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Hamiltonian Approach to Magnetic Fields with Toroidal Surfaces

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Abstract

The equivalence of magnetic field line equations to a one-dimensional time-dependent Hamiltonian system is used to construct magnetic fields with arbitrary toroidal magnetic surfaces I = const. For this purpose Hamiltonians H which together with their invariants satisfy periodicity constraints have to be known. The choice of H fixes the rotational transform $\eta(I)$. Arbitrary axisymmetric fields, and nonaxisymmetric fields with constant $\eta(I)$ are considered in detail.

Configurations with coinciding magnetic and current density surfaces are obtained. The particular approach used is not well suited, however, to satisfying the additional MHD equilibrium condition of constant pressure on magnetic surfaces.

1. Introduction

In plasma physics the topology of magnetic fields is of particular importance. The magnetohydrodynamic equilibrium equation for the current density $\underline{j}$ and the magnetic field $\underline{B}$,

$$\underline{j} \times \underline{B} = \nabla p, \tag{1.1}$$

requires that the field lines be confined to toroidal nested surfaces I = const, provided the pressure p is constant on I = const surfaces and has a finite normal gradient.

It is well known /1/,/2/,/3/ that the field line equations can be written as a one-dimensional time-dependent Hamiltonian system. An integrable Hamiltonian corresponds to the existence of magnetic surfaces. The aim of the paper is to explore to what extent the Hamiltonian structure may be used to construct magnetic fields with desired properties.

In Section 2 the field line Hamiltonian is recapitulated. It is shown that a magnetic field with arbitrary magnetic surfaces I = const can always be obtained from any suitable integrable Hamiltonian. The Hamiltonian alone determines the rotational transform $\eta(I)$. The general axisymmetric field, a particular class of nonaxisymmetric fields with constant rotational transform, and the combination of the two classes are considered in detail.

The necessary equilibrium condition that the magnetic surfaces and the current density surfaces J = const coincide is discussed in Section 3, and appropriate configurations for the examples mentioned are presented. In Section 4 it is attempted to satisfy all three components of eq. (1.1). It turns out that the particular Hamiltonian approach used is not very suitable for this purpose. Conclusions are given in Section 5.

2. Magnetic surfaces

Let r^1, r^2, r^3 be an arbitrary nonsingular coordinate system, such that r^2 and r^3 are a poloidal and a toroidal angle, respectively, with periodicity 2π. Covariant and contravariant components of any vector $\underline{A} = A^i \underline{e}_i = A_i \nabla r^i$ (with summation convention) are defined by

$$A_i = \underline{A}\cdot\underline{e}_i \qquad \text{and} \qquad A^i = \underline{A}\cdot\nabla r^i, \qquad i = 1, 2, 3, \tag{2.1}$$

respectively, where $\underline{e}_i = \partial\underline{r}/\partial r^i$, $\underline{r}$ = position vector. The usual relations

$$\begin{aligned} A^i &= g^{ij} A_j, \qquad & A_i &= g_{ij} A^j, \\ g^{ij} &= \nabla r^i\cdot\nabla r^j, \qquad & g_{ij} &= \underline{e}_i\cdot\underline{e}_j, \\ \sqrt{g} &= ([\nabla r^1 x \nabla r^2] \cdot\nabla r^3)^{-1} \end{aligned} \tag{2.2}$$

will also be used.

Let $\underline{A}$ be the vector potential of the magnetic field $\underline{B}$. The relation $\underline{B}$ = curl $\underline{A}$, which implies div $\underline{B}$ = 0, in component notation is

$$B^1 = \frac{1}{\sqrt{g}} \left(\frac{\partial A_3}{\partial r^2} - \frac{\partial A_2}{\partial r^3}\right), \tag{2.3}$$

etc., with B^2, B^3 from cyclic permutation of eq. (2.3). To the vector potential $\underline{A}$ one can always add the gradient of an arbitrary function without changing $\underline{B}$. This can be used to obtain $A_1 = 0$. With this gauge one obtains

$$B^2 = \frac{-1}{\sqrt{g}} \frac{\partial A_3}{\partial r^1} , \qquad B^3 = \frac{1}{\sqrt{g}} \frac{\partial A_2}{\partial r^1} . \tag{2.4}$$

If the angle r^3 is used as a parameter along the field lines, the equations for the field lines are

$$\frac{dr^1}{dr^3} = \frac{B^1}{B^3} , \qquad \frac{dr^2}{dr^3} = \frac{B^2}{B^3} . \tag{2.5}$$

The equations are equivalent to a one-dimensional time-dependent Hamiltonian system /1/, /2/ if the following identifications are made:

$$p = - A_2(r^1, r^2, r^3) , \qquad q = r^2 , \qquad t = r^3 \tag{2.6}$$

$$H = A_3(r^1, r^2, r^3) \tag{2.7}$$

(a confusion of the momentum p with the pressure p(I) should not occur). Consider, for example, $\dot{q} = \partial H/\partial p$. From eqs. (2.6) and (2.7) it follows that

$$1 = - \frac{\partial A_2}{\partial p} = - \frac{\partial A_2}{\partial r^1} \frac{\partial r^1}{\partial p} \tag{2.8}$$

and

$$\dot{q} = \frac{\partial A_3}{\partial p} = \frac{\partial A_3}{\partial r^1} \frac{\partial r^1}{\partial p} , \tag{2.9}$$

so that

$$\dot{q} = \frac{dr^2}{dr^3} = -\frac{\partial A_3}{\partial r^1} \Big/ \frac{\partial A_2}{\partial r^1} = \frac{B^2}{B^3} , \tag{2.10}$$

which is the second part of eq. (2.5). The first part follows analogously.

Hamiltonians H(p,q,t) are always integrable in the sense that they possess, in general unknown, single-valued invariants $I_H(p,q,t)$, such that

$$\dot{I}_H = \frac{\partial I_H}{\partial p}\dot{p} + \frac{\partial I_H}{\partial q}\dot{q} + \frac{\partial I_H}{\partial t} = 0 . \tag{2.11}$$

This follows from the fact that the initial values p_o and q_o at $t = t_o$ are related to p and q by a succession of local nonsingular canonical transformations /4/. They therefore represent invariants $I_H(p,q,t)$.

If p is replaced by A_2, eq. (2.6), one obtains invariant surfaces $I(r^1,r^2,r^3)$ = const in configuration space:

$$I_H(p,q,t) = I_H(p(r^1,r^2,r^3),r^2,r^3) \equiv I(r^1,r^2,r^3) . \tag{2.12}$$

These invariant surfaces are magnetic surfaces. This follows from eqs. (2.5) (if it is assumed that $B^3 \neq 0$):

$$\dot{I} = \frac{\partial I}{\partial r^1}\dot{r}^1 + \frac{\partial I}{\partial r^2}\dot{r}^2 + \frac{\partial I}{\partial r^3}\dot{r}^3 = \frac{1}{B^3}\underline{B}\cdot\nabla I = 0 . \tag{2.13}$$

In general, however, the invariant surfaces, apart from being unknown, are unphysical because the invariants I_H will not be (2π -) periodic and hence not single-valued in the angles $r^2 = q$, $r^3 = t$.

So, for a given magnetic field its vector potential in the proper gauge defines a corresponding Hamiltonian system, but in most cases one cannot decide whether it is "truly" integrable, i.e. whether the field has single-valued magnetic surfaces. It is possible, however, to reverse the roles of given and of desired quantities and to use the Hamiltonian formulation for the construction of magnetic fields with desired magnetic surfaces. This can be done in the following way:

Two functions have to be specified to define the relation between the magnetic field and the Hamiltonian system. As will be shown presently, one can choose a Hamiltonian $H(p,q,t)$ and the function $I(r^1,r^2,r^3)$, which defines the magnetic surfaces I = const, provided an invariant $I_H(p,q,t)$ of H is known. Different Hamiltonians correspond to different magnetic fields but with the same magnetic surfaces. Knowledge of time-dependent Hamiltonians with known invariant(s) has steadily increased in recent years (see, for example, /5/ - /8/). Both the Hamiltonians and the invariants have to be such that the resulting magnetic field is periodic in r^2 and r^3.

With $H(p,q,t)$ given and an invariant $I_H(p,q,t)$ known, I_H is equal to the desired function I,

$$I_H(p,r^2,r^3) = I(r^1,r^2,r^3) . \tag{2.14}$$

This defines p as a function of r^1, r^2, r^3 and hence defines the vector potential $A_2 = -p$, eq. (2.6). Furthermore, from eq. (2.7) one gets

$$A_3(r^1,r^2,r^3) = H(p(r^1,r^2,r^3),r^2,r^3) , \tag{2.15}$$

so that $A_1 = 0$, A_2 and A_3, i.e. the magnetic field, is indeed determined by prescribing its magnetic surfaces and a Hamiltonian with known invariant. Examples are given below. The method is only practicable provided eq.(2.14) can be solved explicitly for p. This rules out Hamiltonians with invariants of too complicated structure.

It is also possible to express the vector potentials A_i completely in terms of the magnetic surface label I (instead of r^1) and the angles r^2, r^3. This follows from eq. (2.14), which can be written as

$$I_H(-A_2,r^2,r^3) = I \tag{2.16}$$

and defines $A_2 = \tilde{A}_2(I,r^2,r^3)$. Similarly, one has

$$A_3 = \tilde{A}_3(I,r^2,r^3) = H(-\tilde{A}_2,r^2,r^3) , \tag{2.17}$$

where the tilde indicates the dependence on the surface label. The poloidal and toroidal fluxes Φ_p and Φ_t are then easily found:

$$\Phi_p(I) = \iint dr^1 dr^3 \sqrt{g}\, B^2 = -\oint dr^3 \tilde{A}_3(I,r^2,r^3) ,$$

$$\Phi_t(I) = \iint dr^1 dr^2 \sqrt{g}\, B^3 = \oint dr^2 \tilde{A}_2(I,r^2,r^3) . \tag{2.18}$$

It can be shown that the dependence of Φ_p and Φ_t on r^2 and r^3, respectively, is only apparent, as it of course must be.

The rotational transform $\eta(I)$ is defined by

$$\eta(I) = \frac{\Phi_p'(I)}{\Phi_t'(I)} , \tag{2.19}$$

where the prime denotes differentiation with respect to I. It therefore follows that prescribing the Hamiltonian of a magnetic field is equivalent to prescribing its rotational transform. We shall come back to this point during and after the discussion of several examples.

<u>Example</u> 1: axisymmetric case

In the axisymmetric case the Hamiltonian and the invariant surfaces are independent of $t = r^3$. Hence, the Hamiltonian itself is an invariant:

$$I_H(p,q) = H(p,q). \tag{2.20}$$

One can choose any Hamiltonian H(p,q) and any desired function $I(r^1,r^2)$

for the magnetic surfaces I = const and one immediately obtains the vector potential from

$$\begin{aligned} H(-A_2, r^2) &= I(r^1, r^2), \\ A_3 &= I(r^1, r^2). \end{aligned} \tag{2.21}$$

For $H(p,q)$ 2π - periodic in q the magnetic field is automatically periodic in r^2, because the potentials are. According to eq. (2.4) only the derivative $\partial A_2/\partial r^1$ need be periodic. This also allows a more general dependence of $H(p,q)$ on q.

The magnetic field is obtained from eqs. (2.3), (2.4) and (2.21):

$$\begin{aligned} B^1 &= \frac{1}{\sqrt{g}} \frac{\partial I}{\partial r^2} \\ B^2 &= \frac{-1}{\sqrt{g}} \frac{\partial I}{\partial r^1} \\ B^3 &= \frac{-1}{\sqrt{g}} \left(\frac{\partial H}{\partial p} \right)^{-1} \frac{\partial I}{\partial r^1} . \end{aligned} \tag{2.22}$$

The magnetic fluxes follow from eqs. (2.18) and

$$\begin{aligned} H(-\tilde{A}_2, r^2) &= I , \\ \tilde{A}_3 &= I, \end{aligned} \tag{2.23}$$

which imply $\Phi_p = -2\pi I$. Differentiation of eqs. (2.23) yields the rotational transform

$$\eta(I) = \frac{2\pi}{\oint dr^2 \left(\frac{\partial H}{\partial p} \right)^{-1}} = \frac{2\pi}{\oint dr^2 \frac{B^3}{B^2}} , \qquad (2.24)$$

where $\partial H/\partial p$ or B^2 and B^3 have to be expressed in terms of I and r^2. By properly choosing H(p,q) the rotational transform may be given any desired dependence on I.

Example 2: nonaxisymmetric shear-free case

We consider the special class of time-dependent Hamiltonians

$$H(p,q,t) = p\, G(q,t) , \qquad (2.25)$$

where G is 2π - periodic in both arguments. Invariants of H are assumed to have the form

$$I_H(p,q,t) = p\, K(q,t) , \qquad (2.26)$$

where K is also 2π - periodic in both arguments. $I(r^1,r^2,r^3)$ may again be arbitrary (2π - periodic).

A trivial example of the combined structure of eqs. (2.25), (2.26), case A, is

$$G(q,t) = G_1(q)\, G_2(t) \ ,$$
$$K(q,t) = G_1(q) \ , \tag{2.27A}$$

where G_1 and G_2 are arbitrary periodic functions. Another, nontrivial example /5/, /6/, case B, is

$$G(q,t) = 1 + \left[\omega^2(t) - 1 \right] \cos^2 q \ ,$$
$$K(q,t) = \frac{1}{R^2(t)} \cos^2 q + (\dot{R} \cos q - R \sin q)^2 \ , \tag{2.27B}$$

where R(t) is an arbitrary periodic function and

$$\omega^2(t) = \frac{1}{R^4} - \frac{\ddot{R}}{R} \ . \tag{2.28}$$

Not very much is known about integrals of $H = pG(q,t)$ where G is not related to cases A or B. Preliminary results indicate, for example, that generically K(q,t) contains infinitely many harmonics e^{imq} if G contains more than two of them. In numerical studies related to /9/ using Poincaré surface-of-section cuts, integrability with periodic invariants was found for "most" randomly selected functions G(q,t) in finite parameter intervalls. Hamiltonians of type (2.25), chosen for their formal simplicity, are of a special type, the non-KAM type /9/ (see Appendix). This, however, does not affect the subsequent qualitative conclusions.

From $\dot{I} = (pK)^{\bullet} = 0$ it is straightforward to derive the equation

$$\frac{\partial}{\partial q}\frac{G}{K} + \frac{\partial}{\partial t}\frac{1}{K} = 0. \tag{2.29}$$

Combining eqs. (2.25), (2.26), (2.14), (2.15) and (2.3), (2.4), (2.29) yields

$$\begin{aligned} B^1 &= \frac{1}{\sqrt{g}}\frac{1}{K}\left(G\frac{\partial I}{\partial r^2} + \frac{\partial I}{\partial r^3}\right), \\ B^2 &= \frac{-1}{\sqrt{g}}\frac{G}{K}\frac{\partial I}{\partial r^1}, \\ B^3 &= \frac{-1}{\sqrt{g}}\frac{1}{K}\frac{\partial I}{\partial r^1}. \end{aligned} \tag{2.30}$$

As a consequence, the magnetic field is expressed completely in terms of arbitrary surfaces $I(r^1,r^2,r^3) = \text{const}$, of $G(r^2,r^3)$ and of $K(r^2, r^3)$.

The rotational transform η is found to be

$$\eta = \oint dr^3 \frac{G}{K} \;/\; \oint dr^2 \frac{1}{K}. \tag{2.31}$$

This is evidently independent of the surface I, so that Hamiltonians of type (2.25), (2.26) correspond to shear-free magnetic fields as approximately realyzed in heliotron and stellarator configurations /10/. Equation (2.29) ensures that η is a true surface function.

For the special examples (2.27A), (2.27B) one easily obtains

$$\eta = \langle G_2 \rangle / \langle \frac{1}{G_1} \rangle , \tag{2.32A}$$

$$\eta = \langle \frac{1}{R^2} \rangle , \tag{2.32B}$$

where < > denotes the average with respect to r^2 or r^3.

Example 3: axisymmetric shear-free case

This case, with

$$H(p,q) = p\, G(q) , \tag{2.33}$$

$G(q)$ 2π - periodic, is a combination of examples 1 and 2. It is unnecessary to repeat the results for B^i and $\eta(I)$ which simply follow from eqs. (2.30), (2.31) by setting $K(r^2,r^3) = G(r^2,r^3) = G(r^2)$. Rather, the example is chosen to direct attention to the following point which is relevant below:

Looking for axisymmetric configurations with constant rotational transform $\eta(I)$, one obtains from eqs. (2.18), (2.19) and (2.23)

$$\langle \frac{\partial \tilde{A}_2(I,r^2)}{\partial I} \rangle = - \frac{1}{\eta} , \tag{2.34}$$

which is satisfied by

$$\tilde{A}_2(I,r^2) = - I\, M(r^2) + N(I,r^2) , \tag{2.35}$$

provided $\langle M(r^2)\rangle = \eta^{-1}$ and

$$< N(I,r^2) > = 0. \tag{2.36}$$

With $A_2 = - p$ and $H = I$, one finally obtains

$$H(p,q) = p \frac{1}{M(q)} - \frac{N(H,q)}{M(q)} . \tag{2.37}$$

This shows that $H = pG(q)$ is far from being the most general Hamiltonian with constant rotational transform, ($G = M^{-1}$). There is an arbitrary function $N(H,q)$, apart from condition (2.36), which yields all sorts of Hamiltonians with different structure. This is evidently a general result: If one picks an arbitrary class of of Hamiltonians $H(p,q,t)$, the rotational transform is fixed but, generically, the Hamiltonians selected are not the most general ones with the same $\eta(I)$.

3. Current density surfaces

The MHD equation (1.1) implies that the magnetic fields discussed so far can only be in equilibrium with a finite pressure gradient provided the currents are confined to the same surfaces I = const as the field lines are.

The existence and properties of current density surfaces J = const can be treated, in principle, in the same way as that of magnetic surfaces. The lines of current

$$\frac{dr^1}{dr^3} = \frac{j^1}{j^3} , \qquad \frac{dr^2}{dr^3} = \frac{j^2}{j^3} \tag{3.1}$$

may again be written as a conjugate pair of Hamiltonian equations provided div $\underline{j}$ = 0 and

$$B_1 = 0 . \tag{3.2}$$

While div $\underline{j}$ = 0 is automatically satisfied with $\underline{j}$ = curl $\underline{B}$, the second condition, which in the case of field lines is a simple condition on the gauge of the vector potential, is nontrivial. In order to satisfy it, one has to abandon the arbitrariness of the coordinate system: if one chooses r^1 as a function of **the surface label I only**, and the surfaces r^2, r^3 = const orthogonal to I = const, i.e.

$$g^{12} = g^{13} = 0 , \tag{3.3}$$

then $\underline{B}\cdot\nabla I = 0$ implies $B^1 = B_1 = 0$. Such adapted coordinates will be used in the following except where otherwise stated (examples 1 and 3 below).

By analogy with eqs. (2.6), (2.7) the canonical coordinates Q, P and the Hamiltonian H_c will be defined as

$$P = -B_2(r^1,r^2,r^3)\ , \qquad Q = r^2\ , \qquad t = r^3\ , \tag{3.4}$$

$$H_c = B_3(r^1,r^2,r^3)\ . \tag{3.5}$$

B_2 and B_3 are not arbitrary because magnetic surfaces have been assumed to exist. Nevertheless, the Hamiltonian system (3.4), (3.5) is still so general that the existence of invariant surfaces J = const is open. For this reason we again restrict further discussion to the examples 1 - 3 of the previous section.

<u>Example</u> 1:

In the axisymmetric case the Hamiltonian H_c is also independent of t, and $H_c(P,Q)$ itself is an invariant $J_H(P,Q)$. From eq. (3.5) and by analogy with eqs. (2.21) an invariant in configuration space is

$$J(r^1,r^2) = B_3. \tag{3.6}$$

The equilibrium condition that the surfaces of field lines I = const and of current density lines J = const coincide therefore requires that

$$B_3 = F(I) \ , \tag{3.7}$$

where F(I) so far is an arbitrary function. This condition also follows direct from $\underline{j}\cdot\nabla I = 0$, $\underline{j} = \text{curl}\ \underline{B}$ and axisymmetry, which proves that it is valid for arbitrary coordinates to which we revert in this example and example 3.

Equations (2.22) and (3.7) for the magnetic field can be combined into the classical form /11/, /12/

$$\underline{B} = \left[\nabla I \times \nabla r^3\right] + F(I)\ \nabla r^3 \ . \tag{3.8}$$

From eq. (3.7), the third of eqs. (2.22) and axisymmetry one obtains

$$F(I) = \frac{-1}{\sqrt{g}\ g^{33}} \frac{1}{\frac{\partial H}{\partial p}} \frac{\partial I(r^1, r^2)}{\partial r^1} \ , \tag{3.9}$$

where $\partial H/\partial p$ is to be understood at $p = - \tilde{A}_2(I, r^2)$ or $p = - A_2(r^1, r^2)$. Given H(p,q) and F(I), this differential equation determines the shape of the magnetic surfaces $I(r^1, r^2)$ = const. It is unexpected that the magnetic surfaces are already determined (up to free functions which appear in the integration procedure) from the condition that the current density surfaces coincide with I = const. In Section 4 the relation with MHD equilibrium surfaces will be discussed.

From eq. (2.24) and the poloidal average of eq. (3.9) another relation may be derived,

$$F(I) = \frac{1}{\eta(I)} \frac{-1}{\langle \sqrt{g}\, g^{33} / \frac{\partial I}{\partial r^1} \rangle} \tag{3.10a}$$

$$= \frac{1}{\eta(I)} \frac{1}{\langle g^{33} / B^2 \rangle} , \tag{3.10b}$$

which shows that F(I) is determined by the shape of the magnetic surfaces and the rotational transform. This relation is more general than eq. (3.9) because is does not contain H(p,q) explicitly. It is therefore valid for all H(p,q) which have the same $\eta(I)$ (see Sec. 2, example 3). Relation (3.9b) is useful in the discussion of flux-conserving tokamak equilibria /13/.

Example 2:

From eqs. (2.30), (3.4) and (3.5), and keeping in mind the special coordinate system introduced above, one obtains

$$P = (g_{22}\, G + g_{23}) \frac{1}{\sqrt{g}} \frac{1}{K} \frac{dI}{dr^1} , \tag{3.11}$$

$$H_c = - (g_{23}\, G + g_{33}) \frac{1}{\sqrt{g}} \frac{1}{K} \frac{dI}{dr^1} , \tag{3.12}$$

so that the Hamiltonian for the current density may be written as

$$H_c(P,Q,t) = P\,\Gamma(P,Q,t)\ , \qquad (3.13)$$

where

$$\Gamma(P,Q,t) = -\frac{g_{23}\,G(r^2,r^3) + g_{33}}{g_{22}\,G(r^2,r^3) + g_{23}}\ . \qquad (3.14)$$

The dependence of Γ on r^1,r^2,r^3 is determined by $G(r^2,r^3)$ and the surfaces I = const. Inversion of eq. (3.11) for $r^1 = r^1(P,r^2,r^3)$ leads to $\Gamma(P,Q,t)$. In general, however, this cannot be done explicitly.

In order to proceed further, we consider the hypothetical case

$$\Gamma(P,Q,t) = \Gamma(Q,t). \qquad (3.15)$$

As a consequence,

$$H_c(P,Q,t) = P\,\Gamma(Q,t) \qquad (3.16)$$

would have the same structure as H itself, viz. eq. (2.25), and the results of Section 2 could therefore be applied to the current surfaces as well.

The question is, can condition (3.15) be satisfied? A sufficient, purely geometric condition is

$$g_{23} = 0 \qquad (3.17)$$

together with

$$\frac{\partial}{\partial r^1} g_{22}(r^1,r^2,r^3) = \frac{\partial}{\partial r^1} g_{33}(r^1,r^2,r^3) = 0 . \tag{3.18}$$

From $(d\underline{r})^2 = g_{ik}\, dr^i\, dr^k$ it follows that the arc lengths L_p and L_t of poloidally and toroidally closed coordinate lines r^1,r^3 = const and r^1,r^2 = const are given by

$$L_p = \oint dr^2 \sqrt{g_{22}} , \qquad L_t = \oint dr^3 \sqrt{g_{33}} , \tag{3.19}$$

respectively. Equations (3.18) therefore require that the lengths $L_p(r^1,r^3)$ and $L_t(r^1,r^2)$ be the same from one surface $r^1 = I$ to the next. It should be possible to satisfy this requirement by the following scheme:

Starting from a relatively smooth toroidal surface, the successive surfaces towards the inside are progressively corrugated, both in the poloidal and toroidal cross-sections, in such a way that the lengths of the coordinate lines stay constant. This procedure can be continued until the corrugated surfaces reach the magnetic axis or close the hole of the torus. Obviously, there is considerable freedom in the configuration, stemming from the arbitrariness of the initial surface. It should be possible to select it such that the coordinate surfaces are mutually orthogonal in order to satisfy eq. (3.17).

Having "constructed" nested sets of surfaces destined to become current density surfaces J = const, one is free to use them as magnetic surfaces I = const as well, setting

$$J = F(I), \tag{3.20}$$

with F(I) arbitrary. As a final problem, $G(r^2,r^3)$ has to be found such that not only does H = pG(q,t) have doubly periodic invariants but also

$$H_c = -P\,\frac{g_{33}(Q,t)}{G(Q,t)\;g_{22}(Q,t)}\;. \tag{3.21}$$

According to the experience gained in /9/ this should be possible. This requires, however, that an explicit representation of the corrugated surfaces together with the proper coordinate system be at hand.

Example 3:

With $G(r^2,r^3) = K(r^2,r^3) = G(r^2)$ and eq. (2.31) the relation (3.10a) between F(I) and I becomes

$$F(I) = \left\langle \frac{1}{G} \right\rangle \frac{-1}{\left\langle \sqrt{g}\, g^{33} \,/\, \frac{\partial I}{\partial r^1} \right\rangle}\;. \tag{3.22}$$

On the other hand, for given F(I) and H = pG(q) the differential equation (3.9) may be integrated explicitly to give the family of magnetic surfaces

$$I(r^1,r^2) = f\left(I_o(r^2) + G(r^2)\, c(r^1,r^2) \right) , \tag{3.23}$$

where $I_o(r^2)$ is an arbitrary periodic function,

$$c(r^1,r^2) = \int_o^{r^1} dr^1 \sqrt{g}\, g^{33} , \tag{3.24}$$

and the inverse of f is defined by

$$f^{-1}(I) = - \int_o^{I} \frac{dI}{F(I)} . \tag{3.25}$$

The radial integration in eq. (3.24) may be performed in, for example, orthogonal coordinates which are related to Cartesian coordinates x,y,z by

$$x = R \cos r^3 , \qquad y = R \sin r^3 , \qquad z = r^1 \sin r^2 , \tag{3.26}$$

where $R = R_o - r^1 \cos r^2$, and R_o is the radius of the circular magnetic axis. With $\sqrt{g} = r^1 R$, and $g^{33} = 1/R^2$ one obtains

$$c(r^1,r^2) = \frac{-1}{\cos^2 r^2} \left[r^1 \cos r^2 + R_o \ln \left(1 - \frac{r^1}{R_o} \cos r^2 \right) \right] . \tag{3.27}$$

For $|r^1 \cos r^2| \ll R_o$, $c(r^1,r^2) \cong (r^1)^2/(2R_o)$ stays bounded. For f, I_o and $G \neq 0$ the surfaces I = const are nested around the magnetic axis, bounded, and crowded together close to the axis of symmetry.

Again, discussion of eq. (3.23) is deferred to the next section.

4. MHD equilibrium pressure balance

In the last two sections we have either derived or made plausible the existence of configurations with coinciding nested surfaces $I(r^1,r^2,r^3) = \text{const}$ of field lines and current density. This, however, is not sufficient for the existence of MHD equilibria with surfaces of constant pressure. The coinciding surfaces imply (in arbitrary coordinates)

$$[\underline{j} \times \underline{B}] = a(r^1,r^2,r^3)\ \nabla I , \tag{4.1}$$

where $a(r^1,r^2,r^3)$ depends on the details of the field. According to eq. (1.1) however, with $p = p(I)$, the function a has to depend on the coordinates via $I(r^1,r^2,r^3)$ only:

$$a = \frac{dp(I)}{dI} . \tag{4.2}$$

This yiels an additional condition. In axisymmetry it is the Grad-Shafranov condition /11/, /12/ (4.4). In order to discuss the equilibrium condition, we again turn to the previously used examples.

Examples 1 and 3:

The first component of the equilibrium relation is

$$(\nabla p)_1 = \frac{dp}{dI}\frac{\partial I}{\partial r^1} = \sqrt{g}\ (j^2B^3 - j^3B^2) . \tag{4.3}$$

B^3 may be expressed in terms of F(I), eq. (3.7), while all other quantities may be expressed in terms of derivatives of $I(r^1,r^2)$ via the first two of eqs. (2.22). The result is the Grad-Shafranov equation

$$\frac{\partial}{\partial r^1}\left(\frac{1}{\sqrt{g}\,g^{22}}\frac{\partial I}{\partial r^1}\right) + \frac{\partial}{\partial r^2}\left(\frac{1}{\sqrt{g}\,g^{11}}\frac{\partial I}{\partial r^2}\right) + \\ + \sqrt{g}\,\frac{dp}{dI} + \sqrt{g}\,g^{33}\,F(I)\,\frac{dF}{dI} = 0\,, \qquad (4.4)$$

written here for arbitrary orthogonal coordinates r^1,r^2, with r^3 being the angle around the axis of symmetry. The two profiles p(I) and F(I) have to be specified in order to determine the magnetic surfaces $I(r^1,r^2)$ = const.

In Section 3, on the other hand, magnetic surfaces have been dedermined from H(p,q) and F(I) merely by the requirement that the magnetic and current density surfaces be identical. The question is whether these "Bj-surfaces" are always contained among the "GS-surfaces" (Grad-Shafranov) with some pressure profile p(I) to be determined a posteriori.

This can be tested in example 3 with the Bj-surfaces given explicitly in eq. (3.23). Inserting $I(r^1,r^2)$ in eq. (4.4) yields

$$\frac{dp}{dI} = -F\,\frac{dF}{dI}\left\{G^2\,\frac{(r^1)^2}{R^4} + \frac{1}{(r^1R)^2}\left[\frac{\partial}{\partial r^2}(I_o+Gc)\right]^2 + \\ + \frac{1}{R^2}\right\} + F\,\frac{1}{r^1R}\left\{G\frac{\partial}{\partial r^1}\left(\frac{r^1}{R}\right)^2 + \frac{\partial}{\partial r^2}\left[\frac{1}{r^1R}\,\frac{\partial}{\partial r^2}(I_o+Gc)\right]\right\}, \qquad (4.5)$$

where $G(r^2)$, $I_o(r^2)$ are the only free functions. Owing to the given dependence of the r.h.s. on r^1, and there being no free function of r^1 available, it is obviously impossible in general to turn the r.h.s. into a function of $I(r^1,r^2)$ alone by any nontrivial choice of G and I_o. In consequence, it is impossible in general to proceed from Bj-surfaces to GS-surfaces if the Bj-surfaces are derived from an arbitrary pair of functions $H(p,q)$, $F(I)$.

The reason for this result is easy to understand. If instead of the differential eq. (3.9) its poloidal average, eq. (3.10a), is used to express $F(I)$ in terms of $I(r^1,r^2)$, the GS-equation becomes an integro-differential equation for $I(r^1,r^2)$. The profile $\eta(I)$ has to be specified, with $\eta(I)$ = const in example 3. The solution $I(r^1,r^2)$ in turn determines $F(I)$, eq. (3.10a), so that the differential equation (3.9) finally serves to determine $(\partial H/\partial p)^{-1}$. More precisely, since the averaged equation has already been satisfied, the poloidal variation $\delta(\partial H/\partial p)^{-1}$ is determined, where $\delta A = A - \langle A\rangle$. On the other hand, from Section 2, eq. (2.37), one has

$$\delta \left(\frac{\partial H}{\partial p} \right)^{-1} = \delta \frac{1}{G} + \frac{\partial N(H,q)}{\partial H} , \tag{4.6}$$

so that $N(H,q) \not\equiv 0$ in general. This implies that the rotational transform profile $\eta(I)$ = const requires a more general Hamiltonian than $H = pG(q)$ if the configuration is to satisfy the Grad-Shafranov equation. This result is valid for other profiles $\eta(I)$ as well: a particular

Hamiltonian $H(p,q)$ which produces a desired $\eta(I)$ is in general not good enough to encompass a solution to the Grad-Shafranov equation.

It is of course possible to obtain axisymmetric MHD equilibria with desired $p(I)$, $\eta(I)$ without bothering about a Hamiltonian. As indicated above, eqs. (3.10a), (4.4) combine into an integro-differential equation for I which finally yields $F(I)$. The magnetic field follows from eq. (3.8). This approach was used in /13/ for the description of "flux conserving tokamaks". Fixed totoidal and poloidal magnetic fluxes also encompass fixed $\eta(I)$. The integro-differential equation was solved approximately in /13/, but we shall not further discuss it here. Incidentally, $\eta(I)$ = const as in example 3 was assumed in /13/.

Example 2:

The particular magnetic and current density surfaces which were obtained in Section 3 for this example were constructed starting inwards from a boundary surface $r^1 = r^1_o = I = S(r^2,r^3)$ = const which also determines the coordinate system and the metric coefficients $g^{ij}(r^1,r^2,r^3)$. The MHD equilibrium condition, eqs. (4.1), (4.2), again requires that $[\underline{j} \times \underline{B}]_1$, which is a functional of $F(I)$, $G(r^2,r^3)$, $S(r^2,r^3)$, depend on $I = r^1$ only. In view of the many terms which compose $[\underline{j} \times \underline{B}]_1$ (see eq. (4.5) for the much simpler case of axisymmetry) it seems virtually impossible, however, to select $G(r^2,r^3)$ and $S(r^2,r^3)$ in such a way that the dependence on r^2 and r^3 completely disappears. This is particularly so because it is probably only possible to give the g^{ij}

explicitly in terms of $S(r^2,r^3)$ for special cases at most.

In consequence, just as in the axisymmetric case, the Hamiltonian method with fixed Hamiltonian and restricted magnetic surfaces is unsuited in practice to proceeding further from configurations with equal magnetic and current density surfaces to MHD equilibria.

5. Conclusions

The equations for magnetic field lines are equivalent to a Hamiltonian system. This has been used to obtain a representation of magnetic fields in terms of their toroidal magnetic surfaces I = const, which may be arbitrary, and of arbitrary Hamiltonians H(p,q,t) with appropriate invariants $I_H(p,q,t)$. The Hamiltonian uniquely determines the rotational transform $\eta(I)$, but not vice versa. For the general axisymmetric field, for a particular class of nonaxisymmetric fields with constant $\eta(I)$, and for the combination of both classes explicit expressions are given for the magnetic field $\underline{B}$ and for $\eta(I)$.

For the axisymmetric case configurations with coinciding $\underline{B}$-surfaces and $\underline{j}$-surfaces are given explicitly, while for the nonaxisymmetric examples it is shown qualitatively that a particular type of corrugation should make the $\underline{B}$- and $\underline{j}$-surfaces coincide.

The step onwards from necessary equilibrium conditions to a full equilibrium solution, however, proves to be unfeasible. The reason is that in order to make the pressure be a surface function, H(p,q,t) should not be fixed a priori: The above-mentioned freedom in H for given $\eta(I)$ is essential to satisfy the pressure balance. With the Hamiltonian not fixed a priori, however, nothing is known about invariants of H in general, and the method used breaks down. This deficiency is not causally related to the well-known uncertainty whether nonaxisymmetric equilibria exist at all, since it already occurs in the axisymmetric case. It is unlikely that the situation is different for the Hamiltonian system in /3/, which differs from the one used here in that, for example, the toroidal flux is treated as the canonical momentum.

In spite of the shortcomings mentioned, the Hamiltonian approach is nevertheless useful for the equilibrium problem. In /14/, for example, other aspects of this approach were used successfully to minimize deviations from a true MHD equilibrium.

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Appendix

It is easily seen that the Hamiltonian (2.25) is equivalent to a two-dimensional time-independent Hamiltonian system

$$H(p_1,p_2,q_1,q_2) = p_1 G_1(q_1,q_2) + p_2 G_2(q_1,q_2) \tag{A.1}$$

with $G_2 \equiv 1$. For Hamiltonians with arbitrary 2π -periodic G_1, G_2 one has

$$\det \left\{ \frac{\partial^2 H_o}{\partial p_i \, \partial p_j} \right\} = 0 , \tag{A.2}$$

where H_o is any part of H. In this case the theorem of Kolmogorov, Arnold and Moser (KAM) /15/ cannot be applied. For numerical studies see /9/.

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