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A solution to the non-linear equations of D=10 super Yang–Mills theory

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In this letter, we present a formal solution to the non-linear field equations of ten-dimensional super Yang–Mills theory. It is assembled from products of linearized superfields which have been introduced as multiparticle superfields in the context of superstring perturbation theory. Their explicit form follows recursively from the conformal field theory description of the gluon multiplet in the pure spinor superstring. Furthermore, superfields of higher mass dimensions are defined and their equations of motion spelled out.

INTRODUCTION

Super Yang–Mills (SYM) theory in ten dimensions can be regarded as one of the simplest SYM theories, its spectrum contains just the gluon and gluino, related by sixteen supercharges. However, it is well-known that its dimensional reduction gives rise to various maximally supersymmetric Yang–Mills theories in lower dimensions, including the celebrated $\mathcal{N}=4$ theory in $D=4$ [1]. Therefore a better understanding of this theory propagates a variety of applications to any dimension $D \leq 10$.

In a recent line of research [2, 3], scattering amplitudes of ten-dimensional SYM have been determined and simplified using so-called multiparticle superfields [4]. They represent entire tree-level subdiagrams and build up in the conformal field theory (CFT) on the worldsheet of the pure spinor superstring [5] via operator product expansions (OPEs). Multiparticle superfields satisfy the linearized field equations up to inverse off-shell propagators. In this letter we demonstrate that these off-shell modifications can be resummed to capture the non-linearities in the SYM equations of motion. The generating series of multiparticle superfields is shown to solve the non-linear field equations.

We also define superfields of arbitrary mass dimension and reduce their non-linear expressions to the linearized superfields of lower mass dimensions. This framework simplifies the expressions of kinematic factors in higher-loop scattering amplitudes, including the D^6R^4 operator in the superstring three-loop amplitude [6].

REVIEW OF TEN-DIMENSIONAL SYM

The equations of motion of ten-dimensional SYM theory can be described covariantly in superspace by defining supercovariant derivatives [7, 8]

$$\nabla_\alpha \equiv D_\alpha - \mathbb{A}_\alpha(x,\theta), \quad \nabla_m \equiv \partial_m - \mathbb{A}_m(x,\theta). \qquad (1)$$

The connections $\mathbb{A}_\alpha$ and $\mathbb{A}_m$ take values in the Lie algebra associated with the Yang–Mills gauge group. The derivatives are taken with respect to ten-dimensional superspace coordinates (x^m, θ^α) with vector and spinor indices $m,n=0,\ldots,9$ and $\alpha,\beta=1,\ldots,16$ of the Lorentz group. The fermionic covariant derivatives

$$D_\alpha \equiv \partial_\alpha + \tfrac{1}{2}(\gamma^m\theta)_\alpha, \quad \{D_\alpha, D_\beta\} = \gamma^m_{\alpha\beta}\partial_m \qquad (2)$$

involve the 16×16 Pauli matrices $\gamma^m_{\alpha\beta}=\gamma^m_{\beta\alpha}$ subject to the Clifford algebra $\gamma^{(m}_{\alpha\beta}\gamma^{n)\beta\gamma} = 2\eta^{mn}\delta^\gamma_\alpha$, and the convention for (anti)symmetrizing indices does not include $\frac{1}{2}$.

The connections in (1) give rise to field-strengths

$$\mathbb{F}_{\alpha\beta} \equiv \{\nabla_\alpha,\nabla_\beta\} - \gamma^m_{\alpha\beta}\nabla_m, \quad \mathbb{F}_{mn} \equiv -[\nabla_m,\nabla_n]. \qquad (3)$$

One can show that the constraint equation $\mathbb{F}_{\alpha\beta}=0$ puts the superfields on-shell, and Bianchi identities lead to the non-linear equations of motion [8],

$$\begin{aligned}
\{\nabla_\alpha,\nabla_\beta\} &= \gamma^m_{\alpha\beta}\nabla_m\\
[\nabla_\alpha,\nabla_m] &= -(\gamma_m\mathbb{W})_\alpha\\
\{\nabla_\alpha,\mathbb{W}^\beta\} &= \tfrac{1}{4}(\gamma^{mn})_\alpha{}^\beta\mathbb{F}_{mn}\\
[\nabla_\alpha,\mathbb{F}^{mn}] &= [\nabla^{[m},(\gamma^{n]}\mathbb{W})]\ .
\end{aligned} \qquad (4)$$

In the subsequent, we will construct an explicit solution for the superfields $\mathbb{A}_\alpha$, $\mathbb{A}_m$, $\mathbb{W}^\alpha$ and $\mathbb{F}^{mn}$ in (4).

LINEARIZED MULTIPARTICLE SUPERFIELDS

In perturbation theory, it is conventional to study solutions $A_\alpha, A_m,\ldots$ of the *linearized* equations of motion

$$\begin{aligned}
\{D_{(\alpha},A_{\beta)}\} &= \gamma^m_{\alpha\beta}A_m\\
[D_\alpha,A_m] &= k_mA_\alpha + (\gamma_mW)_\alpha\\
\{D_\alpha,W^\beta\} &= \tfrac{1}{4}(\gamma^{mn})_\alpha{}^\beta F_{mn}\\
[D_\alpha,F^{mn}] &= k^{[m}(\gamma^{n]}W)_\alpha\ .
\end{aligned} \qquad (5)$$

Their dependence on the bosonic coordinates x is described by plane waves $e^{k\cdot x}$ with on-shell momentum $k^2=0$. The θ dependence is known in terms of fermionic power series expansions from [9, 10] whose coefficients contain gluon polarizations and gluino wave functions.

As an efficient tool to determine and compactly represent scattering amplitudes in SYM and string theory, multiparticle versions of the linearized superfields have been constructed in [4]. They satisfy systematic modifications of the linearized equations of motion (5), and their significance for BRST invariance was pointed out in [11]. For example, their two-particle version

$$\begin{aligned}
A^{12}_\alpha &\equiv -\tfrac{1}{2}\big[A^1_\alpha(k^1\cdot A^2)+A^1_m(\gamma^m W^2)_\alpha-(1\leftrightarrow 2)\big]\\
A^{12}_m &\equiv \tfrac{1}{2}\big[A^p_1 F^2_{pm}-A^1_m(k^1\cdot A^2)+(W^1\gamma_m W^2)-(1\leftrightarrow 2)\big]\\
W^\alpha_{12} &\equiv \tfrac{1}{4}(\gamma^{mn}W_2)^\alpha F^1_{mn}+W^\alpha_2(k^2\cdot A^1)-(1\leftrightarrow 2) \qquad (6)\\
F^{12}_{mn} &\equiv F^2_{mn}(k^2\cdot A^1)+\tfrac{1}{2}F^2_{[m}{}^p F^1_{n]p}\\
&\quad + k^1_{[m}(W^1\gamma_{n]}W^2)-(1\leftrightarrow 2)\,,
\end{aligned}$$

can be checked via (5) to satisfy

$$\begin{aligned}
D_{(\alpha}A^{12}_{\beta)} &= \gamma^m_{\alpha\beta}A^{12}_m+(k^1\cdot k^2)(A^1_\alpha A^2_\beta+A^1_\beta A^2_\alpha) \qquad (7)\\
D_\alpha A^{12}_m &= (\gamma_m W^{12})_\alpha+k^{12}_m A^{12}_\alpha+(k^1\cdot k^2)(A^1_\alpha A^2_m-A^2_\alpha A^1_m)\\
D_\alpha W^\beta_{12} &= \tfrac{1}{4}(\gamma^{mn})_\alpha{}^\beta F^{12}_{mn}+(k^1\cdot k^2)(A^1_\alpha W^\beta_2-A^2_\alpha W^\beta_1)\\
D_\alpha F^{12}_{mn} &= k^{12}_m(\gamma_n W^{12})_\alpha-k^{12}_n(\gamma_m W^{12})_\alpha\\
&\quad +(k^1\cdot k^2)\big(A^1_\alpha F^2_{mn}+A^1_{[n}(\gamma_{m]}W^2)_\alpha-(1\leftrightarrow 2)\big)\,.
\end{aligned}$$

The modifications as compared to the single-particle equations of motion (5) involve the overall momentum $k_{12}\equiv k_1+k_2$ whose inverse propagator is generically off-shell, $k^2_{12}=2(k_1\cdot k_2)\neq 0$.

The construction of the two-particle superfields in (6) is guided by the OPEs among integrated vertex operators of the gluon multiplet in the pure spinor formalism [5],

$$U^i\equiv \partial\theta^\alpha A^i_\alpha+\Pi^m A^i_m+d_\alpha W^\alpha_i+\tfrac{1}{2}N^{mn}F^i_{mn}\,. \qquad (8)$$

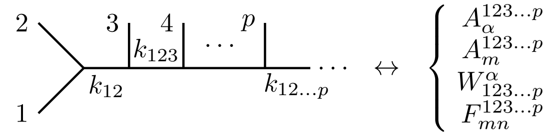
Worldsheet fields $[\partial\theta^\alpha,\Pi^m,d_\alpha,N^{mn}]$ with conformal weight one and well-known OPEs are combined with linearized superfields associated with particle label i. The multiplicity-two superfields in (6) are obtained from the coefficients of the conformal fields in the OPE [4]

$$\begin{aligned}
U^{12} &\equiv -\oint (z_1-z_2)^{\alpha' k^1\cdot k^2}U^1(z_1)U^2(z_2) \qquad (9)\\
&= \partial\theta^\alpha A^{12}_\alpha+\Pi^m A^{12}_m+d_\alpha W^\alpha_{12}+\tfrac{1}{2}N^{mn}F^{12}_{mn}\,,
\end{aligned}$$

where α' denotes the inverse string tension, and total derivatives in the worldsheet variables z_1,z_2 have been discarded in the second line. The CFT-inspired two-particle prescription (6) can be promoted to a recursion leading to superfields of arbitrary multiplicity whose equations of motion generalize along the lines of

$$\begin{aligned}
\{D_{(\alpha},A^{123}_{\beta)}\} &= \gamma^m_{\alpha\beta}A^{123}_m+(k^{12}\cdot k^3)\big[A^{12}_\alpha A^3_\beta-(12\leftrightarrow 3)\big]\\
&\quad +(k^1\cdot k^2)\big[A^1_\alpha A^{23}_\beta+A^{13}_\alpha A^2_\beta-(1\leftrightarrow 2)\big] \qquad (10)
\end{aligned}$$

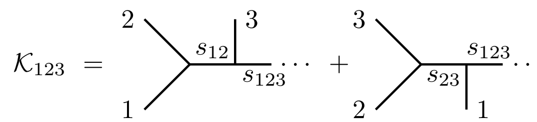
for suitable definitions of A^{123}_α and A^{123}_m [4]. The BCJ symmetries [12, 13] of the multiparticle superfields as well as the momenta $k_{12\ldots j}\equiv k_1+k_2+\ldots+k_j$ in their equations of motion suggest to associate them with tree-level subdiagrams shown in the subsequent figure [4]:



Berends–Giele currents: As a convenient basis of multiparticle fields $K_B\in\{A^B_\alpha,A^m_B,W^\alpha_B,F^{mn}_B\}$ with multiparticle label $B=12\ldots p$, we define Berends–Giele currents $\mathcal{K}_B\in\{\mathcal{A}^B_\alpha,\mathcal{A}^m_B,\mathcal{W}^\alpha_B,\mathcal{F}^{mn}_B\}$, e.g. $\mathcal{K}_1\equiv K_1$ and [4]

$$\mathcal{K}_{12}\equiv\frac{K_{12}}{s_{12}}\,,\quad \mathcal{K}_{123}\equiv\frac{K_{123}}{s_{12}s_{123}}+\frac{K_{321}}{s_{23}s_{123}} \qquad (11)$$

with generalized Mandelstam invariants $s_{12\ldots p}\equiv\frac{1}{2}k^2_{12\ldots p}$. Berends–Giele currents $\mathcal{K}_B$ are defined to encompass all propagator-dressed tree subdiagrams compatible with the ordering of the external legs in B. As shown in the following figure, the three-particle current in (11) is assembled from the s- and t-channels of a four-point amplitude with an off-shell leg (represented by $\ldots$):



A closed formula at arbitrary multiplicity [4, 14] involves the inverse of the momentum kernel $S[\cdot|\cdot]_1$ [15],

$$\mathcal{K}_{1\sigma(23\ldots p)}\equiv\sum_{\rho\in S_{p-1}}S^{-1}[\sigma|\rho]_1\,K_{1\rho(23\ldots p)}\,, \qquad (12)$$

with permutation $\sigma\in S_{p-1}$.

The combination of color-ordered trees as in (11) and (12) simplifies their equations of motion [4]

$$\begin{aligned}
\{D_{(\alpha},\mathcal{A}^B_{\beta)}\} &= \gamma^m_{\alpha\beta}\mathcal{A}^B_m+\sum_{XY=B}\big(\mathcal{A}^X_\alpha\mathcal{A}^Y_\beta-\mathcal{A}^Y_\alpha\mathcal{A}^X_\beta\big) \qquad (13)\\
[D_\alpha,\mathcal{A}^B_m] &= k^B_m\mathcal{A}^B_\alpha+(\gamma_m\mathcal{W}_B)_\alpha+\sum_{XY=B}\big(\mathcal{A}^X_\alpha\mathcal{A}^Y_m-\mathcal{A}^Y_\alpha\mathcal{A}^X_m\big)\\
\{D_\alpha,\mathcal{W}^\beta_B\} &= \tfrac{1}{4}(\gamma^{mn})_\alpha{}^\beta\mathcal{F}^B_{mn}+\sum_{XY=B}\big(\mathcal{A}^X_\alpha\mathcal{W}^\beta_Y-\mathcal{A}^Y_\alpha\mathcal{W}^\beta_X\big)\\
[D_\alpha,\mathcal{F}^{mn}_B] &= k^{[m}_B(\gamma^{n]}\mathcal{W}_B)+\sum_{XY=B}\big(\mathcal{A}^X_\alpha\mathcal{F}^{mn}_Y-\mathcal{A}^Y_\alpha\mathcal{F}^{mn}_X\big)\\
&\quad +\sum_{XY=B}\big(\mathcal{A}^{[n}_X(\gamma^{m]}\mathcal{W}_Y)_\alpha-\mathcal{A}^{[n}_Y(\gamma^{m]}\mathcal{W}_X)_\alpha\big)\,.
\end{aligned}$$

Momenta $k_B\equiv k_1+k_2+\ldots k_p$ are associated with multiparticle labels $B=12\ldots p$, and $\sum_{XY=B}$ instructs to sum over all their deconcatenations into $X=12\ldots j$ and $Y=j+1\ldots p$ with $1\leq j\leq p-1$. For example, the three-particle equation of motion of $\mathcal{A}^{123}_\alpha$ reads

$$\begin{aligned}
\{D_{(\alpha},\mathcal{A}^{123}_{\beta)}\} &= \gamma^m_{\alpha\beta}\mathcal{A}^{123}_m \qquad (14)\\
&\quad +\mathcal{A}^1_\alpha\mathcal{A}^{23}_\beta+\mathcal{A}^{12}_\alpha\mathcal{A}^3_\beta-\mathcal{A}^{23}_\alpha\mathcal{A}^1_\beta-\mathcal{A}^3_\alpha\mathcal{A}^{12}_\beta\,,
\end{aligned}$$

and a comparison with (10) highlights the advantages of the diagram expansions in (11).

The symmetry properties of the $\mathcal{K}_B$ can be inferred from their cubic-graph expansion and summarized as

$$\mathcal{K}_{A\sqcup\!\sqcup B} = 0, \quad \forall A, B \neq \emptyset \ , \tag{15}$$

where $\sqcup\!\sqcup$ denotes the shuffle product [16]. For example,

$$0 = \mathcal{K}_{12}+\mathcal{K}_{21} = \mathcal{K}_{123}-\mathcal{K}_{321} = \mathcal{K}_{123}+\mathcal{K}_{231}+\mathcal{K}_{312}\,. \tag{16}$$

GENERATING SERIES OF SYM SUPERFIELDS

In order to connect multiparticle fields and Berends-Giele currents with the non-linear field equations (4), we define generating series $\mathbb{K} \in \{\mathbb{A}_\alpha, \mathbb{A}^m, \mathbb{W}^\alpha, \mathbb{F}^{mn}\}$

$$\begin{aligned}\mathbb{K} &\equiv \sum_i \mathcal{K}_i t^i + \sum_{i,j} \mathcal{K}_{ij} t^i t^j + \sum_{i,j,k} \mathcal{K}_{ijk} t^i t^j t^k + \ldots \\ &= \sum_i \mathcal{K}_i t^i + \tfrac{1}{2}\sum_{i,j} \mathcal{K}_{ij}[t^i,t^j] + \tfrac{1}{3}\sum_{i,j,k}\mathcal{K}_{ijk}[[t^i,t^j],t^k] + \cdots\end{aligned} \tag{17}$$

where t^i denote generators in the Lie algebra of the non-abelian gauge group. The second line follows from the symmetry (15), which guarantees that $\mathbb{K}$ is a Lie element [16]. As a key virtue of the generating series (17), they allow to rewrite (13) as non-linear equations of motion (where $[\partial^m, \mathbb{K}]$ translates into components $k^m_B \mathcal{K}_B$)

$$\begin{aligned}\big\{D_{(\alpha}, \mathbb{A}_{\beta)}\big\} &= \gamma^m_{\alpha\beta}\mathbb{A}_m + \{\mathbb{A}_\alpha, \mathbb{A}_\beta\} \\ \big[D_\alpha, \mathbb{A}_m\big] &= \big[\partial_m, \mathbb{A}_\alpha\big] + (\gamma_m \mathbb{W})_\alpha + [\mathbb{A}_\alpha, \mathbb{A}_m] \\ \big\{D_\alpha, \mathbb{W}^\beta\big\} &= \tfrac{1}{4}(\gamma^{mn})_\alpha{}^\beta \mathbb{F}_{mn} + \{\mathbb{A}_\alpha, \mathbb{W}^\beta\} \\ \big[D_\alpha, \mathbb{F}^{mn}\big] &= \big[\partial^{[m}, (\gamma^{n]}\mathbb{W})_\alpha\big] + [\mathbb{A}_\alpha, \mathbb{F}^{mn}] \\ &\quad - [\mathbb{A}^{[m}, (\gamma^{n]}\mathbb{W})_\alpha] \ .\end{aligned} \tag{18}$$

They are equivalent to the SYM field equations (4) if the connection in (1) is defined through the representatives $\mathbb{A}_\alpha$ and $\mathbb{A}_m$ of the generating series in (17).

Given that the multiparticle superfields satisfy [4]

$$\begin{aligned}\mathcal{F}^{mn}_B &= k^{[m}_B \mathcal{A}^{n]}_B - \sum_{XY=B}(\mathcal{A}^m_X \mathcal{A}^n_Y - \mathcal{A}^m_Y \mathcal{A}^n_X) \\ k^B_m(\gamma^m \mathcal{W}_B)_\alpha &= \sum_{XY=B}\big[\mathcal{A}^X_m(\gamma^m \mathcal{W}_Y) - \mathcal{A}^Y_m(\gamma^m \mathcal{W}_X)\big] \\ k^B_m \mathcal{F}^{mn}_B &= \sum_{XY=B}\big[2(\mathcal{W}_X \gamma^n \mathcal{W}_Y) + \mathcal{A}^X_m \mathcal{F}^{mn}_Y - \mathcal{A}^Y_m \mathcal{F}^{mn}_X\big]\,,\end{aligned} \tag{19}$$

the above definitions are compatible with (3) and

$$\big[\nabla_m, (\gamma^m \mathbb{W})_\alpha\big] = 0\,, \quad \big[\nabla_m, \mathbb{F}^{mn}\big] = \gamma^n_{\alpha\beta}\big\{\mathbb{W}^\alpha, \mathbb{W}^\beta\big\}\,. \tag{20}$$

A linearized gauge transformation in particle one,

$$\delta_1 A^1_\alpha = D_\alpha \Omega_1, \quad \delta_1 A^1_m = k^1_m \Omega_1, \tag{21}$$

with scalar superfields Ω_1 and $\delta_1 W^\alpha_1 = \delta_1 F^{mn}_1 = 0$ propagates to multiparticle cases with $B = 12\ldots p$ via [17]

$$\begin{aligned}\delta_1 \mathcal{A}^B_\alpha &= \big[D_\alpha, \Omega_B\big] + \sum_{XY=B}\Omega_X \mathcal{A}^Y_\alpha, \quad \delta_1 \mathcal{W}^\alpha_B = \sum_{XY=B}\Omega_X \mathcal{W}^\alpha_Y \\ \delta_1 \mathcal{A}^B_m &= \big[\partial_m, \Omega_B\big] + \sum_{XY=B}\Omega_X \mathcal{A}^Y_m, \quad \delta_1 \mathcal{F}^{mn}_B = \sum_{XY=B}\Omega_X \mathcal{F}^{mn}_Y\end{aligned} \tag{22}$$

The multiparticle gauge scalars $\Omega_{12\ldots p}$ are exemplified in appendix B of [17] and gathered in the generating series

$$\mathbb{L}_1 \equiv \Omega_1 t^1 + \sum_i \Omega_{1i}[t^1,t^i] + \sum_{j,k}\Omega_{1jk}[[t^1,t^j],t^k] + \ldots \tag{23}$$

This allows to cast (22) in the standard form of non-linear gauge transformations:

$$\begin{aligned}\delta_1 \mathbb{A}_\alpha &= \big[\nabla_\alpha, \mathbb{L}_1\big], \quad \delta_1 \mathbb{W}^\alpha = \big[\mathbb{L}_1, \mathbb{W}^\alpha\big] \\ \delta_1 \mathbb{A}_m &= \big[\nabla_m, \mathbb{L}_1\big], \quad \delta_1 \mathbb{F}^{mn} = \big[\mathbb{L}_1, \mathbb{F}^{mn}\big]\ .\end{aligned} \tag{24}$$

HIGHER MASS DIMENSION SUPERFIELDS

The introduction of the Lie elements $\mathbb{K}$ and their associated supercovariant derivatives allow the recursive definition of superfields with higher mass dimensions,

$$\begin{aligned}\mathbb{W}^{m_1\ldots m_k\alpha} &\equiv \big[\nabla^{m_1}, \mathbb{W}^{m_2\ldots m_k\alpha}\big]\,, \\ \mathbb{F}^{m_1\ldots m_k|pq} &\equiv \big[\nabla^{m_1}, \mathbb{F}^{m_2\ldots m_k|pq}\big]\,.\end{aligned} \tag{25}$$

Their component fields are defined by

$$\begin{aligned}\mathbb{W}^{m_1\ldots m_k\alpha} &\equiv \sum_{B\neq\emptyset} t^B \mathcal{W}^{m_1\ldots m_k\alpha}_B\,, \\ \mathbb{F}^{m_1\ldots m_k|pq} &\equiv \sum_{B\neq\emptyset} t^B \mathcal{F}^{m_1\ldots m_k|pq}_B\,,\end{aligned} \tag{26}$$

with $t^B \equiv t^1 t^2 \ldots t^p$ for $B = 12\ldots p$ and identified as

$$\begin{aligned}\mathcal{W}^{m_1\ldots m_k\alpha}_B &= k^{m_1}_B \mathcal{W}^{m_2\ldots m_k\alpha}_B \\ &\quad + \sum_{XY=B}\big(\mathcal{W}^{m_2\ldots m_k\alpha}_X \mathcal{A}^{m_1}_Y - \mathcal{W}^{m_2\ldots m_k\alpha}_Y \mathcal{A}^{m_1}_X\big)\,, \\ \mathcal{F}^{m_1\ldots m_k|pq}_B &= k^{m_1}_B \mathcal{F}^{m_2\ldots m_k|pq}_B \\ &\quad + \sum_{XY=B}\big(\mathcal{F}^{m_2\ldots m_k|pq}_X \mathcal{A}^{m_1}_Y - \mathcal{F}^{m_2\ldots m_k|pq}_Y \mathcal{A}^{m_1}_X\big)\,.\end{aligned} \tag{27}$$

Note from (27) that the non-linearities in the definition of higher mass superfields do not contribute in the single-particle context where $\mathcal{W}^{m_1\ldots m_k\alpha}_i = k^{m_1}_i \mathcal{W}^{m_2\ldots m_k\alpha}_i$.

Equations of motion at higher mass dimension: The equations of motion for the superfields of higher mass dimension (25) follow from $\big[\nabla_\alpha, \nabla_m\big] = -(\gamma_m \mathbb{W})_\alpha$ and $\big[\nabla_m, \nabla_n\big] = -\mathbb{F}_{mn}$ together with Jacobi identities among iterated brackets. The simplest examples are given by

$$\begin{aligned}\big\{\nabla_\alpha, \mathbb{W}^{m\beta}\big\} &= \tfrac{1}{4}(\gamma_{pq})_\alpha{}^\beta \mathbb{F}^{m|pq} - \big\{(\mathbb{W}\gamma^m)_\alpha, \mathbb{W}^\beta\big\}\,, \\ \big[\nabla_\alpha, \mathbb{F}^{m|pq}\big] &= (\mathbb{W}^{m[p}\gamma^{q]})_\alpha - \big[(\mathbb{W}\gamma^m)_\alpha, \mathbb{F}^{pq}\big]\,,\end{aligned} \tag{28}$$

which translate to

$$D_\alpha \mathcal{W}^{m\beta}_B = \tfrac{1}{4}(\gamma_{pq})_\alpha{}^\beta \mathcal{F}^{m|pq}_B + \sum_{XY=B}(\mathcal{A}^X_\alpha \mathcal{W}^{m\beta}_Y - \mathcal{A}^Y_\alpha \mathcal{W}^{m\beta}_X)$$

$$- \sum_{XY=B} \big[(\mathcal{W}_X \gamma^m)_\alpha \mathcal{W}_Y^\beta - (\mathcal{W}_Y \gamma^m)_\alpha \mathcal{W}_X^\beta \big] \, ,$$

$$D_\alpha \mathcal{F}_B^{m|pq} = (\mathcal{W}_B^{m[p} \gamma^{q]})_\alpha + \sum_{XY=B} (\mathcal{A}_\alpha^X \mathcal{F}_Y^{m|pq} - \mathcal{A}_\alpha^Y \mathcal{F}_X^{m|pq})$$

$$- \sum_{XY=B} \big[(\mathcal{W}_X \gamma^m)_\alpha \mathcal{F}_Y^{pq} - (\mathcal{W}_Y \gamma^m)_\alpha \mathcal{F}_X^{pq} \big] . \quad (29)$$

In general, one can prove by induction that

$$\big\{ \nabla_\alpha, \mathbb{W}^{N\beta} \big\} = \tfrac{1}{4} (\gamma_{pq})_\alpha{}^\beta \mathbb{F}^{N|pq} - \sum_{\substack{M \in P(N) \\ M \neq \emptyset}} \big\{ (\mathbb{W}\gamma)_\alpha^M, \mathbb{W}^{(N\setminus M)\beta} \big\}$$

$$\big[\nabla_\alpha, \mathbb{F}^{N|pq} \big] = (\mathbb{W}^{N[p} \gamma^{q]})_\alpha - \sum_{\substack{M \in P(N) \\ M \neq \emptyset}} \big[(\mathbb{W}\gamma)_\alpha^M, \mathbb{F}^{(N\setminus M)pq} \big] . \quad (30)$$

The vector indices have been gathered to a multi-index $N \equiv n_1 n_2 \ldots n_k$. Its power set $P(N)$ consists of the 2^k ordered subsets, and $(\mathbb{W}\gamma)^N \equiv (\mathbb{W}^{n_1 \ldots n_{k-1}} \gamma^{n_k})$.

The higher-mass-dimension superfields obey further relations which can be derived from Jacobi identities of nested (anti)commutators. For example, (3) determines their antisymmetrized components

$$\mathbb{W}^{[n_1 n_2] n_3 \ldots n_k \beta} = \big[\mathbb{W}^{n_3 \ldots n_k \beta}, \mathbb{F}^{n_1 n_2} \big] \quad (31)$$

$$\mathbb{F}^{[n_1 n_2] n_3 \ldots n_k | pq} = \big[\mathbb{F}^{n_3 \ldots n_k | pq}, \mathbb{F}^{n_1 n_2} \big] .$$

Moreover, the definition (25) via iterated commutators together with $\delta_1 \nabla_m = -[\mathbb{L}_1, \nabla_m]$ implies that

$$\mathbb{F}^{[m|np]} = 0, \quad \mathbb{F}^{[mn]|pq} + \mathbb{F}^{[pq]|mn} = 0 \, , \quad (32)$$

$$\delta_1 \mathbb{W}_N^\alpha = \big[\mathbb{L}_1, \mathbb{W}_N^\alpha \big], \quad \delta_1 \mathbb{F}^{N|pq} = \big[\mathbb{L}_1, \mathbb{F}^{N|pq} \big] \, ,$$

and manifold generalizations of (20), (31) and (32) can be generated using these same manipulations.

OUTLOOK AND APPLICATIONS

The representation of the non-linear superfields of ten-dimensional SYM theory described in this letter was motivated by the computation of scattering amplitudes in the pure spinor formalism. Accordingly, they give rise to generating functions for amplitudes. For example, color-dressed tree-level amplitudes $M(1,2,\ldots,n)$ involving particles $1,2,\ldots,n$ are generated by

$$\frac{1}{3} \mathrm{Tr} \langle \mathbb{V}\mathbb{V}\mathbb{V} \rangle = \sum_{n=3}^{\infty} (n-2) \sum_{i_1 < i_2 < \ldots < i_n} M(i_1, i_2, \ldots, i_n) \, . \quad (33)$$

As firstly pointed out in the appendix of [18], the generating series $\mathbb{V} \equiv \lambda^\alpha \mathbb{A}_\alpha$ involving the pure spinor λ^α satisfies the field equations $Q\mathbb{V} = \mathbb{V}\mathbb{V}$ of the action $\mathrm{Tr} \int \mathrm{d}^{10} x \langle \frac{1}{2} \mathbb{V} Q \mathbb{V} - \frac{1}{3} \mathbb{V}\mathbb{V}\mathbb{V} \rangle$ [19] with BRST operator $Q \equiv \lambda^\alpha D_\alpha$. The zero mode prescription of schematic form $\langle \lambda^3 \theta^5 \rangle = 1$ is explained in [5], and the pure spinor representation of SYM amplitudes on the left hand side of (33) is described in [2]. Further details and generalizations to supersymmetrized operators F^4 and $D^2 F^4$ at higher mass dimension will be given elsewhere [20].

The multiparticle superfields of higher mass dimensions can be used to obtain simpler expressions for higher-loop kinematic factors of superstring amplitudes, e.g.

$$T_{12,3,4} \equiv \langle (\lambda \gamma_m W_{12}^n)(\lambda \gamma_n W_{[3}^p)(\lambda \gamma_p W_{4]}^m) \rangle \quad (34)$$

$$T_{1234}^m \equiv \langle A_{(1}^m T_{2),3,4} + (\lambda \gamma^m W_{(1})(\lambda \gamma^n W_{2)}^q)(\lambda \gamma_q W_{[3}^p) F_{4]}^{np} \rangle$$

is an equivalent representation for the complicated three-loop kinematic factors generating the operator $D^6 R^4$ [6].

Finally, it would be interesting to construct formal solutions to supergravity field equations along similar lines.

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