On the Rôles of the Density and Temperature
Gradients in the Theory of Nonlinear Drift Waves

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Abstract

It was recently claimed that in nonlinear coherent drift wave theory the temperature gradient $\kappa_T = \frac{d \ln T}{dx}$ is irrelevant and that, instead of $\kappa_T$, the derivative $d\kappa_n/dx$ of the density gradient $\kappa_n = d \ln n/dx$ (with a factor $T/u$) determines the nonlinearity. It is shown that this claim is erroneous in general and is caused by neglecting the space dependence of the phase velocity $u(x) \approx -\kappa_n T$ of the waves.

The inhomogeneity of the refractive index for drift waves is approximately $\sim \frac{du}{dx} \sim \kappa_T - (d\kappa_n/dx)T/u$. The case $\kappa_n(x)T(x) = \text{const}$, therefore, is special in two respects: the plasma does appear homogeneous to the waves, but also the disparate nonlinear terms coincide. In the generic case, however, $n(x)$ and $T(x)$ are independent profiles, and the temperature gradient nonlinearity is responsible for quasi-one-dimensional soliton-like waves. Their duration, however, is limited by the inhomogeneity of the refractive index.
1. Introduction and conclusions

In the presence of a density gradient \(dn/dx\) perpendicular to a magnetic field \(B = Be_z\) electrostatic drift waves with potential \(\Phi\) can propagate obliquely to \(B\) with phase velocity \(v_{ph}\) of the order of \(v_{se} = -(eTdn/dx)/(enB)\). The waves originate from the \(E \times B\) drift \(v_\perp = v_E = c[E \times B]/B^2\) of the ions in the field \(E = -\nabla \Phi\). In the presence of \(dn/dx \neq 0\) the ion flux \(v_\perp n\) has a non-zero divergence \(\text{div}(nv_\perp) \approx v_\perp \cdot \nabla n\) which produces modulations of the ion density. Electrons tend to follow the ions and at the same time to adjust themselves according to the Boltzmann distribution \(\sim \exp[e\Phi/T(x)]\). The condition that both be compatible determines the phase velocity just mentioned. Depending on the details of the plasma model, this basic picture can be refined in many ways; see, for example, MIKHAILOVSKI (1967).

Here, we want to contribute to the extension of the linear theory into the nonlinear domain, using rather simple model equations. Our aim is to further clarify the recent controversy regarding nonlinear coherent wave propagation, which is discussed further below.

The first nonlinear drift wave equation was derived by TASSO (1967) and included a term proportional to \(\kappa_T \Phi^2\), where \(\kappa_T = dlnT/dx\) is the logarithmic gradient of the electron temperature. Subsequently, ORAEVKII, TASSO and WOBIG (1969) and later HASEGAWA and MIMA (1978) introduced additional nonlinear terms proportional to \(\sim \nabla \Phi \cdot \partial \nabla \Phi/\partial t\) (OTW term) and \(\sim [\nabla \Phi \times \nabla] \cdot e_z \nabla^2 \Phi\) (HM term), respectively, which both arise from the polarization drift \(v_P = -(e\nabla \Phi/\partial t)/(e\Pi)\). Its linear contribution gives rise to a dispersive term. The OTW term is neglected for no specific reasons by most authors, an exception being RAHMAN and SHUKLA (1980), and it is here neglected also. For justification see SALAT (1990), where the relative sizes of all the linear and nonlinear terms occurring from the divergence of the product of \(v_\perp = v_E + v_P\) and \(n(x) = n_0(x) \exp[e\Phi/T(x)]\) are discussed for various scalings of the wave numbers.
In the simplest case, which is also considered here, the waves propagate at right angles to B. If, in addition, any dependence of the waves on the coordinate x in the gradient direction is neglected, the HM term vanishes identically and only the $\kappa_T$ nonlinearity remains. The resulting 1-d equation is found to be the KdV equation (OREFICE and POZZOLI, 1970; MEISS and HORTON, 1982a) or the regularized KdV equation ($\Phi_{xxx}$ is replaced essentially by $\Phi_{xxt}$) (PETVIASHVILI, 1977; MEISS and HORTON, 1982b). They have soliton or soliton-like solutions and also cnoidal wave solutions. Soliton-like solutions, often called solitons as well in the literature, propagate with constant profile and velocity and fall off to infinity but do not survive upon collision (ABDULLOEV, BOGOLUBSKY and MAKHANKOV, 1976).

If the solution is allowed to depend on both x and y, the IIM term $[\nabla \Phi \times \nabla] \cdot e_z \nabla^2 \Phi$ comes into play. HASEGAWA and MIMA (1978) originally used it to study drift wave turbulence. The existence of 2-d ordered structures (drift waves and drift vortices) has been studied by many authors. In some cases the HM term was omitted in favour of the $\kappa_T \Phi^2$ term, e.g. PETVIASHVILI (1980) and LAEDKE and SPATSCHKEK (1986), while in others it was considered dominant, as in MIKHAILOVSKII et al. (1984) and again LAEDKE and SPATSCHKEK (1986).

This now takes us to the controversy mentioned above: Recently, it was claimed that a correct treatment of the HM term and of the space dependence of $n(x)$ and $T(x)$ brings about an exact cancellation of the $\kappa_T$ nonlinearity, replacing it by a term $\sim ((d\kappa_n/dx)T/u)\Phi^2$, where $\kappa_n = dlnn/dx$ and u is the phase velocity; see HORIHATA and SATO (1987), LAKHIN, MIKHAILOVSKII and ONISHCHENKO (1987), (1988) and LAEDKE and SPATSCHKEK (1988). This work has been objected to because the selection of terms from the outset was rather arbitrary and a systematic expansion in terms of $\epsilon = e\Phi/T \ll 1$ reinstalls the $\kappa_T \Phi^2$ term (SALAT, 1990). Objections were raised also by NYCANDER (1989), who pointed out that the space dependence of the density and temperature profiles were, after all, not consistently incorporated.
The controversy just mentioned is connected with the ansatz usually made, $\Phi(x, y, t) = \Phi(x, y - ut)$, where the phase velocity $u$ is treated as a space-independent constant. In the search for an isolated vortex which moves as a whole this ansatz is reasonable. LARICHEV and REZNIK (1976) were the first to obtain such solutions. For wave propagation, however, it seems much more natural to assume a space-dependent phase velocity. For low-amplitude long-wavelength waves, for example, one has approximately $u = u(x) = v_{ec}(x)$, with the $x$-coordinate being singled out by the $x$-dependence of the density and temperature. We therefore find it illuminating to repeat the derivations of the above-mentioned authors with the ansatz $u = u(x)$ ab initio. This space-dependence was taken into account implicitly to lowest order in $\epsilon$ by SALAT (1990), but its relevance was not fully realized.

Section 2 presents the plasma model. The resulting nonlinear p.d.e. is integrated, essentially by following the procedure of LAKHIN et al. (1987), (1988). With the usual ansatz $|\partial^2 \Phi/\partial x^2| < |\partial^2 \Phi/\partial y^2|$ the equation for $\Phi(x, y - u(x)t)$ becomes an o.d.e. with respect to $y$, with $x$ as a parameter only, and a “quasi-1-d” solution is obtained. Cancellation of the $\kappa T \Phi^2$ term occurs in favour of $((d\kappa_n/dx)T/u) \Phi^2$, but a term containing the space derivative of $u$ reverses the cancellation. At the end, therefore, the temperature gradient alone determines the nonlinearity. $e\Phi/T = \epsilon \ll 1$ was used and this is compatible with the plasma model and the procedure of LAKHIN et al. in the long-wavelength region $(k_y\rho_0)^2 \sim \epsilon$ only.

The assumption of small $x$-dependence breaks down after a finite time since $\partial \Phi/\partial x$ contains a term $t(du/dx)\partial \Phi/\partial y$, which grows unboundedly, for $du/dx \neq 0$. The physical reason is obvious: the refractive index $\nu = u(x = 0)/u(x)$ is space-dependent, too, if the phase velocity is. In this case diffraction and refraction ultimately destroy the plane-wave propagation. To dominant order in $\epsilon$ one has $u \approx -\kappa_n T$, in dimensionless units, and one obtains approximately

$$\left| \frac{d\nu}{dx} \right| = \left| \kappa T - \frac{T}{u} \frac{d\kappa_n}{dx} \right| ;$$
see Sec. 3. Thus, the quasi-1-d solution can be indefinitely valid, for \(\frac{d\nu}{dx} = 0\), but this would imply a connection between the density and temperature profiles such that \(\kappa_n(x)T(x) = \text{const}\). In this special case one has \(Td\kappa_n/dx = u\kappa_T\), and the discrepancy between the two \(\Phi^2\) terms proposed disappears. In general, however, \(\kappa_n(x)\) and \(T(x)\) are independent profiles, and hence the refractive index is space-dependent. As mentioned above and as shown subsequently, under these conditions the classical nonlinearity \(\sim \kappa_T\Phi^2\) (TASSO, 1967) is correct as long as the quasi-1-d solution exists.
2. Nonlinear drift waves: derivation

We consider the 'standard' plasma model, namely a plasma with density and temperature gradients in the $x$-direction and a constant magnetic field $\mathbf{B} = B \mathbf{e}_x$ in the $x$-direction. The continuity equation and the equation of motion for the ions, considered to be cold, are

$$\frac{\partial n_i}{\partial t} + \text{div}(n_i \mathbf{v}_i) = 0,$$

$$\left( \frac{\partial}{\partial t} + \mathbf{v}_i \cdot \nabla \right) \mathbf{v}_i = \frac{e}{m_i} \left( -\nabla \Phi + \frac{1}{c} [\mathbf{v}_i \times \mathbf{B}] \right),$$

where $\mathbf{E} = -\nabla \Phi$ and $e$, $m_i$ are the charge and mass of the ions. The electrons with temperature $T'(x)$ are assumed to obey the Boltzmann distribution

$$n_e = \frac{e^\Phi}{n(x) e^T}.$$  

(3)

The system is closed by the quasineutrality condition $n_e = n_i$ and the assumption that $\mathbf{v}_i \cdot \mathbf{e}_x = \partial \Phi / \partial x = 0$. For drift waves with a frequency much less than the ion gyrofrequency $\Omega_i = eB/(m_i c)$ equ. (2) can be solved iteratively, yielding $\mathbf{v}_i \approx \mathbf{v}_E + \mathbf{v}_P$, where

$$\mathbf{v}_E = \frac{c}{B} [\mathbf{e}_z \times \nabla \Phi],$$

$$\mathbf{v}_P = -\frac{c}{B \Omega_i} \left( \frac{\partial}{\partial t} + \mathbf{v}_E \cdot \nabla \right) \nabla \Phi.$$  

(4)

Out of the many terms resulting from div($n_i \mathbf{v}_i$) we only keep those which are considered relevant by many authors; see comments made in the introduction.

It is advantageous to use dimensionless variables. Let $T_0$ be the electron temperature at $x = 0$, say, and $r_0 = L = 1/|\kappa_n(x = 0)|$ the scale length of the density gradient, with $\kappa_n = \text{dlnn}/\text{dx}$ and $\kappa_T = \text{dlnT}/\text{dx}$. The quantity $v_0 = cT_0/(eBL)$ is a velocity of the order of the electron drift velocity $v_{*e} = -cT\kappa_n/(eB)$. We consider the dimensionless quantities $x' = x/r_0$, $y' = y/r_0$, $t' = t/t_0$, where $t_0 = r_0/v_0$, and $T' = T/T_0$, $\Phi' = e\Phi/T_0$, $\kappa_n' = r_0\kappa_n$, $\kappa_T' = r_0\kappa_T$. With the definitions

$$\rho_0^2 = \frac{T_0}{m_i \Omega_i^2}, \quad s = \left( \frac{\rho_0}{L} \right)^2$$

(5)
it follows that \( t_0 = 1/(s \Omega) \). In the following, the primes are again omitted for brevity.

The resulting equation is

\[
\frac{\partial \Phi}{\partial t} - \kappa_n T \frac{\partial \Phi}{\partial y} - sT \frac{\partial}{\partial t} \nabla^2 \Phi - sT[\nabla \Phi \times \nabla] \cdot e_z \nabla^2 \Phi + \kappa_T \frac{\partial \Phi}{\partial y} = 0.
\]  

(6)

We make the wave ansatz

\[
\Phi(x, y, t) = \Phi(x, y - u(x) t).
\]

(7)

With the definition

\[
D = \frac{\partial}{\partial y} - \frac{1}{u} [\nabla \Phi \times \nabla] \cdot e_z
\]

(8)

equ. (6) reads

\[
D \nabla^2 \Phi - \Lambda(x) \frac{\partial \Phi}{\partial y} + 2 S_0(x) \Phi \frac{\partial \Phi}{\partial y} = 0,
\]

(9)

where

\[
\Lambda(x) = \frac{1}{s} \left( \frac{1}{T(x)} + \frac{\kappa_n(x)}{u(x)} \right),
\]

(10)

\[
S_0(x) = \frac{\kappa_T(x)}{2 s u(x) T(x)}.
\]

(11)

With the additional definitions

\[
F = \nabla^2 \Phi - \Lambda(x) \Phi + S(x) \Phi^2
\]

(12)

and

\[
S = S_0 + \frac{1}{2u} \frac{d \Lambda}{dx}
\]

(13)

equ. (9) can be written in the form

\[
D F = \frac{1}{u} \frac{dS}{dx} \Phi^2 \frac{\partial \Phi}{\partial y}.
\]

(14)

The r.h.s. can be neglected provided that

\[
\frac{1}{S} \frac{dS}{dx} \Phi \ll u,
\]

(15)
a condition which follows by comparing the r.h.s. with, for example, the last term in \( F \) on the l.h.s. If \( DF = 0 \) is written in the form \( [\nabla Q \times \nabla F] \cdot e_x = 0 \), it becomes evident that the most general solution is \( F = G(\Phi - \int dxu(x)) \) with arbitrary function \( G \). Not being interested here in vortex solutions we make no use of the freedom in \( G \) and put \( G = \text{const} = 0 \) for simplicity. This finally yields

\[
\nabla^2 \Phi - \Lambda(x) \Phi + S(x) \Phi^2 = 0 .
\]

(16)

The nonlinear term is governed by (see equs. (13) and (10), (11))

\[
S = \frac{1}{2su^2} \frac{d\kappa_n}{dx} - \frac{\kappa_n}{2su^3} \frac{du}{dx} ,
\]

(17)

where two \( \kappa_T \) terms cancelled each other. The first term was derived by, for example, LAKHIN et al. (1987), (1988), where it replaced \( S_0 \); see equ. (11). The second term, however, is missing there and in the other pertinent references.

We look for a quasi-one-dimensional solution defined by

\[
\left| \frac{\partial^2 \Phi}{\partial x^2} \right| \ll \left| \frac{\partial^2 \Phi}{\partial y^2} \right| .
\]

(18)

In this case equ. (16) becomes an o.d.e. With soliton-type boundary conditions at \( y \to \pm \infty \) its solution is

\[
\Phi(x, y, t) = A(x) \sech^2 \left\{ k(x) \left[ y - u(x) t \right] + \delta(x) \right\} ,
\]

(19)

where the amplitude \( A \) and the wave number \( k \) in the \( y \)-direction are related to \( \Lambda \) and \( S \) by

\[
A(x) = \frac{3}{2} \frac{\Lambda(x)}{S(x)} ,
\]

(20)

\[
k^2(x) = \frac{1}{6} S(x) A(x) = \frac{1}{4} \Lambda(x) .
\]

(21)

The amplitude \( A(x) \) and the phase \( \delta(x) \) so far are arbitrary, provided \( k^2(x) \geq 0 \). \( u(x) \) is then determined implicitly by equ. (20).
In order to satisfy the inequality (15), we consider $\Phi \sim A$ to be a small quantity of order $\epsilon \ll 1$. When equ. (20) is written in the form
\[ A \left( u \frac{d\kappa_n}{dx} - \kappa_n \frac{du}{dx} \right) = 3 u^2 \left( \frac{u}{T} + \kappa_n \right), \tag{20a} \]
it follows that either $u \approx -\kappa_n T \sim O(1)$ or $u \sim O(\epsilon)$. Relation (15) can be satisfied only in the first case, as remarked by NYCANDER (1989). This motivates an expansion in $\epsilon$:
\[ u(x) = u_0(x) + u_1(x) + \cdots, \quad u_n \sim \epsilon^n. \tag{22} \]
From equ. (20a) it follows that
\[ u_0(x) = -\kappa_n(x) T(x) \tag{23} \]
and, with a cancellation of two $d\kappa_n/dx$ terms,
\[ u_1(x) = \frac{1}{3} \kappa_T(x) A(x). \tag{24} \]
Similarly, expanding $A$ and $S$ in powers of $\epsilon$ yields
\[ A(x) = \frac{\kappa_T A}{3 s u_0 T} \left( 1 + O(\epsilon) \right), \tag{25} \]
\[ S(x) = \frac{\kappa_T}{2 s u_0 T} \left( 1 + O(\epsilon) \right). \tag{26} \]
Thus, the temperature gradient nonlinearity is back again after all and $S$ coincides with $S_0$ to lowest order. In fact, the solution $\Phi(x,y,t)$, equ. (19), agrees to relevant order with the solution of
\[ \frac{\partial \Phi}{\partial t} - \kappa_n T \frac{\partial \Phi}{\partial y} - sT \frac{\partial}{\partial t} \nabla^2 \Phi + \kappa_T \Phi \frac{\partial \Phi}{\partial y} = 0, \tag{27} \]
i.e. equ. (6) without the HM term $sT[\nabla \Phi \times \nabla] \cdot e_z \nabla^2 \Phi$. This is easily checked by making the ansatz $\Phi = \Phi(x,y-\hat{u}t)$, where $\hat{u} = u_0 + \hat{u}_1$, and $\hat{u}_1 \sim \epsilon$ is yet unknown. One integration is then immediately possible and gives an equation like equ. (16). Proceeding analogously
to the above proves the statement. Alternatively, it is easy to check a posteriori with the obtained solution (19) that the nonlinear HM term is a factor $\epsilon$ smaller than the term $\kappa_T \Phi \partial \Phi / \partial y$, so that it cannot contribute to the order considered. The erroneous claim to the contrary by LAKHIN et al. (1987), (1988) is caused by their neglecting the space dependence of the wave velocity $u(x)$, as pointed out above.

Equations (21) and (23)-(25) show that the solution is in the long-wavelength region $(k_y \rho_0)^2 \sim \epsilon$, where $k_y = k/L$ is the unnormalized wave number in the $y$-direction. The expressions for $u$, $\Lambda$ and $S$ agree with those derived by SALAT (1990), except that the $x$-dependence of $u_1$ was not treated there in a fully consistent way.

It remains to discuss the validity of inequality (18). Since the $x$-dependence enters in various ways, one obtains several independent criteria. The first three of them are straightforward. The fourth criterion, however, offers a new aspect and is treated separately.

1. The profiles $\kappa_n(x)$, $T(x)$ by assumption vary on a scale length $\mathcal{O}(1)$. This yields the condition $k^2 \gg 1$, which corresponds to $s = (\rho_0/L)^2 \ll \epsilon$. For a given plasma this is a lower bound on the soliton amplitude. Its origin is the balancing of the nonlinear term with linear ones.

2. The free profiles $A(x)$, $\delta(x)$ have to vary smoothly enough, the condition being $r^2 \gg (\rho_0/L)^2 / \epsilon$, where $r$ is their dimensionless scale length.

3. Inequality (18) is obviously violated around the two points $y_{\pm}$, where the curvature $\partial^2 \Phi / \partial y^2$ of the soliton hump vanishes. Provided the other conditions are satisfied this is harmless. It only implies a minor $x$-dependent displacement of the points $y_{\pm}$.
3. Refractive index

Since the solution, equ. (19), is of the form \( \Phi(x, \eta = y - u(x)t) \) the \( x \)-derivative is

\[
\frac{\partial \Phi}{\partial x} \bigg|_y = \frac{\partial \Phi}{\partial x} \bigg|_\eta - t \frac{du}{dx} \frac{\partial \Phi}{\partial \eta} .
\]  \hspace{1cm} (28)

The condition \( |\partial^2 \Phi/\partial x^2| \ll |\partial^2 \Phi/\partial y^2| \) for the validity of any quasi-one-dimensional solution therefore requires, in addition to the restrictions discussed above, that

\[
t \left| \frac{du}{dx} \right| \ll 1
\]  \hspace{1cm} (29)

(putting \( t = 0 \) initially, without restriction of generality).

In order to interpret relation (29) it is useful to go back to equ. (6) or (27). To lowest order in \( \varepsilon \) they are reduced to

\[
\frac{\partial \Phi}{\partial t} - \kappa_n(x) T(x) \frac{\partial \Phi}{\partial y} = 0,
\]  \hspace{1cm} (30)

i.e. to a unidirectional version of the wave equation \( \partial^2 \Phi/\partial t^2 = u_0^2 \partial^2 \Phi/\partial y^2 \) with phase velocity \( u_0(x) \). The general solution \( \Phi(x, y - u_0(x)t) \) of equ. (30) is valid for all times although \( |\partial^2 \Phi/\partial x^2| \ll |\partial^2 \Phi/\partial y^2| \) is already violated for \( t \left| du_0/\partial x \right| \gtrsim 1 \). The other terms with \( t \)- and \( y \)-derivatives in equ. (27) make a soliton solution possible. They change the phase velocity into \( u \approx u_0 + u_1 \), which numerically, however, amounts to a small modification only. The term \( \sim \partial^3 \Phi/\partial t \partial x^2 \), finally, brings in diffraction and destroys the quasi-one-dimensional propagation after a finite time.

If \( u(x) \) is Taylor expanded around \( x = 0 \), one obtains from equ. (23) for \( x \ll 1 \)

\[
u(x) \approx u_0(x) \approx u_0(x = 0) \left[ 1 + \left( \kappa_T - \frac{T}{u} \frac{d\kappa_n}{dx} \right) \bigg|_{x=0} x \right],
\]  \hspace{1cm} (31)

with \( u_0(x = 0) = \pm 1 \), where the upper sign corresponds to \( \kappa_n(x = 0) < 0 \). The validity of the quasi-1-d solution therefore depends on \( \Delta \), where

\[
\Delta = \kappa_T - \frac{T}{u} \frac{d\kappa_n}{dx}.
\]  \hspace{1cm} (32)

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The refractive index $\nu$, defined as $\nu = u(x = 0)/u(x)$, to the same approximation is given by $\nu(x) = 1 - \Delta \cdot x$. Relation (29) is therefore equivalent to $t \left| \frac{d\nu}{dx} \right| \ll 1$.

For $\Delta = 0$ the refractive index is spatially uniform and the (nonlinear) wave propagates without limitation and distortion, to the order considered. In this specific case it holds that $\kappa_T = (d\kappa_n/dx)T/u$. The density and temperature profiles are linked in such a way that the distinction between the nonlinear terms of TASSO (1967), PETVIASHVILI (1977), $\sim \kappa_T \Phi^2$, and that of LAKHIN et al. (1987), (1988), $\sim ((d\kappa_n/dx)T/u)\Phi^2$, disappears.

In general, however, the profiles do not satisfy $d(\kappa_n T)/dx = 0$, so that $\Delta \neq 0$ and the effects of the spatially variable refractive index limit the quasi-1-d propagation. Generically, $\nu(x)$ varies on the same scale as the density and temperature profiles so that $\Delta \sim \mathcal{O}(1)$, and relation (29) reduces to $t \ll 1$. In unnormalized units this amounts to $\Omega_t t \ll (L/\rho_0)^2$. The basic frequency of the solution is $\omega = ku$; see equ. (19). Since $k \gg 1$ according to the last section, and $u \sim \mathcal{O}(1)$, the condition $t \ll 1$, however, can still be compatible with $|\omega t| \gg 1$, i.e. with the solution lasting many periods.
References

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