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Subroutine for Series Solutions of Linear Differential Equations

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Abstract

A subroutine for Taylor series solutions of systems of ordinary linear differential equations is described. It uses the old idea of Lie series but allows simple implementation and is time-saving for symbolic manipulations.

Taylor series for solving systems of autonomous differential equations can be obtained by Lie series constructed by iteration of the Lie operator corresponding to the system of equations. [1] The iterations of the Lie operator can be very tedious for general systems.

For linear systems of differential equations with variable coefficients

(1) $$y_i' = \sum_k u_{ik} y_k \qquad i = 1..m, \; k = 1..m$$

it is found more efficient to leave the system non-autonomous and iterate in a more appropriate way, as is described here.

For one differential equation

$$y' = uy$$

it is known that

$$y'' = u'y + u^2y,$$

and, if $y^{(k)} = a_k(x)y$,

then $y^{(k+1)} = (a_k' + a_k u)y$,

so that the Taylor expansions of the solution can be constructed using the recursion formulae:

(2) $$a_1 = u \ , \quad a_{k+1} = a_k' + a_k u$$

and

(3) $$y = y(x_o) \sum_{k=o} \frac{1}{k!} a_k(x_o) \ (x-x_o)^k \ , \ a_o = 1 \ .$$

This can easily be extended to a first-order system

(4) $$Y' = UY,$$

where $Y = \begin{pmatrix} Y_1 \\ \vdots \\ Y_m \end{pmatrix}$ $\quad U = \begin{pmatrix} u_{11} & \cdots & u_{1m} \\ \vdots & & \vdots \\ u_{m1} & \cdots & u_{mm} \end{pmatrix}$ $\quad u_{ij} = u_{ij}(x)$,

and the recursion formula (2) becomes

(5) $$A_1 = U \ , \quad A_{k+1} = A_k' + A_k \cdot U \ ,$$

where the A_k are matrices.

$$(6) \qquad Y = \left[\sum_{k=o} \frac{1}{k!} A_k(x_o)\ (x-x_o)^k \right] Y(x_o)$$

The recursion formula (5) is verified by the following algorithm in Algol-like notation:

```
read (m)                 number of differential equations
matrix (A,U,Y) [1:m,1:m]; vector Z [1,m]
read (p,x0)              order of the Taylor series, starting point
read (U)                 matrix elements
A:=U; f:=(x-x0); Y:=f*sub(x=x0,A);
for k:=2:p  do begin
f:=f(x-x0)/k;  A:=A*U + df(A,x)
Y:=Y+f*sub(x=x0,A) end ;
read (Z);   initial values
Y:=Y*Z
write(Y);   output
```

It is assumed that matrix operations, substitution and differentiation operators are implemented. The computing times are stated in the examples.

In the case of a single equation of n-th order

$$y^{(n)} + \sum_{k=o}^{n-1} u_k y^{(k)} = o$$ the corresponding system

will possess a rather sparse matrix

$$\mathcal{U} = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ \vdots & & \ddots & \ddots & \vdots \\ \vdots & & & \ddots & 0 \\ 0 & \cdots & \cdots & 0 & 1 \\ -u_0 & \cdots & \cdots & \cdots & -u_{n-1} \end{pmatrix}$$

but the iterations A_k will be rapidly populated.

The following examples were calculated with REDUCE 2 [2] on the IBM 360/91 computer at Max-Planck-Institut für Plasmaphysik (Garching). REDUCE is slow for matrix operations.

<u>Example 1</u>

$$y'' + y = o \Rightarrow \mathcal{U} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

let us expand around $x_o = o$ up to order 9.

Computation time for symbolic manipulation: 3 sec.

<u>Example 2</u>: Bessel equation

$$x^2y'' + xy' + (x^2-n^2)y = o$$

for $n = o$, $x_o = 1$ up to order 9.

The Taylor series obtained is evaluated in double precision with Fortran. For $x_o = 2$ y is approached up to the seventh decimal.

Computation time for "symbolic" : 9 sec

Computation time for "numerical": 6 milli sec.

References

[1] W. Gröbner: Die Lie-Reihen und ihre Anwendungen (Deutscher Verlag der Wissenschaften, Berlin 1960)

W. Gröbner, H. Knapp: Contributions to the Method of Lie Series (Hochschultaschenbücher-Verlag 802/802a) (1967)

[2] A.C. Hearn: Reduce 2 User's Manual. Univ. of Utah, Salt Lake City, Utah 84112, U.S.A.

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