New Journal of Physics
An Institute of Physics and Deutsche Physikalische Gesellschaft Journal

Modulational instability of drift waves

K Hallatschek[1] **and P H Diamond**[2]

[1] Centre for Interdisciplinary Plasma Science, Max-Planck-Institut für Plasmaphysik, EURATOM-IPP Association, D-85748 Garching, Germany
[2] Department of Physics, University of California San Diego, La Jolla, CA 92093-0319, USA
E-mail: ksh@rzg.mpg.de

New Journal of Physics **5** (2003) 29.1–29.9 (http://www.njp.org/)
Received 1 November 2002
Published 4 April 2003

Abstract. The instability of drift waves against zonal flows and streamers is discussed. Unlike in previous treatments, we do not make the assumption that their frequency is resonant with drift wave packets. In this more general treatment we find at least two unstable roots even in the simple case of a monochromatic pump drift wave, and potentially an infinite multitude of roots for a more complicated drift wave spectrum. One of them is the well known modulational instability in resonance with the drift wave packets; the other is a new instability corresponding to the inelastic refraction of drift waves at the streamer. It is nontrivial which of the many roots is the most unstable one.

Contents

1. Introduction

The turbulence in a tokamak core is believed to be controlled by its volatility to the secondary instability of zonal flows [1, 2]. Generalizing the notion of a zonal flow, nonlinear streamers can be regarded as radially elongated macro-modes driven by a similar secondary instability

[3]. Unlike zonal flows, the streamers can couple to a background gradient and cause transport. For large scales of the secondary modes compared to the primary drift waves, the action of the growing streamer/zonal flow on the drift wave is a pure modulation. The analogy to the kinetic instability of particle distributions suggests that these are only noticeable (and the instability is only present) when their phase velocity is resonant with the group velocity of the driving part of the drift wave spectrum. In contrast to this, from the fluid perspective, the secondary modes are often treated as Kelvin–Helmholtz instabilities [4], and their instability is usually analysed assuming zero real frequency, precluding a phase velocity of the flow/streamer.

It is insightful to do an analysis without *a priori* assuming any frequency for the secondary mode in a tractable model system in which adiabatic drift waves interact with 2D fluid streamers, to uncover the deviations from the two simple perspectives. It is understood that this eliminates artificially many of the complexities of the 3D nature of the flow involving multiple resonant surfaces [5], the turbulence mode structure [6], the neoclassical effects [7], and additional drive terms apart from the Reynolds stress [8]. As we show below, in general at least two roots exist; indeed, for complex drift wave spectra, potentially many more. As for the transport relevance in tokamaks one is interested in the fastest growing solution, since usually the competition between the primary and the secondary instabilities determines the turbulence saturation amplitude.

2. Equations for the analysis

We use as space units $\rho_s = c_s/\omega_c i$ ($c_s = \sqrt{T/m}$,$\omega_{ci} = eB/(mc)$), and time units L_n/c_s, where L_n is the density gradient length $d\ln(n)/d\ln(r)$. The units for the density are $n_0\rho_s/L_n$ (n_0 is the background density), the unit of the electric potential is $T/e \cdot \rho_s/L_n$, the electrons are isothermal and the ions are cold. Starting from the Hasegawa–Wakatani equations [9] in an unsheared slab we use the same model equations as in [3]. We distinguish between primary drift modes and the secondary streamer mode by means of parallel filtering: the streamer is idealized to have no variation along the field lines ($k_\parallel = 0$) while the drift modes have very large $k_\parallel$ with $v_{\mathrm{the}}^2 k_\parallel^2/(\nu\omega) \gg 1$ (v_{the}: electron thermal velocity, ν: electron collision frequency, ω: characteristic wave frequency) which causes the electrons to be adiabatic (i.e. $n = \phi$). This idealization effectively reduces the 3D problem to the interaction of 2D streamers with 2D drift waves. The appropriate mathematical (orthogonal) projectors will be called P^s and P^d respectively. With the advective time derivative for magnetic field in the $\hat{z}$ direction, $D_t = \partial_t + (\hat{z} \times \nabla_\perp\phi) \cdot \nabla$, the equation for the drift wave potential $\phi^d = P^d\phi$ is

$$P^d\left[D_t(n - \Delta_\perp\phi)\right] + \partial_y\phi^d = 0. \tag{1}$$

For the density fluctuations of the drift waves adiabaticity simply requires $n^d = \phi^d$. The secondary streamer modes obey incompressible fluid equations,

$$P^s\left[D_t\Delta_\perp\phi\right] = 0, \tag{2}$$

and

$$P^s\left[D_t n\right] + \partial_y\phi^s = 0. \tag{3}$$

Linearly, i.e. replacing D_t with ∂_t in equations (1) and (2) we note that the drift waves have the drift frequency $k_y/(1 + k_\parallel^2)$ while the streamers have zero frequency since they obey regular fluid equations.

Expanding the expressions and abbreviating $A^\xi = (\hat{z}\times\nabla_\perp\phi^\xi)\cdot\nabla$, $(\xi = d, s)$, we arrive at

$$\partial_t(1-\Delta_\perp)\phi^d + \partial_y\phi^d + A^s(1-\Delta_\perp)\phi^d + A^d(n^s - \Delta_\perp\phi^s) - P^d A^d\Delta_\perp\phi^d = 0, \tag{4}$$

$$\partial_t\Delta_\perp\phi^s + A^s\Delta_\perp\phi^s + P^s A^d\Delta_\perp\phi^d = 0, \tag{5}$$

$$\partial_t n^s + A^s n^s + \partial_y\phi^s = 0. \tag{6}$$

3. Dispersion relation of streamers and zonal flows for a monochromatic pump drift wave

We specialize to the case of a drift wave with wave vector $\boldsymbol{d}$ interacting with a secondary mode of wave vector $\boldsymbol{s}$ which is assumed to be a small perturbation. In what follows all perturbation quantities are indicated by the index '1'. The first-order perturbations of the three equations then read

$$\partial_t(1-\Delta_\perp)\phi_1^d + \partial_y\phi_1^d + A_1^s(1+\boldsymbol{d}_\perp^2)\phi_0^d + A_0^d(n_1^s + \boldsymbol{s}_\perp^2\phi_1^s) + P^d A_1^d\boldsymbol{d}_\perp^2\phi_0^d - P^d A_0^d\Delta_\perp\phi_1^d = 0, \tag{7}$$

$$-\partial_t\boldsymbol{s}_\perp^2\phi_1^s + P^s A_0^d\Delta_\perp\phi_1^d - P^s A_1^d\boldsymbol{d}_\perp^2\phi_0^d = 0, \tag{8}$$

$$\partial_t n_1^s + \mathrm{i}s_y\phi_1^s = 0. \tag{9}$$

The first two equations can be simplified further:

$$\partial_t(1-\Delta_\perp)\phi_1^d + A_0^d(n_1^s + (\boldsymbol{s}_\perp^2 - \boldsymbol{d}_\perp^2 - 1)\phi_1^s) + \partial_y\phi_1^d - P^d A_0^d(\boldsymbol{d}_\perp^2 + \Delta_\perp)\phi_1^d = 0, \tag{10}$$

$$-\partial_t\boldsymbol{s}_\perp^2\phi_1^s + P^s A_0^d(\boldsymbol{d}_\perp^2 + \Delta_\perp)\phi_1^d = 0, \tag{11}$$

If it were not for the last term in equation (10) these equations could be easily solved, and ϕ_1^d would exhibit only contributions at wave vectors $\boldsymbol{d}+\boldsymbol{s}$ and $\boldsymbol{s}-\boldsymbol{d}$. Due to the three waves involved, we would obtain exactly three roots for the frequency of the perturbation. This last term, however, couples the streamers to an infinite set of drift wave modes with wave numbers $\boldsymbol{s}+n\boldsymbol{d}$, where n is an integer. Therefore in general an infinity of secondary eigenmodes exists. The nonlinear term in question can be neglected in comparison to the term due to the partial time derivative either for small pump wave amplitude $|\boldsymbol{d}_\perp||\phi_0| \ll \omega$ with the drift wave frequency ω, or for stronger pump wave amplitude, if for $|\boldsymbol{s}_\perp|$, $|\boldsymbol{d}_\perp| \ll 1$ we restrict ourselves to fast-growing solutions with growth rate of order of the Kelvin–Helmholtz growth rate $O(|\boldsymbol{d}_\perp||\boldsymbol{s}_\perp||\phi_0^d|)$. We now split ϕ_1^d into ϕ_1^{d+} with wavenumber $\boldsymbol{s}+\boldsymbol{d}$ and ϕ_1^{d-} with $\boldsymbol{s}-\boldsymbol{d}$. The frequency of the primary mode is denoted by ω, the complex frequency of the secondary mode is indicated by Ω. With the abbreviation

$$N_1^s = n_1^s + (\boldsymbol{s}_\perp^2 - \boldsymbol{d}_\perp^2 - 1)\phi_1^s \tag{12}$$

and the nonlinear coefficient

$$C = \hat{z}\cdot(\boldsymbol{d}_\perp\times\boldsymbol{s}_\perp) \tag{13}$$

we obtain:

$$\mathrm{i}\left[s_y + d_y - (\Omega+\omega)(1+(\boldsymbol{s}_\perp+\boldsymbol{d}_\perp)^2)\right]\phi_1^{d+} - C\phi_0^d N_1^s = 0, \tag{14}$$

$$\mathrm{i}\left[s_y - d_y - (\Omega-\omega)(1+(\boldsymbol{s}_\perp-\boldsymbol{d}_\perp)^2)\right]\phi_1^{d-} + C(\phi_0^d)^* N_1^s = 0, \tag{15}$$

$$P^s A_0^d(\boldsymbol{d}_\perp^2 + \Delta_\perp)\phi_1^d = C(\boldsymbol{d}_\perp^2 - (\boldsymbol{s}_\perp+\boldsymbol{d}_\perp)^2)(\phi_0^d)^*\phi_1^{d+} - C(\boldsymbol{d}_\perp^2 - (\boldsymbol{s}_\perp-\boldsymbol{d}_\perp)^2)\phi_0^d\phi_1^{d-}. \tag{16}$$

From equation (9) we derive

$$\Omega n_1^s = s_y \phi_1^s, \tag{17}$$

or

$$\frac{\phi_1^s}{N_1^s} = \frac{\Omega}{s_y + \Omega[s_\perp^2 - d_\perp^2 - 1]}. \tag{18}$$

Putting everything together, equation (11) transforms into the dispersion relation for the secondary mode

$$\Omega \frac{s_\perp^2 \phi_1^s}{C^2 |\phi_0^d|^2 N_1^s} = \frac{d_\perp^2 - (s_\perp + d_\perp)^2}{s_y + d_y - (\Omega + \omega)(1 + (s_\perp + d_\perp)^2)} + \frac{d_\perp^2 - (s_\perp - d_\perp)^2}{s_y - d_y - (\Omega - \omega)(1 + (s_\perp - d_\perp)^2)}, \tag{19}$$

where the term on the left-hand side can also be written as

$$\frac{s_\perp^2 \Omega^2}{C^2 |\phi_0^d|^2 \left(s_y + \Omega[s_\perp^2 - d_\perp^2 - 1]\right)}. \tag{20}$$

For non-zero s_y, i.e., the case of a streamer, and very weak pump wave intensity $|\phi_0^d|^2$, asymptotically the structure of the left-hand side of the dispersion relation, expression (20), reveals a double root at $\Omega = 0$ corresponding to a pure density fluctuation or a streamer secondary eigenmode, while the right-hand side of (19) has two real frequency poles at

$$\Omega_\pm = \mp\omega + \frac{s_y \pm d_y}{1 + (s_\perp \pm d_\perp)^2}. \tag{21}$$

In wave number space, we can approximate for $|s_\perp| \ll |d_\perp|, 1$

$$\Omega_\pm \approx (s_\perp \cdot \nabla_k)\omega(k) \pm \tfrac{1}{2}(s_\perp \cdot \nabla_k)^2, \tag{22}$$

where ∇_k acts only on the wavenumber. We introduce the abbreviations

$$\bar{\Omega} = (\Omega_+ + \Omega_-)/2 \approx (s_\perp \cdot \nabla_k)\omega(k) = s_\perp \cdot v_g, \tag{23}$$

with v_g being the group velocity, and

$$\Delta\Omega = (\Omega_+ - \Omega_-)/2 \approx \tfrac{1}{2}(s_\perp \cdot \nabla_k)^2 \omega(k). \tag{24}$$

From a quantum mechanical perspective, ω would correspond to the energy, $d_\perp$ to the momentum and $\Delta\Omega$ is proportional to the mass of the drift waves for a momentum change in the direction of $s_\perp$ (the mass is in general anisotropic and depends on the direction of momentum change). From the correspondence (23) we conclude that these eigenmodes are pure modulations of the drift wave with either the envelope function $1 + \epsilon \exp(\mathrm{i}k_s \cdot r)$ or $1 + \epsilon \exp(-\mathrm{i}k_s \cdot r)$. Both propagate essentially with the group velocity v_g except for a small deviation due to dispersion.

With the additional definitions

$$\gamma_\pm = \frac{(s_\perp \pm d_\perp)^2 - d_\perp^2}{1 + (s_\perp \pm d_\perp)^2} \tag{25}$$

$$\gamma_0 = \frac{s_\perp^2}{1 - s_\perp^2 + d_\perp^2} \tag{26}$$

$$\Omega_0 = \frac{s_y}{1 - \boldsymbol{s}_\perp^2 + \boldsymbol{d}_\perp^2} \tag{27}$$

$$\gamma = \frac{\gamma_- - \gamma_+}{\gamma_+ + \gamma_-} \tag{28}$$

$$\gamma_3 = \frac{\gamma_0}{\gamma_+ + \gamma_-} \tag{29}$$

and

$$\Omega_M = \bar{\Omega} + \gamma \Delta\Omega \tag{30}$$

the dispersion relation transforms into

$$-\frac{\gamma_3 \Omega^2}{C^2 |\phi_0^d|^2 (\Omega - \Omega_0)} = \frac{(\Omega - \Omega_M)}{(\Omega - \Omega_+)(\Omega - \Omega_-)} \tag{31}$$

or

$$-\frac{\gamma_3 \Omega^2 (\Omega - \Omega_+)(\Omega - \Omega_-)}{(\Omega - \Omega_0)(\Omega - \Omega_M)} = C^2 |\phi_0^d|^2 \tag{32}$$

or

$$-\frac{\gamma_3 \Omega^2 (\Omega - \bar{\Omega} - \Delta\Omega)(\Omega - \bar{\Omega} + \Delta\Omega)}{(\Omega - \Omega_0)(\Omega - \bar{\Omega} - \gamma \Delta\Omega)} = C^2 |\phi_0^d|^2. \tag{33}$$

An instability threshold for a secondary instability is crossed if the right-hand side just exceeds a local *maximum* of the left-hand side (in Ω). Therefore the possible instabilities correspond, in the limit of low pump wave amplitude, exactly to the local maxima of the left-hand side of (33).

4. Discussion of secondary instabilities

For reference we list the values of the various coefficients and frequencies involved in the limit of $|\boldsymbol{s}_\perp| \ll |\boldsymbol{d}_\perp| \ll 1$ to leading order:

$$\Omega_0 = s_y \tag{34}$$

$$\bar{\Omega} = s_y - 2 d_y \boldsymbol{s}_\perp \cdot \boldsymbol{d}_\perp \tag{35}$$

$$\Delta\Omega = -(d_y \boldsymbol{s}_\perp^2 + 2 \boldsymbol{s}_\perp \cdot \boldsymbol{d}_\perp s_y) \tag{36}$$

$$\Omega_M = \bar{\Omega} + \gamma \Delta\Omega \approx s_y \tag{37}$$

$$\gamma_3 = \tfrac{1}{2} \tag{38}$$

$$\gamma = -2 \frac{\boldsymbol{s}_\perp \cdot \boldsymbol{d}_\perp}{\boldsymbol{s}_\perp^2}. \tag{39}$$

For very high pump wave amplitude, asymptotically we obtain three roots, one with very high growth rate at negligible real frequency (corresponding to the strong drive), and two uninteresting eigenmodes with frequencies close to the poles Ω_0, Ω_M of the left-hand side of the dispersion relation (32). For the fast growing mode we can neglect all fixed frequencies on the left-hand side of (32), and the growth rate is for $\gamma_3 > 0$ is just given by

$$\gamma_\infty = |C||\phi_0^d| / \sqrt{\gamma_3}. \tag{40}$$

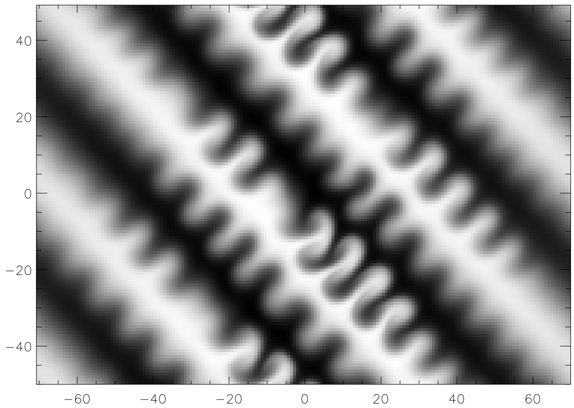

Figure 1. Grey scale plot of the slightly nonlinear state of the secondary instability of a strong drift wave from a numerical simulation of the drift wave and streamer equation including all harmonics. The length units of the plot are ρ_s (as throughout the paper), the wavenumber of the primary mode is 0.089, its peak velocity (in the region of maximum gradient) is $10.7v_{\mathrm{d}}$, the wavenumber of the perturbation is measured to be 0.64.

The criterion for growth is $\gamma_3 > 0$ and $C \neq 0$. It can be shown that $\gamma_3 \approx 1/2$ under the condition that the gyroradius is small compared to all wavelengths $|\boldsymbol{s}_\perp|$, $|\boldsymbol{d}_\perp| \ll 1$ alone, yielding the well known Kelvin–Helmholtz growth rate for large scale perturbations of short scale waves

$$\gamma_\infty \approx \sqrt{2}|C||\phi_0^d|. \tag{41}$$

As distinct from the usual Kelvin–Helmholtz case, it is *not* necessary that the secondary perturbations be large scale compared to the primary flow pattern ($\boldsymbol{s}_\perp < \boldsymbol{d}_\perp$); zonal flows and streamers are even unstable if their wavelengths are *shorter* than the wavelength of the primary drift waves. This case of strong pump wave amplitude corresponds to, e.g., ITG turbulence well above the instability thresholds. The fastest growing perturbations are the ones for which $|C|$ is largest, while the wave vector is still required to be small compared to the gyroradius. This favours perpendicular perturbations with high wave numbers.

In the case of cold ions, the drift wave equation is valid even if $|\boldsymbol{s}_\perp| \sim 1$. It can be shown that for $|\boldsymbol{d}_\perp| \ll 1$ but arbitrary $|\boldsymbol{s}_\perp|$ we have

$$\gamma_3^{-1/2} \approx \sqrt{\frac{2(1-\boldsymbol{s}_\perp^2)}{1+\boldsymbol{s}_\perp^2}}, \tag{42}$$

i.e., for $|\boldsymbol{s}_\perp| \to 1$, the instability is cut off by gyroradius effects. Maximizing $|C|\gamma_3^{-1/2}$ over all $\boldsymbol{s}_\perp$ at fixed $\boldsymbol{d}_\perp$ with $|\boldsymbol{d}_\perp| \ll 1$ yields an optimum wavenumber $|s| = \sqrt{\sqrt{2}-1} \approx 0.64$, at which perpendicular orientation of the wave vector $\boldsymbol{s}_\perp$ with respect to $\boldsymbol{d}_\perp$ produces the maximum growth rate of all secondary perturbations. This fact is approximately confirmed by numerical simulations (figure 1) of the full initial system, including all the higher perturbations, which have been neglected to arrive at the analytical result.

Let us now explore the thresholds for instability in more detail.

For the case of a zonal flow we have by definition $0 = s_y \propto \Omega_0 \propto \Omega_M$. That is, the root $\Omega = 0$ is completely cancelled. The threshold is therefore determined by the local maximum of the quadratic expression $-\gamma_3(\Omega - \Omega_+)(\Omega - \Omega_-)$ on the left side. At the threshold the real frequency of the zonal flow lies exactly in the middle between Ω_+ and Ω_- at $\bar{\Omega}$, i.e., approximately at the group velocity of the drift waves times the wave number of the zonal flows. This is understood because the flows are causing modulations travelling with the group velocity of the drift waves. Using definition (24), zonal flows are driven by the drift wave, if

$$\gamma_3 \Delta\Omega^2 < C^2|\phi_0^d|^2 \Leftrightarrow |\phi_0^d| > |\phi_{0,\text{thresh}}^d| \tag{43}$$

with

$$|\phi_{0,\text{thresh}}^d| \approx \frac{\sqrt{2}|\Delta\Omega|}{|C|} = \frac{|d_y|\boldsymbol{s}_\perp^2}{\sqrt{2}|C|}. \tag{44}$$

The growth rate for the zonal flows is in the same approximation

$$\gamma_{\text{ZF}} = \sqrt{2}|C|\sqrt{|\phi_0^d|^2 - |\phi_{0,\text{thresh}}^d|^2}, \tag{45}$$

which tends towards the Kelvin–Helmholtz growth rate in the limit of strong pump wave amplitude, i.e. asymptotically

$$|\phi_0^d| \to \infty \Rightarrow \gamma_{\text{ZF}} \to \sqrt{2}|C||\phi_0^d|. \tag{46}$$

The threshold condition therefore states exactly the condition that the frequency dispersion of the drift waves due to the modulation ($\Delta\Omega$) has to be smaller than the Kelvin–Helmholtz growth rate.

Now let us turn to the case of more general orientation of the secondary modes $s_y \neq 0$, the streamer case. To first order in the primary wave intensity the frequencies of the two propagating roots ($\approx \Omega_\pm$) have to stay real since they are single roots. However, the double root of the streamer/density fluctuation at $\Omega = 0$ gains an imaginary part indicating growth for one root if $\Omega = 0$ corresponds to a local maximum of the left-hand side, even for arbitrarily small pump wave amplitude. Near $\Omega = 0$ equation (32) can be approximated by

$$-\frac{\gamma_3\Omega^2\Omega_+\Omega_-}{\Omega_0\Omega_M} = C^2|\phi_0^d|^2, \tag{47}$$

which yields a growth rate

$$\gamma_{\text{ST}} = |C\phi_0^d|\sqrt{\frac{\Omega_0\Omega_M}{\gamma_3\Omega_+\Omega_-}}, \tag{48}$$

provided the sign in the square root is positive. For $|\boldsymbol{s}_\perp| \ll |\boldsymbol{d}_\perp| \ll 1$, $s_y \sim |\boldsymbol{s}_\perp|$ this is approximately

$$\gamma_{\text{ST}} = \sqrt{2}|C||\phi_0^d|, \tag{49}$$

the Kelvin–Helmholtz growth rate, although the instability mechanism is quite different and necessarily involves the density perturbation of the test mode, and the pump wave amplitude was assumed to be *weak*. Note that this is clearly not a modulational instability, since the growth rate does not involve the dispersion of the drift waves. The real frequency of the perturbation is zero, the streamer does not travel with the drift waves, as the zonal flow did in the previous case. Instead, the drift waves traversing the streamer are distorted 'on the fly', which extracts a part of their energy, which contributes to the streamer growth. One can say that the drift waves are inelastically scattered by the streamer by undergoing a Doppler reduction of their frequency. The real frequency of the secondary mode (zero) thus corresponds to the resonance frequency of the *streamer*.

Let us now discuss the modification of the other roots ($\Omega_\pm$) corresponding to an envelope modulation of the drift waves, which naturally travels with the drift wave group velocity, and will therefore yield a solution with a real frequency around $\Omega_\pm \approx \boldsymbol{s}_\perp \cdot \boldsymbol{v}_g$.

In the limit $|\boldsymbol{s}_\perp| \ll |\boldsymbol{d}_\perp| \ll 1$, we have $|\bar{\Omega}|, |\Omega_0 - \bar{\Omega}|, |\Omega_0 - \bar{\Omega} - \gamma\Delta\Omega| \ll |\Delta\Omega|$. This is because $\Delta\Omega$ is entirely due to dispersion of the drift wave phase velocity, i.e. it is of second order in the wave vectors, while the other frequencies are proportional to the group velocity, i.e. first order in the wave vectors. Therefore the left-hand side of equation (33) has one local extremum between Ω_+ and Ω_-, which gives rise to instability, if it has positive sign, and is just exceeded by the right-hand side of the equation due to a sufficiently large pump wave intensity.

The associated threshold in drift wave amplitude is just given by the extremal value. Because of the large distance between the frequencies Ω_0, Ω_M from $\Omega_\pm$, we can simply assume the maximum to be at $\Omega = \bar{\Omega}$. The formula for the threshold turns out to be

$$\begin{aligned} |\phi_0^d|^2_{\text{thresh}} &= \frac{\gamma_3\bar{\Omega}^2\Delta\Omega}{\gamma C^2(\Omega_0 - \bar{\Omega})} \\ &= \frac{\Delta\Omega}{C^2}\frac{s_y^2\boldsymbol{s}_\perp^2}{8d_y(\boldsymbol{s}_\perp\cdot\boldsymbol{d}_\perp)^2}. \\ &=: \frac{\Delta\Omega}{d_y}\alpha^2. \end{aligned} \tag{50}$$

which is of order $O(\epsilon^2)$. Of course, instability can only occur if the threshold actually exists. The condition for the existence of a threshold is $\Delta\Omega d_y > 0$. Interpreting the dispersion relation in a quantum mechanical sense, this means that the drift waves have positive (negative) mass ($\Delta\Omega$) in the direction of the wave vector of the modulation ($\boldsymbol{s}_\perp$), if their energy is positive (negative) ($d_y > 0 \Leftrightarrow \omega > 0$). For the correct sign of mass and energy, the streamer acts as an attractive force field on the drift waves. If it is sufficiently strong (as determined by the dispersion of the drift waves) it can trap some of them and cause growing modulations. If the sign is opposite, the attractive (repulsive) potential of the streamer will actually push away the modulation of the drift waves with negative (positive) mass instead of trapping them and carrying them with it.

The growth rate of the modulational streamer instability is

$$\begin{aligned} \gamma_{\text{MD}} &= |\Delta\Omega|\sqrt{|\phi_0^d|^2/|\phi_0^d|^2_{\text{thresh}} - 1} \\ &= \alpha^{-1}\sqrt{\Delta\Omega d_y\left(|\phi_0^d|^2 - |\phi_0^d|^2_{\text{thresh}}\right)}. \end{aligned} \tag{51}$$

The constants can be approximated to

$$\alpha^{-1}\sqrt{\Delta\Omega d_y} = \sqrt{8}|C||\boldsymbol{s}_\perp\cdot\boldsymbol{d}_\perp|\sqrt{-\frac{d_y^2}{s_y^2} - 2\frac{\boldsymbol{s}_\perp\cdot\boldsymbol{d}_\perp d_y}{\boldsymbol{s}_\perp^2 s_y}}. \tag{52}$$

Overall this is of order

$$O(|C||\boldsymbol{d}_\perp|^2) \tag{53}$$

which is by $O(|\boldsymbol{d}_\perp|^2)$ smaller than the factor of non-modulational streamer instability, which makes it difficult to find in numerical simulations.

5. Discussion

We have derived the growth rates of zonal flow of the streamer mode and of the streamer-like modulational instability for a monochromatic drift wave spectrum for weak and strong primary drift wave amplitude. The instability of the zonal flow or the modulational instability is controlled by the coupling of the secondary modes to the drift wave *modulations* (not the drift waves themselves), and propagates therefore with the group velocity of the drift waves for weak primary mode amplitude. These two modes have the character of modulational instabilities, including the fact that the dispersion of the drift waves has to be overcome by the growth rate of the instability, which leads to a threshold in the primary mode amplitude for secondary growth. In addition to the modulational instabilities, there exists a non-modulational streamer mode which is driven by the inelastic scattering of drift waves. It has a real frequency close to the natural frequency of the streamer (zero). Since the streamer itself is not eroded by dispersion (as are the drift wave modulations), this secondary mode does not have a threshold. In the case of strong drift wave amplitude, both the (modulational) zonal flow and the non-modulation streamer growth rate tend towards the Kelvin–Helmholtz growth rate, but *without* the cutoff condition that the secondary wavenumber be smaller than the primary wavenumber. There exists no equivalent to the dispersion-free non-modulational streamer instability in the zonal flow case (with zero poloidal wavenumber), since the drive of density perturbations vanishes in this limit. The complex frequency of these density perturbations (which correspond to diamagnetic zonal flows) is simply zero.

References

[1] Hammett G W, Beer M A, Dorland W, Cowley S C and Smith S A 1993 *Plasma Phys. Control. Fusion* **35** 973

[2] Dimits A M, Williams T J, Byers J A and Cohen B I 1996 *Phys. Rev. Lett.* **77** 71

[3] Champeaux S and Diamond P H 2001 *Phys. Lett.* A **288** 214

[4] Finn J M, Drake J F and Guzdar P N 1992 *Phys. Fluids* B **4** 2758

[5] Hallatschek K 2000 *Phys. Rev. Lett.* **84** 5145

[6] Diamond P H, Rosenbluth M M, Hinton F L, Malkov M, Fleischer J and Smolyakov A 2000 *Proc. 17th IAEA Fusion Energy Conf. (19–24 October 1998, Yokohama)* paper IAEA-CN-69/TH3/1

[7] Rosenbluth M N and Hinton F L 1998 *Phys. Rev. Lett.* **80** 724

[8] Hallatschek K and Biskamp D 2001 *Phys. Rev. Lett.* **86** 1223

[9] Hasegawa A and Wakatani M 1987 *Phys. Rev. Lett.* **59** 1581