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Axiomatizing bounded rationality: the priority heuristic

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Abstract This paper presents an axiomatic framework for the priority heuristic, a model of bounded rationality in Selten's (in: Gigerenzer and Selten (eds.) Bounded rationality: the adaptive toolbox, 2001) spirit of using empirical evidence on heuristics. The priority heuristic predicts actual human choices between risky gambles well. It implies violations of expected utility theory such as common consequence effects, common ratio effects, the fourfold pattern of risk taking and the reflection effect. We present an axiomatization of a parameterized version of the heuristic which generalizes the heuristic in order to account for individual differences and inconsistencies. The axiomatization uses semiorders (Luce, Econometrica 24:178–191, 1956), which have an intransitive indifference part and a transitive strict preference component. The axiomatization suggests new testable predictions of the priority heuristic and makes it easier for theorists to study the relation between heuristics and other axiomatic theories such as cumulative prospect theory.

Keywords Bounded rationality · Axiomatization · Priority heuristic · EUT · EVT

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1 Introduction

Expected utility theory remains to this day the dominant decision theoretic framework. Much of the appeal of expected utility theory lies in its elegant axiomatic characterizations (von Neumann and Morgenstern 1944; Savage 1954), which lend themselves to a normative reading. It has, however, been shown empirically that the axioms of expected utility are systematically violated by people. For instance, Allais (1953) has demonstrated violations of the independence axiom, and Kahneman and Tversky have empirically identified a number of violations of expected utility theory, including framing effects, the reflection effect and the fourfold pattern of risk-taking (Kahneman and Tversky 1979; Tversky and Kahneman 1992). For a review of non-expected utility theories, see Starmer (2000).

An alternative approach for studying human decision making is to start by investigating how human beings make choices in the real world. To obtain a more realistic account of human decision making, Selten (2001) and, before, Simon (1991), have called for a theory of bounded rationality that is based on an empirical analysis of the cognitive processes that lead to choice. The formal study of simple heuristics provides one approach towards this end (Gigerenzer and Selten 2001). A heuristic is a strategy that relies on limited search for information and does not involve sophisticated calculations. Instead, it is composed of rules for search, stopping and decision making consistent with the observation that people often search for information sequentially in time and stop search at some point rather than engaging in exhaustive search. A limitation of this approach is that it has so far not been characterized axiomatically.

This paper gives an axiomatic characterization of a family of lexicographic theories of choice, which includes the priority heuristic as a special case. The priority heuristic concerns binary decisions between gambles. It is remarkable because it predicts the choices between gambles of the majority well (Brandstätter et al. 2006). Furthermore, it can account for a number of violations of expected utility theory, in particular the common consequence and common ratio effects, reflection effects and the fourfold pattern of risk taking (Katsikopoulos and Gigerenzer 2008). An axiomatization will be helpful in at least two ways: First, it will make it possible for theorists to study the relation of the priority heuristic to other axiomatic theories, such as cumulative prospect theory (see Wakker and Tversky 1993). Second, it will allow for new empirical tests via the axioms of the heuristic. Our axiomatization builds on Luce (1978).

The representation given here is for a parameterized version of the priority heuristic. While the version of the heuristic with fixed parameters predicts the data well, there is a need for a parameterized version as well. For example, parameters are needed in order to account for individual differences, and for inconsistencies in choice (Rieskamp 2008). The axiomatization suggested here makes no claims with respect to parameters. Studying this generalization of the priority heuristic does not mean that we advocate a research program in which heuristics are populated with parameters, to be fitted anew to each data set. Rather, we see the generalization as covering other possible fixed parameters of the priority heuristic, in case that independent theory or evidence suggests such fixed values in some situations.

The representation uses semiorders (Luce 1956), which have a transitive strict preference part and an intransitive indifference part. In particular, the concept of intransitive indifference entails that an agent may be indifferent between any two elements that are too similar for the agent to form a strict preference between them. However, while several elements may be judged indifferent, beyond a certain threshold indifference may switch to strict preference. This seems reasonable and consistent with real world evidence, since utility may not be perfectly discriminable.

We proceed as follows. Section 2 introduces the priority heuristic and reviews relevant analytical and empirical results. Section 3 presents a representation theorem for the heuristic in choices where gambles differ on two attributes (an outcome and a probability). Section 4 concludes with a discussion of the present contribution to the foundations of a theory of bounded rationality in the sense of Selten (2001) and Simon (1991).

2 The priority heuristic

The priority heuristic is a model of how people make choices between gambles. Its domain concerns difficult risky-choice problems, that is, problems in which no alternative dominates the other and expected values are close (ratio $\leq$2). A large part of the evidence on people's choice behaviour derives from simple monetary gambles. The priority heuristic proposes that people make choices by using at most three attributes: The minimum outcome, the probability of the minimum outcome and the maximum outcome. For choosing between two gambles with nonnegative outcomes (then called gains), the priority heuristic has a search rule, stopping rule and decision rule (Brandstätter et al. 2006):

Search rule: Go through attributes in the order: Minimum gain, probability of minimum gain, maximum gain.

Stopping rule: Stop search if the minimum gains differ by 1/10, or more, of the maximum gain (across the two gambles); otherwise, stop search if probabilities of the minimum gains differ by .1 or more.

Decision rule: Choose the gamble that is more attractive in the attribute (gain or probability) that stopped search.

The more attractive gamble is the one with the higher (minimum or maximum) gain or with the lower probability of minimum gain. For negative outcomes (the minimum and maximum outcomes are then called losses), the difference in the statement of the heuristic is that "gain" is replaced by "loss". The more attractive loss is the lower one and the more attractive probability of minimum loss is the higher one. The logic of the priority heuristic also applies to mixed gambles: the priority heuristic then holds that first the minimum gains (losses) are compared, and if the difference between minimum gains exceeds 1/10 of the maxmimum outcome across the two gambles, maximum outcomes are compared and the gamble with the higher maximum outcome is chosen. Our axiomatization refer to gambles with gains and it will be obvious how they would be restated for gambles with losses.

Formally, we axiomatize a relation $\succcurlyeq$, defined on $X \times P \times Y$, where X is the set of minimum outcomes, P is the set of probabilities of minimum outomes and Y is the set of maximum outcomes, such that $(x_1, p_1, y_1) \succcurlyeq (x_2, p_2, y_2)$ iff

$$
\begin{aligned}
&\text{(i)} \quad x_1 - x_2 > \frac{\max\{y_1, y_2\}}{10} \text{ , or} \\
&\text{(ii)} \quad |x_1 - x_2| \leq \frac{\max\{y_1, y_2\}}{10} \text{ and} \\
&\quad p_2 - p_1 > .1 \text{ , or} \\
&\text{(iii)} \quad |x_1 - x_2| \leq \frac{\max\{y_1, y_2\}}{10} \text{ and} \\
&\quad |p_2 - p_1| \leq .1 \text{ and} \\
&\quad y_1 \geq y_2
\end{aligned}
$$

The priority heuristic is lexicographic in the sense that an attribute is used for making a choice only if the attributes that precede it in the search order do not allow making a choice (see also Luce 1956). For more discussion on the heuristic, for example, on why the aspiration levels for stopping were fixed to .1, see Brandstätter et al. (2006) and Katsikopoulos and Gigerenzer (2008).

To illustrate how the heuristic works, consider one of the problems posed by Allais (1953, p. 527), known as the Allais paradox, where people first choose between gambles A and B:

$$
\begin{aligned}
\text{A}: &\ 100,000,000 \text{ with probability } 1.00 \\
\text{B}: &\ 500,000,000 \text{ with probability } .10 \\
&\ 100,000,000 \text{ with probability } .89 \\
&\ 0 \text{ with probability } .01
\end{aligned}
$$

By subtracting a .89 probability to win 100 million from both gambles A and B, Allais obtained the following gambles, C and D:

$$
\begin{aligned}
\text{C}: &\ 100,000,000 \text{ with probability } .11 \\
&\ 0 \text{ with probability } .89 \\
\text{D}: &\ 500,000,000 \text{ with probability } .10 \\
&\ 0 \text{ with probability } .90
\end{aligned}
$$

The majority of people chose gamble A over B and D over C (MacCrimmon 1968), and this constitutes a violation of the independence axiom. Expected utility theory does not predict whether A or B will be chosen; it only makes conditional predictions such as "if A is chosen from A and B, then C is chosen from C and D." The priority heuristic, in contrast, makes stronger predictions: It predicts whether A or B will be chosen, and whether C or D will be chosen. Consider the choice between A and B. The maximum gain across the two gambles is 500 million, and therefore the aspiration level for gains is 50 million. The difference between the minimum gains equals $100 - 0 = 100$ million, which exceeds the aspiration level, and search is stopped. The gamble with the more attractive minimum gain is A. Thus, the heuristic predicts the majority choice correctly. In the choice between C and D, minimum gains are equal. Thus, the next attribute is looked up. The difference between the probabilities of minimum gains equals $.90 - .89 = .01$, which is smaller than the aspiration level for probabilities of

.1. Thus, the choice is decided by the last attribute, maximum gain, in which gamble D is more attractive. This prediction is again consistent with the choice of the majority.

More generally, Katsikopoulos and Gigerenzer (2008) have shown mathematically that the priority heuristic implies common consequence effects, common ratio effects, reflection effects and the fourfold pattern of risk attitude. In fact, because the parameters of the heuristic (the order in which attributes are searched, and the aspiration levels that stop attribute search) are fixed, the priority heuristic implies the effects simultaneously.

On the other hand, modifications of expected utility theory, such as cumulative prospect theory (Tversky and Kahneman 1992), that are consistent with the effects by appropriately setting parameters, cannot simultaneously account for the empirical evidence (Neilson and Stowe 2002). Neilson and Stowe (2002) also showed that no parameter combinations allow for these two behaviours and a series of choices made by a large majority of participants and reasonable risk premia. Similarly, Blavatskyy (2010) showed that the conventional parameterizations of cumulative prospect theory do not explain the St. Petersburg paradox. Overall, in multi-parameter models, the parameter values fitted to one set of data are not necessarily robust, in the sense of generating accurate predictions for new sets of data. For more on the importance of distinguishing between fitting and prediction in economic modelling, see Harless and Camerer (1994); Binmore and Shaked (2010); Erev et al. (2010).

No model of risky choice can predict people's behaviour in every pair of gamble correctly; therefore, it is crucial that researchers refrain from constructing pairs that fit their model when testing it against competing theories. To avoid such a possible bias, Brandstätter et al. (2006) tested the predictive power of the priority heuristic exclusively against sets of gambles designed by the authors of competing theories (Kahneman and Tversky 1979; Tversky and Kahneman 1992; Lopes and Oden 1999) as well as randomly generated gambles (Erev et al. 2002; Brandstätter et al. 2006). These test sets included two-outcome gambles, five-outcome gambles and choices based on certainty equivalents. Across 260 pairs of gambles, the priority heuristic predicted 87 % of majority choices correctly, while cumulative prospect theory predicted 77 % (the second most predictive theory was the security-potential/aspiration theory of Lopes and Oden, with 79 % of majority choices).

The limits of the predictive power of the priority heuristic were analyzed using 450 pairs of gambles designed by Mellers et al. (1992). The priority heuristic was more predictive than the modifications of expected utility theory when the problems were difficult (i.e. the ratio of the expected values of the two gambles was ≤ 2) but not when problems were easy (ratio >2) or dominated. For easy problems, however, none of the modifications of expected utility theory could outperform the simple theory of expected value (for a discussion of the evidence, see Birnbaum 2008 and Brandstätter et al. 2006). These studies suggest that non-linear transformations of probabilities or monetary values may be needed neither for easy problems nor for difficult ones. Difficult problems can be modeled by the priority heuristic and easy ones by expected value theory, each of which is based on non-transformed values and probabilities. This result clarifies that "overweighting of small probabilities and underweighting of large probabilities", which is often evoked to account for anomalies, is in fact not necessary.

Leland (2010) distinguishes three approaches towards descriptive theories of choice. What he calls the "road taken" is the representation of lotteries as prospects that leads to a preoccupation with explaining violations of independence and has led to a plethora of modifications of expected utility theories, such as prospect theory. A representation of lotteries in terms of Savage's action-by-state matrices instead of prospects, however, makes violations of independence transparent, infrequent, and not the main problem. In this approach, the "road less travelled", more substantial violations such as transitivity and preference for dominated-alternatives become more central, as in regret theory (Loomes and Sugden 1987). Common to both approaches, nevertheless, is that choices are interpreted as revealing properties of preferences. In the third approach, the "road not taken", choices do not reveal properties of the preferences, but instead properties of the decision processes that individuals use to satisfy their preferences. The priority heuristic is a formal model of this third approach, as are the similarity models by Rubinstein (1988) and Leland (1994, 2002).

3 Axiomatization of two-attribute lexicographic heuristics

3.1 Preliminaries

This section assumes that the two gambles have equal minimum gains.[1] This means that this section ignores the first step of the priority heuristic, where minimum gains are compared.

Let P and Y be sets containing the attributes of the gambles. A gamble is a pair (p, y) with $p \in P$ and $y \in Y$, where p denotes the probability of the maximum[2] outcome and y the value of the maximum outcome. Let $\succcurlyeq$ be a binary relation on $P \times Y$, the preference relation over gambles. The relation $\succcurlyeq$ is not assumed to be transitive.

Assume that $\succcurlyeq$ is *independent* in the following sense: For all p_1, p_2 in P and for all y_1, y_2 in Y,

$$(p_1, y_1) \succcurlyeq (p_2, y_1) \quad \text{iff} \quad (p_1, y_2) \succcurlyeq (p_2, y_2) \tag{1}$$

$$(p_1, y_1) \succcurlyeq (p_1, y_2) \quad \text{iff} \quad (p_2, y_1) \succcurlyeq (p_2, y_2) \tag{2}$$

Statement 1 induces an unambiguous order on P, denoted $\succcurlyeq_P$, and statement 2 induces the unambiguous order on Y, denoted $\succcurlyeq_Y$.

Furthermore, we define strict preference, $\succ$, and indifference, $\sim$, in terms of $\succcurlyeq$ in the usual sense: For all p_1, p_2 in P, and for all y_1, y_2 in Y,

[1] Some important empirical evidence, such as the possibility effect of Kahneman and Tversky (1979), refers to zero minimum outcomes; Rubinstein (1988) also makes this assumption.

[2] The priority heuristic, as stated in Sect. 2, compares probabilities of minimum outcomes. Given the additivity of probabilities, for gambles with two outcomes the probability of the maximum outcome is the complement of the probability of minimum outcomes. For convenience, we consider the mathematically equivalent case where the probabilities of maximum outcomes are compared.

$$(p_1, y_1) \succ (p_2, y_2) \quad \text{iff} \quad (p_1, y_1) \succcurlyeq (p_2, y_2) \quad \text{and not} \quad (p_2, y_2) \succcurlyeq (p_1, y_1) \tag{3}$$

$$(p_1, y_1) \sim (p_2, y_2) \quad \text{iff} \quad (p_1, y_1) \succcurlyeq (p_2, y_2) \quad \text{and} \quad (p_2, y_2) \succcurlyeq (p_1, y_1) \tag{4}$$

Note that neither $\succ$ nor $\sim$ can be assumed to be transitive, since the weak preference relation $\succcurlyeq$ is not assumed to be transitive. The strict preference and indifference relations on P and Y, $\succ_P$, $\sim_P$, $\succ_Y$ and $\sim_Y$ are defined similarly.

Next we define another relation on R, denoted R_P, which expresses the lexicographic nature of the decision rule of the priority heuristic. The following definition of R_P expresses the fact that the first attribute searched, probabilities, dominates the second attribute searched, maximum outcomes: For all p_1, p_2 in P

$$p_1 R_P p_2 \quad \text{iff for every } y_1, y_2 \quad \text{in} \quad Y, \ (p_1, y_1) \succ (p_2, y_2) \tag{5}$$

Suppose R is a binary relation on P. Then we can define two other relations, $I(R)$ and $W(R)$ in terms of it. The interpretation we would like to give to these relations is that $I(R)$ is an indifference relation on P, and $W(R)$ is a weak preference relation on P which is defined in a non-standard way (Luce 1956, 1978).

$$p_1 I(R) p_2 \quad \text{iff not } p_1 R p_2 \quad \text{and not } p_2 R p_1 \tag{6}$$

$p_1 W(R) p_2$ iff either

$$\begin{aligned} (i) \quad & p_1 R p_2 \text{ or} \\ (ii) \quad & p_1 I(R) p_2 \text{ and there exists a } p_3 \text{ in } P \\ & \text{such that } p_1 I(R) p_3 \text{ and } p_3 R p_2, \text{ or} \\ (iii) \quad & p_1 I(R) p_2 \text{ and there exists a } p_4 \text{ in } R \\ & \text{such that } p_1 R p_4 \text{ and } p_4 I(R) p_2 \end{aligned} \tag{7}$$

This definition expresses the intuition that one probability is weakly preferred to a second probability if (i) the first probability is strictly preferred to the second, or (ii) the first and second probabilities are indistinguishable, and there exists a third probability that is indistinguishable from the first and strictly preferred to the second, or (iii) the first and second probabilities are indistinguishable, and there exists a fourth probability such that the first probability is strongly preferred to the fourth, and the fourth probability is indistinguishable from the second.

This intuition is expressed by the stopping rule of the priority heuristic: A user of the heuristic may weakly prefer obtaining the maximum outcome with a probability of $p_1 = .23$ to obtaining it with a probability of $p_2 = .22$. This weak preference may arise not because s/he has a strong preference for .23 over .22, but rather because s/he cannot discriminate between the two probabilities of .23 and .22, and there exists a third probability, e.g. $p_3 = .33$, such that s/he has a strict preference for .33 over .22 and cannot distinguish between .33 and .23; this is an example of case (ii) just above.

Let us now turn to the definition of a semiorder, as presented by Luce (1956). A semiorder is characterized by the properties of having a transitive strict preferences part, and an intransitive indifference part. These properties make semiorders particularly well suited to modelling the behaviour of people who may express indifference between two elements they can essentially not distinguish. Nevertheless, there may exist a threshold beyond which indifference switches to strict preference. A semiorder is defined as follows:

A binary relation R on P is a *semiorder* iff, for all p_1, p_2 in R

$$\begin{aligned} &(i) \quad \text{not } p_1 R p_1 \\ &(ii) \quad p_1 R p_2 \text{ and } p_3 R p_4 \text{ imply either } p_1 R p_4 \text{ or } p_3 R p_2 \\ &(iii) \quad p_1 R p_2 \text{ and } p_2 R p_3 \text{ imply either } p_1 R p_4 \text{ or } p_4 R p_3 \end{aligned} \tag{8}$$

The first part of the definition holds that the relation R on P is irreflexive. The second and third parts of the definition convey the intuitive idea that an indifference interval should never span a strict preference interval (see Luce 1956).

We now turn to the concept of an *indifference interval*. This formalizes the idea that even though one attribute value may be weakly preferred to another attribute value, they may not be sufficiently different to induce a strict preference for one of them. The two attribute values will then span an indifference interval, such that all elements of it are considered indifferent, and such that b_1 and b_2 delimit the interval from above and below. For example, in the context of the priority heuristic, the probabilities $p_1 = .33$ and $p_2 = .23$ would span an indifference interval. The concept is formalized as follows:

If R is a semiorder on P, and if $p_1 W(R_P) p_2$ and $p_1 I(R_P) p_2$, the set

$$P(p_1, p_2) = \{p_3 \mid b_1 W(R) p_3 \quad \text{and} \quad p_3 W(R) p_2\} \tag{9}$$

is called an indifference interval.

Finally, we introduce the relation R_Y on Y which is designed to single out that part of $\succcurlyeq$, where the dominant component, P, does not discriminate. Let $P(p_3, p_4)$ be an indifference interval, and let p_1, p_2 be elements of it. Then for all indifference intervals $P(p_3, p_4)$, and for all y_1, y_2 in Y

$$y_1 R_Y y_2 \quad \text{iff} \quad \text{for every } p_1, p_2 \text{ in } P(p_3, p_4),\ (p_1, y_1) \succcurlyeq (p_2, y_2) \tag{10}$$

3.2 Axioms

Consider a binary relation $\succcurlyeq$ on $P \times Y$, with the derived concepts $\succcurlyeq_P$, R_P, $W(R_P)$, $P(p_3, p_4)$ and R_Y defined above.

Axiom 1 $\succcurlyeq$ is reflexive, complete, and independent
Axiom 2 R_P is a semiorder
Axiom 3 $W(R_P)$ is identical to $\succcurlyeq_P$
Axiom 4 R_Y is a simple order
Axiom 5 R_Y is identical to $\succcurlyeq_Y$

Axiom 6 There exists a finite or countable subset of P,
$Q = \{..., q_{-2}, q_{-1}, q_0, q_1, q_2, ...\}$ such that for all q_{i-1}, q_i, q_{i+1} in Q
(i) $q_i \succ_R q_{i-1}$,
(ii) $P(q_{i-1}, q_{i+1})$ is an indifference interval,
(iii) for p_1 in P, there exists an q_{i-1}, q_i in Q with $q_i \succcurlyeq_P p_1 \succcurlyeq_P q_{i-1}$

Axiom 7 For every p_1 in P, there exists some p_2 in P such that $p_2 I(R_P) p_1$, and for any p_3 in P with $p_3 \succ_P p_2$, then $p_3 R_P p_1$

Axioms 1–5 are necessary, while Axioms 6 and 7 are structural. This makes Axioms 6 and 7 less suited for constructing empirical tests. Axiom 6 states that the indifference intervals on P span all of P (for the priority heuristic, the entire probability scale), and overlap one another. This axiom ensures that the scales over P and Y agree. Axiom 7 requires that the set of elements indistinguishable from a given element be closed from above. Together, Axioms 6 and 7 ensure the existence of a supremum (see Sect. 3.3).

Axiom 1 is standard except for the assumption that the preference relation over gambles $\succcurlyeq$ is not necessarily transitive, a property that the priority heuristic does not always satisfy. The property of independence implies that each attribute in (p_1, y_1) affects the relation $\succcurlyeq$ independently of the other attribute.

Axiom 2 requires that the strictly dominating part of $\succcurlyeq$, R_P, is transitive; this follows from the conjunction of statements (i) and (iii) of the definition of a semiorder. The definition of a semiorder implies that indifference intervals cannot cover elements between which there exists a strict preference. Also, the indifference relation $I(R_P)$ will not be transitive, as is the case for the priority heuristic. Axiom 4 is used to impose an order on that part of $\succcurlyeq$, where the first component of the tuples (p_1, y_1) does not dominate. Thus if two elements p_3 and p_4 are indistinguishable, then only elements of Y should determine choice, and these should be ordered according to a simple order. In particular, Axiom 4 ensures that the restriction of the set $P \times Y$ to $P(p_3, p_4) \times Y$ agrees with the order between elements of Y, which is a simple order.

Axiom 3 forces the order $\succcurlyeq_P$ on P induced by independence to be identical to the weak order $W(R_P)$ on P, which was defined in terms of the relation R_P. This implies that both $\succcurlyeq_P$ and R_P will be representable using the same numerical scale. Axiom 5 forces the order $\succcurlyeq_Y$ on Y induced by independence to be identical to the simple order R_Y on Y, which was defined on the indifference intervals only. This implies that both $\succcurlyeq_Y$ and R_Y will be representable using the same numerical scale.

This axiom system is similar to the one used by Luce (1978) for axiomatizing a two-attribute lexicographic model. Luce's (1978) model produces trade-offs between attributes in its second step, whereas this is not the case for the priority heuristic, which considers the second attribute alone when the first attribute does not determine choice.

3.3 Representation Theorem

Theorem 1 Suppose $\langle P \times Y, \succcurlyeq \rangle$ satisfies Axioms 1–7. Then there exist real-valued functions ϕ_P and δ_P on P, and ϕ_Y on C such that for all p_1, p_2 in P, and y_1, y_2 in Y,

1. $\delta_P(p_1) = \sup_{\substack{p_2 \\ p_2 I(R_P) p_1}} [\phi_P(b_2) - \phi_P(p_1)] > 0$

2. $p_1 R_P p_2$ iff $\phi_P(p_1) > \phi_P(p_2) + \delta_P(p_2)$
3. $p_1 W(R_P) p_2$ iff $\phi_P(p_1) > \phi_P(p_2)$
4. $y_1 R_Y y_2$ iff $\phi_Y(y_1) \geq \phi_Y(y_2)$
5. $(p_1, y_1) \succcurlyeq (p_2, y_2)$ iff either

   (i) $\phi_P(p_1) > \phi_P(p_2) + \delta_P(p_2)$, or
   (ii) $-\delta_P(p_1) \leq \phi_P(p_1) - \phi_P(p_2) \leq \delta_P(p_2)$, and $\phi_Y(y_1) \geq \phi_Y(y_2)$

If $f(.)$ is a strictly increasing and continuous function, and α, $\beta_Y > 0$ are constants, then ϕ'_P, δ'_P and ϕ'_Y form another representation such that:

$$\phi'_P = f(\phi_P) \quad \delta'_P = f(\phi_P + \delta_P) - f(\phi_P) \quad \phi'_Y = \alpha\phi_Y + \beta_Y$$

If such a representation exists, then Axioms 1–5 must hold. For the proof, see Appendix.

3.4 Comments

Jointly, Axiom 2 and 4 imply that empirically people find it hard to distinguish between probabilities that are close (Axiom 2), but they can distinguish very well between maximum outcomes (Axiom 4). This prediction about people having different abilities distinguishing outcomes and probabilities is a strong prediction and, to the best of our knowledge, a new one that should be tested empirically. This prediction is indirectly supported by research indicating that (i) people spend more time on outcomes than on probabilities suggesting that outcomes are more important than probabilities (Schkade and Johnson 1989), (ii) in the extreme, people neglect probabilities altogether, and instead base their choices on the immediate feelings elicited by the gravity or benefit of future events (Loewenstein et al. 2001), (iii) highly emotional outcomes tend to override the impact of probabilities (Sunstein 2003), (iv) anxiety is largely influenced by the intensity of the shock, not by its probability of occurrence (Deane 1969), and heuristics have been reported that rely on outcomes while ignoring probabilities, but not vice versa (Brandstätter et al. 2006, Table 3).

Theorem 1 axiomatizes a family of heuristics of which the priority heuristic is a special case. By setting ϕ_B and ϕ_C to the identity functions and δ_B to .1, the representation expresses the priority heuristic for the case of equal minimum outcomes. Note that while setting the function ϕ_B to the identity function, the representation theorem yields a lexicographic structure with linear transformations of the probabilities, the theorem can also yield structures that use non-linear transformations of probabilities.

4 Towards a theory of bounded rationality

The term "bounded rationality" has been used for at least three different research programs: Optimization under constraints (Sargent 1993), deviations from optimization (Kahneman 2003) and for the study of decision processes in situations, where optimization may be out of reach (Gigerenzer and Selten 2001; Simon 1955; Selten 2001). Note that these three uses are not the same. The first two emphasize rationality and irrationality, respectively, but share optimization as a reference point. The third program models the process of decision rather than optimization or deviations from optimization. As mentioned before, in this program, choices reveal decision processes (Leland 2010). The priority heuristic is such a formal model of the decision process. The three building blocks—rules for search, stopping and decision—are also part of other heuristics in what is termed the "adaptive toolbox" of humans (Gigerenzer and Selten 2001). To date, the study of bounded rationality has accumulated converging evidence that heuristics can model decision making in both experts and laypeople, and that heuristics can often make more accurate predictions than can complex forecasting models, including linear regression, neural networks, Bayesian models and classification trees (Katsikopoulos 2011). Yet as Selten (2001, p. 14) noted, a comprehensive, coherent theory of bounded rationality is not yet available.

This paper is a step in the direction of providing a theory of bounded rationality, in particular, by providing greater conceptual clarity through the use of an axiomatic representation. In general, axioms give exact behavioural characterizations which can be tested empirically. Also, by using axioms on (unobservable) preference relations and thereby yielding a representation result which models decisions consistent with heuristics, this paper provides a link between existing axiomatic theories of decision making and bounded rationality.

The contribution made here can be seen as an exercise consistent with the empiricist school of thought: Starting from observable phenomena, by abstraction a theory is derived—the priority heuristic—, and from the theory, we obtain mathematical concepts—the axiomatization. Our approach is, in fact, consistent with the origins of probability and decision theory: Decision theory was first studied by Blaise Pascal and Pierre de Fermat as an attempt to understand gambling behaviour. The priority heuristic is a theory which, as explained above, predicts just these choices between gambles well, and is therefore a good starting point for the derivation of axiomatic characterizations of bounded rationality.

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5 Appendix

5.1 Proof of Theorem 1

5.1.1 Sufficiency

Statement 1 and 2: Define δ_P by Statement 1. Axioms 6 and 7 insure the existence of the supremum, and by Axioms 6(ii) and 6(iii), $\delta_P > 0$. By the definition of δ_P and Axiom 7, Statement 2 follows.

Statement 3: By Axioms 2 and 6, and Theorem 16.7 of Suppes et al. (1989), there exists a real-valued function ϕ_P on P such that $p_1 W(R_P) p_2$ iff $\phi_P(p_1) > \phi_P(p_2)$, and with the asserted uniqueness properties.

Statement 4: By Axiom 4 and Theorem 2.1 of Krantz et al. (1971), there exists a real-valued function ϕ_Y on Y such that $y_1 R_Y y_2$ iff $\phi_Y(y_1) \geq \phi_Y(y_2)$, and with the asserted uniqueness properties.

Statement 5: Statement 4 says that ϕ_Y preserves R_Y. By the definition of R_Y, it is identical to $\succcurlyeq$ when $\succcurlyeq$ is applied to $P(p_3, p_4) \times Y$ and restricted to Y. So, ϕ_Y also preserves the order $\succcurlyeq$ when it is applied to $P(p_3, p_4) \times Y$ and restricted to Y. By Axiom 6 (ii), there are successive indifference intervals on P with nontrivial regions of overlap. Forcing the local scales to agree yields a global scale on $P \times Y$. The restriction of this scale to Y, ϕ_Y preserves R_Y as well. Statement 5 follows from this, together with the other four statements and the whole construction.

5.1.2 Necessity of Axioms 1–5

Axiom 1 The reflexivity and completeness of $\succcurlyeq$ follow immediately from Statement 5. To show independence of the first attribute from the second, consider a y_1 in Y and assume $(p_1, y_1) \succcurlyeq (p_2, y_1)$. By Statement 5, this means $\phi_P(p_1) > \phi_P(p_2) + \delta_P(p_2)$, which in turn means that $(p_1, y_2) \succcurlyeq (p_2, y_2)$ for any y_2 in Y. To show independence of the second attribute from the first, consider a p_1 in P and assume $(p_1, y_1) \succcurlyeq (p_1, y_2)$. By Statement 5, this means that $\phi_Y(y_1) > \phi_Y(y_2)$, which in turn means that $(p_2, y_1) \succcurlyeq (p_2, y_2)$ for any p_2 in P.

Axiom 2 **Part (i)** of the definition of a semiorder follows immediately from Statement 2.

For **Part (ii)** of the definition, we assume $p_1 R_P p_2$, $p_3 R_P p_4$ and show that if also not $p_1 R_P p_4$, then $p_3 R_P p_2$. By Statement 2, $p_1 R_P p_2$ implies $\phi_P(p_2)+\delta_P(p_2) < \phi_P(p_1)$, and not $p_1 R_P p_4$ implies $\phi_P(p_1) \leq \phi_P(p_4)+\delta_P(p_4)$. Thus, also $\phi_P(p_2)+\delta_P(p_2) < \phi_P(p_4)+\delta_P(p_4)$. This, together with $\phi_P(p_4)+\delta_P(p_4) < \phi_P(p_3)$ (which holds from $p_3 R_P p_4$ and Statement 2), means that $\phi_P(p_2) + \delta_P(p_2) < \phi_P(p_3)$, or, by Statement 2, $p_3 R_P p_2$.

For **Part (iii)** of the definition of a semiorder, we assume $p_1 P_P p_2$ and $p_2 R_P p_3$, and considering a p_4 in P, we show that either $p_4 R_P p_3$ or $p_1 R_P p_4$. Specifically, we show that, if **(a)** $\phi_P(p_4) \geq \phi_P(p_2)$, then $p_4 R_P p_3$, and if **(b)** $\phi_P(p_4) < \phi_P(p_2)$, then $p_1 R_P p_4$.

For **(a)**, $p_2 R_P p_3$ implies, by Statement 2, that $\phi_P(p_2) > \phi_P(p_3) + \delta_P(p_3)$. Together with $\phi_P(p_4) \geq \phi_P(p_2)$, this means $\phi_P(p_4) > \phi_P(p_3) + \delta_P(p_3)$, or, by Statement 2, $p_4 R_P p_3$.

For **(b)**, we first show that $p_1 R_P p_4$ holds if additionally $\phi_P(p_4) + \delta_P(p_4) \leq \phi_P(p_2) + \delta_P(p_2)$. This, together with $\phi_P(p_2) + \delta_P(p_2) < \phi_P(p_1)$ (by $p_1 R_P p_2$ and Statement 2), means that $\phi_P(p_4) + \delta_P(p_4) < \phi_P(p_1)$, or, by Statement 2, $p_1 R_P p_4$ as required.

To complete the argument, we show by contradiction that $\phi_P(p_4) + \delta_P(p_4) \leq \phi_P(p_2)+\delta_P(p_2)$. Suppose $\phi_P(p_4)+\delta_P(p_4) > \phi_P(p_2)+\delta_P(p_2)$. Then it is possible to find a p_5 in P such that: $\phi_P(p_4) + \delta_P(p_4) = \phi_P(p_5) > \phi_P(p_2) + \delta_P(p_2)$. By Statement 2, $\phi_P(p_5) > \phi_P(p_2) + \delta_P(p_2)$ implies $p_5 R_P p_2$.

By Statement 2, $\phi_P(p_4) + \delta_P(p_4) = \phi_P(p_5)$ implies that not $p_5 R_P p_4$. Also, by Statement 1,$\phi_P(p_4)+\delta_P(p_4) = \phi_P(p_5)$ implies that $\phi_P(p_4) < \phi_P(p_5) < \phi_P(p_5)+$

$\delta_P(p_5)$. By Statement 2, this implies that not $p_4 R_P p_5$. Together, not $p_5 R_P p_4$ and not $p_4 R_P p_5$ imply that $p_5 I(R_P) p_4$.

By the assumption of **(b)**, $\phi_P(p_4) < \phi_P(p_2)$ and by Statement 1, $\phi_P(p_4) < \phi_P(p_2) + \delta_P(p_2)$. By Statement 2 this implies that not $p_4 R_P p_2$. Furthermore, from $\phi_P(p_4) + \delta_P(p_4) > \phi_P(p_2)$, which we assumed for contradiction, it follows that not $p_2 R_P p_4$. From not $p_4 R_P p_2$ and not $p_2 R_P p_4$ it follows that $p_4 I(R_P) p_2$.

Having established $p_5 I(R_P) p_4$, $p_4 I(R_P) p_2$ and $p_5 R_P p_2$, by the definition of weak preference $p_4 W(R_P) p_2$.

By Statement 3, $p_4 W(R_P) p_2$ implies $\phi_P(p_4) > \phi_P(p_2)$ which is inconsistent with the assumption of **(b)**, $\phi_P(p_4) < \phi_P(p_2)$. Whence, $\phi_P(p_4) + \delta_P(p_4) \leq \phi_P(p_2) + \delta_P(p_2)$ as required.

Axiom 3 By Statement 5, ϕ_P preserves the order $\succsim_P$ and by Statement 3, ϕ_P preserves the order $W(R_P)$, so $\succsim_P$ and $W(R_P)$ are identical.

Axiom 4 By Statement 4 and Theorem 2.1 of Krantz et al. (1971), Axiom 4 follows.

Axiom 5 By Statement 5, ϕ_Y preserves the order $\succsim_Y$ and by Statement 4, ϕ_Y also preserves R_Y, so $\succsim_Y$ and R_Y are identical.

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