfrak{psu}(2,2|4)$ [5]. For the Graßmannian integral formulation this was achieved in [6,7]. The appearance of this infinite-dimensional Yangian algebra is synonymous with integrability. Later, it was observed that the tree-level amplitudes allow for multi-parameter deformations while maintaining Yangian invariance. These deformations are of considerable interest as they relate the four-dimensional scattering problem to the two-dimensional quantum inverse scattering method. Furthermore, they might regulate infrared divergences at loop-level [8,9].

As in the undeformed case, the deformed tree-level amplitudes can be nicely packaged as Graßmannian integrals [10,11]. Let us briefly review this formulation. The *Graßmannian* $\mathrm{Gr}(N, K)$ is the space of all K-dimensional linear subspaces of $\mathbb{C}^N$. The entries of a $K \times N$ matrix C provide "homogeneous" coordinates on this space. The transformation $C \mapsto VC$ with $V \in GL(K)$ corresponds to a change of basis within a given subspace, and thus it does not change the point in the Graßmannian. This allows us to describe a generic point in $\mathrm{Gr}(N, K)$ by the "gauge fixed" matrix

$$C = \left(1_{K\times K} \,\middle\vert\, \mathcal{C} \right) \quad \text{with} \quad \mathcal{C} = \begin{pmatrix} C_{1K+1} & \cdots & C_{1N} \\ \vdots & & \vdots \\ C_{KK+1} & \cdots & C_{KN} \end{pmatrix}. \tag{1.1}$$

The amplitudes are labeled by the number of particles N and the degree of helicity violation K. Amplitudes with $K = 2$ are maximally helicity violating (MHV). The deformed N-point N^{K-2}MHV tree-level amplitude is given by the *Graßmannian integral*

$$\mathcal{A}_{N,K} = \int \mathrm{d}\mathcal{C} \frac{\delta^{4K|4K}(C\mathcal{W})}{(1,\ldots,K)^{1+v_K^+-v_1^-} \cdots (N,\ldots,K-1)^{1+v_{K-1}^+-v_N^-}} \tag{1.2}$$

with the holomorphic $K(N-K)$-form $\mathrm{d}\mathcal{C} = \bigwedge_{k,l} \mathrm{d}C_{kl}$. In this formula $(i,\ldots,i+K-1)$ denotes the minor of the matrix C consisting of the consecutive columns $i,\ldots,i+K-1$. These are counted modulo N such that they are in the range $1,\ldots,N$. The kinematics of the j-th particle is encoded in a supertwistor with components $\mathcal{W}_A^j$, where A is a fundamental $\mathfrak{gl}(4|4)$ index. The $2N$ deformation parameters $\{v_i^+, v_i^-\}$ have to obey the constraints

$$v_{i+K}^+ = v_i^- \tag{1.3}$$

for $i = 1,\ldots,N$. Then the Graßmannian integral (1.2) is invariant under the Yangian of $\mathfrak{psu}(2,2|4)$, where the generators of the algebra act on the supertwistors. In the undeformed case $v_i^\pm = 0$, the proper integration contour for (1.2) is known and the integral can be evaluated by means of a multi-dimensional residue theorem [2,4]. In the deformed case, the evaluation is much more involved due to branch cuts of the integrand. Most notably there are partial results on the 6-point NMHV amplitude [10]. However, finding an appropriate multi-dimensional integration contour for the evaluation of (1.2) is still a pressing open problem.

In the present work, we establish a connection between the Graßmannian integral formulation of Yangian invariants and unitary matrix models. We follow a systematic approach and do not focus on the particular supertwistor realization of the algebra $\mathfrak{psu}(2,2|4)$ that is often employed for amplitudes. Instead, we work with a class of harmonic oscillator representations of the non-compact algebra $\mathfrak{u}(p,q)$, where we restrict to the bosonic case for clarity. We find that also in this setting Yangian invariants can be formulated as Graßmannian integrals. The *only* change compared to (1.2) is that the delta function of the supertwistors gets replaced by an exponential function of oscillators,

$$\delta^{4K|4K}(\mathcal{C}\mathcal{W}) \mapsto (\det\mathcal{C})^{-q} e^{\operatorname{tr}(\mathcal{C}\mathbf{I}_\bullet^t+\mathbf{I}_\circ\mathcal{C}^{-1})}|0\rangle\,. \tag{1.4}$$

For the moment, we restrict for simplicity to the "split helicity" case $N=2K$ in order for $\mathcal{C}^{-1}$ to exist. The $K\times K$ matrices $\mathbf{I}_\bullet$ and $\mathbf{I}_\circ$ contain certain oscillator invariants associated with the compact subalgebras $\mathfrak{u}(p)$ and $\mathfrak{u}(q)$, respectively. The integration contour is still unspecified. We observe that the deformation parameters $v_i^\pm$ can be chosen such that the exponents of all minors in (1.2) vanish. If we restrict in addition the range of integration to unitary matrices $\mathcal{C}$, the Graßmannian integral reduces to an intensively studied unitary matrix model, the Brezin–Gross–Witten model [12,13]. Similarly, we may also obtain the Leutwyler–Smilga model [14]. This motivates us to conjecture that the "unitary contour" works as well for general deformation parameters. This would mean that the Graßmannian integrals can be considered as novel types of unitary matrix models. We provide a non-trivial example of this conjecture by investigating the invariant with $(N,K)=(4,2)$. In this example the Graßmannian integral becomes a $U(2)$ matrix model that correctly evaluates to the $\mathfrak{u}(p,q)$ R-matrix, which is known to be Yangian invariant. This R-matrix generates non-compact integrable spin chains.

The connection between Graßmannian integrals and matrix models opens exciting possibilities. In particular, advanced matrix model technology such as character expansions, see e.g. the concise review [15], might become applicable for the evaluation of Graßmannian integrals. We expect our results to generalize straightforwardly from $\mathfrak{u}(p,q)$ to superalgebras $\mathfrak{u}(p,q|r)$ and thus to $\mathfrak{psu}(2,2|4)$. Hence our matrix model approach should also be of utility for the open problem mentioned above, the evaluation of deformed $\mathcal{N}=4$ SYM amplitudes. There are further fascinating prospects which we elaborate on in the outlook of Section 7.

2. Yangian and non-compact oscillators

In this preparatory section, we introduce the Yangian of the Lie algebra $\mathfrak{gl}(n)$ and the notion of Yangian invariants. In addition, we define the classes of oscillator representations of the algebra $\mathfrak{u}(p,q)\subset\mathfrak{gl}(p+q=n)$ that we will use to build up representations of the Yangian.

The *Yangian* of $\mathfrak{gl}(n)$ is defined by the relation, see e.g. [16],

$$R(u-u')(M(u)\otimes 1)(1\otimes M(u'))=(1\otimes M(u'))(M(u)\otimes 1)R(u-u')\,. \tag{2.1}$$

Here $R(u)$ acts on the tensor product $\mathbb{C}^n\otimes\mathbb{C}^n$ and solves the Yang–Baxter equation. It is built from $n\times n$ matrices with components $(e_{AB})_{CD}=\delta_{AC}\delta_{DB}$ and reads

$$R(u)=1+u^{-1}\sum_{A,B=1}^{n} e_{AB}\otimes e_{BA}\,. \tag{2.2}$$

The operator valued monodromy matrix $M(u)$ contains the infinitely many Yangian generators $M_{AB}^{(r)}$ with $r = 1, 2, \ldots$. They are obtained from an expansion in the complex spectral parameter u

$$M(u) = \sum_{A,B=1}^{n} e_{AB} M_{AB}(u) , \quad M_{AB}(u) = M_{AB}^{(0)} + u^{-1} M_{AB}^{(1)} + u^{-2} M_{AB}^{(2)} + \ldots \tag{2.3}$$

with $M_{AB}^{(0)} = \delta_{AB}$. Expressed in terms of these generators, the defining relation (2.1) becomes

$$[M_{AB}^{(r)}, M_{CD}^{(s)}] = \sum_{q=1}^{\min(r,s)} \left(M_{CB}^{(r+s-q)} M_{AD}^{(q-1)} - M_{CB}^{(q-1)} M_{AD}^{(r+s-q)} \right) . \tag{2.4}$$

From this formula one easily deduces that all generators $M_{AB}^{(r)}$ with $r > 2$ can be expressed via $M_{AB}^{(1)}$ and $M_{AB}^{(2)}$. In our study we are interested in states that are *Yangian invariant* [17]

$$M_{AB}(u)|\Psi\rangle = \delta_{AB}|\Psi\rangle . \tag{2.5}$$

With the help of the expansion in (2.3) this condition translates into

$$M_{AB}^{(1)}|\Psi\rangle = 0 , \quad M_{AB}^{(2)}|\Psi\rangle = 0 . \tag{2.6}$$

From now on we specialize on realizations of the Yangian where the monodromy is that of an inhomogeneous spin chain with N sites. Thus

$$M(u) = L_1(u - v_1) \cdots L_N(u - v_N) \tag{2.7}$$

is the product of N Lax operators

$$L_i(u - v_i) = 1 + (u - v_i)^{-1} \sum_{A,B=1}^{n} e_{AB} J_{BA}^{i} . \tag{2.8}$$

Here the meaning of the word *inhomogeneous* is twofold. First, we associate a complex inhomogeneity parameter v_i with each site. Second, each site carries a different representation of the $\mathfrak{gl}(n)$ algebra with generators J_{AB}^{i} that satisfy

$$[J_{AB}^{i}, J_{CD}^{i}] = \delta_{CB} J_{AD}^{i} - \delta_{AD} J_{CB}^{i} \tag{2.9}$$

and act on a space $\mathcal{V}^i$. Consequently the matrix elements of the monodromy $M(u)$ act on the tensor product $\mathcal{V}^1 \otimes \cdots \otimes \mathcal{V}^N$. The Yangian generators introduced in (2.3) can be expressed in terms of the $\mathfrak{gl}(n)$ generators,

$$M_{AB}^{(1)} = \sum_{i=1}^{N} J_{BA}^{i} , \quad M_{AB}^{(2)} = \sum_{i=1}^{N} v_i J_{BA}^{i} + \sum_{\substack{i,j=1 \\ i<j}}^{N} \sum_{C=1}^{n} J_{CA}^{i} J_{BC}^{j} , \quad \ldots . \tag{2.10}$$

Next, we introduce the representations of the $\mathfrak{gl}(n)$ algebra which we will employ at the sites of the spin chain monodromy (2.7). We work with certain classes of unitary representations of the *non-compact* Lie algebra $\mathfrak{u}(p,q) \subset \mathfrak{gl}(p+q)$ that are constructed in terms of a single family of *harmonic oscillator algebras*. These are sometimes referred to as “ladder representations”, see e.g. [18]. Consider the family of oscillator algebras

$$[\mathbf{a}_A, \bar{\mathbf{a}}_B] = \delta_{AB} , \quad \mathbf{a}_A^{\dagger} = \bar{\mathbf{a}}_A , \quad \mathbf{a}_A|0\rangle = 0 \tag{2.11}$$

with $A, B = 1, 2, \dots n$. These oscillators are realized on a Fock space $\mathcal{F}$ that is spanned by monomials of creation operators $\bar{\mathbf{a}}_A$ acting on the vacuum $|0\rangle$. We split the index $A = (\alpha, \dot{\alpha})$ into a pair of indices $\alpha = 1, \dots, p$ and $\dot{\alpha} = p+1, \dots, p+q$ with $p+q = n$. Employing this notation, we define generators

$$(\mathbf{J}_{AB}) = \left(\begin{array}{c:c} \mathbf{J}_{\alpha\beta} & \mathbf{J}_{\alpha\dot{\beta}} \\ \hdashline \mathbf{J}_{\dot{\alpha}\beta} & \mathbf{J}_{\dot{\alpha}\dot{\beta}} \end{array}\right) = \left(\begin{array}{c:c} \bar{\mathbf{a}}_\alpha \mathbf{a}_\beta & -\bar{\mathbf{a}}_\alpha \bar{\mathbf{a}}_{\dot{\beta}} \\ \hdashline \mathbf{a}_{\dot{\alpha}} \mathbf{a}_\beta & -\mathbf{a}_{\dot{\alpha}} \bar{\mathbf{a}}_{\dot{\beta}} \end{array}\right) \tag{2.12}$$

that satisfy the $\mathfrak{gl}(n)$ algebra (2.9). Let $\mathcal{V}_c \subset \mathcal{F}$ be the eigenspace of the central element $\mathbf{C} = \sum_{A=1}^{n} \mathbf{J}_{AA}$ with eigenvalue c. For each $c \in \mathbb{Z}$ this infinite-dimensional space forms a unitary irreducible representation of $\mathfrak{u}(p, q)$. Hence we may interpret c as a representation label. The space $\mathcal{V}_c$ contains a lowest weight state, which by definition is annihilated by all $\mathbf{J}_{AB}$ with $A > B$. Notice that in the special case $q = 0$ or $p = 0$ the space $\mathcal{V}_c$ is finite-dimensional and forms a unitary irreducible representation of the compact Lie algebra $\mathfrak{u}(n)$. According to (2.6), Yangian invariants are in particular $\mathfrak{gl}(n)$ singlet states. For such states to exist, we need also spin chain sites with representations that are dual to the class of representations $\mathcal{V}_c$. Its generators are obtained from (2.12) by $\bar{\mathbf{J}}_{AB} = -\mathbf{J}^\dagger_{AB}$. This yields

$$(\bar{\mathbf{J}}_{AB}) = \left(\begin{array}{c:c} \bar{\mathbf{J}}_{\alpha\beta} & \bar{\mathbf{J}}_{\alpha\dot{\beta}} \\ \hdashline \bar{\mathbf{J}}_{\dot{\alpha}\beta} & \bar{\mathbf{J}}_{\dot{\alpha}\dot{\beta}} \end{array}\right) = \left(\begin{array}{c:c} -\bar{\mathbf{a}}_\beta \mathbf{a}_\alpha & \mathbf{a}_{\dot{\beta}} \mathbf{a}_\alpha \\ \hdashline -\bar{\mathbf{a}}_\beta \bar{\mathbf{a}}_{\dot{\alpha}} & \mathbf{a}_{\dot{\beta}} \bar{\mathbf{a}}_{\dot{\alpha}} \end{array}\right) \tag{2.13}$$

satisfying the $\mathfrak{gl}(n)$ algebra (2.9). The element $\bar{\mathbf{C}} = \sum_{A=1}^{n} \bar{\mathbf{J}}_{AA}$ is central. We denote the eigenspace of $\bar{\mathbf{C}}$ with eigenvalue c by $\bar{\mathcal{V}}_c \subset \mathcal{F}$. For each $c \in \mathbb{Z}$ this space forms a unitary irreducible representation of $\mathfrak{u}(p, q)$. The representation $\bar{\mathcal{V}}_c$ is dual to $\mathcal{V}_{-c}$. It contains a highest weight state, which is annihilated by all $\bar{\mathbf{J}}_{AB}$ with $A < B$. In case of $q = 0$ or $p = 0$ the representation $\bar{\mathcal{V}}_c$ is a unitary irreducible representation of $\mathfrak{u}(n)$. Having defined the two classes of non-compact oscillator representations allows us to use them at the sites of the monodromy $M(u)$ in (2.7). At each site we chose either a representation $\mathcal{V}_{c_i}$ with generators $J^i_{AB} = \mathbf{J}^i_{AB}$ or $\bar{\mathcal{V}}_{c_i}$ with $J^i_{AB} = \bar{\mathbf{J}}^i_{AB}$. The monodromy $M(u)$, and hence the representation of the Yangian, is completely specified by $2N$ parameters, i.e. N inhomogeneities $v_i \in \mathbb{C}$ and N representation labels $c_i \in \mathbb{Z}$. We remark that the tensor product decomposition of the oscillator representations employed at the spin chain sites has been studied in [19], see also e.g. [20,21] for exemplary results.

3. Simple sample invariant

Before formulating a Graßmannian integral for the just defined oscillator representations of $\mathfrak{u}(p, q)$, it is instructive to construct a simple solution of the Yangian invariance condition (2.5) "by hand".

We consider a monodromy with two sites. To be able to construct a $\mathfrak{gl}(n)$ singlet state, we choose for the first site a "dual" representation and for the second site an "ordinary" one. Hence the monodromy elements $M_{AB}(u)$ act on the space $\bar{\mathcal{V}}_{c_1} \otimes \mathcal{V}_{c_2}$. The $\mathfrak{gl}(n)$ generators, which appear in the Lax operators (2.8) and consequently also in the Yangian generators (2.10), become $J^1_{AB} = \bar{\mathbf{J}}^1_{AB}$ and $J^2_{AB} = \mathbf{J}^2_{AB}$. To proceed we will make an ansatz for the Yangian invariant state $|\Psi_{2,1}\rangle$, which is labeled by the total number of sites $N = 2$ and the number of "dual" sites $K = 1$. For this ansatz we introduce $\mathfrak{u}(p)$ and $\mathfrak{u}(q)$ invariant contractions of oscillators, respectively,

$$(1 \bullet 2) = \sum_{\alpha=1}^{p} \bar{\mathbf{a}}^1_\alpha \bar{\mathbf{a}}^2_\alpha \,, \quad (1 \circ 2) = \sum_{\dot{\alpha}=p+1}^{p+q} \bar{\mathbf{a}}^1_{\dot{\alpha}} \bar{\mathbf{a}}^2_{\dot{\alpha}} \,. \tag{3.1}$$

We assume $|\Psi_{2,1}\rangle$ to be a power series in $(1 \bullet 2)$ and $(1 \circ 2)$ acting on the Fock vacuum $|0\rangle$. Next, we demand Yangian invariance (2.5) of this ansatz. Furthermore, we impose that each site carries an irreducible representation of $\mathfrak{u}(p,q)$, i.e. $\bar{\mathbf{C}}_1|\Psi_{2,1}\rangle = c_1|\Psi_{2,1}\rangle$ and $\mathbf{C}_2|\Psi_{2,1}\rangle = c_2|\Psi_{2,1}\rangle$. This fixes the invariant, up to a normalization constant, to be

$$|\Psi_{2,1}\rangle = 2\pi i \sum_{\substack{g,h=0\\ g-h=c_2+q}}^{\infty} \frac{(1\bullet 2)^g}{g!}\frac{(1\circ 2)^h}{h!}|0\rangle = 2\pi i\, \frac{I_{c_2+q}\big(2\sqrt{(1\bullet 2)(1\circ 2)}\big)}{\sqrt{(1\bullet 2)(1\circ 2)}^{\,c_2+q}}(1\bullet 2)^{c_2+q}|0\rangle, \tag{3.2}$$

where we identified the sum with the series expansion of the modified Bessel function of the first kind $I_\nu(x)$.[1] The parameters of the monodromy have to obey

$$v_1 - v_2 = 1 - n - c_2\,, \quad c_1 = -c_2 \in \mathbb{Z}\,. \tag{3.3}$$

We observe that the invariant (3.2) can be expressed as a complex contour integral

$$|\Psi_{2,1}\rangle = \int \mathrm{d}C_{12}\frac{e^{C_{12}(1\bullet 2)+C_{12}^{-1}(1\circ 2)}|0\rangle}{C_{12}^{1+c_2+q}}\,. \tag{3.4}$$

Here the contour is a counterclockwise unit circle around the essential singularity at $C_{12}=0$. It can be interpreted as group manifold of the unitary group $U(1)$. The integral is easily evaluated by using the residue theorem. This yields the series representation in (3.2). As we will see in the next section, (3.4) can already be considered as a simple Graßmannian integral.

We finish this section with some remarks. The two-site invariant (3.2) can be thought of as the oscillator analogue of the twistor intertwiner that has been essential for the construction of Yangian invariants in [22–24]. This intertwiner already appeared in the early days of twistor theory, cf. [25,26]. We also note that recently a two-site Yangian invariant for oscillator representations of $\mathfrak{psu}(2,2|4)$ was used in [27] based on a construction in [28]. It takes the form of an exponential function instead of a Bessel function as in (3.2). This difference occurs because the invariant of [27] is not an eigenstate of the central element of the symmetry algebra at each site.[2] Furthermore, we remark that employing the identity

$$\frac{I_\nu(2\sqrt{x})}{\sqrt{x}^{\,\nu}} = \frac{{}_0F_1(\nu+1;x)}{\Gamma(\nu+1)}\,, \tag{3.5}$$

cf. [29], the invariant (3.2) can alternatively be expressed in terms of a generalized hypergeometric function ${}_0F_1(a,x)$. Sometimes this form is more convenient because it avoids the "spurious" square roots, which are absent in the series expansion. Additionally, the invariant in (3.2) has infinite norm and thus is technically speaking not an element of the Hilbert space $\bar{\mathcal{V}}_{c_1}\otimes\mathcal{V}_{c_2}$. As a last aside, let us consider the special case of the compact algebra $\mathfrak{u}(p,0)$, i.e. we set $q=0$. The sum in (3.2) simplifies to a single term

$$|\Psi_{2,1}\rangle = 2\pi i\frac{(1\bullet 2)^{c_2}}{c_2!}|0\rangle \tag{3.6}$$

with $c_2 \geq 0$, where we used $(1\circ 2)^h = \delta_{0h}$. This form of the compact two-site Yangian invariant is known from [17].

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1 In the double sum in (3.2) c_2+q can also manifestly take negative values. The validity of the Bessel function formulation in this case is easily verified using the series expansion.

2 We thank Ivan Kostov and Didina Serban for clarifying this point.

4. Graßmannian integral formula

At this point everything is set up to state our main formula, a Graßmannian integral for Yangian invariants with oscillator representations of the non-compact algebra $\mathfrak{u}(p,q)$. We motivate it by combining our knowledge of the Graßmannian integral for scattering amplitudes (1.2) with that of the simple sample invariant (3.4). In this section we merely state the resulting formula. A proof of its Yangian invariance is deferred to Appendix A.

A Yangian invariant for a monodromy with $N = 2K$ sites, out of which the first K are "dual" sites and the remaining $K = N - K$ sites are "ordinary", is given by the *Graßmannian integral formula*

$$|\Psi_{N,K}\rangle = \int \mathrm{d}\mathcal{C} \frac{e^{\operatorname{tr}(\mathcal{C}\mathbf{I}_\bullet^t + \mathbf{I}_\circ \mathcal{C}^{-1})}|0\rangle}{(\det \mathcal{C})^q (1,\dots,K)^{1+v_K^+ - v_1^-} \cdots (N,\dots,K-1)^{1+v_{K-1}^+ - v_N^-}} . \tag{4.1}$$

Here the numerator can be understood as a matrix generalization of the sample invariant (3.4). The single contractions of oscillators in the exponent are replaced by the matrices

$$\mathbf{I}_{\substack{\bullet\\\circ}} = \begin{pmatrix} (1 \substack{\bullet\\\circ} K+1) & \cdots & (1 \substack{\bullet\\\circ} N) \\ \vdots & & \vdots \\ (K \substack{\bullet\\\circ} K+1) & \cdots & (K \substack{\bullet\\\circ} N) \end{pmatrix} . \tag{4.2}$$

These $K \times K$ matrices $\mathbf{I}_\bullet$ and $\mathbf{I}_\circ$ contain, respectively, all possible $\mathfrak{u}(p)$ and $\mathfrak{u}(q)$ invariant contractions of the type (3.1) between a "dual" and an "ordinary" site. The denominator of (4.1) is analogous to the Graßmannian integral for scattering amplitudes (1.2) and contains the minors of the $K \times N$ matrix C defined in (1.1). Notice however the extra factor of $(\det \mathcal{C})^q$. The gauge fixing of the matrix C corresponds to the order of "dual" and "ordinary" sites. Furthermore, the measure is the same as in (1.2). Finally, the $2N$ parameters $\{v_i^+, v_i^-\}$ have to obey the N relations in (1.3).

Next, we specify in detail the monodromy $M(u)$ with which the Graßmannian integral for $|\Psi_{N,K}\rangle$ in (4.1) satisfies the Yangian invariance condition (2.5). The elements $M_{AB}(u)$ of this monodromy act on the space $\bar{\mathcal{V}}_{c_1} \otimes \cdots \otimes \bar{\mathcal{V}}_{c_K} \otimes \mathcal{V}_{c_{K+1}} \otimes \cdots \otimes \mathcal{V}_{c_N}$. The $\mathfrak{gl}(n)$ generators in the Lax operators (2.8) and in the Yangian generators (2.10) become

$$J_{AB}^i = \begin{cases} \bar{\mathbf{J}}_{AB}^i & \text{for} \quad i = 1,\dots,K\,, \\ \mathbf{J}_{AB}^i & \text{for} \quad i = K+1,\dots,N\,. \end{cases} \tag{4.3}$$

In the formula (4.1), the $2N$ parameters $\{v_i, c_i\}$ describing the monodromy were traded for a different set of $2N$ parameters $\{v_i^+, v_i^-\}$. They are related by, cf. [30],[3]

$$v_i^\pm = v_i' \pm \frac{c_i}{2}\,, \qquad v_i' = v_i - \frac{c_i}{2} + \begin{cases} n-1 & \text{for} \quad i = 1,\dots K\,, \\ 0 & \text{for} \quad i = K+1,\dots N\,. \end{cases} \tag{4.4}$$

The monodromy is equivalently described by either $\{v_i, c_i\}$ or $\{v_i^+, v_i^-\}$. Notice, however, that for the oscillator representations under consideration the deformation parameters $v_i^\pm$ *cannot* be any complex numbers. They have to be such that the corresponding c_i are integers. This completes the specification of the monodromy.

[3] This redefinition of parameters has also been discussed in [23] for the $\mathfrak{u}(2)$ case, i.e. $n = 2$. The equation for v_i' differs from the corresponding equation (40) in [23] by a shift of 1 at the dual sites. This shift originates from a shift of the inhomogeneities of the Lax operators at those sites.

Let us remark that imposing the condition $N = 2K$ guarantees $\mathcal{C}$ to be a square matrix. Thus it is sensible to use its inverse in (4.1). In the compact special case $\mathfrak{u}(p, 0)$ we have $\mathbf{I}_\circ = 0$, thus $\mathcal{C}^{-1}$ is absent from (4.1) and the Graßmannian integral yields Yangian invariants also for $N \neq 2K$. However, we do not elaborate on the compact case in this work. We note that because of $\mathbf{I}_\circ = 0$, the *compact* case of (4.1) is reminiscent of the link representation of scattering amplitudes, cf. [2]. It is different though, as the amplitudes transform under the *non-compact* algebra $\mathfrak{psu}(2, 2|4)$. Another remark concerns the multi-dimensional contour of integration in (4.1), which we did not specify so far. The proof in Appendix A only assumes that the boundary terms vanish upon integration by parts, which is satisfied in particular for closed contours. The choice of the integration contour will be an issue in the following sections.

5. Unitary matrix models

In this section we choose a "unitary contour" and special values of the deformation parameters $v_i^\pm$ in the Graßmannian integral (4.1). Thereby this integral reduces to the Brezin–Gross–Witten matrix model or even a slight generalization thereof, the Leutwyler–Smilga model. In this special case, the Graßmannian integral can be computed easily by applying well established matrix model techniques. In this way, we obtain a representation of these Yangian invariants in terms of Bessel functions.

In order to reduce (4.1) with $N = 2K$ to the Leutwyler–Smilga integral, we restrict to a special solution of the constraints in (1.3) on the deformation parameters $v_i^\pm$. The solution has to be such that all minors in (4.1), except for $(1, \dots, K) = 1$ and $(N - K + 1, \dots, N) = \det\mathcal{C}$, have a vanishing exponent. A short calculation shows that this solution depends only on two parameters $v \in \mathbb{C}$, $c \in \mathbb{Z}$. It is given by

$$\begin{aligned} v_i &= v - c - n + 1 + (i - 1)\,, & c_i &= -c & \text{for} \quad & i = 1, \dots, K\,, \\ v_i &= v + (i - K - 1)\,, & c_i &= c & \text{for} \quad & i = K + 1, \dots, 2K\,. \end{aligned} \tag{5.1}$$

Here we used (4.4) to change from the variables $\{v_i^+, v_i^-\}$ employed in (4.1) to the variables $\{v_i, c_i\}$. Let us now focus on the measure $\mathrm{d}\mathcal{C} = \bigwedge_{k,l} \mathrm{d}\mathcal{C}_{k,l}$ in (4.1). One readily verifies that

$$[\mathrm{d}\mathcal{C}] = \chi \, \frac{\mathrm{d}\mathcal{C}}{(\det\mathcal{C})^K}\,, \tag{5.2}$$

with a constant number $\chi \in \mathbb{C}$, is invariant under $\mathcal{C} \mapsto \mathcal{V}\mathcal{C}$ and $\mathcal{C} \mapsto \mathcal{C}\mathcal{V}$ for any constant matrix $\mathcal{V} \in GL(K)$. Hence for unitary $\mathcal{C}$ the expression $[\mathrm{d}\mathcal{C}]$ defined in (5.2) is the Haar measure on the unitary group $U(K)$. The normalization χ is chosen such that $\int_{U(K)} [\mathrm{d}\mathcal{C}] = 1$. We select a "unitary contour" in the Graßmannian integral (4.1) by demanding $\mathcal{C}^\dagger = \mathcal{C}^{-1}$. This allows us to express the Yangian invariant with the special choice of deformation parameters (5.1) as

$$|\Psi_{2K,K}\rangle = \chi^{-1} \int\limits_{U(K)} [\mathrm{d}\mathcal{C}] \frac{e^{\operatorname{tr}(\mathcal{C}\mathbf{I}_\bullet^t + \mathbf{I}_\circ\mathcal{C}^\dagger)}|0\rangle}{(\det\mathcal{C})^{c+q}}\,, \tag{5.3}$$

where $c \in \mathbb{Z}$ is a free parameter. Eq. (5.3) is known as *Leutwyler–Smilga model* [14], where the matrices $\mathbf{I}_\bullet^t$ and $\mathbf{I}_\circ$ are considered as sources. For $c = -q$ it becomes the *Brezin–Gross–Witten* *model* [12,13]. Remarkably, the integral (5.3) can be computed exactly. For two *independent* source matrices $\mathbf{I}_\bullet^t$ and $\mathbf{I}_\circ$ this was achieved in [31] using the character expansion methods of [32],

$$|\Psi_{2K,K}\rangle = \chi^{-1}\prod_{j=0}^{K-1} j!\,\frac{(\det \mathbf{I}_\bullet^t)^{c+q}}{\Delta(\mathbf{I}_\circ\mathbf{I}_\bullet^t)}\det\left(\frac{I_{k+c+q-K}\left(2\sqrt{(\mathbf{I}_\circ\mathbf{I}_\bullet^t)_l}\right)}{\sqrt{(\mathbf{I}_\circ\mathbf{I}_\bullet^t)_l}^{\,k+c+q-K}}\right)_{k,l}|0\rangle\,. \tag{5.4}$$

Assuming the matrix $\mathbf{I}_\circ\mathbf{I}_\bullet^t$ to be diagonalizable, we denote its l-th eigenvalue by $(\mathbf{I}_\circ\mathbf{I}_\bullet^t)_l$. Furthermore, $\Delta(\mathbf{I}_\circ\mathbf{I}_\bullet^t) = \det\big(({\mathbf{I}_\circ\mathbf{I}_\bullet^t})_l{}^{k-1}\big)_{k,l}$ is the Vandermonde determinant. The formula (5.4) involving a determinant of Bessel functions generalizes the single Bessel function that we found for the sample Yangian invariant $|\Psi_{2,1}\rangle$ in (3.2).

In this section we showed that the choice of a "unitary contour" in the Graßmannian integral (4.1) is appropriate for the *special* deformation parameters $v_i^\pm$ given by (5.1). We conjecture that this contour can also be used for the Graßmannian integral (4.1) with *general* deformation parameters. In this case one is lead to a novel unitary matrix model of the type (5.3) containing powers of principal minors of the matrix $\mathcal{C}$ in addition to $\det\mathcal{C}$. In the next section we illustrate for a non-trivial example that this model indeed produces the correct Yangian invariant.

6. Another sample invariant: R-matrix

Let us now apply the "unitary contour" to the Graßmannian integral (4.1) for the sample invariant $|\Psi_{4,2}\rangle$. This invariant is of special importance because its Yangian invariance condition (2.4) can be translated into the Yang–Baxter equation, cf. [17]. Therefore $|\Psi_{4,2}\rangle$ is equivalent to the $\mathfrak{u}(p,q)$ R-matrix.

We begin by choosing $\mathcal{C}$ to be unitary which transforms the integral (4.1) into

$$|\Psi_{4,2}\rangle = \chi^{-1}\int_{U(2)}[\mathrm{d}\mathcal{C}]\,\frac{e^{\operatorname{tr}(\mathcal{C}\mathbf{I}_\bullet^t+\mathbf{I}_\circ\mathcal{C}^\dagger)}|0\rangle}{(-\mathcal{C}_{13})^{1+z}(\det\mathcal{C})^{-1+q-z+c_3}(-\mathcal{C}_{24})^{1+z-c_3+c_4}} \tag{6.1}$$

with the abbreviation $z = v_3 - v_4$. The constraints on the deformation parameters in (1.3) read explicitly

$$v_1 - v_3 = 1 - n - c_3\,,\quad c_1 = -c_3 \in \mathbb{Z}\,,\quad v_2 - v_4 = 1 - n - c_4\,,\quad c_2 = -c_4 \in \mathbb{Z}\,. \tag{6.2}$$

Notice that (6.1) is a generalization of the Leutwyler–Smilga model (5.3), as it contains in addition the principal minors $\mathcal{C}_{13}$ and $\mathcal{C}_{24}$ of the unitary 2×2 matrix $\mathcal{C}$. This currently hinders the direct application of matrix model techniques to evaluate (6.1). Therefore we resort to an explicit parameterization,

$$\mathcal{C} = \begin{pmatrix} \mathcal{C}_{13} & \mathcal{C}_{14}\\ \mathcal{C}_{23} & \mathcal{C}_{24}\end{pmatrix} = c\begin{pmatrix} a\cos\theta & -b\sin\theta\\ b^{-1}\sin\theta & a^{-1}\cos\theta\end{pmatrix}, \tag{6.3}$$

where $\theta\in[0,\frac{\pi}{2}]$ and $a = e^{i\alpha}$, $b = e^{i\beta}$, $c = e^{i\gamma}$ with $\alpha,\beta\in[0,2\pi]$ and $\gamma\in[0,\pi]$. With this the Haar measure (5.2) becomes

$$[\mathrm{d}\mathcal{C}] = \chi\,\frac{4\sin\theta\cos\theta}{abc}\mathrm{d}a\wedge\mathrm{d}b\wedge\mathrm{d}c\wedge\mathrm{d}\theta\,. \tag{6.4}$$

We observe that the exponents of a, b, c in denominator of (6.1) combine into integers, where for the moment we ignore that this rearrangement is not allowed for generic values of the exponent

$z \in \mathbb{C}$. Thus the integrals in the variables a, b, c can be performed by means of the residue theorem,

$$|\Psi_{4,2}\rangle = (-1)^{c_4-c_3}(2\pi i)^3 \sum_{\substack{g_{13},\dots,g_{24}=0\\ h_{13},\dots,h_{24}=0\\ \text{with (6.6)}}}^{\infty} \frac{(1\bullet 3)^{g_{13}}}{g_{13}!}\frac{(1\bullet 4)^{g_{14}}}{g_{14}!}\frac{(2\bullet 3)^{g_{23}}}{g_{23}!}\frac{(2\bullet 4)^{g_{24}}}{g_{24}!} \times \frac{(1\circ 3)^{h_{13}}}{h_{13}!}\frac{(1\circ 4)^{h_{14}}}{h_{14}!}\frac{(2\circ 3)^{h_{23}}}{h_{23}!}\frac{(2\circ 4)^{h_{24}}}{h_{24}!}|0\rangle \times (-1)^{g_{14}+h_{14}}\mathrm{B}(g_{14}+h_{23}+1, h_{13}+g_{24}-z+c_3-c_4)\,. \tag{6.5}$$

In this formula the constraints

$$\begin{aligned} g_{13}-h_{13}+g_{14}-h_{14} &= -c_1+q\,, \quad & g_{23}-h_{23}+g_{24}-h_{24} &= -c_2+q\,,\\ g_{13}-h_{13}+g_{23}-h_{23} &= c_3+q\,, \quad & g_{14}-h_{14}+g_{24}-h_{24} &= c_4+q \end{aligned}$$

on the summation range guarantee that $|\Psi_{4,2}\rangle$ is an eigenstate of $\bar{\mathbf{C}}^1$, $\bar{\mathbf{C}}^2$, $\mathbf{C}^3$, $\mathbf{C}^4$ with eigenvalues c_1, c_2, c_3, c_4, respectively. The remaining θ-integration yields the Euler beta function

$$\mathrm{B}(x,y) = 2\int_0^{\frac{\pi}{2}} \mathrm{d}\theta\,(\sin\theta)^{2x-1}(\cos\theta)^{2y-1} = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}\,, \tag{6.6}$$

which is valid for $\mathrm{Re}\,x, \mathrm{Re}\,y > 0$, cf. [29]. This means $c_3 - c_4 > \mathrm{Re}\,z$ for the arguments of the beta function in (6.5). The expression (6.5) is our final form of the Yangian invariant $|\Psi_{4,2}\rangle$, i.e. the $\mathfrak{u}(p,q)$ R-matrix for oscillator representations. The parameter z is the spectral parameter of this R-matrix. A formula analogous to (6.5), however derived in a completely different way, can be obtained by specializing the $\mathfrak{u}(p,q|r)$ R-matrix expression found in [9] to the bosonic case. At this point we remark that the integrand in (6.1) is multi-valued for generic z and thus the $U(2)$ contour is not closed. Hence in principle the formal proof in Appendix A does not directly apply. Therefore we verified the Yangian invariance of $|\Psi_{4,2}\rangle$ explicitly on the level of the series expansion (6.5). Finally, it is worth noting that in the compact case $\mathfrak{u}(p,0) = \mathfrak{u}(n)$ the invariant (6.5) simplifies to

$$|\Psi_{4,2}\rangle = (-1)^{c_4-c_3}2(2\pi i)^3 \sum_{g_{14}=0}^{\infty} \frac{(1\bullet 3)^{c_3-g_{14}}}{(c_3-g_{14})!}\frac{(1\bullet 4)^{g_{14}}}{g_{14}!}\frac{(2\bullet 3)^{g_{14}}}{g_{14}!}\frac{(2\bullet 4)^{c_4-g_{14}}}{(c_4-g_{14})!}|0\rangle \times (-1)^{g_{14}}\mathrm{B}(g_{14}+1, -z+c_3-g_{14})\,. \tag{6.7}$$

This agrees with the compact invariant $|\Psi_{4,2}\rangle$ obtained in [17] up to a normalization factor.

7. Conclusions and outlook

In this work we showed that the Graßmannian integral, commonly used in the realm of $\mathcal{N}=4$ SYM scattering amplitudes, can be applied to construct Yangian invariants for oscillator representations of the non-compact algebra $\mathfrak{u}(p,q)$. We found that in this setting the integral takes the form of a matrix model which generalizes the Brezin–Gross–Witten and the Leutwyler–Smilga model. Our results also imply that these two well-known matrix models are Yangian invariant in the external source fields!

Our work calls for a series of further investigations, both on a technical and on a conceptual level. Technically, the generalization to superalgebras $\mathfrak{u}(p,q|r)$ should impose no obstacles. This is of importance to cover the $\mathfrak{psu}(2,2|4)$ case relevant for amplitudes. In addition, the applicability of the "unitary contour" has to be investigated further. In particular, replacing $\mathcal{C}^{-1}$ by the conjugate transpose $\mathcal{C}^\dagger$ in the Graßmannian integral formula (4.1) should also provide a way to avoid the "split helicity" constraint $N = 2K$. Here the issue is to use an appropriate measure on the complex Stiefel manifold of *rectangular* $K \times (N-K)$ matrices $\mathcal{C}$ with $\mathcal{C}\mathcal{C}^\dagger = 1_{K\times K}$, see e.g. [33]. This generalizes the unitary group manifold to the case of rectangular matrices. Moreover, we want to apply matrix model technology for the evaluation of the Graßmannian integral (4.1) beyond the case of the Leutwyler–Smilga model (5.3). One might wonder whether the Bessel function formula (5.4) generalizes to the case of Yangian invariants with general deformation parameters $v_i^\pm$. This formula would include the R-matrix constructed "by hand" in Section 6. One promising technique for this endeavor is a character expansion, which was successfully employed for the Leutwyler–Smilga model (5.3), see [31,32]. Another auspicious method is the use of Gelfand–Tzetlin coordinates, which has been applied to compute correlation functions of the Itzykson–Zuber model [34]. In our setting these coordinates might be well adapted to the minors appearing in the Graßmannian integral (4.1). A further interesting point to be addressed in the future is the precise relation between the Graßmannian integral for twistors and that for oscillator representations (1.4). There should exist a change of basis transforming the delta function of twistors into the exponential function of oscillators. A twistorial description of the $\mathfrak{u}(p,q)$ oscillator representations, a.k.a. "ladder representations", is discussed e.g. in [35].

Even more exciting questions arise on the conceptual level. It is well known that matrix models possess an integrable structure, see e.g. [36] and references therein. Their partition functions, like e.g. (5.3), correspond to solutions, so-called τ-functions, of classically integrable hierarchies. There should be a relation between this *classical* integrable structure and *quantum* integrability in the sense of Yangian invariance. One might even ask if there is an integrable hierarchy governing (tree-level) $\mathcal{N}=4$ SYM scattering amplitudes. Finally, let us speculate that our matrix model approach might also provide a conceptually clear route to loop-amplitudes. The $\mathfrak{psu}(2,2|4)$ analogues of the oscillator representations, which we are using in this work, feature prominently in the spectral problem of $\mathcal{N}=4$ SYM. There it is understood how to introduce the coupling constant of the theory as a central extension of the algebra. Appealing to a common integrable structure of the $\mathcal{N}=4$ model, we suspect that in the oscillator basis such a coupling can also be introduced in the Graßmannian integral.

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Appendix A. Proof of Yangian invariance

In this appendix we prove the Yangian invariance (2.6) of the Graßmannian integral (4.1) for the invariant $|\Psi_{N,K}\rangle$ with $N = 2K$ sites and representations of the non-compact algebra $\mathfrak{u}(p,q)$. With straightforward modifications this proof also applies to the compact case, i.e. $q = 0$, where, in particular, $\mathbf{I}_\circ = 0$ and $N \neq 2K$ is possible.

Let us start with the ansatz

$$e^{\mathrm{tr}(\mathcal{C}\mathbf{I}_\bullet^t + \mathbf{I}_\circ \mathcal{C}^{-1})}|0\rangle\,, \tag{A.1}$$

which we recognize as the exponential function in (4.1). We want to show that this ansatz satisfies the first equation of (2.6), that is to say $\mathfrak{gl}(n)$ invariance. With the $\mathfrak{gl}(n)$ generators of our monodromy defined in (4.3), the Yangian generators appearing in this equation read

$$M^{(1)}_{AB} = \sum_{k=1}^{K} \bar{\mathbf{J}}^k_{BA} + \sum_{l=K+1}^{N} \mathbf{J}^l_{BA}\,. \tag{A.2}$$

To evaluate the action of this operator on the ansatz (A.1) we compute

$$\begin{aligned}
(\bar{\mathbf{J}}^k_{AB})\, e^{\mathrm{tr}(\mathcal{C}\mathbf{I}_\bullet^t + \mathbf{I}_\circ \mathcal{C}^{-1})}|0\rangle &= \left(\begin{array}{c:c} -\sum_w \bar{\mathbf{a}}^w_\alpha \bar{\mathbf{a}}^k_\beta C_{kw} & \sum_{w,w'} \bar{\mathbf{a}}^w_\alpha \bar{\mathbf{a}}^{w'}_{\dot\beta} D_{w'k} C_{kw} \\ \hdashline -\bar{\mathbf{a}}^k_{\dot\alpha} \bar{\mathbf{a}}^k_\beta & \sum_w \bar{\mathbf{a}}^k_{\dot\alpha} \bar{\mathbf{a}}^w_{\dot\beta} D_{wk} + \delta_{\dot\alpha\dot\beta} \end{array}\right) e^{\mathrm{tr}(\mathcal{C}\mathbf{I}_\bullet^t + \mathbf{I}_\circ \mathcal{C}^{-1})}|0\rangle\,, \\
(\mathbf{J}^l_{AB})\, e^{\mathrm{tr}(\mathcal{C}\mathbf{I}_\bullet^t + \mathbf{I}_\circ \mathcal{C}^{-1})}|0\rangle &= \left(\begin{array}{c:c} \sum_v \bar{\mathbf{a}}^l_\alpha \bar{\mathbf{a}}^v_\beta C_{vl} & -\bar{\mathbf{a}}^l_\alpha \bar{\mathbf{a}}^l_{\dot\beta} \\ \hdashline \sum_{v,v'} \bar{\mathbf{a}}^v_{\dot\alpha} \bar{\mathbf{a}}^{v'}_\beta C_{v'l} D_{lv} & -\sum_v \bar{\mathbf{a}}^v_{\dot\alpha} \bar{\mathbf{a}}^l_{\dot\beta} D_{lv} - \delta_{\dot\alpha\dot\beta} \end{array}\right) \\
&\quad \times e^{\mathrm{tr}(\mathcal{C}\mathbf{I}_\bullet^t + \mathbf{I}_\circ \mathcal{C}^{-1})}|0\rangle\,,
\end{aligned} \tag{A.3}$$

where the components of the matrix $\mathcal{C}^{-1}$ are denoted by D_{lk}. Here and in the rest of this proof the indices k, v, v' always take the values $1, \dots, K$ while l, w, w' are in the range $K+1, \dots, N$. Now one immediately obtains

$$M^{(1)}_{AB}\, e^{\mathrm{tr}(\mathcal{C}\mathbf{I}_\bullet^t + \mathbf{I}_\circ \mathcal{C}^{-1})}\,|0\rangle = 0\,. \tag{A.4}$$

Hence the first equation of (2.6) holds for the ansatz (A.1).

However, each site of the ansatz (A.1) does not yet transform in an irreducible representation of the algebra $\mathfrak{u}(p,q)$. In fact, (A.1) is not an eigenstate of the central elements $\mathbf{C}^l = \sum_{A=1}^n \mathbf{J}^l_{AA}$ and $\bar{\mathbf{C}}^k = \sum_{A=1}^n \bar{\mathbf{J}}^k_{AA}$ that were defined in the context of (2.12) and (2.13), respectively. To obtain eigenstates we have to pick special linear combinations of the ansatz (A.1),

$$|\Psi_{N,K}\rangle = \int \mathrm{d}\mathcal{C}\, f(\mathcal{C})\, e^{\mathrm{tr}(\mathcal{C}\mathbf{I}_\bullet^t + \mathbf{I}_\circ \mathcal{C}^{-1})}|0\rangle\,. \tag{A.5}$$

It turns out to be suitable to choose an integrand that contains only consecutive minors of the matrix C defined in (1.1),

$$f(\mathcal{C}) = \frac{1}{(1,\dots,K)^{1+\alpha_1}\cdots(N,\dots,K-1)^{1+\alpha_N}} \tag{A.6}$$

with arbitrary complex constants α_i. With this integrand the ansatz (A.5) is an eigenstate of the central elements,

$$\bar{\mathbf{C}}^k\,|\Psi_{N,K}\rangle = \left(q - \sum_{i=k+1}^{k+N-K} \alpha_i\right)|\Psi_{N,K}\rangle\,, \quad \mathbf{C}^l\,|\Psi_{N,K}\rangle = \left(-q + \sum_{i=l-K+1}^{l} \alpha_i\right)|\Psi_{N,K}\rangle\,.$$

(A.7)

To show this property we assumed that upon integration by parts the boundary terms vanish. Furthermore, we employed the identity

$$\frac{\mathrm{d}}{\mathrm{d}\,C_{kl}} e^{\mathrm{tr}(\mathcal{C}\mathbf{I}_\bullet^t+\mathbf{I}_\circ\mathcal{C}^{-1})}|0\rangle = \Big((k\bullet l) - \sum_{v,w} D_{wk}D_{lv}(v\circ w)\Big) e^{\mathrm{tr}(\mathcal{C}\mathbf{I}_\bullet^t+\mathbf{I}_\circ\mathcal{C}^{-1})}|0\rangle\,, \tag{A.8}$$

which is easily verified taking into account $\frac{\mathrm{d}}{\mathrm{d}C_{kl}} D_{wv} = -D_{wk}D_{lv}$. In addition, in evaluating derivatives of the minors in $f(\mathcal{C})$ we used, cf. [6,7],

$$\sum_w C_{kw}\frac{\mathrm{d}}{\mathrm{d}C_{kw}}(i,\ldots,i+K-1)^{1+\alpha_i} = (1+\alpha_i)\,(i,\ldots,i+K-1)^{1+\alpha_i} \tag{A.9}$$

for $i = k+1,\ldots,k+N-K$. For other values of i the left hand side in (A.9) vanishes due to the gauge fixing of C in (1.1).

Next, we turn our attention to the second equation of the Yangian invariance condition (2.6), which involves the generators $M^{(2)}_{AB}$. From (2.4) with $r=2$ and $s=1$ one sees that if a state $|\Psi\rangle$ is annihilated by all $M^{(1)}_{AB}$ and by one of the generators $M^{(2)}_{AB}$, e.g. by $M^{(2)}_{11}$, then it is annihilated by all $M^{(2)}_{AB}$. Thus in our case it is sufficient to verify the second equation of (2.6) for one of the four blocks of generators, say for $M^{(2)}_{\alpha\beta}$. Expressions for these generators can be found in (2.10). We compute the action of all terms appearing therein on our ansatz (A.1),

$$\begin{aligned}
&\sum_{I=1}^{n} \bar{\mathbf{J}}^k_{I\alpha}\mathbf{J}^l_{\beta I}\, e^{\mathrm{tr}(\mathcal{C}\mathbf{I}_\bullet^t+\mathbf{I}_\circ\mathcal{C}^{-1})}|0\rangle = -\bar{\mathbf{a}}^k_\alpha\,\bar{\mathbf{a}}^l_\beta\Big(\sum_{v,w} C_{vl}C_{kw}\,\frac{\mathrm{d}}{\mathrm{d}\,C_{vw}} + p\,C_{kl}\Big) e^{\mathrm{tr}(\mathcal{C}\mathbf{I}_\bullet^t+\mathbf{I}_\circ\mathcal{C}^{-1})}|0\rangle\,,\\
&\sum_{I=1}^{n} \bar{\mathbf{J}}^k_{I\alpha}\bar{\mathbf{J}}^{k'}_{\beta I}\, e^{\mathrm{tr}(\mathcal{C}\mathbf{I}_\bullet^t+\mathbf{I}_\circ\mathcal{C}^{-1})}|0\rangle = \sum_{w,w'}\bar{\mathbf{a}}^k_\alpha\,\bar{\mathbf{a}}^w_\beta\, C_{k'w}\,C_{kw'}\frac{\mathrm{d}}{\mathrm{d}\,C_{k'w'}} e^{\mathrm{tr}(\mathcal{C}\mathbf{I}_\bullet^t+\mathbf{I}_\circ\mathcal{C}^{-1})}|0\rangle\,,\\
&\sum_{I=1}^{n} \mathbf{J}^l_{I\alpha}\mathbf{J}^{l'}_{\beta I}\, e^{\mathrm{tr}(\mathcal{C}\mathbf{I}_\bullet^t+\mathbf{I}_\circ\mathcal{C}^{-1})}|0\rangle = \sum_{v,v'}\bar{\mathbf{a}}^v_\alpha\,\bar{\mathbf{a}}^{l'}_\beta\, C_{vl}\,C_{v'l'}\,\frac{\mathrm{d}}{\mathrm{d}\,C_{v'l}}\, e^{\mathrm{tr}(\mathcal{C}\mathbf{I}_\bullet^t+\mathbf{I}_\circ\mathcal{C}^{-1})}|0\rangle\,,
\end{aligned} \tag{A.10}$$

for $k \neq k'$ and $l \neq l'$, and furthermore

$$\Big(\sum_k v_k\,\bar{\mathbf{J}}^k_{\beta\alpha} + \sum_l v_l\,\mathbf{J}^l_{\beta\alpha}\Big) e^{\mathrm{tr}(\mathcal{C}\mathbf{I}_\bullet^t+\mathbf{I}_\circ\mathcal{C}^{-1})}|0\rangle = \sum_{k,l}\bar{\mathbf{a}}^k_\alpha\,\bar{\mathbf{a}}^l_\beta\, C_{kl}\,(v_l - v_k)\, e^{\mathrm{tr}(\mathcal{C}\mathbf{I}_\bullet^t+\mathbf{I}_\circ\mathcal{C}^{-1})}|0\rangle\,. \tag{A.11}$$

Making use of these formulas we can evaluate the action on (A.5),

$$\begin{aligned}
M^{(2)}_{\alpha\beta}\,|\Psi_{N,K}\rangle = &\sum_{k,l}\Big(v_l - v_k - p + 1 - \sum_{i=l-K+1}^{k+N-K}\alpha_i\Big)\,\bar{\mathbf{a}}^k_\alpha\,\bar{\mathbf{a}}^l_\beta\\
&\times\int \mathrm{d}\mathcal{C}\, f(\mathcal{C})\, C_{kl}\, e^{\mathrm{tr}(\mathcal{C}\mathbf{I}_\bullet^t+\mathbf{I}_\circ\mathcal{C}^{-1})}|0\rangle\,.
\end{aligned} \tag{A.12}$$

Here we assumed once more that the boundary terms of the integration by parts vanish. Furthermore, we used (A.8) and properties of the minors in $f(\mathcal{C})$ similar to (A.9). To ensure Yangian invariance of the ansatz the parameters α_i have to be chosen such that the bracket in (A.12) vanishes.

In conclusion, for the ansatz (A.5) to be Yangian invariant, the parameters v_i, c_i of the monodromy and the α_i appearing in this ansatz have to obey the equations obtained from (A.7) and (A.12),

$$c_k = q - \sum_{i=k+1}^{k+N-K} \alpha_i \,, \quad c_l = -q + \sum_{i=l-K+1}^{l} \alpha_i \,, \quad v_k - v_l = -p + 1 - \sum_{i=l-K+1}^{k+N-K} \alpha_i \,, \tag{A.13}$$

for $k = 1, \dots K$ and $l = K + 1, \dots, N$. These equations are conveniently addressed after changing from $\{v_i, c_i\}$ to $\{v_i^+, v_i^-\}$ with (4.4). In these variables they are solved by

$$\alpha_i = v_{i+K-1}^+ - v_i^- + q\, \delta_{i,N-K+1} \tag{A.14}$$

and imposing the N constraints in (1.3). Eq. (A.14) turns the ansatz (A.5) into the Graßmannian integral formula (4.1). This concludes the proof of its Yangian invariance.

References

[1] N. Beisert, C. Ahn, L.F. Alday, Z. Bajnok, J.M. Drummond, L. Freyhult, N. Gromov, R.A. Janik, V. Kazakov, T. Klose, G.P. Korchemsky, C. Kristjansen, M. Magro, T. McLoughlin, J.A. Minahan, R.I. Nepomechie, A. Rej, R. Roiban, S. Schäfer-Nameki, C. Sieg, M. Staudacher, A. Torrielli, A.A. Tseytlin, P. Vieira, D. Volin, K. Zoubos, Review of AdS/CFT integrability: an overview, Lett. Math. Phys. 99 (2012) 3, arXiv:1012.3982.

[2] N. Arkani-Hamed, F. Cachazo, C. Cheung, J. Kaplan, A duality for the S matrix, J. High Energy Phys. 1003 (2010) 020, arXiv:0907.5418.

[3] L.J. Mason, D. Skinner, Dual superconformal invariance, momentum twistors and Grassmannians, J. High Energy Phys. 0911 (2009) 045, arXiv:0909.0250.

[4] N. Arkani-Hamed, J.L. Bourjaily, F. Cachazo, A.B. Goncharov, A. Postnikov, J. Trnka, Scattering amplitudes and the positive Grassmannian, arXiv:1212.5605.

[5] J. Drummond, J. Henn, J. Plefka, Yangian symmetry of scattering amplitudes in N = 4 super Yang–Mills theory, J. High Energy Phys. 0905 (2009) 046, arXiv:0902.2987.

[6] J.M. Drummond, L. Ferro, Yangians, Grassmannians and T-duality, J. High Energy Phys. 1007 (2010) 027, arXiv:1001.3348.

[7] J.M. Drummond, L. Ferro, The Yangian origin of the Grassmannian integral, J. High Energy Phys. 1012 (2010) 010, arXiv:1002.4622.

[8] L. Ferro, T. Łukowski, C. Meneghelli, J. Plefka, M. Staudacher, Harmonic R matrices for scattering amplitudes and spectral regularization, Phys. Rev. Lett. 110 (2013) 121602, arXiv:1212.0850.

[9] L. Ferro, T. Łukowski, C. Meneghelli, J. Plefka, M. Staudacher, Spectral parameters for scattering amplitudes in N = 4 super Yang–Mills theory, J. High Energy Phys. 1401 (2014) 094, arXiv:1308.3494.

[10] L. Ferro, T. Łukowski, M. Staudacher, N = 4 scattering amplitudes and the deformed Graßmannian, Nucl. Phys. B 889 (2014) 192, arXiv:1407.6736.

[11] T. Bargheer, Y. Huang, F. Loebbert, M. Yamazaki, Integrable amplitude deformations for N = 4 super Yang–Mills and ABJM theory, arXiv:1407.4449.

[12] D.J. Gross, E. Witten, Possible third order phase transition in the large N lattice gauge theory, Phys. Rev. D 21 (1980) 446–453.

[13] E. Brezin, D.J. Gross, The external field problem in the large N limit of QCD, Phys. Lett. B 97 (1980) 120.

[14] H. Leutwyler, A.V. Smilga, Spectrum of Dirac operator and role of winding number in QCD, Phys. Rev. D 46 (1992) 5607.

[15] A.Y. Morozov, Unitary integrals and related matrix models, Theor. Math. Phys. 162 (2010) 1, arXiv:0906.3518.

[16] A. Molev, Yangians and Classical Lie Algebras, American Mathematical Society, 2007.

[17] R. Frassek, N. Kanning, Y. Ko, M. Staudacher, Bethe ansatz for Yangian invariants: towards super Yang–Mills scattering amplitudes, Nucl. Phys. B 883 (2014) 373, arXiv:1312.1693.

[18] I.T. Todorov, Discrete series of hermitian representations of the Lie algebra of u(p, q), preprint IC-66-71, 1966.

[19] M. Kashiwara, M. Vergne, On the Segal–Shale–Weil representations and harmonic polynomials, Invent. Math. 44 (1978) 1.

[20] R.L. Anderson, J. Fischer, R. Rączka, Coupling problem for U(p, q) ladder representations. I, Proc. R. Soc. A 302 (1968) 491.
[21] R.L. Anderson, R. Rączka, Coupling problem for U(p, q) ladder representations. II. Limitations on the coupling of U(p, q) ladders, Proc. R. Soc. A 302 (1968) 501.
[22] D. Chicherin, S. Derkachov, R. Kirschner, Yang–Baxter operators and scattering amplitudes in N = 4 super-Yang–Mills theory, Nucl. Phys. B 881 (2014) 467, arXiv:1309.5748.
[23] N. Kanning, T. Łukowski, M. Staudacher, A shortcut to general tree-level scattering amplitudes in N = 4 SYM via integrability, Fortschr. Phys. 62 (2014) 556, arXiv:1403.3382.
[24] J. Broedel, M. de Leeuw, M. Rosso, A dictionary between R-operators, on-shell graphs and Yangian algebras, J. High Energy Phys. 1406 (2014) 170, arXiv:1403.3670.
[25] R. Penrose, M.A.H. MacCallum, Twistor theory: an approach to the quantization of fields and space-time, Phys. Rep. 6 (1972) 241.
[26] A.P. Hodges, A twistor approach to the regularization of divergences, Proc. R. Soc. Lond. A 397 (1985) 341.
[27] Y. Jiang, I. Kostov, A. Petrovskii, D. Serban, String bits and the spin vertex, arXiv:1410.8860.
[28] L.F. Alday, J.R. David, E. Gava, K.S. Narain, Towards a string bit formulation of N = 4 super Yang–Mills, J. High Energy Phys. 0604 (2006) 014, arXiv:hep-th/0510264.
[29] M. Abramowitz, I.A. Stegun, Handbook of Mathematical Functions, Dover Publications, 1964.
[30] N. Beisert, J. Broedel, M. Rosso, On Yangian-invariant regularization of deformed on-shell diagrams in N = 4 super-Yang–Mills theory, J. Phys. A 47 (2014) 365402, arXiv:1401.7274.
[31] B. Schlittgen, T. Wettig, Generalizations of some integrals over the unitary group, J. Phys. A 36 (2003) 3195, arXiv:math-ph/0209030.
[32] A.B. Balantekin, Character expansions, Itzykson–Zuber integrals, and the QCD partition function, Phys. Rev. D 62 (2000) 085017, arXiv:hep-th/0007161.
[33] A. Mathai, Jacobians of Matrix Transformations and Functions of Matrix Arguments, World Scientific Publishing, 1997.
[34] S.L. Shatashvili, Correlation functions in the Itzykson–Zuber model, Commun. Math. Phys. 154 (1993) 421, arXiv:hep-th/9209083.
[35] E.G. Dunne, M.G. Eastwood, The twistor transform, in: T.N. Bailey, U.J. Baston (Eds.), Twistors in Mathematics and Physics, Cambridge University Press, 1990, p. 110.
[36] A. Morozov, Integrability and matrix models, Phys. Usp. 37 (1994) 1, arXiv:hep-th/9303139.