

Better Answers to Real Questions

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Abstract

We consider existential problems over the reals. Extended quantifier elimination generalizes the concept of regular quantifier elimination by providing in addition answers, which are descriptions of possible assignments for the quantified variables. Implementations of extended quantifier elimination via virtual substitution have been successfully applied to various problems in science and engineering. So far, the answers produced by these implementations included infinitesimal and infinite numbers, which are hard to interpret in practice. We introduce here a post-processing procedure to convert, for fixed parameters, all answers into standard real numbers. The relevance of our procedure is demonstrated by application of our implementation to various examples from the literature, where it significantly improves the quality of the results.

1 Introduction

We consider existential problems over the reals. Extended quantifier elimination generalizes the concept of regular quantifier elimination by providing in addition answers, which are descriptions of possible assignments for the quantified variables (Weispfenning, 1994b, 1997b). Implementations of extended quantifier elimination (Dolzmann and Sturm, 1997a, 1996) via virtual substitution (Loos and Weispfenning, 1993; Weispfenning, 1997a, 1994a, 1988; Dolzmann et al., 1999) have been successfully applied to various problems in science and engineering (Sturm and Weispfenning, 1997; Sturm, 1999b; Weispfenning, 2001; Sturm and Weber, 2008; Sturm et al., 2009; Errami et al., 2011; Weber et al., 2011; Errami et al., 2013; Sturm, 1999a).

So far, the answers produced by these implementations included infinitesimal and infinite numbers, which are hard to interpret in practice. This has been explicitly criticized in the literature, e.g., by Collard (2003). In the present article, we introduce a complete post-processing procedure to convert, for fixed values

of parameters, all answers into standard real numbers. We furthermore demonstrate the successful application of an implementation of our method within Redlog (Dolzmann and Sturm, 1997a) to a number of extended quantifier elimination problems from the scientific literature including computational geometry (Sturm and Weispfenning, 1997), motion planning (Weispfenning, 2001), bifurcation analysis for models of genetic circuits and for mass action (Sturm and Weber, 2008; Sturm et al., 2009), and sizing of electrical networks (Sturm, 1999a).

The plan of the paper is as follows: In Section 2 we make ourselves familiar with the concept of extended quantifier elimination. In Section 3 we give an introduction of virtual substitution for extended quantifier elimination to the extent necessary to understand how nonstandard values enter the answers and what information is available for fixing them to standard values. Section 4 is the technical core; we describe and prove our method and illustrate it by discussing one example in detail. In Section 5 we revisit *degree shifts*, a successful heuristics for reducing the degree of quantified variables before their elimination. We reinterpret these degree shifts as quantifier eliminations by virtual substitution. This allows us in Section 6 to generalize our method to cover also possible degree shifts during elimination. In Section 7 we revisit examples from the scientific literature where the application of extended quantifier elimination to various problems from planning, modeling, science, and engineering had yielded nonstandard answers. In all cases we can efficiently fix all nonstandard symbols to standard values using our implementation of the method as it was introduced in Section 4 and generalized in Section 6. This significantly improves the quality of the results from a practical point of view. Finally, in Section 8 we summarize our results and discuss possible extensions of our method.

2 The Concept of Extended Quantifier Elimination

For our purposes here, we restrict ourselves to existential problems

$$\varphi(u_1, \dots, u_m) = \exists x_n \dots \exists x_1 \psi(x_1, \dots, x_n, u_1, \dots, u_m)$$

in the Tarski language $L = (0, 1, +, -, \cdot, \leq, <, \geq, >, \neq)$ interpreted in the Tarski algebra $(\mathbb{R}, 0, 1, +, -, \cdot, \leq, <, \geq, >, \neq)$. As usual in algebraic model theory, the symbol “=” and its interpretation as equality is part of first-order logic so that it does not occur explicitly in the language L . Without loss of generality, ψ is an \wedge - \vee -combination of atomic constraints, and we agree that all right hand sides of the atomic constraints are 0.

Extended quantifier elimination applied to φ yields an *extended quantifier elimination result (EQR)*

$$\left[\begin{array}{c|ccc} \beta_1(\mathbf{u}) & x_1 = e_{11}(\mathbf{u}) & \dots & x_n = e_{1n}(\mathbf{u}) \\ \vdots & \vdots & \ddots & \vdots \\ \beta_k(\mathbf{u}) & x_1 = e_{k1}(\mathbf{u}) & \dots & x_n = e_{kn}(\mathbf{u}) \end{array} \right].$$

The *conditions* $\beta_i(\mathbf{u})$ are quantifier-free Tarski formulas such that $\mathbb{R} \models \varphi \iff \bigvee_{i=1}^k \beta_i$. In other words, $\bigvee_{i=1}^k \beta_i$ is a regular quantifier elimination result for φ ,

and extended quantifier elimination generalizes regular quantifier elimination. The *answers* $\mathbf{e}_i(\mathbf{u})$ are terms in an extension language of the Tarski language. For $\mathbf{a} \in \mathbb{R}^m$, if $\varphi(\mathbf{a})$ holds, then at least one $\beta_i(\mathbf{a})$ holds, and so does $\psi(\mathbf{e}_i(\mathbf{a}), \mathbf{a})$. We agree that “false” never occurs as a condition. If φ itself is equivalent to “false,” we possibly obtain the empty scheme [1].

As an example, consider the input formula

$$\varphi = \exists x \exists y \psi, \quad \psi = ay + 3x^2 + 4x \leq a \wedge x \geq a \geq y.$$

A possible extended quantifier elimination result for φ is given by

$$\left[\begin{array}{l|l} a \neq 0 \wedge 4a + 3 \geq 0 & y = -3a - 3 \quad x = a \\ a \leq 0 \wedge 3a^2 - 3a - 4 \leq 0 & y = a \quad x = \frac{\sqrt{-3a^2 + 3a + 4} - 2}{3} \end{array} \right].$$

From this extended quantifier elimination result we can derive a regular quantifier elimination result

$$(a \neq 0 \wedge 4a + 3 \geq 0) \vee (a \leq 0 \wedge 3a^2 - 3a - 4 \leq 0),$$

which can be simplified to $a \geq 0 \vee 3a^2 - 3a - 4 \leq 0$. Hence, φ holds if and only if $a \geq \alpha$, where $\alpha \approx -0.758306$ is the smaller root of $3a^2 - 3a - 4$.

In the extended quantifier elimination result, the first row covers the case that $-0.75 \leq a$ and $a \neq 0$, while the second row covers $\alpha \leq a \leq 0$. Let us consider some particular interpretations of a :

- For $a = 2$, the condition in the first row holds and the corresponding answers yield $x = 2$ and $y = -9$. In fact, these three values satisfy ψ . The condition in the second row, in contrast, does not hold. If we plug $a = 2$ into the corresponding answers anyway, then we obtain a negative argument for the square root, which cannot be interpreted over the reals.
- For $\alpha < a = -0.7525 < -0.75$, the condition in the second row holds and the corresponding answers yield $x = \frac{\sqrt{6997} - 800}{1200}$ and $y = -0.7525$. Again, these three values satisfy ψ . Now the condition in the first row does not hold. If we plug $a = -0.7525$ into the corresponding answers anyway, then we obtain $x = -0.7525$ and $y = -0.7425$, which does not satisfy ψ .
- For $a = -0.5$, both conditions hold and yield two different sets of values satisfying ψ , viz. $x = -0.5$, $y = -1.5$ and $x = \frac{\sqrt{7} - 4}{6}$, $y = -0.5$, respectively.
- For $a = 0$ only the condition in the second row holds, but the answers in the first row happen to work as well. This shows that the conditions are sufficient but not necessary for the answers to be valid.

3 The Method of Virtual Substitution

Given $\varphi(\mathbf{u}) = \exists x \psi(x, \mathbf{u})$, we compute a finite *elimination set*

$$E = \{ \dots, (\gamma(\mathbf{u}), e(\mathbf{u})), \dots \} \quad \text{such that} \quad \exists x \psi \iff \bigvee_{(\gamma, e) \in E} \gamma \wedge \psi[x // e]. \quad (1)$$

In the elimination set E the $e(\mathbf{u})$ are *elimination terms* substituted for the quantified variable x via a *virtual substitution* $[x // e]$. The γ are quantifier-free Tarski formulas serving as *substitution guards*. Equation (1) formally describes regular quantifier elimination of one quantifier $\exists x$ from φ . For the elimination of several quantifiers, one assumes without loss of generality that the formula is prenex and processes the prenex quantifier block from the inside to the outside.

We are now going to give an idea of the exact shape and computation of elimination sets that is sufficiently precise to understand our main contribution here. For a more thorough introduction into the theory of quantifier elimination by virtual substitution we refer to the original publications by Weispfenning (1988, 1994a, 1997a), Loos and Weispfenning (1993), and Dolzmann et al. (1999). In Section 3.1 we restrict ourselves to input formulas φ not containing any strict inequalities “ $<$,” “ $>$,” or “ \neq .” In that course it will become clear how exactly we derive extended quantifier elimination from the virtual substitution procedure.

Later, in Section 3.2, we generalize to formulas containing also strict inequalities. While with regular quantifier elimination the techniques used in the course of the generalization to strict inequalities remain completely transparent to the user, with extended quantifier elimination they leave visible traces in the answers by possibly introducing certain nonstandard elements, which do not have a straightforward interpretation over the reals. The purpose of this paper is to convert these answers to real numbers, given fixed values for the parameters.

3.1 Virtual Substitution for Weak Inequalities

Recall that our constraints are normalized such that their right hand sides are 0. Assume that all occurrences of x in $\varphi(\mathbf{u}) = \exists x \psi(x, \mathbf{u})$ are at most quadratic. Consider fixed real interpretations $\mathbf{u} = \mathbf{a} \in \mathbb{R}^m$ for all parameters. Then all constraints in $\psi(x, \mathbf{a})$ become univariate, and the set $\{c \in \mathbb{R} \mid \mathbb{R} \models \psi(c, \mathbf{a})\}$ is a finite union of real intervals, where the interval endpoints are zeros of the univariate left hand side polynomials.

Our goal is to include at least one such interval endpoint into our elimination set E : For each constraint $f_2(\mathbf{u})x^2 + f_1(\mathbf{u})x + f_0(\mathbf{u}) \varrho 0$, with a weak relation $\varrho \in \{=, \leq, \geq\}$ and discriminant $\Delta = f_1^2 - 4f_2f_0$ we add to E three pairs (γ, e) as follows:

$$\left(f_2 \neq 0 \wedge \Delta \geq 0, \frac{-f_1 + \sqrt{\Delta}}{2f_2} \right),$$

$$\left(f_2 \neq 0 \wedge \Delta \geq 0, \frac{-f_1 - \sqrt{\Delta}}{2f_2} \right), \quad \left(f_2 = 0 \wedge f_1 \neq 0, -\frac{f_0}{f_1} \right).$$

In order to obtain a quantifier-free equivalent for φ , such pairs have to be plugged into φ according to (1). To start with, note that the substitution guards γ make the substitution terms e meaningful by ensuring that denominations are not zero and arguments to square roots are not negative.

Next, observe that our elimination terms e are not terms in the Tarski language as they contain division as well as root symbols. It is one central idea of the virtual substitution approach that the substitution operator does not map L -terms to L -terms but atomic L -formulas to quantifier-free L -formulas:

$$[x // t] : \text{atomic } L\text{-formulas} \rightarrow \text{quantifier-free } L\text{-formulas}$$

Note that when there is more than one quantifier, it is crucial to obtain L -formulas in (1) in order to be able to proceed.

To give an impression of virtual substitution, we describe here the substitution $(f = 0)[x // \frac{g_1 + g_2\sqrt{g_3}}{g_4}]$ of a *root expression*

$$\frac{g_1 + g_2\sqrt{g_3}}{g_4}, \quad g_i \in \mathbb{Z}[\mathbf{u}]$$

into an equation $f = 0$, where $f \in \mathbb{Z}[\mathbf{u}][x]$ of arbitrary degree: It is easy to see that there are g_1^*, g_2^*, g_4^* such that

$$f\left(\frac{g_1 + g_2\sqrt{g_3}}{g_4}\right) = \frac{g_1^* + g_2^*\sqrt{g_3}}{g_4^*}$$

is again a root expression. Using this intermediate result, we transform

$$\begin{aligned} \frac{g_1^* + g_2^*\sqrt{g_3}}{g_4^*} = 0 &\doteq g_1^* + g_2^*\sqrt{g_3} = 0 \\ &\doteq |g_1^*| = |g_2^*\sqrt{g_3}| \wedge \\ &\quad (\text{sgn}(g_1^*) \neq \text{sgn}(g_2^*) \vee \text{sgn}(g_1^*) = \text{sgn}(g_2^*) = 0) \\ &\doteq g_1^{*2} - g_2^{*2}g_3 = 0 \wedge g_1^*g_2^* \leq 0. \end{aligned}$$

Technical details and formal descriptions of virtual substitutions for all our relations have been given by Weispfenning (1997a).

Let us now apply these ideas to extended quantifier elimination of several existential quantifiers via virtual substitution. Given

$$\exists x_n \dots \exists x_1 \psi(x_1, \dots, x_n, u_1, \dots, u_m)$$

our intended result is a scheme as described in Section 2:

$$\left[\begin{array}{c|ccc} \beta_1(\mathbf{u}) & x_1 = e_{11}(\mathbf{u}) & \dots & x_n = e_{1n}(\mathbf{u}) \\ \vdots & \vdots & \ddots & \vdots \\ \beta_k(\mathbf{u}) & x_1 = e_{k1}(\mathbf{u}) & \dots & x_n = e_{kn}(\mathbf{u}) \end{array} \right].$$

We successively apply (1) to x_1, \dots, x_n using elimination sets $E_1(x_2, \dots, x_n, \mathbf{u}), \dots, E_n(\mathbf{u})$ to obtain β_1, \dots, β_k as follows:

$$\bigvee_{(\gamma_n, e_n) \in E_n} \dots \bigvee_{(\gamma_1, e_1) \in E_1} \underbrace{\gamma_n \wedge (\dots \wedge (\gamma_1 \wedge \psi[x_1 // e_1]) \dots)}_{\beta_i(\mathbf{u})} [x_n // e_n]. \quad (2)$$

The index i of β_i describes one choice $((\gamma_n, e_n), \dots, (\gamma_1, e_1))$ from the Cartesian product $E_n \times \dots \times E_1$. In practice, the β_i obtained this way undergo sophisticated simplification methods such as those described by Dolzmann and Sturm (1997b). Recall from the previous section that β_i that become “false” are ignored. It is important to understand that for the computation of the β_i we are using exclusively *virtual* substitution.

The corresponding $e_i(\mathbf{u})$, in contrast, are obtained from $e_n(\mathbf{u}), e_{n-1}(x_n, \mathbf{u}), \dots, e_1(x_2, \dots, x_n, \mathbf{u})$ via *regular back-substitution* of terms in an suitable extension language of L :

$$e_{i,n} = e_n, \quad e_{i,n-1} = e_{n-1}[x_n/e_{i,n}], \quad e_{i,1} = e_1[x_1/e_{i,2}] \dots [x_n/e_{i,n}]. \quad (3)$$

Note once more that the back-substitution possible creates objects like

$$u_1 + \frac{3\sqrt{\sqrt{5}u_1 - u_2} - 2}{2},$$

which are not L -terms or even root expressions. It is thus suitable for obtaining the \mathbf{e}_i but not for the β_i . Vice versa, virtual substitution requires atomic formulas as input. Thus it is suitable for obtaining the β_i while for the \mathbf{e}_i it is not applicable at all. Virtual substitution and regular term substitution are independent concepts, which complement each other.

3.2 Virtual Substitution with Strict Inequalities

Let us return to the elimination of a single quantifier from $\exists x\psi(x, \mathbf{u})$. Recall from the beginning of the previous Section 3.1 how we considered fixed interpretation $\mathbf{u} = \mathbf{a} \in \mathbb{R}^m$ causing the set $S = \{c \in \mathbb{R} \mid \mathbb{R} \models \psi(c, \mathbf{a})\}$ to be a finite union of intervals.

When considering in addition strict inequalities, the intervals in S are possibly open. Consequently, for a *strict constraint*

$$\psi_i(x, \mathbf{u}) \doteq f_{i2}(\mathbf{u})x_i^2 + f_{i1}(\mathbf{u})x_i + f_{i0}(\mathbf{u}) \varrho_i 0, \quad \varrho_i \in \{<, >, \neq\}, \quad (4)$$

contained in $\psi(x, \mathbf{u})$ we cannot use the zeros $z_i(\mathbf{u})$ of the left hand side but need a point from *inside* the corresponding interval.

In early versions of virtual substitution methods for the linear case, Weispfenning (1988) used *arithmetic means* $\frac{1}{2}(z_i + z_j)$ for all pairs (ψ_i, ψ_j) of strict constraints. However, the size of the elimination set then grows quadratically in the number of constraints, which turned out to be critical for the practical performance of the method (Burhenne, 1990; Loos and Weispfenning, 1993). For the quadratic case, observe that expressions

$$\frac{1}{2} \left(\frac{-g_{i1} \pm \sqrt{\Delta_i}}{2g_{i2}} + \frac{-g_{j1} \pm \sqrt{\Delta_j}}{2g_{j2}} \right)$$

are not root expression of the form

$$\frac{g_1^* + g_2^* \sqrt{\Delta^*}}{g_3^*},$$

so that Weispfenning's (1997a) virtual substitution rules sketched in the previous Section 3.1 cannot be used.

The established approach for strict inequalities uses nonstandard extensions of \mathbb{R} : Let $\varepsilon \in \mathbb{R}^* \supset \mathbb{R}$ be a positive infinitesimal number, i.e., $0 < \varepsilon < x$ for all $0 < x \in \mathbb{R}$. Then for a strict constraint as defined in (4) we use four test points

$$\frac{-f_{i1} \pm \sqrt{\Delta_i}}{2f_{i2}} \pm \varepsilon.$$

As an optimization, it suffices to consider only upper bounds using $-\varepsilon$. For solution sets that are unbounded from above we have to add only one more point $\infty := 1/\varepsilon \in \mathbb{R}^*$ for the entire problem.

For the application of the elimination set as described in (1) and thus for the computation of the β_i with extended quantifier elimination, both ε and ∞ are equivalently translated into the Tarski language L via virtual substitution. For instance, let t be a standard term. Then

$$(x^3 + x^2 - x - 1 < 0)[x // t - \varepsilon] \doteq (x^3 + x^2 - x - 1 < 0 \vee (x^3 + x^2 - x - 1 = 0 \wedge (3x^2 + 2x - 1 > 0 \vee (3x^2 + 2x - 1 = 0 \wedge (6x + 2 < 0 \vee (6x + 2 = 0 \wedge 6 > 0)))))))[x // t].$$

In a subsequent step, t can be virtually substituted for x as discussed in the previous section. For understanding the principal idea, notice that $3x^2 + 2x - 1$, $6x + 2$, and 6 are the first, second, and third derivatives of $x^3 + x^2 - x - 1$, respectively. For the substitution of ∞ we have, e.g.,

$$(ax^2 + bx + c < 0)[x // \infty] \doteq a < 0 \vee (a = 0 \wedge b < 0) \vee (a = 0 \wedge b = 0 \wedge c < 0).$$

Again, precise definitions and proofs have been given by Weispfenning (1997a).

When again performing regular back-substitution of terms on the side of the \mathbf{e}_i , nonstandard symbols cannot be removed but are propagated along the way. In the final result a single answer \mathbf{e}_i can even contain several of such nonstandard symbols. For example, on input of

$$\varphi = \exists x \exists y \psi, \quad \psi \doteq ay + 3x^2 + 4x < 0 \wedge x > y > a \quad (5)$$

we obtain a nonstandard extended quantifier elimination result:

$$\left[\begin{array}{l|ll} a + 4 < 0 & y = \frac{-a - 3\varepsilon_1 - 3\varepsilon_2 - 4}{3} & x = \frac{-a - 3\varepsilon_1 - 4}{3} \\ a < 0 \wedge a + 4 > 0 & y = -\varepsilon_1 - \varepsilon_2 & x = -\varepsilon_1 \end{array} \right]. \quad (6)$$

Given such answers containing nonstandard symbols, it is not hard to non-constructively prove from the elimination procedure that for fixed real interpretations of the parameter a there are positive real choices $\varepsilon_1, \varepsilon_2 \in \mathbb{R}$ so that the answers satisfy ψ . Infinitesimals introduced at different stages of the elimination are indexed accordingly. It is noteworthy that they have to be chosen differently in general: Fixing $a = -2$ in the example, it is easy to see that ε_1 has to be chosen different from ε_2 because otherwise we would obtain $y = 2x$, which together with $a = -2$ does not satisfy $ay + 3x^2 + 4x < 0$.

Until now users of extended quantifier elimination were left alone with results as in (6). In spite of the difficulties discussed above, there is a considerable record of applications of extended quantifier elimination in the literature. We are going to discuss some of these with our examples in Section 7.

We conclude this section with an important observation that for unfixed parameters it is not possible in general to determine suitable real choices for nonstandard symbols:

Proposition 1 (No Standard Answers for Unfixed Parameters). Consider the formula $\varphi = \exists x(a < x < 1)$ and the nonstandard extended quantifier elimination result

$$\left[a < 1 \mid x = 1 - \varepsilon_1 \right].$$

There is no standard choice $\tilde{\varepsilon}_1 \in \mathbb{R}$ such that $\left[a < 1 \mid x = 1 - \tilde{\varepsilon}_1 \right]$ is an extended quantifier elimination result for φ as well.

Proof. Assume for a contradiction that $\tilde{\varepsilon}_1 \in \mathbb{R}$ is a suitable choice. Then by definition of extended quantifier elimination it follows for all $a \in]-\infty, 1[$ that $a < 1 - \tilde{\varepsilon}_1 < 1$, in particular $\tilde{\varepsilon}_1 > 0$. On the other hand, for $a_0 = 1 - \frac{\tilde{\varepsilon}_1}{2}$ we have $a_0 \in]-\infty, 1[$ and $1 - \tilde{\varepsilon}_1 < a_0 < 1$, a contradiction. \square

4 Elimination of Nonstandard Symbols from Answers

Given an extended quantifier elimination result and prescribed values for all parameters, our goal is to compute answers containing only standard real numbers. For instance, given (5) and (6), and fixing $a = -2$ we are going to obtain

$$\left[\text{true} \mid y = -\frac{9}{256} \quad x = -\frac{1}{32} \right].$$

From the point-of-view of our method, it makes no difference whether the parameters are fixed after extended quantifier elimination or in advance. For the sake of a concise description, we are thus going to restrict to existential decision problems from now on. Recall that if the regular quantifier elimination result is “false,” then the extended quantifier elimination result is $[]$, i.e., empty. If the result is “true,” then we assume for simplicity that the extended quantifier elimination result contains only one row, like

$$\left[\text{true} \mid x_1 = e_{1,1} \quad \dots \quad x_n = e_{1,n} \right]. \quad (7)$$

Recall from (3) in Section 3.1 that here the $e_{1,1}, \dots, e_{1,n}$ have been obtained from elimination terms e_1, \dots, e_n via regular back-substitution. Our method is going to use not the back-substituted answers but these original elimination terms. In addition, we are going to use the substitution guards $\gamma_1, \dots, \gamma_n$ substituted with those elimination terms in (2). Hence the input for our method is not an EQR like in (7) but a *pre-EQR* as follows:

$$\left[\text{true} \mid \begin{array}{cccc} x_1 = e_1(x_2, \dots, x_n) & \dots & x_{n-1} = e_{n-1}(x_n) & x_n = e_n() \\ \gamma_1(x_2, \dots, x_n) & \dots & \gamma_{n-1}(x_n) & \gamma_n() \end{array} \right].$$

Lemma 2 (Semantics of Virtual Substitution). Let $\varphi(x_1, \dots, x_n)$ be a Tarski formula, and let $a_2, \dots, a_n \in \mathbb{R}$.

- (i) Assume that $\mathbb{R} \models \varphi[x_1 // x_2 - \varepsilon](a_2, \dots, a_n)$. Then there is $\tilde{\varepsilon}_0 \in \mathbb{R}$, $0 < \tilde{\varepsilon}_0$, such that for all $\tilde{\varepsilon} \in \mathbb{R}$, $0 < \tilde{\varepsilon} < \tilde{\varepsilon}_0$, we have $\mathbb{R} \models \varphi(a_2 - \tilde{\varepsilon}, a_2, \dots, a_n)$.
- (ii) Assume that $\mathbb{R} \models \varphi[x_1 // \infty](a_2, \dots, a_n)$. Then there is $\tilde{a}_0 \in \mathbb{R}$, such that for all $\tilde{a}_1 \in \mathbb{R}$, $\tilde{a}_0 < \tilde{a}_1$, we have $\mathbb{R} \models \varphi(\tilde{a}_1, a_2, \dots, a_n)$. In particular, the set $T = \{a \in \mathbb{R} \mid 0 < a \text{ and } \mathbb{R} \models \varphi(a, a_2, \dots, a_n)\}$ is unbounded from above.

Proof. Consider $L_\varepsilon = L \cup \{\varepsilon, \infty\}$. Using the compactness theorem for first-order logic, there is a real closed field \mathbb{R}^* where the interpretation of ε is a positive infinitesimal and $\infty = \varepsilon^{-1}$. The L -restriction of \mathbb{R}^* is a proper extension field of \mathbb{R} and, by the Tarski principle, elementary equivalent to \mathbb{R} . Formally, $\mathbb{R}^*|L \supset \mathbb{R}$ and $\mathbb{R}^*|L \cong \mathbb{R}$.

- (i) Using $\mathbb{R} \cong \mathbb{R}^*|L$ and expanding to \mathbb{R}^* it follows from $\mathbb{R} \models \varphi[x_1 // x_2 - \varepsilon](a_2, \dots, a_n)$ that also $\mathbb{R}^* \models \varphi[x_1 // x_2 - \varepsilon](a_2, \dots, a_n)$. It has been discussed by Loos and Weispfenning (1993) and Weispfenning (1997a) that for our $a_2, \dots, a_n \in \mathbb{R}$ the virtual substitution of ε has the following property:

$$\mathbb{R}^* \models \varphi[x_1 // x_2 - \varepsilon](a_2, \dots, a_n) \longleftrightarrow \varphi[x_1/x_2 - \varepsilon](a_2, \dots, a_n).$$

It follows that $\mathbb{R}^* \models \varphi[x_1/x_2 - \varepsilon](a_2, \dots, a_n)$. Let $n \in \mathbb{N} \setminus \{0\}$. Then we can conclude $\mathbb{R}^* \models \psi[x_0/\varepsilon](a_2, \dots, a_n)$, where

$$\psi = (0 < x_0 \wedge nx_0 < 1 \wedge \varphi)[x_1/x_2 - x_0].$$

Now we can generalize $\mathbb{R}^* \models \exists x_0 \psi(a_2, \dots, a_n)$. Since ε does not occur anymore, we restrict from \mathbb{R}^* to $\mathbb{R}^*|L$ and then use the elementary equivalence to obtain $\mathbb{R} \models \exists x_0 \psi(a_2, \dots, a_n)$. We have just shown that for any $n \in \mathbb{N} \setminus \{0\}$ there exists $a_0 \in \mathbb{R}$, $0 < a_0 < \frac{1}{n}$, such that $\mathbb{R} \models \varphi(a_2 - a_0, a_2, \dots, a_n)$. It follows that $\inf S = 0$, where

$$S = \{a \in \mathbb{R} \mid 0 < a \text{ and } \mathbb{R} \models \varphi(a_2 - a, a_2, \dots, a_n)\} \subseteq \mathbb{R}.$$

On the other hand, S is a semialgebraic set and thus a finite union of intervals and points. Hence there is $\tilde{\varepsilon}_0 \in \mathbb{R}$, $0 < \tilde{\varepsilon}_0$, such that $]0, \tilde{\varepsilon}_0[\subseteq S$.

- (ii) The argument is essentially the same as in (i) above: We conclude that $\mathbb{R}^* \models \varphi[x_1 // \infty](a_2, \dots, a_n)$. According to Loos and Weispfenning (1993) and Weispfenning (1997a) we know that

$$\mathbb{R}^* \models \varphi[x_1 // \infty](a_2, \dots, a_n) \longleftrightarrow \varphi[x_1/\infty](a_2, \dots, a_n)$$

so that for $n \in \mathbb{N}$ we can conclude $\mathbb{R}^* \models \psi[x_0/\infty](a_2, \dots, a_n)$, where

$$\psi = (n < x_0 \wedge \varphi)[x_1/x_0].$$

Again, we generalize, restrict, and apply elementary equivalence to obtain $\mathbb{R} \models \exists x_0 \psi(a_2, \dots, a_n)$. We thus know that for any $n \in \mathbb{N}$ there exists $a_0 \in \mathbb{R}$, $n < a_0$, such that $\mathbb{R} \models \varphi(a_0, a_2, \dots, a_n)$. It follows that the set T is unbounded from above. On the other hand, T is a semialgebraic set and thus a finite union of intervals and points. Hence there is $\tilde{a}_0 \in \mathbb{R}$, such that $] \tilde{a}_0, \infty[\subseteq T$. \square

Lemma 3. Consider a quantifier-free Tarski formula $\psi(x_1, \dots, x_n)$. Assume that for each $i \in \{2, \dots, n\}$ we have a root expression $\tilde{e}_i = \frac{a_i + b_i \sqrt{c_i}}{d_i}$ with $a_i, b_i, c_i, d_i \in \mathbb{Z}[x_{i+1}, \dots, x_n]$. Assume furthermore that $\mathbb{R} \models \psi \longrightarrow c_i \geq 0 \wedge d_i \neq 0$, and let $\alpha_i \in \mathbb{R}$ be the interpretation of $e_i = \tilde{e}_i[x_{i+1}/\tilde{e}_{i+1}] \dots [x_n/\tilde{e}_n]$. Then

$$\{\alpha \in \mathbb{R} \mid \mathbb{R} \models \psi[x_2 // \tilde{e}_2] \dots [x_n // \tilde{e}_n](\alpha)\} = \{\alpha \in \mathbb{R} \mid \mathbb{R} \models \psi(\alpha, \alpha_2, \dots, \alpha_n)\}.$$

Proof. Consider $L' = L \cup \{\sqrt{},^{-1}\}$ and the L' -expansion \mathbb{R}' of \mathbb{R} where $\sqrt{}$ and $^{-1}$ have the usual semantics if defined and an arbitrary but fixed value otherwise. Let $\nu = (f \varrho 0)$, where $f \in \mathbb{Z}[x, u_1, \dots, u_m]$, $\varrho \in \{\leq, <, \geq, >, \neq, =\}$, and let $e = \frac{a+b\sqrt{c}}{d}$, where $a, b, c, d \in \mathbb{Z}[u_1, \dots, u_m]$. Let $\mathbf{s} \in \mathbb{R}^m$ such that $c(\mathbf{s}) \geq 0$

and $d(\mathbf{s}) \neq 0$. Then according to Weispfenning (1997a) the following holds for $\mathbf{s} \in \mathbb{R}^m$:

$$\mathbb{R}' \models (\nu[x // e] \longleftrightarrow \nu[x/e])(\mathbf{s}).$$

For our formula ψ and all $\alpha \in \mathbb{R}$ we obtain

$$\mathbb{R}' \models (\psi[x_2 // \tilde{e}_2] \dots [x_n // \tilde{e}_n] \longleftrightarrow \psi[x_2/\tilde{e}_2] \dots [x_n/\tilde{e}_n] \longleftrightarrow \psi)(\alpha, \alpha_2, \dots, \alpha_n).$$

It follows that $\mathbb{R} \models (\psi[x_2 // \tilde{e}_2] \dots [x_n // \tilde{e}_n] \longleftrightarrow \psi)(\alpha, \alpha_2, \dots, \alpha_n)$ for all $\alpha \in \mathbb{R}$. \square

Lemma 4 (Commutation of Virtual Substitutions). Consider a quantifier-free Tarski formula $\psi(x_1, \dots, x_n)$. Let $e_1 = \frac{a_1 + b_1 \sqrt{c_1}}{d_1}$, with $a_1, b_1, c_1, d_1 \in \mathbb{Z}$, $c_1 \geq 0, d_1 \neq 0$. Furthermore, let $i \in \{2, \dots, n\}$, let $e_i = \frac{a_i + b_i \sqrt{c_i}}{d_i}$ with $a_i, b_i, c_i, d_i \in \mathbb{Z}[x_{i+1}, \dots, x_n]$, and let $\gamma_i(x_{i+1}, \dots, x_n)$ be a quantifier-free formula such that $\mathbb{R} \models \gamma_i \longrightarrow c_i \geq 0 \wedge d_i \neq 0$. Then

$$\mathbb{R} \models (\gamma_i \wedge \psi)[x_i // e_i][x_1 // e_1] \longleftrightarrow (\gamma_i \wedge \psi)[x_1 // e_1][x_i // e_i].$$

Proof. Let L', \mathbb{R}' be as in the proof of Lemma 3. Recall the equivalence $\mathbb{R}' \models (\nu[x // e] \longleftrightarrow \nu[x/e])(\mathbf{s})$ and observe that, on the other hand, if $\mathbb{R} \models (c_i < 0)(\mathbf{s})$ or $\mathbb{R} \models (d_i = 0)(\mathbf{s})$ for $\mathbf{s} \in \mathbb{R}^{n-i}$, then $\mathbb{R} \models (\gamma_i \wedge \psi \longleftrightarrow \text{false})(\mathbf{s})$. It follows that

$$\begin{aligned} \mathbb{R}' \models (\gamma_i \wedge \psi)[x_i // e_i][x_1 // e_1] &\longleftrightarrow (\gamma_i \wedge \psi)[x_i/e_i][x_1/e_1] \\ &\longleftrightarrow (\gamma_i \wedge \psi)[x_1/e_1][x_i/e_i] \\ &\longleftrightarrow (\gamma_i \wedge \psi)[x_1 // e_1][x_i // e_i]. \end{aligned}$$

Since virtual substitution eliminates all occurrences of $\sqrt{}$ and $^{-1}$, we can conclude that $\mathbb{R} \models (\gamma_i \wedge \psi)[x_i // e_i][x_1 // e_1] \longleftrightarrow (\gamma_i \wedge \psi)[x_1 // e_1][x_i // e_i]$. \square

Theorem 5 (Computation of Standard Answers). Consider a closed Tarski formula $\varphi = \exists x_n \dots \exists x_1 \psi(x_1, \dots, x_n)$. Assume that

$$\left[\begin{array}{c|ccc} \text{true} & x_1 = e_1 & \dots & x_n = e_n \\ & \gamma_1 & \dots & \gamma_n \end{array} \right]$$

is a pre-EQR for φ such that each e_i is of one of the following forms:

(a) a root expression $\frac{a+b\sqrt{c}}{d}$, where $a, b, c, d \in \mathbb{Z}[x_{i+1}, \dots, x_n]$,

(b) ∞ ,

(c) $x_{i+1} - \varepsilon$.

Then we can compute root expressions $\tilde{e}_1, \dots, \tilde{e}_n$ meeting the specification (a) above and $\tilde{\gamma}_1, \dots, \tilde{\gamma}_n$ such that the following is a pre-EQR for φ as well:

$$\left[\begin{array}{c|ccc} \text{true} & x_1 = \tilde{e}_1 & \dots & x_n = \tilde{e}_n \\ & \tilde{\gamma}_1 & \dots & \tilde{\gamma}_n \end{array} \right]. \quad (8)$$

Proof. For the sake of the proof, we are going to show that in addition to the required $\widetilde{e}_1, \dots, \widetilde{e}_n$ and $\widetilde{\gamma}_1, \dots, \widetilde{\gamma}_n$ we can compute real algebraic numbers $\alpha_1, \dots, \alpha_n$ for the values of $\widetilde{e}_1, \dots, \widetilde{e}_n$ after back-substitution. We represent these real algebraic numbers as pairs of univariate defining polynomials and open isolating intervals with rational bounds. Given $k \in \{1, \dots, n\}$, it suffices to show that from

$$\left[\text{true} \mid \begin{array}{cccccc} x_1 = e_1 & \dots & x_k = e_k & x_{k+1} = \widetilde{e}_{k+1} & \dots & x_n = \widetilde{e}_n \\ & & \gamma_1 & \dots & \gamma_k & \dots & \widetilde{\gamma}_n \end{array} \right]$$

and $\alpha_{k+1}, \dots, \alpha_n$ we can compute suitable $\widetilde{e}_k, \widetilde{\gamma}_k$, and α_k . Define

$$\begin{aligned} \varphi_k(x_k, \dots, x_n) &= (\gamma_{k-1} \wedge \dots \wedge \gamma_1 \wedge \psi)[x_1 // e_1] \dots [x_{k-1} // e_{k-1}], \\ \varphi_{k+1}(x_{k+1}, \dots, x_n) &= (\gamma_k \wedge \varphi_k)[x_k // e_k], \end{aligned}$$

and observe that

$$\left[\text{true} \mid \begin{array}{cccc} x_k = e_k & x_{k+1} = \widetilde{e}_{k+1} & \dots & x_n = \widetilde{e}_n \\ & \gamma_k & \dots & \widetilde{\gamma}_n \end{array} \right] \quad (9)$$

and

$$\left[\text{true} \mid \begin{array}{ccc} x_{k+1} = \widetilde{e}_{k+1} & \dots & x_n = \widetilde{e}_n \\ & \widetilde{\gamma}_{k+1} & \dots & \widetilde{\gamma}_n \end{array} \right] \quad (10)$$

are pre-EQRs for $\exists x_n \dots \exists x_k \varphi_k$ and $\exists x_n \dots \exists x_{k+1} \varphi_{k+1}$, respectively. On the basis of these definitions it is sufficient for our proof to compute suitable $\widetilde{e}_k, \widetilde{\gamma}_k$, and α_k such that

$$\left[\text{true} \mid \begin{array}{cccc} x_k = \widetilde{e}_k & x_{k+1} = \widetilde{e}_{k+1} & \dots & x_n = \widetilde{e}_n \\ & \widetilde{\gamma}_k & \dots & \widetilde{\gamma}_n \end{array} \right]$$

is a pre-EQR for $\exists x_n \dots \exists x_k \varphi_k$ as well. We define furthermore

$$\begin{aligned} \xi(x_k, \dots, x_n) &= \widetilde{\gamma}_n \wedge \dots \wedge \widetilde{\gamma}_{k+1} \wedge \varphi_k, \\ \xi'(x_k) &= \xi[x_{k+1} // \widetilde{e}_{k+1}] \dots [x_n // \widetilde{e}_n]. \end{aligned}$$

Lemma 3 applied to the quantifier-free formula ξ , the root expressions $\widetilde{e}_{k+1}, \dots, \widetilde{e}_n$, and the real algebraic numbers $\alpha_{k+1}, \dots, \alpha_n$ yields

$$\{r \in \mathbb{R} \mid \mathbb{R} \models \xi'(r)\} = \{r \in \mathbb{R} \mid \mathbb{R} \models \xi(r, \alpha_{k+1}, \dots, \alpha_n)\}. \quad (11)$$

We distinguish three cases depending on the type of e_k .

- (a) We have $e_k = \frac{a_k + b_k \sqrt{c_k}}{d_k}$, and γ_k equals $d_k \neq 0 \wedge c_k \geq 0$. We set $\widetilde{e}_k = e_k$ and $\widetilde{\gamma}_k = \gamma_k$. Since (10) is a pre-EQR for $\exists x_n \dots \exists x_{k+1} \varphi_{k+1}$ and $\alpha_{k+1}, \dots, \alpha_n$ correspond to the values of $\widetilde{e}_{k+1}, \dots, \widetilde{e}_n$ after back-substitution, we have $\mathbb{R} \models \varphi_{k+1}(\alpha_{k+1}, \dots, \alpha_n)$. It follows that in particular $\mathbb{R} \models \gamma_k(\alpha_{k+1}, \dots, \alpha_n)$ and furthermore $\mathbb{R} \models d_k(\alpha_{k+1}, \dots, \alpha_n) \neq 0$ and $\mathbb{R} \models c_k(\alpha_{k+1}, \dots, \alpha_n) \geq 0$. This allows us to compute $\alpha_k = \widetilde{e}_k(\alpha_{k+1}, \dots, \alpha_n)$ from $a_k, b_k, c_k, d_k, \alpha_{k+1}, \dots, \alpha_n$.

- (b) We have $e_k = \infty$, and γ_k is “true.” Since (9) is a pre-EQR for formula $\exists x_n \dots \exists x_k \varphi_k$ we have

$$\mathbb{R} \models \xi[x_k // \infty][x_{k+1} // \widetilde{e_{k+1}}] \dots [x_n // \widetilde{e_n}].$$

Using the fact that $\alpha_{k+1}, \dots, \alpha_n$ are real algebraic numbers corresponding to the values of $\widetilde{e_{k+1}}, \dots, \widetilde{e_n}$ after back-substitution we conclude that

$$\mathbb{R} \models \xi[x_k // \infty](\alpha_{k+1}, \dots, \alpha_n).$$

Lemma 2(ii) now guarantees that the set $\{r \in \mathbb{R} \mid \mathbb{R} \models \xi(r, \alpha_{k+1}, \dots, \alpha_n)\}$ is unbounded from above. Thus by (11) the set $\{r \in \mathbb{R} \mid \mathbb{R} \models \xi'(r)\}$ is unbounded from above as well. Using well-known bounds (Akritas, 2009) on the roots of the univariate polynomials contained in ξ' , we compute a sufficiently large $\frac{p}{q} \in \mathbb{Q}$ satisfying ξ' . We set $\widetilde{e}_k = \frac{p+0\sqrt{0}}{q}$ and construct a corresponding real algebraic number α_k , and we set $\widetilde{\gamma}_k = \text{true}$. Then $\mathbb{R} \models (\widetilde{\gamma}_k \wedge \xi')[x_k // \widetilde{e}_k]$, and $n - k$ applications of Lemma 4 yield

$$\mathbb{R} \models (\widetilde{\gamma}_n \wedge \dots \wedge \widetilde{\gamma}_k \wedge \varphi_k)[x_k // \widetilde{e}_k][x_{k+1} // \widetilde{e_{k+1}}] \dots [x_n // \widetilde{e_n}].$$

- (c) We have $e_k = x_{k+1} - \varepsilon$, and γ_k is “true.” Similarly to case (b), we observe that (9) is a pre-EQR for $\exists x_n \dots \exists x_k \varphi_k$ and obtain

$$\mathbb{R} \models \xi[x_k // x_{k+1} - \varepsilon][x_{k+1} // \widetilde{e_{k+1}}] \dots [x_n // \widetilde{e_n}],$$

and conclude that $\mathbb{R} \models \xi[x_k // x_{k+1} - \varepsilon](\alpha_{k+1}, \dots, \alpha_n)$. Lemma 2(i) now guarantees the existence of some $\widetilde{\varepsilon}_0 \in \mathbb{R}$, $0 < \widetilde{\varepsilon}_0$, such that

$$\mathbb{R} \models \xi(\alpha_{k+1} - \widetilde{\varepsilon}, \alpha_{k+1}, \dots, \alpha_n) \quad \text{for } 0 < \widetilde{\varepsilon} < \widetilde{\varepsilon}_0.$$

By (11) it follows that $\mathbb{R} \models \xi'(\alpha_{k+1} - \widetilde{\varepsilon})$ for all $0 < \widetilde{\varepsilon} < \widetilde{\varepsilon}_0$. Therefore, after finitely many refinements of the isolating interval $]\frac{p}{q}, u[$ of α_{k+1} we obtain $\mathbb{R} \models \xi'(\frac{p}{q})$. We set $\widetilde{e}_k = \frac{p+0\sqrt{0}}{q}$ and construct a corresponding real algebraic number, and we set $\widetilde{\gamma}_k = \text{true}$. Exactly as in case (b), $\mathbb{R} \models (\widetilde{\gamma}_k \wedge \xi')[x_k // \widetilde{e}_k]$, and $n - k$ applications of Lemma 4 yield

$$\mathbb{R} \models (\widetilde{\gamma}_n \wedge \dots \wedge \widetilde{\gamma}_k \wedge \varphi_k)[x_k // \widetilde{e}_k][x_{k+1} // \widetilde{e_{k+1}}] \dots [x_n // \widetilde{e_n}].$$

Note that instead of using the lower bound $\frac{p}{q}$ one can heuristically try and find a satisfying integer. \square

A careful inspection of our proof reveals that in all cases $\widetilde{\gamma}_k = \gamma_k$. However, this is going to change in Corollary 7, which generalizes our theorem.

Notice that the constructive proof of Theorem 5 suggests to recompute the intermediate quantifier elimination results φ_k . In practice, there are arguments for saving these φ_k during the quantifier elimination run. Consider, e.g., the following common optimization: Whenever some φ_k heuristically simplifies to a disjunction $\varphi_{k,1} \vee \dots \vee \varphi_{k,s}$, then the virtual substitution procedure would treat each $\varphi_{k,j}$ separately, i.e., like originating from several elimination set elements. In general, in the course of the application of Theorem 5 such transformations cannot be reconstructed exclusively from the pre-EQR.

To illustrate the theorem we revisit our example given in (5) on page 7 for the choice $a = -2$. In that case ψ in (5) specializes to

$$\psi \doteq -2y + 3x^2 + 4x < 0 \wedge x > y > -2.$$

For our theorem we have to consider the following pre-EQR corresponding to the specialization of the EQR (6) to $a = -2$:

$$\left[\begin{array}{l|lll} \text{true} & y = x - \varepsilon & x = h - \varepsilon & h = 0 \\ & \text{true} & \text{true} & \text{true} \end{array} \right]. \quad (12)$$

Notice the introduction of an artificial variable h to meet the requirement of the theorem that infinitesimals occur only in expressions of the form $x_i = x_{i+1} - \varepsilon$.

To apply the theorem to the pre-EQR (12), we consider

$$\begin{aligned} \varphi_1(y, x, h) &\doteq -2y + 3x^2 + 4x < 0 \wedge x > y > -2, \\ \varphi_2(x, h) &\doteq \varphi_1[y // x - \varepsilon] \\ &\doteq x + 2 > 0 \wedge 3x^2 + 2x < 0, \\ \varphi_3(h) &\doteq \varphi_2[x // h - \varepsilon] \\ &\doteq h + 2 > 0 \wedge (h = 0 \vee 3h^2 + 2h < 0). \end{aligned}$$

As in the theorem, we proceed from the right to the left, i.e., our first step is fixing h and computing a respective algebraic number α_h . Since $h = 0$, we are in case (a). Now

$$\left[\begin{array}{l|l} \text{true} & h = 0 \\ & \text{true} \end{array} \right]$$

is a pre-EQR for $\exists h \varphi_3$. Therefore, α_h is the root of the polynomial h in the interval $] -1, 1[$, i.e., α_h is the rational number 0.

We continue with $x = h - \varepsilon$, which is case (c). Now

$$\left[\begin{array}{l|ll} \text{true} & x = h - \varepsilon & h = 0 \\ & \text{true} & \text{true} \end{array} \right]$$

is a pre-EQR for $\exists x \exists h \varphi_2$. Lemma 2(i) ensures that there exists $\tilde{\varepsilon}_0 \in \mathbb{R}$, $0 < \tilde{\varepsilon}_0$ such that for all $\tilde{\varepsilon} \in \mathbb{R}$, $0 < \tilde{\varepsilon} < \tilde{\varepsilon}_0$, we have $\mathbb{R} \models \varphi_2(\alpha_h - \tilde{\varepsilon}, \alpha_h)$. Refining α_h we compute that $\mathbb{R} \models \varphi_2(\alpha_x, \alpha_h)$, where α_x is the root of $32x + 1$ in the interval $] -\frac{1}{16}, \frac{1}{16}[$, i.e., α_x is the rational number $-\frac{1}{32}$.

Finally consider $y = x - \varepsilon$, which is again case (c). Now

$$\left[\begin{array}{l|lll} \text{true} & y = x - \varepsilon & x = -\frac{1}{32} & h = 0 \\ & \text{true} & \text{true} & \text{true} \end{array} \right]$$

is a pre-EQR for $\exists y \exists x \exists h \varphi_1$. Lemma 2(i) ensures that there exists $\tilde{\varepsilon}_0 \in \mathbb{R}$, $0 < \tilde{\varepsilon}_0$ such that for all $\tilde{\varepsilon} \in \mathbb{R}$, $0 < \tilde{\varepsilon} < \tilde{\varepsilon}_0$, we have $\mathbb{R} \models \varphi_1(\alpha_x - \tilde{\varepsilon}, \alpha_x, \alpha_h)$. Refining α_x we compute that $\mathbb{R} \models \varphi_1(\alpha_y, \alpha_x, \alpha_h)$, where α_y is the root of $256y + 9$ in the interval $] -\frac{10}{256}, \frac{10}{256}[$, i.e., α_y is the rational number $-\frac{9}{256}$. To conclude we state that

$$\left[\begin{array}{l|lll} \text{true} & y = -\frac{9}{256} & x = -\frac{1}{32} & h = 0 \\ & \text{true} & \text{true} & \text{true} \end{array} \right]$$

is a pre-EQR for $\exists y \exists x \exists h \varphi_1$, which does not contain any nonstandard symbols. Since h does not occur in $\varphi_1 = \psi$,

$$\left[\text{true} \mid \begin{array}{cc} y = -\frac{9}{256} & x = -\frac{1}{32} \\ \text{true} & \text{true} \end{array} \right]$$

is a pre-EQR for $\exists y \exists x \psi$.

Finally, note that all quantified variables have to be present in a pre-EQR before Theorem 5 can be applied. This has a consequence, which we illustrate on an example. Consider a valid sentence $\exists x \exists a (x < a)$. An extended quantifier elimination result containing a nonstandard symbol for this formula is

$$\left[\text{true} \mid a = \infty \right].$$

Since x does not occur in this result, it can be “freely chosen,” i.e., the result is independent of the value of x . Put another way, this means that

$$\left[\text{true} \mid x = t \quad a = \infty \right]$$

is an extended quantifier elimination result for any standard term t as well. This degree of freedom disappears when computing standard answers in the following sense: The term t has to be fixed before the computation of standard answers. Fixing $t = 2$ and using Theorem 5 we obtain a standard EQR

$$\left[\text{true} \mid x = 2 \quad a = 3 \right].$$

The computed standard answer for a depends on this choice of t , i.e., it is possibly invalid for other choices. Fixing for example $x = 4$, we see that $a = 3$ is not admissible anymore, because substituting these terms into $(x < a)$ yields “false.” To compute a standard term for a when $x = 4$, we have to start with

$$\left[\text{true} \mid x = 4 \quad a = \infty \right],$$

and apply Theorem 5 again.

5 Degree Shifts by Virtual Substitution

We have already discussed that the feasibility of the virtual substitution method strongly depends on the degrees of the quantified variables. Among the heuristics for decreasing these degrees there is an observation, which was essentially made already by Weispfenning (1997a), and which was refined and named *degree shift* by Dolzmann et al. (1998). The following lemma restates the result by Dolzmann et al.:

Lemma 6 (Degree Shift). Consider a quantifier-free Tarski formula ψ . Let g be the GCD of all exponents of x in ψ . We divide all exponents of x in ψ by g yielding ψ' . If g is odd, we have $\exists x \psi \longleftrightarrow \exists x \psi'$, if g is even we have $\exists x \psi \longleftrightarrow \exists x (x \geq 0 \wedge \psi')$. For $g > 1$ this reduces the degree of x in ψ . In order to obtain larger GCDs and hence a better degree reduction, we may in advance “adjust” the degree $n > 0$ of x in polynomials of the form $x^n f$, where x does not occur in f as follows: In equations and disequations, n may be equivalently replaced by any $m > 0$. In ordering inequalities we may choose any $m > 0$ of the same parity as n . \square

We now want to reanalyze this result as a special case of virtual substitution. For this, we have to slightly generalize the framework by introducing *shadow quantifiers*. Recall that we are considering existential problems of the form

$$\varphi(u_1, \dots, u_m) = \exists x_n \dots \exists x_1 \psi(x_1, \dots, x_n, u_1, \dots, u_m).$$

As a first step we now switch to the equivalent problem

$$\hat{\varphi}(u_1, \dots, u_m) = \exists \hat{x}_n \exists x_n \dots \exists \hat{x}_1 \exists x_1 \psi(x_1, \dots, x_n, u_1, \dots, u_m),$$

where $\{\hat{x}_1, \dots, \hat{x}_n\} \cap \{x_1, \dots, x_n, u_1, \dots, u_m\} = \emptyset$. That is, the variables \hat{x}_i do not occur in ψ . Consequently, proceeding with the elimination as discussed in Section 3, each shadow quantifier $\exists \hat{x}_i$ imposes a trivial elimination problem.

Strictly following the virtual substitution framework, one would not simply drop those quantifiers $\exists \hat{x}_i$ but eliminate them via trivial elimination sets like $\{(\text{true}, 0)\}$. Notice that one cannot use \emptyset as an elimination set here because $\bigvee \emptyset = \text{false}$. Furthermore, from the point of view of extended quantifier elimination the use of $\{(\text{true}, 0)\}$ formally provides answers also for $\hat{x}_1, \dots, \hat{x}_n$.

Consider now w.l.o.g. the elimination of $\exists x_1$, and assume that we are in the situation of Lemma 6, where $g > 1$ is the GCD of the, possibly adjusted, degrees of x_1 in ψ . We use an elimination set that depends on the parity of g :

$$E = \left\{ \left(\gamma, \sqrt[g]{\hat{x}_1} \right) \right\},$$

where γ is $x_1 \geq 0$ if g is even and “true” otherwise. Of course, we have to define a suitable virtual substitution of $\sqrt[g]{\hat{x}_1}$ for x_1 within ψ :

$$\left(\sum_{j=1}^k a_j x_1^j \varrho 0 \right) \left[x_1 // \sqrt[g]{\hat{x}_1} \right] = \left(\sum_{j=1}^k a_j \hat{x}_1^{\lfloor \frac{\max(j,g)}{g} \rfloor} \varrho 0 \right),$$

where $a_1, \dots, a_k \in \mathbb{Z}[x_2, \dots, x_n, u_1, \dots, u_m]$ and $\varrho \in \{=, \leq, <, \geq, >, \neq\}$. The floor function is applied to make the definition complete; with our elimination sets we will always have divisibility $g \mid \max(j, g)$. The max operator takes care of possible degree adjustments made for the computation of E .

Observe that in contrast to the elimination sets studied so far we introduce here a variable \hat{x}_1 which was not present in ψ before. That variable is bound by shadow quantifier $\exists \hat{x}_1$. Intuitively, for the elimination of $\exists \hat{x}_1 \exists x_1$ we switch from one hard plus one trivial elimination step to two nontrivial elimination steps.

The termination of quantifier elimination with shadow quantifiers follows from the termination of the underlying quantifier elimination method plus the fact that there are only finitely many shadow quantifiers for each regular quantifier.

To keep the notation simple, we will in the sequel not formally introduce shadow quantifiers for all quantifiers. Instead, our procedure will silently assume their presence whenever it performs a degree shift. In the corresponding pre-EQR this can be recognized by assignments of the form $x_i = \sqrt[g]{\hat{x}_i}$, which cannot come into existence otherwise.

6 Generalization and Extensions of the Method

In this section we generalize Theorem 5 to admit more general pre-EQRs as input. Furthermore, we discuss heuristics for obtaining rational numbers or even integers instead of root expressions in our standard answers.

Corollary 7 (Generalized Computation of Standard Answers). *Consider a closed Tarski formula $\varphi = \exists x_n \dots \exists x_1 \psi(x_1, \dots, x_n)$. Assume that the following is a pre-EQR for φ :*

$$\left[\begin{array}{c|ccc} \text{true} & x_1 = e_1 & \dots & x_n = e_n \\ & \gamma_1 & \dots & \gamma_n \end{array} \right]$$

such that each e_i is of one of the following forms:

- (a) $\frac{a+b\sqrt{c}}{d}$, where $a, b, c, d \in \mathbb{Z}[x_{i+1}, \dots, x_n]$,
- (b) $\frac{a+b\sqrt{c}}{d} \pm \varepsilon$, where $a, b, c, d \in \mathbb{Z}[x_{i+1}, \dots, x_n]$,
- (c) $\pm\infty$,
- (d) $\sqrt[g]{x_{i+1}}$, where $g \in \mathbb{N} \setminus \{0\}$,

where as usual “ \pm ” denotes “ $+$ ” or “ $-$.” Then we can compute root expressions $\tilde{e}_1, \dots, \tilde{e}_n$ each meeting either the specification (a) or the specification (d) above and $\tilde{\gamma}_1, \dots, \tilde{\gamma}_n$ such that the following is a pre-EQR for φ as well:

$$\left[\begin{array}{c|ccc} \text{true} & x_1 = \tilde{e}_1 & \dots & x_n = \tilde{e}_n \\ & \tilde{\gamma}_1 & \dots & \tilde{\gamma}_n \end{array} \right].$$

Proof. From a theoretical point of view, the treatment of $e_i = \frac{a+b\sqrt{c}}{d} - \varepsilon$ can be reduced to Theorem 5 via the introduction of artificial variables as we did for our example in (12) above. From a practical point of view, it is not hard to see how to algorithmically treat such expressions directly. Notice that then our corrected answer \tilde{e}_i will generally be a rational number. As already mentioned in the proof of Theorem 5(c), one might heuristically even find an integer. In both cases the possibly non-trivial guard γ_i has to be replaced by $\tilde{\gamma}_i = \text{true}$.

Next, the treatment of $\frac{a+b\sqrt{c}}{d} + \varepsilon$ and $-\infty$ in analogy to $\frac{a+b\sqrt{c}}{d} - \varepsilon$ and ∞ , respectively, is straightforward.

Finally, having obtained algebraic numbers for x_{i+1}, \dots, x_n , one can compute an algebraic number also for $\sqrt[g]{x_{i+1}}$ with $g \in \mathbb{N} \setminus \{0\}$. \square

The proofs for both Theorem 5 and Corollary 7 are constructive. Recall that the ordering of the variables within the given pre-EQR is such that quantifier elimination has taken place from the left to the right, while the construction of the standard answers proceeds from the right to the left.

Consider the computation of \tilde{e}_k for some e_k . Here, the quantifier elimination direction mentioned above has played an important role in our proofs: Although e_{k+1}, \dots, e_n have been replaced with $\tilde{e}_{k+1}, \dots, \tilde{e}_n$, the expression e_k is still valid. Taking that idea a bit further, we may replace e_k with any valid expression without affecting the validity of either e_1, \dots, e_{k-1} or $\tilde{e}_{k+1}, \dots, \tilde{e}_n$.

In fact, it is sometimes possible to convert a root expression e_k into a rational number or even an integer as follows: Before processing e_k , we check whether changing it to one of $e_k \pm \varepsilon$ yields a valid pre-EQR for φ as well. This can be done by means of the virtual substitution

$$\begin{aligned} & (\tilde{\gamma}_n \wedge \dots \wedge \tilde{\gamma}_{k+1} \wedge \gamma_k \wedge \dots \wedge \gamma_1 \wedge \psi) \\ & [x_1 // e_1] \dots [x_{k-1} // e_{k-1}] [x_k // e_k \pm \varepsilon] [x_{k+1} // \tilde{e}_{k+1}] \dots [x_n // \tilde{e}_n]. \end{aligned}$$

In the positive case, we process $e_k \pm \varepsilon$ instead of e_k . In terms of the proofs of Theorem 5 and Corollary 7 this leads to the cases (c) and (b), respectively, where we generally obtain a rational solutions \tilde{e}_k and heuristically even integers.

Finally, it is quite helpful in general to recognize rational numbers among all occurring real algebraic numbers. This holds in particular for the final $\alpha_n, \dots, \alpha_1$, as they correspond to the values of the back-substituted $\tilde{e}_n, \dots, \tilde{e}_1$, which may be complicated nested root expressions. For this one can use the following lemma.

Lemma 8 (Rational Algebraic Numbers). Consider a real algebraic number $\alpha = (a_n x^n + \dots + a_0,]l, u[)$, where $a_0, \dots, a_n \in \mathbb{Z}$, $a_0 > 0$, $l, u \in \mathbb{Q}$, $l > 0$. Assume furthermore that $] \frac{a_0}{u}, \frac{a_0}{l} [\cap \mathbb{Z} = \{z\}$. Then $\alpha \in \mathbb{Q}$ if and only if $\alpha = \frac{a_0}{z}$.

Proof. Let $\alpha \in \mathbb{Q}$. From $l > 0$ it follows that $\alpha > 0$, say $\alpha = \frac{p}{q}$, where $p, q \in \mathbb{Z}$, $p > 0$, $q > 0$. This admits the following factorization:

$$q \cdot \sum_{i=0}^n a_i x^i = (qx - p) \cdot \sum_{i=0}^{n-1} a_i x^i.$$

It follows that $p \mid a_0$, say $pp' = a_0$, and we obtain $\alpha = \frac{pp'}{qp'} = \frac{a_0}{qp'}$. On the other hand, $l < \frac{a_0}{qp'} < u$, which is equivalent to $\frac{a_0}{u} < qp' < \frac{a_0}{l}$, and it follows that $qp' = z$. Together $\alpha = \frac{a_0}{qp'} = \frac{a_0}{z}$. The converse implication is obvious. \square

The lemma can be straightforwardly generalized to arbitrary intervals $]l, u[$.

7 Implementation and Application Examples

We have implemented our method in Redlog, which is a part of the computer algebra system Reduce. Reduce is freely available under a modified BSD license.¹ Technically, our implementation is an extension of Redlog's extended quantifier elimination `rlqea` by a switch `rlqestdans`, which toggles the computation of standard answers.

In the following subsections we are going to revisit a number of applications of extended quantifier elimination that have been documented in the scientific literature. In each case we are going to briefly explain the underlying problem, recompute the solutions with nonstandard answers, and finally compute solutions with standard answers using our approach as described in this article.

Since Redlog is very actively developed and improved, and the considered applications date back up to more than 15 years, the nonstandard answers obtained here partly differ from those reported in the literature. Of course, in such cases both variants are correct.

All computations have been carried out with the CSL version of Reduce, revision 2465, using 4 GB RAM on a 2.4 GHz Intel Xeon E5-4640 running 64 bit Debian Linux 7.3.

7.1 Computational Geometry

Besides many standard problems from computational geometry, Sturm and Weispfenning (1997) consider in their Example 10 the reconstruction of a cuboid

¹<http://reduce-algebra.sourceforge.net>

wireframe from a photography taken from the origin along the x_3 -axis with a lens of focal length 5.

The answers obtained by extended quantifier elimination is going to describe vectors $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3 \in \mathbb{R}^3$ generating the cuboid together with a vector $\mathbf{v} \in \mathbb{R}^3$ describing its translation from the origin. The input formula, which contains in addition points $\mathbf{i} \in \mathbb{R}^2$ on the camera sensor, contains 15 quantifiers:

$$\exists \mathbf{e}_1 \exists \mathbf{e}_2 \exists \mathbf{e}_3 \exists \mathbf{v} \forall \mathbf{i} ((\iota' \leftarrow \pi_0) \wedge \exists k (59k\mathbf{v} = (100, 200, 295k + 295))).$$

The formula $\iota'(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{v}, \mathbf{i})$, which has been obtained by regular quantifier elimination earlier, generically describes that a point \mathbf{i} lies in the image of a cuboid generated by $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$, and translated by \mathbf{v} . The formula $\pi_0(\mathbf{i})$ is a quantifier-free description of one concrete image. The remaining part of the input formula fixes $\mathbf{i} = (\frac{100}{59}, \frac{200}{59})$ to be the image of the origin of the cuboid.

Extended quantifier elimination yields “true” if and only if π_0 is a picture of a cuboid at all. In the positive case, the answers will provide suitable vectors $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$, and \mathbf{v} .

For π_0 as considered by Sturm and Weispfenning (1997) in Example 10, the extended quantifier elimination yields “true” together with the following nonstandard answers:

$$\begin{aligned} \mathbf{e}_1 &= \left(5\infty_1, \frac{7\infty_1}{2}, \frac{5\infty_1}{2} \right), & \mathbf{e}_2 &= \left(\infty_1, 2\infty_1, -\frac{24\infty_1}{5} \right), \\ \mathbf{e}_3 &= \left(-\frac{109\infty_1}{65}, \frac{53\infty_1}{26}, \frac{\infty_1}{2} \right), & \mathbf{v} &= \left(5\infty_1, 10\infty_1, \frac{59\infty_1 + 20}{4} \right). \end{aligned}$$

Our method fixes $\infty_1 = 1$, which yields the following standard answers:

$$\begin{aligned} \mathbf{e}_1 &= \left(5, \frac{7}{2}, \frac{5}{2} \right), & \mathbf{e}_2 &= \left(1, 2, -\frac{24}{5} \right), \\ \mathbf{e}_3 &= \left(-\frac{109}{65}, \frac{53}{26}, \frac{1}{2} \right), & \mathbf{v} &= \left(5, 10, \frac{79}{4} \right). \end{aligned}$$

The entire computation takes 189s, of which the computation of the standard answers takes less than 1 ms.

7.2 Motion Planning

Weispfenning (2001) has studied motion planning problems in dimension two. Both the object to be moved and the free space between given obstacles are semilinear sets. Extended quantifier elimination is used to decide whether a geometrical object can be moved from an initial to a final destination in at most n moves, where the trajectory of each move is a line segment. In the positive case, the answers describe the coordinates $\mathbf{u}_1, \dots, \mathbf{u}_n \in \mathbb{R}^2$ of the object after each of the n moves. Accordingly, the input formulas contain $2n$ variables in the prenex existential block.

We have applied our answer correction to three of the examples discussed by Weispfenning (2001). For the concrete input formulas and pictures of the scenery we refer to that publication.

Example	\mathbf{u}_1	\mathbf{u}_2	\mathbf{u}_3	time
6.4	$(5 - \varepsilon_1, 5 - \varepsilon_1)$	$(5 - \varepsilon_1, \frac{-2\varepsilon_1+1}{2})$	$(9, \frac{9}{2})$	0.06 s
6.8	$(0, 3 + \varepsilon_1)$	$(3 - \varepsilon_1, 6)$	$(7, 6)$	9.5 s
6.9	$(0, 3 + \varepsilon_1)$	$(3 - \varepsilon_1, 6)$		0.28 s

Table 1: Summary of nonstandard answers and computation times for motion planning examples considered by Weispfenning (2001).

Example	ε_1	\mathbf{u}_1	\mathbf{u}_2	\mathbf{u}_3
6.4	$\frac{3}{16}$	$(\frac{77}{16}, \frac{77}{16})$	$(\frac{77}{16}, \frac{5}{16})$	$(9, \frac{9}{2})$
6.8	1	$(0, 4)$	$(2, 6)$	$(7, 6)$
6.9	1	$(0, 4)$	$(2, 6)$	

Table 2: Summary of standard answers for motion planning examples considered by Weispfenning (2001). In all cases the time spent for the computation of ε_1 was less than 1 ms.

For Example 6.4, we obtain the following nonstandard answers:

$$\mathbf{u}_1 = (5 - \varepsilon_1, 5 - \varepsilon_1), \quad \mathbf{u}_2 = \left(5 - \varepsilon_1, \frac{-2\varepsilon_1 + 1}{2}\right), \quad \mathbf{u}_3 = \left(9, \frac{9}{2}\right).$$

Our method fixes $\varepsilon_1 = \frac{3}{16}$, which yields the following standard answers:

$$\mathbf{u}_1 = \left(\frac{77}{16}, \frac{77}{16}\right), \quad \mathbf{u}_2 = \left(\frac{77}{16}, \frac{5}{16}\right), \quad \mathbf{u}_3 = \left(9, \frac{9}{2}\right).$$

The entire computation takes 60 ms, of which the computation of the standard answers takes less than 1 ms. Tables 1 and 2 summarize these results along with the two other examples.

7.3 Models of Genetic Circuits

Recently, symbolic methods for the identification of Hopf bifurcations in vector fields arising from biological networks or chemical reaction networks have received considerable attention in the literature (Sturm and Weber, 2008; Sturm et al., 2009; Errami et al., 2011; Weber et al., 2011; Errami et al., 2013). Given a polynomial vector field, El Kahoui and Weber (2000) introduced a method, which automatically generates first-order Tarski formulas describing the existence of a Hopf bifurcation in terms of the parameters. Then real quantifier elimination is applied to obtain corresponding necessary and sufficient conditions. For efficiency reasons, one often existentially quantifies all parameters and applies extended quantifier elimination. In the positive case, the answers provide one set of parameter values giving rise to a Hopf bifurcation.

Based on models introduced by Boulier et al. (2007), Sturm and Weber (2008) and Sturm et al. (2009) used the approach sketched above to automatically derive the existence of Hopf bifurcations for the gene regulatory network

controlling the circadian clock of a certain unicellular green alga. The input formula is

$$\begin{aligned} \exists!(0 < \mathbf{v}_1 \wedge 0 < \mathbf{v}_3 \wedge 0 < \mathbf{v}_2 \wedge 0 < \vartheta \wedge 0 < \gamma_0 \wedge 0 < \mu \wedge 0 < \delta \wedge 0 < \alpha \\ \wedge \vartheta \cdot (\gamma_0 - \mathbf{v}_1 - \mathbf{v}_1 \mathbf{v}_3^9) = 0 \wedge \lambda_1 \mathbf{v}_1 + \gamma_0 \mu - \mathbf{v}_2 = 0 \\ \wedge 9\alpha(\gamma_0 - \mathbf{v}_1 - \mathbf{v}_1 \mathbf{v}_3^9) + \delta(\mathbf{v}_2 - \mathbf{v}_3) = 0 \wedge 0 < \vartheta \delta + \vartheta \mathbf{v}_3^9 \delta + 9\lambda_1 \vartheta \mathbf{v}_1 \mathbf{v}_3^8 \delta \\ \wedge 162\vartheta \mathbf{v}_3^{17} \alpha \mathbf{v}_1 + 162\vartheta \alpha \mathbf{v}_1 \mathbf{v}_3^8 + 162\alpha \mathbf{v}_1 \mathbf{v}_3^8 \delta + \vartheta + 2\vartheta \mathbf{v}_3^9 \delta + \vartheta^2 \mathbf{v}_3^{18} \delta \\ + \vartheta \mathbf{v}_3^9 + 2\vartheta \delta + 81\alpha \mathbf{v}_1 \mathbf{v}_3^8 \vartheta \delta + 81\alpha \mathbf{v}_1 \mathbf{v}_3^{17} \vartheta \delta + \delta^2 + \vartheta \delta^2 + \vartheta^2 \delta + \vartheta^2 \\ + 2\vartheta^2 \mathbf{v}_3^9 + \vartheta^2 \mathbf{v}_3^{18} + 6561\alpha^2 \mathbf{v}_1^2 \mathbf{v}_3^{16} + 2\vartheta^2 \mathbf{v}_3^9 \delta + \delta + 81\alpha \mathbf{v}_1 \mathbf{v}_3^8 + \vartheta \mathbf{v}_3^9 \delta^2 \\ - 9\lambda_1 \vartheta \mathbf{v}_1 \mathbf{v}_3^8 \delta = 0), \end{aligned}$$

for which we obtain the following nonstandard answers:

$$\begin{aligned} \gamma_0 &= \frac{8\sqrt[9]{\infty_2^{-19}} \infty_3 + 16\sqrt[9]{\infty_2^{-10}} \infty_3 + 8\sqrt[9]{\infty_2} \infty_3}{729\infty_1^2 \infty_2^3 + 1458\infty_1^2 \infty_2^2 + 729\infty_1^2 \infty_2 - 486\infty_1 \infty_2^2 \infty_3 - 486\infty_1 \infty_2 \infty_3 + 9\infty_2 \infty_3^2}, \\ \mu &= \frac{729\infty_1^2 \infty_2^3 + 1458\infty_1^2 \infty_2^2 + 729\infty_1^2 \infty_2 - 486\infty_1 \infty_2^2 \infty_3 - 486\infty_1 \infty_2 \infty_3 + \infty_2 \infty_3^2 - 8\infty_3^2}{8\infty_2^2 \infty_3 + 16\infty_2 \infty_3 + 8\infty_3}, \\ \vartheta &= \frac{-6561\infty_1^4 \infty_2^4 - 26244\infty_1^4 \infty_2^3 - 39366\infty_1^4 \infty_2^2 - 26244\infty_1^4 \infty_2 - 6561\infty_1^4 + 4374\infty_1^3 \infty_2^3 \infty_3 \\ + 13122\infty_1^3 \infty_2^2 \infty_3 + 13122\infty_1^3 \infty_2 \infty_3 + 4374\infty_1^3 \infty_3 - 54\infty_1 \infty_2 \infty_3^3 - 54\infty_1 \infty_3^3 + \infty_3^4}{6561\infty_1^4 \infty_2^5 + 32805\infty_1^4 \infty_2^4 + 65610\infty_1^4 \infty_2^3 + 65610\infty_1^4 \infty_2^2 + 32805\infty_1^4 \infty_2 + 6561\infty_1^4 \\ - 8748\infty_1^3 \infty_2^4 \infty_3 - 34992\infty_1^3 \infty_2^3 \infty_3 - 52488\infty_1^3 \infty_2^2 \infty_3 - 34992\infty_1^3 \infty_2 \infty_3 - 8748\infty_1^3 \infty_3 \\ + 3078\infty_1^2 \infty_2^3 \infty_3 + 9234\infty_1^2 \infty_2^2 \infty_3 + 9234\infty_1^2 \infty_2 \infty_3 + 3078\infty_1^2 \infty_3 - 108\infty_1 \infty_2^2 \infty_3^3 \\ - 216\infty_1 \infty_2 \infty_3^3 - 108\infty_1 \infty_3^3 + \infty_2 \infty_3^4 + \infty_3^4}, \\ \mathbf{v}_1 &= \frac{8\sqrt[9]{\infty_2^{-10}} \infty_3 + 8\sqrt[9]{\infty_2} \infty_3}{729\infty_1^2 \infty_2^3 + 1458\infty_1^2 \infty_2^2 + 729\infty_1^2 \infty_2 - 486\infty_1 \infty_2^2 \infty_3 - 486\infty_1 \infty_2 \infty_3 + 9\infty_2 \infty_3^2}, \\ \mathbf{v}_2 &= \sqrt[9]{\infty_2}, \quad \mathbf{v}_3 = \sqrt[9]{\infty_2}, \quad \alpha = \infty_1, \quad \delta = 1, \quad \lambda_1 = \infty_3. \end{aligned}$$

Our method fixes $\infty_1 = 1$, $\infty_2 = 9$, and $\infty_3 = 87481$, which yields the following standard solution:

$$\begin{aligned} \gamma_0 &= \frac{69984800 \cdot \sqrt[9]{9}}{616061191401}, & \mu &= \frac{3827162521}{69984800}, & \vartheta &= \frac{7652917261}{76056937210}, \\ \mathbf{v}_1 &= \frac{6998480 \cdot \sqrt[9]{9}}{616061191401}, & \mathbf{v}_2 &= \sqrt[9]{9}, & \mathbf{v}_3 &= \sqrt[9]{9}, \\ \alpha &= 1, & \delta &= 1, & \lambda_1 &= 87481. \end{aligned}$$

The entire computation takes 370 ms, of which the computation of the standard answers takes 140 ms.

7.4 Mass Action Systems

We are now going to discuss another example about Hopf bifurcation. This time, the considered system is a chemical reaction system, viz. the famous and well-studied phosphofructokinase reaction. It has been firstly analyzed with symbolic methods by Gatermann et al. (2005, Example 2.1). We adopt here the first-order formulation discussed by Sturm and Weber (2008) and Sturm et al. (2009) following the approach sketched in the previous subsection.

We obtain nonstandard answers of the following form:

$$\begin{aligned}
k_{21} &= K_{21}(\infty_1, \infty_2, \infty_3, \infty_4, \varepsilon_1), & k_{34} &= \infty_1, & k_{43} &= \infty_2, \\
k_{46} &= K_{46}(\infty_1, \infty_2, \infty_3, \infty_4, \infty_5, \varepsilon_1), & k_{64} &= \infty_5, & k_{65} &= \infty_3, \\
k_{56} &= K_{56}(\infty_1, \infty_2, \infty_3, \infty_4, \infty_5, \varepsilon_1), & \mathbf{v}_1 &= \frac{\infty_2 \infty_4}{\infty_1}, \\
\mathbf{v}_2 &= V_2(\infty_1, \infty_2, \infty_3, \infty_4, \infty_5, \varepsilon_1), & \mathbf{v}_3 &= \infty_4.
\end{aligned}$$

The nonstandard terms $K_{21}, \dots, K_{56}, V_2$ are so large that we cannot explicitly display them here. To give an idea, K_{46} would fill more than 16 pages in this document.

Our method fixes $\infty_1 = \infty_2 = \infty_3 = \infty_4 = 1$, $\infty_5 = 20$, and $\varepsilon_1 = 2(\sqrt{2} - 1)$, which yields the following standard solution:

$$\begin{aligned}
k_{21} &= 3, & k_{34} &= 1, & k_{43} &= 1, \\
k_{46} &= \frac{\sqrt{3457} + 1}{8}, & k_{64} &= 20, & k_{65} &= 1, \\
k_{56} &= \frac{-\sqrt{3457} + 159}{6}, & \mathbf{v}_1 &= 1, \\
\mathbf{v}_2 &= \frac{-\sqrt{3457} + 159}{24}, & \mathbf{v}_3 &= 1.
\end{aligned}$$

The entire computation takes 13.2s, of which the computation of the standard answers takes 0.1s.

7.5 Sizing of Electrical Networks

Sturm (1999a, Section 5) has applied generic quantifier elimination to the sizing of a BJT amplifier. Description of a circuit is given as a set of operating point equations E_1 and a set of AC conditions E_2 . For the concrete equations we refer to the mentioned publication. The system $E_1 \wedge E_2$ has to be solved w.r.t. the main variables $M = \{r_1, \dots, r_8, c_3\}$ in terms of parameter variables $P = \{v_{cc}, a_{\text{high}}, a_{\text{low}}, p, z_{\text{in}}, z_{\text{out}}\}$. Fixing values of the parameters to

$$v_{cc} = 3, \quad a_{\text{high}} = 3, \quad a_{\text{low}} = 2, \quad p = 12, \quad z_{\text{in}} = 5, \quad z_{\text{out}} = 5,$$

the answers contain one nonstandard term:

$$\begin{aligned}
r_1 &= \frac{4457058395}{5180672}, & r_2 &= \frac{4457058395}{2590336}, & r_3 &= -\frac{4457058395}{1295168}, \\
r_4 &= -\frac{4182864929375836679}{128066211840000}, & r_5 &= \infty_1, & r_6 &= \frac{282999424999}{804520000}, \\
r_7 &= 5, & r_8 &= \frac{25509595605337086755}{20836792295619328}, & c_3 &= \frac{647584}{13371175185}.
\end{aligned}$$

Our method fixes $\infty_1 = 1$, which yields a standard answer for r_5 . The entire computation takes less than 2ms, of which the computation of the standard answer takes less than 1ms.

7.6 A Linear Feasibility Example

Korovin et al. (2009, Section 9) have considered a small linear existential problem to demonstrate the difference between their conflict resolution method and the Fourier–Motzkin elimination method. The following are nonstandard answers for that problem computed by Redlog:

$$\begin{aligned}x_1 &= -\frac{8}{13}, & x_2 &= \frac{1 - 65\varepsilon_1}{65}, & x_3 &= \frac{-14 + 13\varepsilon_2}{13}, \\x_4 &= \frac{-302 - 195\varepsilon_1 + 65\varepsilon_2}{130}, & x_5 &= \frac{-30 + 26\varepsilon_2}{39}.\end{aligned}$$

Our method fixes $\varepsilon_1 = \frac{1}{65}$ and $\varepsilon_2 = \frac{1}{13}$, which yields the following standard answers:

$$x_1 = -\frac{8}{13}, \quad x_2 = 0, \quad x_3 = -1, \quad x_4 = -\frac{30}{13}, \quad x_5 = -\frac{28}{39}.$$

The entire computation takes 3 ms, of which the computation of the standard answers takes less than 1 ms.

8 Conclusions and Future Work

We have introduced extended quantifier elimination as a general concept, and focused on virtual substitution as one possible method for its realization. Successful applications of extended quantifier elimination via virtual substitution have been documented in the literature over the past two decades. One problem there was that the answers obtained via virtual substitution in general contain nonstandard symbols, which can be hard to interpret. For fixed parameters the present work resolves this issue by providing a complete post-processing method for fixing all answers to standard real numbers. We have implemented our method, and applied it to various extended quantifier elimination problems from the literature. In these experiments we have generally obtained standard answers that are meaningful in terms of the modeled problems. In most cases our post-processing method does not significantly contribute to the overall computation time. It is noteworthy that our method is compatible with our recent work on combining virtual substitution with learning techniques (Korovin et al., 2014).

Recall from our discussion in Section 6 on finding integer and rational answers that there is often a considerable degree of freedom in the choice of standard answers. In future this can be further exploited in many interesting ways: For instance, using extended quantifier elimination methods as a theory solver in the context of Satisfiability Modulo Theory (SMT) solving (Nieuwenhuis et al., 2006), in particular when combining several theories in a Nelson–Oppen (1979) style, one is specifically interested in avoiding identical answers for different variables. Alternatively, one can try to identify certain variables, which might be interesting in certain contexts. As another option, there is only a small step to automatically generating for a given pre-EQR code for an interactive procedure that suggests ranges and finds possible choices for certain variables in cooperation with the user. In cooperation projects with researchers from the sciences we

have had the experience that those researchers often have a surprisingly precise idea about reasonable choices for certain variables.

A theoretically way more challenging step would be the generalization of our method to the parametric case. Recall that Proposition 1 has shown that it is not possible in general to determine real standard values for infinitesimals and infinities without fixing values for the parameters beforehand. Nevertheless, it might well be possible to devise on the basis of our work here a complete method for *symbolically* replacing nonstandard symbols with standard terms. In the example in Proposition 1 the infinitesimal ε_1 could be replaced e.g. with $\frac{1-a}{2}$ yielding a standard extended quantifier elimination result.

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