On $B_j$ Stabilization in the Electron Ring Accelerator

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Abstract

The focusing and stabilizing properties of a superposed $B_y$-field in the electron ring accelerator are discussed. With electron rings of high particle density the $B_y$-field can contribute to focusing during compression and acceleration. The $B_y$-field makes it possible, moreover, to shift the betatron frequencies, thus avoiding betatron resonances during compression.

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Introduction

It has been proposed [1, 2] for electron ring accelerators that a magnetic \( B_y \) field be superposed to focus and stabilize the electron ring on transition from the compression phase to the acceleration phase and during the actual acceleration. This paper discusses the possibility of focusing the electron ring and avoiding betatron resonances during compression by applying a \( B_y \) field. It is then investigated whether the ring can also be focused during acceleration and, finally, the influence of the \( B_y \) field on the dynamics of the acceleration is considered.

In electron ring experiments [3, 4, 5] relativistic electron rings are first compressed in a magnetic mirror field rising with time and then accelerated in an electric or non-uniform magnetic field after being loaded with ions. During compression the electrons are focused in the mirror field with field index \( n : 0 < n < 1 \). Focusing is limited by the space charge of the rings. Superposing a \( B_y \) field, produced by a current along the axis of the mirror field, extends the stable region and allows rings with higher space charge to be focused. The field index \( n \) is not constant in time during compression, but generally traverses a large part of the stable region to become \( n = \sigma' \) at the end of compression. The electrons thereby pass through betatron resonances, which may enlarge the minor ring radius or even destroy the ring. An additional \( B_y \) field now affords the possibility of avoiding dangerous resonances, particularly the \( q-1 \) resonance at \( n = \sigma' \), by shifting the betatron frequencies. These questions are dealt with in Section 1.
Section 2 treats the influence of the $B_y$ field on the dynamics of acceleration and the $B_y$ focusing during acceleration. The field index during acceleration is $n \approx 0$, and so the ring is not focused axially by the external field. If one does not want to rely solely on the self-focusing of the ion loaded ring, an external focusing field has to be provided. Focusing can be achieved with a $B_y$ field for sufficiently large space charge of the rings. Here and in [2], however, it is obvious that neglecting the space charge reducing influence of the ions makes the results doubtful. In order to avoid ion losses, moreover, it should be ensured that the holding power of the ring exceeds the force $K = e v_2 B_y$ acting on the ions. The effect of the $B_y$-field on the acceleration itself is essentially that it reduces the acceleration, just as if the mass of the ring were increased a factor of $1 + \alpha^2$, where $\alpha = B_y/B_z$ is the ratio of the $B_y$-field to the main field $B_z$.

1. Betatron oscillations with superposed $B_y$-field

If a $B_y$-field, produced by a current along the symmetry axis, is superposed on a magnetic mirror field and the electric and magnetic self-fields of the electron ring are taken into account, the linear betatron oscillations of the particles about the closed orbit are modified. This is investigated here. For this purpose we start by deriving the equations for the linear betatron oscillations.

The relativistic equations of motion for a particle of mass $m$ and charge $e$ in an electromagnetic field are expressed in cylindrical coordinates $(\tau, \psi, z)$ as follows:
\( (\gamma \dot{\gamma^2} - \ddot{\gamma} \dot{\gamma}) = \frac{e}{m} \left( E_x + \dot{\gamma} B_z - \ddot{z} B_y \right) \),

\( \dot{\gamma} \dot{\gamma} + r (\dot{\gamma} \dot{\gamma}) = \frac{e}{m} \left( E_y - \dot{\gamma} B_z + \ddot{z} B_x \right) \),

\( \dot{\gamma} \dot{\gamma} = \frac{e}{m} \left( E_z + \ddot{\gamma} B_x + \dot{\gamma} B_y \right) \),

\( \ddot{\gamma} \dot{\gamma} = \frac{e}{m} \left( \dot{\gamma} E_x + \dot{\gamma} B_y + \ddot{\gamma} E_z \right) \).

where the velocities are subject to the condition

\( r^2 + \dot{\gamma}^2 + \ddot{\gamma}^2 = 1 - \frac{\gamma^2}{c^2} \).

In these equations the dot denotes the derivative with respect to time \( (\dot{} = \frac{d}{dt}) \) and the velocity of light is \( c = 1 \). The condition (1.2) is an integral of the equations (1.1). Equation (1.2) thus replaces one of the equations in (1.1).

The electromagnetic field is composed of a magnetic mirror field, given by its vector potential \( A = (0, A_y(r, z), 0) \), and a \( B_y \) field, produced by a current along the \( z \)-axis. The components of the magnetic field are then

\( B_y = -\frac{\partial}{\partial z} A_y(r, z), \quad B_y = \frac{\partial}{\partial r} A_y(r, z), \quad B_z = \frac{\partial}{\partial r} A_y(r, z) \).

We also have the electric and magnetic self-fields of the electron ring, their contributions being formulated a few lines further on.
We shall begin with the equilibrium solution of the system (1.1). Let the vector potential be given such that eqs. (1.1) and (1.2) have the following steady-state solution: \( \tau = R_0, \varphi = 0 \quad \tau - \dot{\tau} = 0 \), and \( \gamma^\varphi = \nu^\varphi, \), with the external magnetic field and the radius \( R_0 \) being related as follows:

\[
(1.4) \quad m \gamma^\varphi \nu^\varphi = -e R_0 B_2, \quad \gamma^\varphi = (1 - \nu^\varphi)^{\frac{1}{2}}
\]

Equilibrium does not depend on \( B_\varphi \) and the self-fields of the ring in so far as toroidal effects are ignored, as will be the case here.

In order to investigate solutions in the vicinity of equilibrium, we expand eqs. (1.1) and (1.2) about the equilibrium. We then introduce the variables \( x, \dot{z}, \delta \nu^\varphi, \delta \dot{\gamma} \), which are related to the original variables as follows:

\[
(1.5) \quad x = \tau - R_0, \quad \dot{z} = \dot{\tau}, \quad \delta \nu^\varphi = \nu^\varphi - \nu^\varphi_0, \quad \delta \dot{\gamma} = \dot{\gamma} - \dot{\gamma}_0,
\]

where \( \nu^\varphi = \tau \dot{\varphi} \).

Expansion of the external magnetic field about the equilibrium radius \( R_0 \) yields

\[
(1.6) \quad B_\varphi(x, z) = B_\varphi(1 - \frac{x}{R_0} \, n),
\]

\[
B_2(x, z) = -B_2(1 - \frac{x}{R_0} \, n),
\]

\[
B_\gamma = B_\gamma(1 - \frac{x}{R_0}),
\]

where \( B_\varphi = B_\varphi(\tau = R_0, \dot{z} = 0) \) and \( B_\gamma = B_\gamma(\tau = R_0, \dot{z} = 0) \). The field index used in eqs. (1.6) is defined by

\[
(1.7) \quad n = -\frac{R_0}{B_2} \frac{\partial}{\partial \tau} \bigg|_{\tau = R_0, \dot{z} = 0}
\]
The electric and magnetic self-fields of the electron ring are given by

\[ E_z = \frac{4\pi e}{\pi a^2} \frac{N}{\alpha^2} \frac{x}{z}, \quad B_z = \frac{4\pi e}{\pi a^2} \frac{N}{\alpha^2} \frac{z}{z}, \]

\[ E_z = \frac{4\pi e}{\pi a^2} \frac{N}{\alpha^2} \frac{2}{z}, \quad B_\phi = -\frac{4\pi e}{\pi a^2} \frac{N}{\alpha^2} \frac{x}{z}, \]

(1.8) where \( a \) denotes the minor ring radius and \( N \) the particle number per cm circumference. The electron density is assumed to be constant over the cross section of the ring; toroidal effects are neglected.

We now expand the equations of motion (1.1) and (1.2) and insert the expressions (1.6) and (1.8) for the external fields. From the last equation of (1.1) and eq. (1.2) it then follows that

\[ \delta y = \frac{e}{m} \frac{4\pi e}{\pi a^2} \frac{N}{\alpha^2} \frac{1}{z} \left( x \ddot{x} + z \ddot{z} \right), \]

\[ \delta v_y = -\frac{1}{v_y, \gamma_0^3} \delta y \]

(1.9)

The right-hand side of the first equation is of second order. In the linear approximation considered here it thus follows that \( \delta y = \text{const} \) and, since attention is to be confined to orbits with initial conditions \( \delta y = 0 \) at \( \xi = 0 \), eqs. (1.9) yields

\[ \delta v_y = \sigma, \quad \delta y = \sigma \]

(1.10)

\( 1 \)
Using the results (1.10) and linearising the first and third equations of (1.1) gives the following equations for $x$ and $z$:

\[
\ddot{x} + \left( \frac{V_y}{R_0} (1-n) - \frac{2 e^2 N}{m a^2 y^3} \right) x = - e \frac{B_y}{m y_0} \frac{V_y}{R_0} \frac{\dot{z}}{n} \\
\ddot{z} + \left( \frac{V_y}{R_0} n - \frac{2 e^2 N}{m a^2 y^3} \right) z = e \frac{B_y}{m y_0} \frac{V_y}{R_0} \frac{\dot{x}}{n}.
\]

Equations (1.10) and (1.11) thus constitute a complete system of equations of motion.

If the angle $\phi = \frac{V_y}{R_0} t$ is chosen as independent variable instead of the time $t$ and the following definitions are used:

\[
\alpha = \frac{B_y}{B_z}, \quad Q = \frac{2 \pi R_0 N}{\delta_0^3 v_{y_0}} \left( \frac{R_0}{\alpha} \right)^3,
\]

where $\tau_0 = 3.8 \cdot 10^{-7}$ is the classical electron radius, eqs. (1.11) transform to

\[
\ddot{x} + (1-n-Q^2) x = \alpha \dot{z}, \\
\ddot{z} + (n-Q^2) z = -\alpha x,
\]

where $'= \frac{d}{d\phi}$. In order to solve this coupled homogeneous system of differential equations for $x$ and $z$, we introduce into the equations the ansatz $x = A e^{i q y}$, $z = B e^{i q y}$ and obtain

\[
(-q^2 + 1-n-Q^2) A - i q \alpha B = \sigma, \\
i q \alpha A + (n-q^2-Q^2) B = \sigma.
\]
The solubility condition yields the characteristic polynomial for $q^2$:

\[(1.15) \quad q^4 - (1 - 2q^2 + \alpha^2)q^2 + (1 - n - Q^2)(n - Q^2) = 0,\]

with the solutions

\[(1.16) \quad q_{4,2}^2 = \frac{1}{2} \left[ (1 - 2q^2 + \alpha^2) \pm \sqrt{(1 - 2q^2 + \alpha^2)^2 - 4(n - Q^2)(1 - n - Q^2)} \right].\]

Simple relations are obtained for the sums and products of the squares $q_1^2, q_2^2$ of the eigenfrequencies:

\[(1.17) \quad q_1^2 + q_2^2 = (1 - 2q^2 + \alpha^2), \]

\[q_4^2 q_5^2 = (q^2 - \frac{1}{2})^2 (n - \frac{1}{2}).\]

We now have to find the parameter range in which the solutions are stable. The oscillations are stable if $q_4^2 > 0$ and $q_5^2 > 0$. Equation (1.16) thus leads directly to the following stability conditions:

\[(1.18) \quad 1 - 2q^2 + \alpha^2 > 0, \quad (1 - 2q^2 + \alpha^2)^2 - 4(n - Q^2)(1 - n - Q^2) > 0, \]

\[(1 - n - Q^2)(n - Q^2) > 0,\]

These inequalities can be expressed in a more convenient form. The betatron oscillations are more stable if it holds that

\[(1.19) \quad 0 \leq Q^2 < \frac{1}{2}, \quad 1 - q^2 > n > Q^2, \quad \alpha^2 \text{ arbitrary},\]

or

\[\frac{1}{2} < Q^2 \quad q^2 > n > 1 - q^2, \quad \alpha^2 (2Q^2 - 1) + \{ (2Q^2 - 1)^2 - (2n - 1)^2 \}^{\frac{1}{2}}.\]
The stable region is plotted in Fig. 1 as a function of the parameters $\alpha^2$, $n$, and $Q^2$. As can be seen, it is thus possible to achieve two things with the additional $B_y$-field, i.e. $\alpha^2 \neq 0$. Firstly, the stable parameter range of $n$ and $Q^2$ can be extended. It is true that for values of $Q^2 < \frac{4}{3}$, the stability region is independent of $\alpha^2$, but for values $Q^2 > \frac{4}{3}$ stability can only be achieved with $\alpha^2 \neq 0$. Secondly, the magnitude of the eigenfrequencies of the stable betatron oscillations can be influenced by the $B_y$-field, as can be seen from eqs. (1.16) and (1.17).

Both points could be of significance for the stability of the electron rings during compression in the magnetic mirror field. For electron rings with particle densities so high that $Q^2 > \frac{4}{3}$ it follows for $\alpha^2 = 0$ that there is instability for all values of $n$. Stabilization will only be achieved when a $B_y$-field is superposed according to eq. (1.19). In the present experiments the values of $Q^2$ are still well below $1/2$, and so there is no need as yet for a $B_y$-field.

The possibility of frequency shifting by means of a $B_y$-field may also be of advantage. During compression in the mirror field the field index at the location of the particle assumes characteristic values from $n = 0.5$ to $n \approx 0$ at the end of compression. The particles thereby pass through a series of betatron resonances, which may lead to expansion of the minor ring radius or even to destruction of the ring. Particularly dangerous is the $q^2 = 1$ resonance at $n = 0$, which the ring enters at the end of the compression phase. Superposing a $B_y$-field makes it possible to arrange that a frequency $\nu_y$ is always $q_y^2 > 1$, while the second $\nu_y > 0$ is always small. The parameter range of $n$, $Q^2$ and $\alpha^2$ for which $q_y^2 > 1$ and $\nu_y > 0$ follows from eq. (1.16). It holds that $q_y^2 > 1$ and $\nu_y > 0$ if
\[ 0 < n(1-n) + Q^2(Q^2-1) < 1, \quad \alpha^2 > Q^2(Q^2-1) + n(1-n) \] 

(1.20) \[ 1 < n(1-n) + Q^2(Q^2-1), \quad \alpha^2 > 2Q^2-1 + 2\{Q^2(Q^2-1) + n(1-n)\}^\frac{1}{2} \]

For the case \( Q^2 = 0 \) the frequencies \( q_4^z \) and \( q_6^z \) are plotted in Fig. 2 as functions of \( \alpha^1 \) for the parameter range of \( n : 0 < n < 1 \). The criterion (1.20) transforms for \( Q^2 = 0 \) to

(1.21) \[ 0 < n < 1, \quad \alpha^1 > n(1-n) \]

and for \( q_4^z \) and \( q_6^z \) eq. (1.16) yields

(1.22) \[ q_4^z = \frac{\lambda + \alpha^2}{z^2} \left[ 1 \pm \left\{ 1 - \frac{4n(1-n)}{(1+\alpha^1)^2} \right\}^\frac{1}{2} \right] \]

For \( \alpha^1 > 1 \) the inequality \( q_4^z > 1 \) is always satisfied.

Discussion of the use of a \( B_y \)-field in the compression phase raises the question of the focusing properties of the \( B \) field during acceleration of the ring, when the field index is \( n = 0 \). The stability criterion (1.19) may be used for the start of acceleration if the influence of the axial velocity can still be neglected. Substituting \( n = 0 \) here, we get stability of the electron orbits

(1.23) \[ q^z > 1, \quad \alpha^z > (2 Q^2-1) + \left\{ (2 Q^2-1) - 1 \right\}^\frac{1}{2} \]

with the appropriate frequencies according to eq. (1.16)

(1.24) \[ q_4^z = \frac{z}{x} \left[ (1-2Q^2+\alpha^2) \pm \left\{ (1+\alpha^1)^2 - 4 \alpha^2 Q^2 \right\}^\frac{1}{2} \right] \]

For sufficiently large space charge there is stability. For high axial velocities the relations have to be modified. Furthermore, the dynamics of acceleration are affected.

These points are dealt with in the next section.
2. Ring acceleration with $B_\gamma$-field

If a $B_\gamma$-field is superposed to achieve stability during acceleration, a condition is imposed on the holding power of the ring to avoid losing the ions. The ions and electrons are acted on in the radial direction by opposing forces of magnitude $K = e_v \nu_2 B_\gamma$. To avoid ion loss, the internal electric force of the rings, which traps the ions, has to exceed the above force:

\begin{equation}
E_\nu > |\nu_2 \, B_\gamma| \tag{2.1}
\end{equation}

Using eqs. (1.8) and taking the approximation $R_0 = -\frac{m_\nu \nu}{e B_\gamma}$ for the major ring radius, we obtain from the above inequality the following relation:

\begin{equation}
\frac{2 \tau \nu N}{\delta} \frac{R_0}{\alpha} > |\alpha \nu_2 \nu_\gamma| \tag{2.2}
\end{equation}

where $\alpha$ is the minor ring radius and $\alpha = \frac{B_\gamma}{B_\gamma}$ is at the location of the ring.

The ring acceleration is also modified by the $B_\gamma$-field. To study this in detail, the ring motion is now calculated in a constant electric field and a non-uniform magnetic field.

2.1 Ring acceleration in a non-uniform magnetic field

The electromagnetic field in which the electrons are accelerated is a weakly non-uniform magnetic field with an additional $B_\gamma$ field, given by the vector potential in cylindrical coordinates:

\begin{equation}
A_r = 0, \quad A_\gamma = \frac{B_\gamma}{2} (1 - \varepsilon \zeta), \quad A_\zeta = -B_\gamma R_0 \frac{\varphi}{R_0} \tag{2.3}
\end{equation}

This yields for the magnetic field components...
\[ B_r = B \left( 1 - \varepsilon^2 \right) \]
\[ B_z = \overline{B}_y \frac{R_0}{r} \]
\[ B_y = \overline{B}_y \frac{R_0}{r} \]

The parameter \( \varepsilon \) is a measure of the non-uniformity and \( R_0 \) is the ring radius at the start of acceleration.

Inserting this magnetic field in the equations of motion (1.1) for the electrons of the ring, it follows from the last equation of (1.1) that \( \gamma = \text{const} \). The second equation of (1.1) can be integrated once, yielding the law of conservation of canonical angular momentum.

The first two equations of (1.1) and the eq. (1.2) - an integral of eq. (1.1) - thus constitute a complete set of equations for determining \( r(t), z(t), v_y(t)=v_y \). These are

\[ \ddot{r} - \frac{v_y^2}{r} = \frac{e}{m} \left( v_y B_z - z \dot{B}_y \right) \]
\[ \ddot{z} + \frac{v_y^2}{r} + \dot{z} \dot{r} = 1 - \frac{A}{r^2} \]

\[ r \dot{v}_y + \frac{e}{m} \dot{R}_y = \text{const}. \]

In the first equation of (2.5) the radial inertia term is now neglected relative to the centrifugal term, i.e. \( \ddot{r} \ll \frac{v_y^2}{r} \)

and, consistently, so is the term \( \dot{z}^2 \) in the last equation of (2.5). This system of equations is solved for the initial conditions \( t = 0, \dot{z} = z = 0, r = R_0, v_y = v_{y0} \), where the following relation between \( R_0 \) and \( v_{y0} \) holds:

\[ e R_0 B = m \gamma v_{y0} \quad \therefore \quad \gamma = (1 - \frac{v_{y0}^2}{R_0^2})^{-\frac{1}{2}} \]

Substituting the expressions (2.4) for the magnetic field in the equations of motion (2.5) and using eqs. (2.6) yields the equations

\[ \alpha \dot{z} = v_y \left[ \frac{\gamma}{R_0} (1 - \varepsilon z) - \frac{v_y}{v_{y0}} \right] \]
\[ v_y = v_{y0} \left[ \frac{R_0}{r} + \frac{\gamma}{R_0} (1 - \varepsilon z) \right] \]
\[ v_y^2 = v_{y0}^2 - \dot{z}^2 \]
From these relations we first obtain the relation between $v_y$ and $\varepsilon$:

$$v_y^2 = v_y^2 \left[ 1 - \varepsilon^2 - \alpha^2 \right] \left\{ \left( 1 - \varepsilon^2 - \alpha^2 \right)^2 + 4 \alpha^4 \right\}^{\frac{1}{4}} \tag{2.8}$$

Substituting in the last equation of (2.7) yields a differential equation for $z(t)$

$$\ddot{z}^2 = v_y^2 \left[ 1 + \varepsilon^2 + \alpha^2 \right] \left\{ \left( 1 - \varepsilon^2 - \alpha^2 \right)^2 + 4 \alpha^4 \right\}^{\frac{3}{4}} \tag{2.9}$$

The solution is only of interest for values of $z$ in which the magnetic field has not changed much, i.e. for $z$ with $\varepsilon \ll 1$. The right-hand side of eq. (2.9) is therefore expanded in $\varepsilon \ll 1$, which yields

$$\ddot{z}^2 = v_y^2 \left( \frac{\varepsilon^2}{1 + \alpha^2} - \frac{\varepsilon^2 \alpha^2}{(1 + \alpha^2)^3} \right) \tag{2.10}$$

Integration of this equation with allowance for the first expansion term only gives

$$z = \frac{v_y}{4(1 + \alpha^2)} \varepsilon^2 \tag{2.11}$$

This equation states that the $B_y$ field increases the inertia of the ring in the axial acceleration, the mass being increased by a factor $1 + \alpha^2$. The dependence of the velocity $v_y$ and the major ring radius $R_o$ on $z$ is given by eq. (2.8) and the second equation of (2.7). Expanding this equation likewise in $\varepsilon \ll 1$ gives

$$v_y = v_y \left( 1 - \frac{1}{2} \frac{\varepsilon^2}{1 + \alpha^2} \right) \tag{2.12}$$

and

$$\frac{z}{R_o} = 1 + \alpha \frac{\varepsilon^2}{\sqrt{1 + \alpha^2}} \tag{2.13}$$
A measure of the realistic values of ε, i.e. the non-uniformity of the field, can be obtained with the following estimate. The accelerating force should be smaller than the holding power of the ring with respect to the ions. Together with eq. (2.11) this leads to the inequality

\[
\frac{v_0^2 \varepsilon}{2(1+\alpha^2)} < \frac{e}{M} E_M
\]

or

\[
\varepsilon < \frac{v_0^2}{v_0^2} \frac{\tau \alpha N}{\tau_a^2} \frac{m}{M}
\]

where m and M are electron and ion mass respectively.

2.2 Electric acceleration of the ring

Instead of accelerating the ring in a non-uniform magnetic field, an electric field can be used, the magnetic field being composed of a constant $B_z$ field and the $B_y$ field. The field components are then

\[
E_x = \sigma, \quad B_x = 0, \\
E_y = 0, \quad B_y = \frac{B_y}{\tau} \frac{R_y}{\tau}, \\
E_z = E(z = \text{const.}), \quad B_z = \text{const.}
\]

The $y$-component of the vector potential in this case is $A_y = \frac{\tau B_z}{2}$. The equations of motion are obtained by substituting (2.16) in eqs. (1.1). The second equation of (1.1) can again be integrated. Together with eq. (1.2) we then get the following system of equations for $\tau(t), z(t)$, and $y(t), v(t) = v_y$:

\[
\begin{align*}
(\dot{y} + \frac{v_y^2}{\tau}) &= \frac{e}{m} \left( n \frac{B_z}{\tau} - \frac{\ddot{B}_y}{\tau} R_y \right), \\
\tau \ddot{y} + \frac{e}{m} \frac{R_y}{\tau} &= \text{const.} \\
\dot{z} &= \frac{e}{m} E_z \\
\dot{z}^2 + \dot{z}^2 + v_y^2 &= 1 - \frac{\tau}{\tau_z},
\end{align*}
\]
In the first equation the radial inertia term is now neglected relative to the centrifugal term and, consistently, so is the term \( \dot{r}^2 \) in the last equation. The equations are then solved for the initial conditions \( t = \sigma, \; z = \dot{z} = 0, \; \tau = R_0, \; v_y = v_{y_0}, \) where the following relation between \( R_0 \) and the magnetic field \( B_z \) holds:

\[
(2.18) \quad -eR_0B_z = m_0 v_{y_0} \gamma, \quad \gamma = \left\{ \frac{1 - v_{y_0}^2}{\gamma} \right\}^{\frac{1}{2}}.
\]

With these initial conditions we obtain after integrating the third equation of (2.17)

\[
\alpha \ddot{z} + \frac{\gamma v_y^2}{y_0 v_{y_0}} = v_y \frac{\tau}{R_0}
\]

\[
\gamma v_y = y_0 v_{y_0} \frac{1}{2} \left( \frac{\tau}{R_0} + \frac{R_0}{\gamma} \right)
\]

\[
(2.19)
\]

\[
y = \frac{eE_z}{m} \dot{z} + y_0
\]

\[
\ddot{z} + v_y^2 = 1 - \frac{1}{\gamma^2}
\]

Further rearrangement then gives the following system of equations:

\[
v_y^2 = \frac{y_0 v_{y_0}}{y} \frac{1}{2} \left[ 1 - \alpha^2 + \left\{ (1 - \alpha^2) + y \frac{\gamma^2 - \alpha^2}{y_0 v_{y_0}^2} \right\}^{\frac{1}{2}} \right]
\]

\[
(2.20)
\]

\[
\tau = R_0 \left[ \frac{\gamma v_y}{y_0 v_{y_0}} + \alpha \frac{1 - v_y^2 - \frac{1}{\gamma}}{\gamma^{\frac{1}{2}}} \right]
\]

\[
y = \frac{eE_z}{m} \dot{z} + y_0
\]

\[
\ddot{z} = 1 - v_y^2 - \frac{1}{\gamma^2}
\]
where \( \nu \) and \( \gamma \) are functions of \( \gamma \). The third equation gives the relation between the total energy \( E \) and \( z \); \( \gamma \) and \( \nu \) are thus known as functions of \( z \). All that is now needed for a complete solution is to determine \( \gamma = \gamma(t) \). From eqs. (2.20) we get for \( \gamma(t) \) the differential equation

\[
(2.21) \quad \frac{m}{2\epsilon E^2} \frac{d\gamma}{dt} \gamma^2 = \left[ 1 + \frac{y^2 - y_0^2}{y_0^2 \nu_0^2} - \frac{1}{3} \left( 1 - \alpha^2 + (1 + \alpha^2)^2 + \frac{1}{2} \frac{y^2 - y_0^2}{y_0^2 \nu_0^2} \alpha^2 \right) \right] \frac{\gamma_0}{\nu_0}
\]

This equation can be integrated exactly. As interest here is confined to acceleration to non-relativistic velocities, i.e. energies \( E \) with \( \frac{y^2 - y_0^2}{y_0^2 \nu_0^2} \ll 1 \), the right-hand side of eq. (2.21) can be simplified by expanding in this quantity, thus giving

\[
(2.22) \quad \frac{m}{2\epsilon E^2} \frac{d\gamma}{dt} \gamma^2 = \frac{\gamma_0 \nu_0}{(1 + \alpha^2)^2} \left( \frac{y^2 - y_0^2}{y_0^2 \nu_0^2} \right)^2
\]

Integration yields

\[
(2.23) \quad \gamma^2 - y_0^2 = \frac{e^2 E^2}{m^2 (1 + \alpha^2)} t^2
\]

Solving this equation for \( \gamma \) and expanding in small \( t \) gives the following result:

\[
(2.24) \quad \gamma = \gamma_0 \left( 1 + \frac{1}{2} \frac{e^2 E^2}{m^2 y_0^2 (1 + \alpha^2)} t^2 \right)
\]
Together with eqs. (2.19) this yields for $z(t)$

$$Z = \frac{2eE_z}{m\gamma_0(1+\alpha^2)} \frac{\gamma^2}{c^2} z^2.$$ 

(2.25)

In the non-relativistic region the same result is obtained for the mass increase with electric acceleration as with magnetic acceleration. Using eq. (2.20) we obtain for the transverse velocity

$$\mathcal{V}_y = \frac{\gamma_0^2 \gamma \gamma_0}{(1+\alpha^2)^{1/2}} + \frac{\gamma_0^2 \gamma \gamma_0}{(1+\alpha^2)^{1/2}} \left(1 - \frac{1}{\beta^2} \right)$$

(2.26)

and for the ring radius $r$

$$\frac{r}{R_0} = 1 + \frac{\alpha}{(1+\alpha^2)^{1/2}} \left( \frac{\gamma^2}{\gamma_0^2} \gamma_0^2 \right)^{1/2}$$

(2.27)

or as a function of $z$

$$\frac{r}{R_0} = 1 + \frac{\alpha}{(1+\alpha^2)^{1/2}} \left( \frac{2eE_z}{m\gamma_0^2} \right)^{1/2}$$

(2.28)
2.3 Stability of the electron ring with axial velocity

For the electron ring with axial velocity $v_z$ and superposed $B_y$-field it is possible as in Section 1 to calculate the stability of the electron orbits in linear approximation. The $B_z$-field in this case is assumed to be uniform, and hence the field index is $n = 0$.

It is convenient to do the calculation in the rest system of the ring. In this system the electrons are acted on by the following forces: Firstly, there are the electric and magnetic self-fields of the ring, which are given by \((1.8)\). Then there is the external electromagnetic field, which results from the $B_z$-field and $B_y$-field by Lorentz transformation. The individual components of these fields are

\[
\begin{align*}
E_x &= -y_z v_z \frac{\vec{B}_y R_0}{\tau}, \quad B_x = 0, \\
E_y &= 0, \quad B_y = y_z \frac{\vec{B}_y R_0}{\tau}, \\
E_z &= 0, \quad B_z = B_z.
\end{align*}
\]

(2.29)

where $y_z = (1 - v_z^2)^{1/2}$ and $R_0$ is the later defined equilibrium radius without $v_z$. The equilibrium solution about which we expand is $\tau = \bar{R}, \tau = 0, \vec{z} = \vec{z}_0 = 0, v_x = v_y$. The ring radius $R$ and the external field are related as follows:

\[
R = R_0 \left(1 + \alpha \frac{\vec{z}_0 v_z}{v_y^2}\right), \quad \text{with} \quad R_0 = -\frac{m v_y v_x}{\vec{B}_z},
\]

(2.30)

where $\alpha = \frac{\vec{B}_y}{\vec{B}_z}$ and $y_y = (1 - v_y^2)^{1/2}$. It should be noted that $\vec{B}_y$ is the value of the $B_y$-field not at the location $R$ of the ring, but at the radius $R_0$, as can be seen from eq. (2.29).

Linearization of the equations of motion is just the same as in Section 1, and the radial and axial deviations from equilibrium are also expressed by a coupled system of linear differential equations as follows:
\[ x'' + \left(1 + \alpha^2 \frac{v_x \gamma_x}{v_y \gamma_y} - Q^2\right) x = \alpha^2 y z \]

(2.31)

\[ z'' + (-Q^2) z = -\alpha^2 y z x' \]

where \( ' = \frac{\alpha}{\partial y} \)

(2.32)

\[ Q^2 = \frac{2 \gamma_o N}{\gamma^2} \left( \frac{R}{a} \right)^2 \]

The solution of this system yields the eigenfrequencies \( \omega_1^2, \omega_2^2 \)

and

(2.33)

\[ \omega_{12}^2 = \frac{i}{2} \left[ \left( 1 - 2 \frac{v_x \gamma_x}{v_y \gamma_y} \frac{v_x}{v_y} \alpha^2 \right) \pm \sqrt{\left( 1 + \frac{v_x \gamma_x}{v_y \gamma_y} \alpha^2 \right)^2 - 4 \left( \frac{v_x \gamma_x}{v_y \gamma_y} \omega_{12}^2 \right)^2} \right] \]

The solutions are stable when \( \alpha^2 > 0 \) and \( \alpha^2 > 0 \) are satisfied. This leads to the following stability criterion:

(2.34)

\[ \alpha^2 \left( \frac{v_x \gamma_x}{v_y \gamma_y} \right) > 1, \quad 1 + \frac{v_x \gamma_x}{v_y \gamma_y} < Q^2 < \frac{1 + \alpha^2 \left( \frac{v_x \gamma_x}{v_y \gamma_y} \right)^2}{4 \alpha^2 \frac{v_x \gamma_x}{v_y \gamma_y}} \]

For \( v_x \gamma_x^2 \) the criterion transforms to the expression from Section 1.

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Fig. 1  The squares of the betatron frequencies \( q_1^2, q_2^2 \) are plotted versus \( \alpha \) for different \( n : 0 \leq n \leq 1 \).
Fig. 2 The stable region of the betatron oscillations is plotted as a function of the field index $n$, the space charge term $Q^2$, and the $B_y$ field term.