Standard Model Fermions and $K(\mathrm{E}_{10})$

Axel Kleinschmidt[1,2] and Hermann Nicolai[1]
[1] *Max-Planck-Institut für Gravitationsphysik (Albert-Einstein-Institut)*
Mühlenberg 1, DE-14476 Potsdam, Germany
[2] *International Solvay Institutes, ULB-Campus Plaine,*
CP 231, BE-1050 Bruxelles, Belgium

In recent work [1] it was shown how to modify Gell-Mann's proposal for identifying the 48 quarks and leptons of the Standard Model with the 48 spin-$\frac{1}{2}$ fermions of maximal SO(8) gauged supergravity that remain after the removal of eight Goldstinos, by deforming the residual U(1) symmetry at the SU(3) × U(1) stationary point of $N = 8$ supergravity to also achieve agreement of the electric charge assignments. In this Letter we show that the required deformation, while not in SU(8), does belong to $K(\mathrm{E}_{10})$, the 'maximal compact' subgroup of E_{10} which is a possible candidate symmetry underlying M theory. The incorporation of infinite-dimensional Kac–Moody symmetries of hyperbolic type, apparently unavoidable for the present scheme to work, opens up completely new perspectives on embedding Standard Model physics into a Planck scale theory of quantum gravity.

The question whether or not the maximally extended $N = 8$ supergravity theory [2, 3] can be related to Standard Model physics has been under debate for a long time. Very recent work [1] has taken up an old proposal of Gell-Mann's [4] on how to match the 48 quarks and leptons (including right-chiral neutrinos) of the Standard Model with the 48 spin-$\frac{1}{2}$ fermions of maximal SO(8) gauged supergravity that remain after the removal of eight Goldstinos (as required by the complete breaking of $N = 8$ supersymmetry). This scheme, which was subsequently shown to be realised at the SU(3) × U(1) stationary point of maximal gauged SO(8) supergravity [5], relies on identifying the residual SU(3) of supergravity with the diagonal subgroup of the colour group $\mathrm{SU}(3)_c$ and a new family symmetry $\mathrm{SU}(3)_f$. Intriguingly, in this way complete agreement is found in the SU(3) charge assignments of quarks and leptons and the spin-$\frac{1}{2}$ fermions of $N = 8$ supergravity, but there remained a systematic mismatch in the electric charges by a spurion charge of $q = \pm\frac{1}{6}$. The main advance reported in [1] was to identify the 'missing' $\mathrm{U}(1)_q$ that rectifies this mismatch, and that was found to take a surprisingly simple form. However, this deformation cannot be explained from 'within' $N = 8$ supergravity (nor from a hypothetical embedding of maximal gauged supergravity into the known superstring theories), as $\mathrm{U}(1)_q$ is not contained in its R symmetry group SU(8). In this Letter we show that the required deformation is, however, contained in an infinite dimensional extension of SU(8), namely the involutory 'maximal compact' subgroup $K(\mathrm{E}_{10})$ of the hyperbolic Kac–Moody group E_{10}, which has been proposed as a possible candidate symmetry of M theory [6, 7]. This, we believe, places the question stated above, and also the eventual incorporation of the chiral electroweak gauge interactions (not considered in [4, 5]), in an entirely new context, by embedding (at least a subset of) the Standard Model symmetries into an infinite-dimensional extension of the exceptional duality symmetries of maximal supergravity. This approach, never tried before to the best of our knowledge, offers completely new perspectives on the possible Planck scale origin of Standard Model physics.

For the rest of this text we will concentrate on the fermionic sector of $N=8$ supergravity, which consists of eight gravitinos ψ^i_μ transforming in the **8**, and a tri-spinor of spin-$\frac{1}{2}$ fermions χ^{ijk} transforming in the **56** of SU(8), whence χ^{ijk} is fully antisymmetric in the SU(8) indices i, j, k. According to [3] one would have to distinguish between upper and lower indices (positive and negative chiralities) with $\chi^{ijk} = \big(\chi_{ijk}\big)^*$ in order to realise the full chiral SU(8) R symmetry. Here we will, however, restrict attention to the vector-like SO(8) subgroup of SU(8), for which the distinction between upper and lower indices $i, j, k, \ldots$ is immaterial, whence we will not distinguish between χ^{ijk} and χ_{ijk} in the remainder. The residual vector-like SO(8) transformations act as

$$\chi^{ijk} \;\to\; U^i{}_l U^j{}_m U^k{}_n \chi^{lmn} \qquad \text{with } U \in \mathrm{SO}(8). \tag{1}$$

In order to obtain the correct electric charge assignments of the quarks and leptons it was found in [1] that the U(1) subgroup of SU(3) × U(1) must be deformed by a new (still vector-like) $\mathrm{U}(1)_q$ whose action on the tri-spinor χ^{ijk} is generated by the following 56-by-56 matrix

$$\mathcal{I} := \frac{1}{2}\Big(T\wedge\mathbf{1}\wedge\mathbf{1} + \mathbf{1}\wedge T\wedge\mathbf{1} + \mathbf{1}\wedge\mathbf{1}\wedge T + T\wedge T\wedge T\Big) \tag{2}$$

acting in the $\mathbf{8}\wedge\mathbf{8}\wedge\mathbf{8}$ representation of SO(8). Here

$$T = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0\\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0\\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1\\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \end{pmatrix}, \tag{3}$$

represents the imaginary unit in the breaking of SO(8) to SU(3) × U(1). We note that, from $T^2 = -\mathbf{1}$ we have $\mathcal{I}^2 = -\mathbf{1}$, whence (2) can be trivially exponentiated to a $\mathrm{U}(1)_q$ phase rotation. The combination (2) differs from the usual co-product obtained from (1) with

$U = \exp(\omega T)$ by the 56-by-56 matrix $T \wedge T \wedge T$. Importantly, the latter is not in SU(8), although it does commute with the $\mathrm{SU}(3) \times \mathrm{U}(1)$ subgroup of SO(8), and hence merely *deforms* this subgroup, but does not enlarge it. We will now show how to accommodate the triple wedge product $T \wedge T \wedge T$ by enlarging the R symmetry SU(8) of $N = 8$ supergravity to the bigger, and in fact, infinite-dimensional R symmetry $K(\mathrm{E}_{10})$, in accordance with the anticipated enlargement of the finite-dimensional exceptional dualities of maximal supergravities to infinite-dimensional groups.

To proceed we recall how the fermions of $D = 11$ supergravity [8] are related to those of $N = 8$ supergravity [2, 9]. Denoting the (spatial) $D = 11$ gravitino components by Ψ^a_A (with $a, b, ... = 1, ..., 10$ and $D = 11$ spinor indices $A, B, ... = 1, ..., 32$) and adopting the temporal supersymmetry gauge $\Psi^0_A = (\Gamma^0\Gamma_a)_{AB}\psi^a_B$ as in [11], we split the $D = 11$ gravitino into four-dimensional spatial and internal components as follows

$$\Psi^a_A = \left(\Psi^{\hat{a}}_{\alpha i}\, ,\, \Psi^{\bar{a}}_{\alpha i}\right) \tag{4}$$

with flat spatial indices $\hat{a}, \hat{b}, ... = 1, 2, 3$ and flat internal indices $\bar{a}, \bar{b}, ... = 4, \ldots, 10$, whose position again does not matter as they are pulled up and down with δ_{ab}. The $D = 11$ spinor indices $A, B, ...$ are split as $A \equiv (\alpha, i)$ into $D = 4$ spinor indices $\alpha, \beta, ... = 1, ..., 4$ and internal SO(8) indices $i, j, ... = 1, ..., 8$ (whose position likewise does not matter here as we restrict attention to a vector-like symmetry). Ignoring a Weyl rescaling factor and a chiral redefinition, and not making a split into left-chiral and right-chiral components as in [9], we have

$$\psi^i_{\hat{a}\alpha} \;\propto\; \Psi^i_{\hat{a}\alpha} - \frac{1}{2}\sum_{\bar{c}=4}^{10} \Gamma^{\bar{c}}_{ij}\left(\gamma^5\gamma_{\hat{a}}\Psi^j_{\bar{c}}\right)_\alpha \tag{5}$$

$$\chi^{ijk}_\alpha \;\propto\; \sum_{\bar{a}=4}^{10} \Gamma^{\bar{a}}_{[ij}\Psi^{\bar{a}}_{k]\alpha} \tag{6}$$

where we temporarily suspend the summation convention for the indices $a, b, ...$ (the summation convention remains, however, in force for all other indices). For the implementation of the action of $T \wedge T \wedge T$ we also need the following redefinition of the $D = 11$ gravitino [12]:

$$\Phi^{\mathtt{a}}_A = \Gamma^a_{AB}\Psi^a_B \qquad \text{(no sum on } a\text{!)} \tag{7}$$

Because there is *no* summation on the spatial index a, manifest SO(10) covariance is lost. To emphasise this point we adopt a different font $(\mathtt{a}, \mathtt{b}, ...)$ although these indices have the same range as $a, b, ...$ before [13]. Importantly, however, the position of the indices $\mathtt{a}, \mathtt{b}, ...$ now *does* matter, as they are to be raised and lowered with the Lorentzian (DeWitt) metric and its inverse

$$G_{\mathtt{ab}} = \delta_{\mathtt{ab}} - 1 \quad \Leftrightarrow \quad G^{\mathtt{ab}} = \delta^{\mathtt{ab}} - \frac{1}{9}. \tag{8}$$

With the redefinition (7) the formula (6) becomes

$$\chi_{ijk\,\alpha} \;\propto\; \sum_{\mathtt{a}=4}^{10} \Gamma^{\mathtt{a}}_{[ij}\Gamma^{\mathtt{a}}_{kl]}\Phi^{\mathtt{a}}_{l\alpha}. \tag{9}$$

The action of $T \wedge T \wedge T$ is therefore realised via (now suppressing $D = 4$ spinor indices)

$$\begin{aligned}\chi_{ijk} \quad &\rightarrow \quad T_{il}\,T_{jm}\,T_{kn}\,\chi_{lmn} \\ &\propto \quad \sum_{\mathtt{a}=4}^{10} (T\Gamma^{\mathtt{a}}T)_{[ij}(T\Gamma^{\mathtt{a}}T)_{kl]}\,T_{lm}\Phi^{\mathtt{a}}_m\end{aligned} \tag{10}$$

where we have inserted a factor $TT = -\mathbf{1}$ and have used the antisymmetry of T. Next we recall that there is a representation of the SO(7) Γ-matrices where

$$T_{ij} \;=\; \Gamma^{45}_{ij} \tag{11}$$

(see e.g. appendix E of [15]); it is then easy to see that

$$(T\Gamma^{\mathtt{a}}T)_{[ij}(T\Gamma^{\mathtt{a}}T)_{kl]} = \Gamma^{\mathtt{a}}_{[ij}\Gamma^{\mathtt{a}}_{kl]} \tag{12}$$

even without summation over $\mathtt{a}$. Using this formula we conclude from (10) that the desired action takes a very simple form on the redefined spinors (7), to wit,

$$\Phi^{\mathtt{a}}_{i\alpha} \;\longrightarrow\; T_{ij}\Phi^{\mathtt{a}}_{j\alpha} \tag{13}$$

which leaves the $D = 4$ spinor indices unaffected. Of course, one could also (though less elegantly) express this action in terms of the original spinors Ψ^a_A. We stress that in order to preserve the relation (5), (13) must hold *for all* $\mathtt{a} = 1, ..., 10$. From this follows the action of the new generator on the $D = 4$ gravitino, an insight that the arguments in [1] could not provide. Observe that the redefinition $\psi^i_{\hat{a}} \to \gamma^{\hat{a}}\psi^i_{\hat{a}}$ implied by (7) does not affect this conclusion, as $\gamma^{\hat{a}}$ commutes with T.

We now want to show that the action (13) is contained in $K(\mathrm{E}_{10})$, the supposed R symmetry of M theory. We refer to our previous work [10, 11, 13, 14] for detailed explanations on $K(\mathrm{E}_{10})$, and here simply summarise some salient results (see also [16, 17] for related work). The group $K(\mathrm{E}_{10})$ is the involutory subgroup of E_{10} which is left invariant by the Cartan-Chevalley involution defined on E_{10} in terms of its Chevalley-Serre presentation. As such, it contains the R symmetries of all $D \geq 2$ maximal supergravities as subgroups (and thus also *chiral* transformations for even D); more specifically, we have

$$\mathrm{SU}(8) \subset \mathrm{SO}(16) \subset K(\mathrm{E}_9) \subset K(\mathrm{E}_{10}) \tag{14}$$

The fermions transform in spinorial (double-valued) representations of $K(\mathrm{E}_{10})$. A remarkable property of the algebra $K(\mathrm{E}_{10})$ is that, though infinite-dimensional, it admits *finite-dimensional*, hence *unfaithful* representations [11, 16]. These are the Dirac [17] and vector-spinor representations [11, 16], which can be directly deduced from $D = 11$ supergravity (in addition, two 'higher spin' realisations are known [13]). As a consequence, $K(\mathrm{E}_{10})$ is *not* simple, because it has nontrivial (finite codimension) ideals $\mathcal{J}$ which are associated with the unfaithful representations in the way explained in [11]. Accordingly, the quotient $K(\mathrm{E}_{10})/\mathcal{J}$ is a finite-dimensional group; more specifically, denoting the vector-spinor ideal by $\mathcal{J}_{\rm vs}$, evidence was presented in [13] that

$$K(\mathrm{E}_{10})/\mathcal{J}_{\rm vs} \;=\; \mathrm{SO}(288,32) \tag{15}$$

The fact that the 'compact' subgroup $K(\mathrm{E}_{10}) \subset \mathrm{E}_{10}$ in this way gives rise to a *non-compact* quotient group is another unusual feature of $K(\mathrm{E}_{10})$.

A convenient realization of the $K(\mathrm{E}_{10})$ Lie algebra generators in the vector-spinor representation was found in [13, 14] (following earlier work on $K(\mathrm{AE}_3)$ in [12, 18, 19]). Like the generators of E_{10}, the generators k^r_α of $K(\mathrm{E}_{10})$ can be labeled by E_{10} roots α and the associated multiplicity index r, but such that [14]

$$\mathrm{k}^r_\alpha = -\mathrm{k}^r_{-\alpha}\;, \quad \text{for all } \mathrm{E}_{10} \text{ roots } \alpha. \tag{16}$$

As shown in [13], for the vector spinor representation there is a concrete realization of these generators in terms of 320-by-320 matrices. For all *real* roots α of E_{10} (for which the multiplicity label r is not needed) we have

$$(\mathrm{k}_\alpha)_{\mathtt{a}A,\mathtt{b}B} = \frac{1}{2}\mathrm{X}_{\mathtt{ab}}(\alpha)\tilde{\Gamma}(\alpha)_{AB} \tag{17}$$

where the symmetric matrix $\mathrm{X}_{\mathtt{ab}}$ is given by

$$\mathrm{X}_{\mathtt{ab}}(\alpha) = -\frac{1}{2}\alpha_{\mathtt{a}}\alpha_{\mathtt{b}} + \frac{1}{4}G_{\mathtt{ab}} \tag{18}$$

in terms of the root components $\alpha_{\mathtt{a}}$ in the 'wall basis' used in [13]; indices $\mathtt{a},\mathtt{b}$ are raised and lowered by means of (8). As explained in [13] there is a map from the E_{10} root lattice into the SO(10) Clifford algebra that associates to each root α of E_{10} a particular element $\tilde{\Gamma}(\alpha) = -\tilde{\Gamma}(-\alpha)$ of the Clifford algebra; furthermore the matrices $\tilde{\Gamma}(\alpha)$ are anti-symmetric for $\alpha^2 \in 4\mathbb{Z}+2$ and symmetric for $\alpha^2 \in 4\mathbb{Z}$. Because the SO(10) Clifford algebra is finite-dimensional, and because there are infinitely many real and imaginary roots of E_{10}, it follows that infinitely many E_{10} roots α are mapped to the same element of the Clifford algebra.

To prove that (17) indeed generates the algebra $K(\mathrm{E}_{10})$, one substitutes the ten simple roots of E_{10} into (17) and verifies the defining relations for $K(\mathrm{E}_{10})$ [13] (the latter characterise the involutory subalgebra in a manner analogous to the Chevalley–Serre presentation for general Kac–Moody algebras [20]). The Lie algebra $K(\mathrm{E}_{10})$ in the vector spinor representation is thus generated by taking commutators of the above real root generators in all possible ways. In this way one 'reaches' all imaginary root spaces with $\alpha^2 \le 0$. However, due to the unfaithfulness of the representation the image of the root space elements consist of linear combinations of finitely many basis elements. The generating elements, and thus $K(\mathrm{E}_{10})$, leave invariant the Lorentzian bilinear form

$$(V,W) \equiv G_{\mathtt{ab}}V^{\mathtt{a}}_A W^{\mathtt{b}}_A \quad \text{(of signature } (288,32)). \tag{19}$$

For general imaginary roots the formula (17) is no longer valid. What is clear, however, is that all matrices k^r_α generated in this way are antisymmetric under interchange of the index pairs $(\mathtt{a}A)$ and $(\mathtt{b}B)$, that is,

$$(\mathrm{k}^r_\alpha)_{\mathtt{a}A,\mathtt{b}B} = -(\mathrm{k}^r_\alpha)_{\mathtt{b}B,\mathtt{a}A}. \tag{20}$$

Any such matrix can be written as a linear combination of matrices of the form (17), with either $\mathrm{X}_{\mathtt{ab}}$ symmetric in $(\mathtt{ab})$ and $\tilde{\Gamma}(\alpha)_{AB}$ anti-symmetric in $[AB]$, or antisymmetric in $[\mathtt{ab}]$ and symmetric in (AB). Because all such matrices leave invariant the Lorentzian bilinear form (19) they all belong to the Lie algebra of $\mathfrak{so}(288,32)$ [13]. The equality (15) then follows because there exists no other quadratic invariant — unlike for $K(\mathrm{AE}_3)$, cf. [18].

Although we do not have a general formula for arbitrary imaginary roots, explicit formulas do exist for null roots δ, and for certain time-like roots Λ [14]. For null roots δ, we have

$$(\mathrm{k}^r_\delta)_{\mathtt{a}A,\mathtt{b}B} = \varepsilon^r_{[\mathtt{a}}\delta_{\mathtt{b}]}\tilde{\Gamma}(\delta)_{AB} \tag{21}$$

with eight transversal polarisation vectors ε^r. For time-like roots Λ with $\Lambda^2 = -2$, the corresponding k^r_Λ can be realised in the form (17) by choosing a decomposition $\Lambda = \alpha+\beta$ with $\alpha^2 = \beta^2 = 2$ and $\alpha\cdot\beta = -3$; this gives

$$\mathrm{X}^{(\alpha)}_{\mathtt{ab}}(\Lambda) = -\frac{1}{2}(\alpha-\beta)_{\mathtt{a}}(\alpha-\beta)_{\mathtt{b}} + \frac{1}{4}G_{\mathtt{ab}} \tag{22}$$

Letting the decomposition range over all pairs of real roots (α,β) with $\Lambda = \alpha+\beta$ one thus re-constructs the full root space, of dimension $\mathrm{mult}(\Lambda) = 44$.

Returning to our initial problem we note that

$$k_{\mathtt{a}A,\mathtt{b}B} = G_{\mathtt{ab}}T_{AB} \equiv G_{\mathtt{ab}}\delta_{\alpha\beta}T_{ij} \;\in\; \mathfrak{so}(288,32) \tag{23}$$

whence this matrix can be generated by a linear combination of matrices obtained by multiple commutation of the basic $K(\mathrm{E}_{10})$ generators (because a linear combination may be required, we omit the root and multiplicity labels on k). The desired result thus directly follows from (15), but it is nevertheless instructive to see how one can arrive more directly at the requisite linear combination.

To this end we note that there are infinitely many roots α (both real and imaginary) that satisfy $\tilde{\Gamma}(\alpha) = T = \Gamma^{45}$. The task of finding a $K(\mathrm{E}_{10})$ generator that implements $T\wedge T\wedge T$ of (13) is then reduced to finding a combination of tensors $\mathrm{X}_{\mathtt{ab}}$ that equals $G_{\mathtt{ab}}$. We are not aware of a single root that achieves this but establishing the existence of a linear combination can be achieved as follows. In accordance with (22) one considers the set of all $\mathrm{X}_{\mathtt{ab}}$ that can arise from the commutation of two real root generators $\mathrm{X}_{\mathtt{ab}}(\alpha)$ and $\mathrm{X}_{\mathtt{cd}}(\beta)$ (given as in (18)) such that $\alpha+\beta = \Lambda$ is an imaginary root that satisfies $\tilde{\Gamma}(\Lambda) = \Gamma^{45}$. Similarly, one can perform the same analysis for odd multiples of Λ given by $(2k+1)\Lambda$ since then $\tilde{\Gamma}\big((2k+1)\Lambda\big) = \tilde{\Gamma}(\Lambda) = \Gamma^{45}$. We have shown by an explicit computer analysis that one can find a linear combination of the generated $\mathrm{X}_{\mathtt{ab}}\big((2k+1)\Lambda\big)$ that equals $G_{\mathtt{ab}}$ and therefore the desired realization of $T\wedge T\wedge T$ on the spinors of $D=11$ supergravity within $K(\mathrm{E}_{10})$. The generator just constructed only extends the R symmetry

$SU(8) \subset SO(3) \times SU(8) \subset K(E_{10})$ and thus leaves the spatial rotation SO(3) symmetry untouched.

The above argument demonstrates the existence of an element of $K(E_{10})$ that acts according to (13), but the combination identified above does not necessarily have a simple algebraic interpretation. Because the spinors ϕ_A^{a} form an unfaithful representation of $K(E_{10})$ there are infinitely many elements that act in this way, and it is thus possible that an alternative realization of $T \wedge T \wedge T$ exists that has a simple physical origin. For the realisation found here one already has to go up to level $\ell = 18$ in a level decomposition of $K(E_{10})$ (that follows directly from the corresponding tables for E_{10} given in [22]); there is thus no easy way of reproducing this result by simple iteration of the low level $K(E_{10})$ transformation rules given in [11]. The explicit realization of the charge shifting $U(1)_q$ generator above relies on the existence of time-like imaginary roots and their integer multiples, but there may be other possibilities, in particular, using only real roots. In any case, it does not appear possible to construct the requisite element without use of the 'hyperbolic' over-extended root of E_{10}, since the structure of the root system of the affine subalgebra $\mathfrak{e}_9$ is too restricted. In this sense, the extension to the full *hyperbolic* Kac–Moody algebra and its involutory subalgebra could be essential for linking $N=8$ supergravity to the real world.

Finally, we note that the embedding of $T \wedge T \wedge T$ into $K(E_{10})$ in principle also allows for a realisation of this transformation on the bosonic fields of the spinning $E_{10}/K(E_{10})$ model studied in [11], although the ambiguities related to the unfaithfulness of the fermionic realisation of $K(E_{10})$ remain to be resolved.

The results of [1] and this Letter represent a significant shift away from the standard paradigm of how to understand the possible emergence of the Standard Model fermions from a Planck scale unified theory, as for instance embodied in currently popular superstring inspired scenarios of low energy ($N = 1$) supergravity. There one starts from a finite-dimensional *compact* Yang-Mills gauge group (such as $E_8 \times E_8$), with the fermions transforming in a standard representation. This symmetry is assumed to be present *as a space-time-based symmetry* already at the Planck scale, and then assumed to be broken in a cascade of symmetry reductions as one descends to the electroweak scale. By contrast, the present scheme proceeds from an infinite-dimensional group that can be fully present as a symmetry only in a phase of the theory *prior to the emergence of classical space and time*, in accord with the proposal of [6], and *crucially relies on the infinite-dimensionality of this group* (and the associated Kac–Moody algebra) [23]. We emphasise once again that $K(E_{10})$ does possess chirality, offering new perspectives for the incorporation of chiral gauge symmetries, such that the electroweak sector of the Standard Model may eventually be understood in a way very different from currently prevailing views.

Acknowledgments: HN thanks the University of Western Australia, and especially Ian McArthur and Sergei Kuzenko for their hospitality and support during a most enjoyable stay while this work was being carried out. The authors are also grateful to Thibault Damour for useful discussions.

[1] K.A. Meissner and H. Nicolai, Phys. Rev. **D 91** (2015) 065029

[2] E. Cremmer and B. Julia, Nucl. Phys. **B159** (1979) 141.

[3] B. de Wit and H. Nicolai, Nucl. Phys. **B208** (1982) 323.

[4] M. Gell-Mann, in Proceedings of the Shelter Island Meeting II (1983), Caltech Preprint CALT-68-1153.

[5] H. Nicolai and N.P. Warner, Nucl. Phys. **B259** (1985) 412.

[6] T. Damour, M. Henneaux and H. Nicolai, Phys. Rev. Lett. **89** (2002) 221601.

[7] For an earlier and conceptually different proposal based on the non-hyperbolic Kac–Moody algebra E_{11} see: P.C. West, Class.Quant.Grav. **18** (2001) 4443.

[8] E. Cremmer, B. Julia and J. Scherk, Phys. Lett. **B76** (1978) 409.

[9] B. de Wit and H. Nicolai, Nucl. Phys. **B274** (1986) 36.

[10] T. Damour, A. Kleinschmidt and H. Nicolai, Phys. Lett. **B 634** (2006) 319.

[11] T. Damour, A. Kleinschmidt and H. Nicolai, JHEP **0608** (2006) 046.

[12] T. Damour and C. Hillmann, JHEP **0908** (2009) 100.

[13] A. Kleinschmidt and H. Nicolai, JHEP **1308** (2013) 041.

[14] A. Kleinschmidt, H. Nicolai and N.K. Chidambaram, `arXiv:1411.5893`, to appear in Phys. Rev. D.

[15] H. Godazgar, M. Godazgar, O. Krüger, H. Nicolai and K. Pilch, JHEP **1501** (2015) 056.

[16] S. de Buyl, M. Henneaux and L. Paulot, JHEP **0602** (2006) 056.

[17] S. de Buyl, M. Henneaux and L. Paulot, Class. Quant. Grav. **22** (2005) 3595.

[18] T. Damour and P. Spindel, Class. Quant. Grav. **30** (2013) 162001.

[19] T. Damour and P. Spindel, Phys. Rev. D **90** (2014) 10, 103509.

[20] S. Berman, Commun. Algebra **17** (1989) 3165.

[21] V. Kac, *Infinite dimensional Lie Algebras*, 3rd edition, Cambridge University Press (1990).

[22] H. Nicolai and T. Fischbacher, `hep-th/0301017`.

[23] The finite group $PSL_2(7)$, a maximal discrete subgroup of the family symmetry $SU(3)_f$ which was recently invoked to explain the quark mass hierarchy [G. Chen, M. J. Pérez and P. Ramond, `arXiv:1412.6107`] sits naturally in the Weyl group of E_7. The latter is contained in the Weyl group of E_{10}, and thus also inside $K(E_{10})$.