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Abstract

Ideal plasma equilibria in axisymmetric geometry can be described by the Grad-Shafranov equation which is a strictly elliptic differential equation. In contrast, non-axisymmetric problems in stellarator geometry lead to elliptic equations which are not strictly elliptic. As a consequence no general existence theorem can be established since singularities of the plasma current may arise. Collisional processes remove these singularities and the issue of existence can be solved by the Leray-Schauder fixed point theorem which is widely used in hydrodynamics to study viscous laminar flow. This procedure requires to compute magnetic field and plasma flow together and to identify the conditions of convergence. Several plasma models will be discussed and the conditions of convergence will be discussed.

1 Introduction

The favourite model to describe a plasma equilibrium in stellarator configurations is the one-fluid model of ideal magneto-hydrodynamics. The gradient of a scalar pressure is balanced by the force $\boldsymbol{j} \times \boldsymbol{B}$.

$$0 \, = \, -\nabla p + \boldsymbol{j} \times \boldsymbol{B} \tag{1}$$

As it is well-known, in systems with symmetries like axisymmetry in tokamaks or helical symmetry in linear stellarators the solution of Eq. (1) can be reduced to a two-dimensional quasilinear elliptic equation, the Schlüter-Grad-Shafranov equation. In a 3-dimensional equilibrium, however, such equation does not exists. As pointed out by Kruskal and Kulsrud [1] Eq. (1) is the Euler equation of the variational principle

$$\delta U \, = \, \delta \int (\frac{B^2}{2} + \frac{p}{(\gamma - 1)}) \, dV \, = \, 0 \tag{2}$$

where the minimization is subject to several constraints. Minimization of the functional U is the method how to compute 3-equilibria numerically however, it should be pointed out that the variational formulation of the problem does not imply the existence of a solution, at least not in the classical sense, where $\boldsymbol{j}(x)$ and $\boldsymbol{B}(x)$ are continuously differentiable functions. Nevertheless, several numerical codes have been developed in the past to solve Eq. (1) by starting from minimization of the energy U. An excellent review over these approaches has been published by J. Johnson [2]. A major difference between the axisymmetric equilibrium and the non-axisymmetric equilibrium is the extra condition $\oint dl/B \, = \, const$ on all rational magnetic surfaces. As has been pointed out by H. Grad [3] this may lead to a very pathological pressure distribution, which could be continuous radially, but has no continuous derivatives. This extra global condition $\oint \frac{dl}{B} = const$ has been strongly criticized by H. Grad as something "completely alien to any concept which has ever arisen in connection with the theory of partial differential equations". Another feature of the ideal model is the existence of islands and stochasticity, which is a generic property of 3-dimensional magnetic fields. Usually it is argued to keep the pressure constant in such a region, however, islands and stochasticity depend on the plasma currents and these on the pressure distribution. If the plasma pressure is large, islands overlap and create stochasticity all over the plasma column. This effect already arises, if one tries to compute the stellarator equilibrium following the iterative scheme proposed by L. Spitzer [4]. Starting from Eq. (1) a sequence $\{\boldsymbol{B}_n\}$ of magnetic fields is constructed, which are assumed to converge towards a self-consistent equilibrium. However, as pointed out by A. Boozer [5] magnetic surfaces can be destroyed, even if the process begins with a set of nested surfaces. Numerical codes following this procedure have been developed by a A. Reiman and H. Greenside [6] and J. Kisslinger, H. Wobig[7]. At low β this procedure yields results which look acceptable, however, at higher β the occurrence of islands prohibits the convergence.

A fundamental drawback of the ideal model is the singularity of the current density on rational magnetic surfaces. The theory of magnetic fields requires a continuous current density, a theory taking into account singularities does not exist. Therefore any reasonable theory of plasma equilibrium should avoid these singularities.

A further objection against the ideal model is its incapability to satisfy boundary conditions, neither in plasma pressure nor in density or electric potentials, which even do not occur in the model. In the ideal model, plasma flow - either diffusion velocity or $\boldsymbol{E} \times \boldsymbol{B}$ - drifts of parallel flows have no feedback on the force balance. The flow velocity $\boldsymbol{v}$ must be calculated from Ohm's law

$$-\nabla\Phi + \boldsymbol{v} \times \boldsymbol{B} = \eta \boldsymbol{j} \quad ; \quad \eta > 0 \tag{3}$$

and the equation of continuity

$$\nabla \cdot \rho \boldsymbol{v} = S \tag{4}$$

S is the density source term. In the frame of this model the electric potential ϕ and parallel flow velocity are calculated from a magnetic differential equation. Again, on rational magnetic surfaces conditions of integrability arise, and it remains uncertain, whether they can be satisfied in all cases. In summarizing the objections against the ideal model of equilibrium in stellarators we find

- Mathematically the model may be not well-posed and a solution in the classical sense may not exist
- The existence of densely nested magnetic surfaces may be a too strong condition for non-axisymmetric equilibria.
- Boundary conditions on plasma parameters cannot be satisfied.
- Magnetic fields without symmetry in general exhibit island formation and stochastic regions

Even if an ideal equilibrium exists the computation of the plasma flow velocity can lead to singularities. Using Ohm's law with finite resistivity one computes the perpendicular diffusion velocity and the equation of continuity yields a magnetic differential equation for the parallel velocity. Again a integral condition ensuring periodicity must be satisfied on rational magnetic surfaces. Nevertheless, there remains the opinion that the ideal model is a good approximation to the real plasma equilibrium with selfconsistent currents and flow velocity. Perpendicular to the magnetic field all other forces like inertial and viscous forces are small and may be neglected. Parallel to the magnetic field, however, even small additional forces can lead to a decoupling of pressure surfaces and magnetic surfaces. Instead of the strong condition $\boldsymbol{B} \cdot \nabla p = 0$, which follows from Eq. (1) any parallel inertial or viscous force leads to

$$\boldsymbol{B} \cdot \nabla p \approx 0 \tag{5}$$

Since finite plasma flow velocity decouples magnetic surfaces and pressure surfaces this effect removes the reason of the criticism against the ideal model. Therefore, from a physics point of view, one could argue, that an "approximate" solution of the ideal MHD - model may still be a good description of the plasma behaviour, since corrections to the equilibrium conditions arising from the finite flow velocity are small. How good this approximation is, can only be answered after a well-posed equilibrium has been established and the properties of its solution are understood.

2 Grad-Shafranov Equation

In mathematics a significant difference between axi-symmetric and non-symmetric configurations exists. Starting from the general ansatz $\boldsymbol{B} = \nabla s \times \nabla \chi$ the conditions $\boldsymbol{j} \cdot \nabla s = 0$ and $\boldsymbol{j} \cdot \nabla \chi = \mu_0 p'(s)$ lead to the coupled equations

$$\nabla \cdot B[s] \nabla \chi = 0 \quad ; \quad \nabla \cdot A[\chi] \nabla s = \mu_0 p'(s) \tag{6}$$

with the matrices

$$A[\chi] = (\nabla \chi)^2 \mathbf{I} - \nabla \chi : \nabla \chi \quad ; \quad B[s] = (\nabla s)^2 \mathbf{I} - \nabla s : \nabla s \tag{7}$$

These equations (6) are two coupled elliptic equations, however they are not strictly elliptic, the matrices A and B are positive but not positive definite. Therefore the solutions may be singular exhibiting discontinuous derivatives. In the case of axi-symmetry, however, the ansatz $\chi = \varphi + f(r,z)$ and $s = s(r,z)$ in cylindrical coordinates r, z, φ leads to the Lüst-Schlüter-Grad-Shafranov equation which is strictly elliptic and therefore all theorems of the theory of strictly elliptic equations apply. Because of the orthogonality of the coordinate system this ansatz implies $\nabla s \cdot \nabla \varphi = 0$; $\nabla f \cdot \nabla \varphi = 0$ and the following representation of the magnetic field

$$\boldsymbol{B} = \nabla s \times \nabla f + \nabla s \times \nabla \varphi \tag{8}$$

The first term is a toroidal field and represented as $\nabla s \times \nabla f = \mu_0 J(r,z) \nabla \varphi$. The plasma current density is

$$\boldsymbol{j} = \mu_0 \nabla \varphi \times \nabla J + \nabla \times (\nabla s \times \nabla \varphi) \tag{9}$$

and because of $\boldsymbol{j} \cdot \nabla s = 0$ and $\nabla s \cdot \nabla \times (\nabla s \times \nabla \varphi) = 0$ we get

$$\mu_0 (\nabla \varphi \times \nabla J) \cdot \nabla s = 0 \implies J = J(s) \tag{10}$$

J is an arbitrary function of s. The relation between J and $f(r,z)$ is

$$(\nabla s \times \nabla f) \cdot \nabla \varphi = \mu_0 J(s) \, (\nabla \varphi)^2 \tag{11}$$

The equation $\boldsymbol{j} \cdot \nabla \chi = \mu_0 p'(s)$

$$\nabla \cdot ((\nabla s \times \nabla (f + \varphi)) \times \nabla (f + \varphi)) = \mu_0 p'(s) \tag{12}$$

is the desired equation for the flux function s. Because of the axi-symmetry

$$\nabla \cdot ((\nabla s \times \nabla f) \times \nabla \varphi) = 0 \; ; \; \nabla \cdot ((\nabla s \times \nabla \varphi) \times \nabla f) = 0 \tag{13}$$

this equation is

$$\nabla \cdot ((\nabla s \times \nabla f) \times \nabla f) + \nabla \cdot ((\nabla s \times \nabla \varphi) \times \nabla \varphi) = \mu_0 p'(s) \tag{14}$$

With the aid of

$$\begin{aligned}\nabla \cdot ((\nabla s \times \nabla f) \times \nabla f) &= \nabla \cdot (\mu_0 J \nabla \varphi \times \nabla f) \\ &= -\mu_0^2 J'(s) J(s) (\nabla \varphi)^2 \end{aligned} \tag{15}$$

the result is

$$\nabla \cdot ((\nabla s \times \nabla \varphi) \times \nabla \varphi) - \mu_0^2 J'(s) J(s) (\nabla \varphi)^2 = \mu_0 p'(s) \tag{16}$$

or

$$-\nabla \cdot (\nabla \varphi)^2 \nabla s = \mu_0^2 J'(s) J(s) (\nabla \varphi)^2 + \mu_0 p'(s) \quad ; \quad (\nabla \varphi)^2 = \frac{1}{r^2} \tag{17}$$

This is the Lüst-Schlüter-Grad-Shafranov-equation. In any domain with $r \neq 0$ this is a quasilinear strictly elliptic equation.

3 Non-axisymmetric equilibria

In special conditions also the existence of solutions can be proven in case of non-axisymmetric configurations. D. Lortz [8] has shown that solutions with zero rotational transform and up-down symmetry exist. However, in this case unique magnetic surfaces created by one field line do not exist.

Furthermore, as it is known from hydrodynamics, small additional terms in the force balance can introduce new phenomena like bifurcations. The solutions are no longer unique and several solutions can exists with the same boundary conditions. A well-known example in hydrodynamics are the Taylor vortices in a rotating fluid. Therefore it cannot be excluded a priori, that small additional terms in Eq. (1) - especially those containing higher order derivatives of the velocity - have a strong impact on the solution. Several models to incorporate the feedback of the plasma flow on the momentum balance have been developed in the past, the most simple ones are those with a friction term -$\alpha \boldsymbol{v}$ in Eq. (1) which leads to dissipation of energy. Such a friction term has been used in the Chodura - Schlüter code [9] to accelerate relaxation towards an equilibrium. In equilibrium, however, the flow velocity is zero, and all objections raised against Eq. (1) remain valid. In the frictional model proposed by H. Wobig [10] a friction term -$\alpha \boldsymbol{v}$ and finite resistivity in Ohm's law have been added. This model allows one to calculate plasma flow velocity and plasma currents in terms of ∇p and $\nabla \Phi$. Next, $\nabla \cdot \boldsymbol{j}$ and the equation of continuity provide a quasilinear elliptic system for p and Φ. This system holds for any magnetic field and no topological restrictions are imposed. Navier Stokes equations of an one-fluid model have been investigated by Spada and Wobig [11] where the Braginskii viscosity and the inertial terms detrmine the mathematical character of the problem. It is a basic feature of elliptic systems, that its solutions have continuous derivatives up to second order, if the coefficients of the equations are sufficiently regular. Therefore the derivatives of p and Φ remain bounded, which is distinct from the singular behaviour of plasma currents and flow velocity in the ideal model close to rational magnetic surfaces. Furthermore, the model does not require the existence of magnetic surfaces, however, radial losses and plasma pressure will strongly depend on the quality of the magnetic surfaces. A similar procedure has been introduced in the HINT-code [12] where finite resistivity allows one to decouple plasma and magnetic field so that magnetic islands can arise. However, since there are no sources to compensate plasma losses the final stationary state is one with zero plasma velocity and $\boldsymbol{B} \cdot \nabla p = 0$.

The problem of a self-consistent equilibrium can be divided in two steps: 1) Definition of a plasma model with a given magnetic field and computation of plasma parameters. 2) Computation of a magnetic field generated by currents which result from the plasma model. The existence of a self-consistent equilibrium can be reduced to the convergence of this iterative procedure or to a fixed point theorem. In both cases appropriate function spaces are needed and by making use of standard methods in the theory of partial differential equations the conditions for self-consistent equilibria can be identified. In this context the important function spaces are the Hölder space, the Hilbert space and the Sobolev spaces. The main tools to proof the existence of solutions are the theorem of Riesz and the fixed point theorem of Leray-Schauder, which are frequently used in hydrodynamic theory.

3.1 The fixed-point theorem

The procedure which will be discussed in the following makes use of the fixed-point theorem of Schauder [13]: If a compact map T maps a convex and bounded subset of a Banach space (or Hilbert space) onto itself, the map has at least one fixed-point. This theorem is widely used in hydrodynamics to prove the existence of a laminar flow [14][15]. The fixed point theorem only provides a weak solution, it does not give information about the higher derivatives of the solution. The regularity of the solutions depends on the properties of the boundary.

Let Ω be a bounded toroidal domain with a sufficiently smooth surface $\partial\Omega$. The magnetic field is the sum of an external field $\boldsymbol{B}_0$ and the field $\boldsymbol{B}$ generated by plasma currents. Plasma currents only exist in this domain Ω and we define the set $M \subset C^{1,\alpha}(\Omega)$ of all Hölder-continuous magnetic fields which have finite current density in this domain and zero current density outside.

$$M := \{\boldsymbol{B}|\ \nabla\cdot\boldsymbol{B} = 0 \quad ; \quad \mathbf{j} = \nabla\times\boldsymbol{B} = 0 \quad \forall\boldsymbol{x} \neg \in \Omega\} \tag{18}$$

The magnetic field is Hölder-continuous and has Hölder-continuous derivatives. On the boundary $\partial\Omega$ the current density may be discontinuous and the magnetic field is continuous but not differentiable. Outside the domain Ω the magnetic field is a vacuum field and $\boldsymbol{B} \to 0$ for $|\boldsymbol{x}| \to \infty$. Obviously this is a linear space and defining a scalar product (,) and the norm $\|..\|_{L_2}$ this space is a Hilbert space $L_2(\Omega)$.

$$\|\boldsymbol{B}\|^2_{L_2} = \int_\Omega \boldsymbol{B}\cdot\boldsymbol{B}\, d^3\boldsymbol{x} \tag{19}$$

The Hölder norm or C_1-norm is defined by

$$\|\boldsymbol{B}\|_{C_{1,\alpha}} = \sup|\boldsymbol{B}| + \sup|\boldsymbol{j}| + \sup\frac{|\boldsymbol{j}(\boldsymbol{x}) - \boldsymbol{j}(\boldsymbol{y})|}{|\boldsymbol{x}-\boldsymbol{y}|^\alpha} \quad ; \quad 0 < \alpha \leq 1 \tag{20}$$

The set of magnetic fields M equipped with this norm represents a Banach space. Any bounded subset of this space is a convex set.

The next step is to specify a plasma model which computes all plasma parameters, density, temperatures, plasma flow and current density etc. at a given magnetic field $\boldsymbol{B} \in M$. The current density is the only function of interest in this context and we require that the plasma model provides a current density which is Hölder-continuous. The mapping $P : \boldsymbol{B} \mapsto \boldsymbol{j}$, in general, is nonlinear and not unique, there may be more than one solution for a given magnetic field. For any Hölder-continuous current density Biot-Savart's law computes the vector potential of the magnetic field

$$\boldsymbol{A}(\boldsymbol{x}) = \frac{1}{4\pi}\int_\Omega \frac{\boldsymbol{j}(\boldsymbol{y})\, d^3\boldsymbol{y}}{|\boldsymbol{x}-\boldsymbol{y}|} \quad ; \quad \boldsymbol{B} = \nabla\times\boldsymbol{A} \quad ; \quad \boldsymbol{j} \in C^{0,\alpha} \tag{21}$$

The theory of the Newton potential says that the vector potential is a $C^{2,\alpha}$-function if the current density is a $C^{0,\alpha}$-function. The vector potential has Hölder-continuous second order derivatives if the current density is Hölder continuous. The magnetic field $\boldsymbol{B}$ is a $C^{1,\alpha}$-function, it is continuous and has Hölder-continuous derivatives. The mapping $K : \boldsymbol{j} \mapsto \boldsymbol{B}$ is linear and one to one. The composition $T = KP : \boldsymbol{B} \mapsto \boldsymbol{j} \mapsto \boldsymbol{B}$ maps the Banach space M onto itself, and if a self-consistent magnetic field exists, this is a fixed point of the map T. According to the fixed point theorem of Leray-Schauder the mapping T has a fixed point, if T is compact and maps a bounded and convex subset of M onto itself. More precisely, if all solutions of $\boldsymbol{B} = \lambda T\boldsymbol{B},\ \lambda \in [0,1]$ are bounded by some constant M_0, the mapping T has a fixed point.

The vector potential is bounded and it holds

$$|\boldsymbol{A}| \leq C(\Omega)\,\sup|\boldsymbol{j}| \quad ; \quad C(\Omega) = \sup\frac{1}{4\pi}\int_\Omega \frac{d^3\boldsymbol{y}}{|\boldsymbol{x}-\boldsymbol{y}|} \tag{22}$$

The constant $C(\Omega)$ depends only on the dimensions of the domain Ω. Integral operators with the kernel $|\boldsymbol{x}-\boldsymbol{y}|^{-1}$ are compact and map $L_2(\Omega)$ onto $L_2(\Omega)$. By virtue of Schwarz's inequality we get

$$|\boldsymbol{A}(\boldsymbol{x})| \leq C_1\|\boldsymbol{j}\|_{L_2} \quad ; \quad C_1 = \left[\frac{1}{4\pi}\int_\Omega \frac{d^3\boldsymbol{y}}{|\boldsymbol{x}-\boldsymbol{y}|^2}\right]^{1/2} \tag{23}$$

which implies $\|\boldsymbol{A}\|_{L_2} \leq C_2\|\boldsymbol{j}\|_{L_2}$. The magnetic field is

$$\boldsymbol{B} = \nabla\times\boldsymbol{A} = \frac{1}{4\pi}\int_\Omega \frac{(\boldsymbol{x}-\boldsymbol{y})\times\boldsymbol{j}(\boldsymbol{y})\,d^3\boldsymbol{y}}{|\boldsymbol{x}-\boldsymbol{y}|^3} := K\boldsymbol{j} \tag{24}$$

which defines a linear integral operator K. The kernel of the operator has a singularity at $\boldsymbol{x} = \boldsymbol{y}$, the polarity is of the order $|\boldsymbol{x}-\boldsymbol{y}|^{-2}$. If the current density is bounded in Ω the magnetic field is also bounded

$$|\boldsymbol{B}| \leq C_1(\Omega)\,\sup|\boldsymbol{j}| \quad ; \quad C_1(\Omega) = \sup\frac{1}{4\pi}\int_\Omega \frac{d^3\boldsymbol{y}}{|\boldsymbol{x}-\boldsymbol{y}|^2} \tag{25}$$

The magnetic field ist bounded if the current density is bounded. Using the properties of the magnetic field and the vector potential at large $\boldsymbol{x}$ the total magnetic energy is

$$\int_\Omega B^2\,d^3\boldsymbol{x} \leq \int_{R^3} B^2\,d^3\boldsymbol{x} = \int_\Omega \boldsymbol{j}\cdot\boldsymbol{A}\,d^3\boldsymbol{x} \tag{26}$$

R^3 is the whole 3-dimensional space. Using the Schwarz' inequality and Eq. (22) yields the estimate

$$\|\boldsymbol{B}\|^2_{L_2} \leq \|\boldsymbol{j}\|_{L_2}\|\boldsymbol{A}\|_{L_2} \leq C_2\|\boldsymbol{j}\|^2_{L_2} \tag{27}$$

The main problem is to construct plasma models where the current density remains continuous and bounded.

4 The plasma model

The mapping $P : \boldsymbol{B} \mapsto \boldsymbol{j}$ depends on the details of the plasma model. A plasma model consists of a set of differential equations describing plasma density $n(\boldsymbol{x})$, temperature $T(\boldsymbol{x})$, electric potential $\phi(\boldsymbol{x})$, plasma velocity $\boldsymbol{v}$ and current density $\boldsymbol{j}$ at a given magnetic field. In a multi-species plasma every constituent has its own density, temperature and velocity. The magnetic field consists of a vacuum field $\boldsymbol{B}_0$ generated by external coils and a field $\boldsymbol{B}$ generated by the plasma currents. In all macroscopic models the magnetic field occurs in the term $q_j n_j \boldsymbol{v}_j \times (\boldsymbol{B}_0 + \boldsymbol{B})$ and the current density is $\sum_j q_j n_j \boldsymbol{v}_j$. q_j is the charge of the particle species with index j. Also the viscosity depends on the magnetic field. Transport processes lead to loss of particles and energy which are compensated by particle and energy sources S, Q. In tokamaks with toroidal current the toroidal loop voltage V_L is another external parameter which acts as a source term. In order to describe the experimental case as closely as possible we assume that all plasma parameters go to zero if the source terms go to zero: $S, Q, V_L \Longrightarrow 0 \succ \nabla n, \nabla T, \boldsymbol{v}, \boldsymbol{j} \Longrightarrow 0$. The conditions on the plasma model are the following

1. The current density in the plasma is a continuous and growing function of the sources and the Hölder-norm is bounded by

$$\|\boldsymbol{j}\|_{C^{0,\alpha}} = f(S, Q, V_L, (\boldsymbol{B}_0 + \boldsymbol{B})) \tag{28}$$

2. Without source terms the gradients and plasma currents are zero.

$$f(0, 0, 0, \boldsymbol{B}_0) = 0 \tag{29}$$

Thus, the map $P : \boldsymbol{B} \mapsto \boldsymbol{j}$ is continuous. Let $\boldsymbol{B}$ be bounded in $C^{1,\alpha}$: $\|\boldsymbol{B}\|_{C^{1,\alpha}} \leq M_0$.This is a subset of a Banach space. Since f continuously depends on the source terms we can make the term $\sup|\boldsymbol{j}|$ arbitrarily small and the condition

$$\|\boldsymbol{B}\|_{C^{1,\alpha}} \leq C_1(\Omega) \sup|\boldsymbol{j}| \leq M_0 \tag{30}$$

can be satisfied. Thus by a proper choice of the sources the map $T = KP$ maps the Banach space M_0 onto itself and a self-consistent solution exists.

5 The Stokes model

The main task in 3-dimensional geometry is to establish a model where the current density remains bounded and continuous. For this purpose we consider a slowly diffusing plasma and neglect the inertial forces. Let us call this model the Stokes model of plasma equilibrium since it looks like the Stokes model of hydrodynamics where viscous forces but no inertial forces are retained. In view of these assumptions the momentum balance equations are

$$\begin{aligned} 0 &= -\nabla p_i - en\nabla\phi + en\boldsymbol{v}_i \times \boldsymbol{B} - \alpha_{i0} n\boldsymbol{v}_i - \alpha_{ei} n(\boldsymbol{v}_i - \boldsymbol{v}_e) - \nabla \cdot \pi_i\left[\boldsymbol{v}_i\right] \\ 0 &= -\nabla p_e + en\nabla\phi - en\boldsymbol{v}_e \times \boldsymbol{B} - \alpha_{e0} n\boldsymbol{v}_e + \alpha_{ei} n(\boldsymbol{v}_i - \boldsymbol{v}_e) - \nabla \cdot \pi_e\left[\boldsymbol{v}_e\right] \end{aligned} \tag{31}$$

and the equations of continuity

$$\nabla \cdot n\boldsymbol{v}_e = Q_e \quad ; \quad \nabla \cdot n\boldsymbol{v}_i = Q_i, \quad Q_i = Q_e \tag{32}$$

Here $\boldsymbol{B}$ is the sum of the external field and the field of plasma currents. Quasi-neutrality leads to $n = n_e = n_i$. The pressure is $p_i = nkT_i$ and $p_e = nkT_e$. The friction coefficients are

$$\alpha_{ei} = m_e \nu_{ei}; \quad \alpha_{i0} = m_i \nu_{i0}; \quad \alpha_{e0} = m_e \nu_{e0} \tag{33}$$

ν_{ei}, ν_{e0}; ν_{i0} are the collision frequencies between electrons and ions, ions and neutrals and electrons and neutrals. Since the neutrals strongly interact with the wall and exchange momentum with the wall we set the average velocity of the neutrals equal to zero. $\pi_i[\boldsymbol{v}_i], \pi_e[\boldsymbol{v}_e]$ are the viscous tensors described by Braginskii [16]. The viscous operator $\nabla \cdot \pi_i\left[\boldsymbol{v}_i\right]$ of a collisional plasma is a second order differential operator. It is a positive definite operator which implies that viscosity always slows down the plasma velocity. Its properties and its relevance with respect to the solvability of the differential equations have analysed in [17].

Neglecting the viscous forces would leads to an algebraic system which can be inverted with respect to the velocities. Inserting these equations into the continuity equations yields a coupled system for the density and the electric potential. The viscous forces modify the character of the momentum balance and - as in hydrodynamics - a system of differential equations must be solved. In order to elaborate the relevance of the terms in the equations we write the system in dimensionless variables. For this purpose we introduce a reference magnetic field B_0, a reference density n_0 and a temperature T_0. The radial length scale is a. The plasma pressure is replaced by the plasma beta $\beta = 2\mu_0 p/B_0^2$. Furthermore, we make use of the plasma frequency ω_p and the classical skin depth δ_e

$$\omega_p^2 = \frac{e^2 n_0}{\epsilon_0 m_e} \quad \text{and} \quad \delta_e^2 = \frac{c^2}{\omega_p^2} = \frac{m_e}{\mu_0 e^2 n_0} \tag{34}$$

We introduce the non-dimensional velocity $\boldsymbol{u}$ and the non-dimensional electric field $\boldsymbol{E}$ by

$$\boldsymbol{u} := \frac{a\boldsymbol{v}}{\delta_e^2 \Omega_e} \quad ; \quad \boldsymbol{E} := \frac{a\boldsymbol{E}}{\delta_e^2 B_0 \Omega_e} \quad ; \quad \boldsymbol{b} = \frac{\boldsymbol{B}}{B_0} \quad ; \quad \Omega_e = \frac{eB_0}{m_e} \quad ; \quad \frac{n}{n_0} \longrightarrow n \tag{35}$$

m_e is the electron mass and e the charge of electrons. In the following $\boldsymbol{E}$ is the non-dimensional electric field and n is the normalized density. $\beta = \beta_i + \beta_e$, $\beta_i = \beta_e = \beta_0 n/2$, $\beta_0 = 2\mu_0 n_0 T_0/B_0^2$.

The collisional interaction between charged particles is characterized by the non-dimensional collision frequency

$$f_{ei} = \frac{\nu_{ee}(T_0, n_0)}{\nu_{ee}(T_0, n)} = \frac{n}{n_0} \longrightarrow n \tag{36}$$

$\nu_{ee}(T_0, n_0)$ is the electron collision frequency at the reference point n_0, T_0. The dimensionless viscous tensor is

$$\frac{\mu_0 \pi_i(\boldsymbol{v})}{B_0^2} \longrightarrow \pi_i(\boldsymbol{u}) \tag{37}$$

Using these definitions the momentum balance and the equations of continuity in non-dimensional formulation are

$$\begin{aligned} \nabla \cdot \pi_i(\boldsymbol{u}_i) &= -\nabla\beta_i + n\{\boldsymbol{E} + \boldsymbol{u}_i \times \boldsymbol{b}\} - \frac{\nu_{ee}}{\Omega_e} n^2(\boldsymbol{u}_i - \boldsymbol{u}_e) - \frac{\nu_{io}}{\Omega_i} n\boldsymbol{u}_i \\ \nabla \cdot \pi_e(\boldsymbol{u}_e) &= -\nabla\beta_e - n\{\boldsymbol{E} + \boldsymbol{u}_e \times \boldsymbol{b}\} - \frac{\nu_{ee}}{\Omega_e} n^2(\boldsymbol{u}_e - \boldsymbol{u}_i) - \frac{\nu_{eo}}{\Omega_e} n\boldsymbol{u}_e \end{aligned} \tag{38}$$

The two scalar equations are in non-dimensional form

$$\nabla \cdot n\boldsymbol{u}_i = \frac{a^2}{\delta_e^2} \frac{S_i}{n_0 \Omega_e} := s_i \quad ; \quad \nabla \cdot n\boldsymbol{u}_e = \frac{a^2}{\delta_e^2} \frac{S_e}{n_0 \Omega_e} := s_e \tag{39}$$

n is the normalized density and $\sigma_i = -\sigma_e = 1$ is the charge of the particles divided by the elementary charge e. Given the magnetic field the two systems 38 and 39 allow to compute the velocities, the current density, the density profile and the electric potential. Exept for continuity there are no conditions on the topology of magnetic surfaces and therefore the equations are also applicable to the divertor region. Boundary conditions on velocity, density and electric potential can be imposed. Based on the momentum equations and the equations of continuity to special case can be identified: The constant-density approach and the constant temperature approximation. If density and temperature are inhomogeneous energy balance equations must be retained.

The existence of solutions of the constant-density model has been investigated in [17]. The source terms of the constant-density model can be eleminated by introducing $\boldsymbol{u}_i = \boldsymbol{u}_i^0 + \boldsymbol{w}_i, \boldsymbol{u}_e = \boldsymbol{u}_e^0 + \boldsymbol{w}_e$ and

$$\nabla \cdot n\boldsymbol{u}_i^0 = s_i \quad ; \quad \nabla \cdot n\boldsymbol{u}_e^0 = s_e \quad ; \quad \nabla \cdot n\boldsymbol{w}_i = 0, \, \nabla \cdot n\boldsymbol{w}_e = 0 \tag{40}$$

which reduces the problem on the computation of the incompressible flow $\boldsymbol{w}_i, \boldsymbol{w}_e$. The ansatz $n\boldsymbol{u}_i^0 = \nabla U_i, n\boldsymbol{u}_e^0 = \nabla U_e$ leads to Poisson's equation for U_i, U_e. Together with Dirichlet boundary conditions these equations can be uniquely solved. To shorten the notation we introduce the abbreviations $\boldsymbol{w} = (\boldsymbol{w}_i, \boldsymbol{w}_e)$, $\boldsymbol{u}^0 = (\boldsymbol{u}_i^0, \boldsymbol{u}_e^0)$ and $P = (p_i - n\Phi, p_e + n\Phi)$ and write the system in short

$$\nabla P = \boldsymbol{L}\boldsymbol{w} + \boldsymbol{L}\boldsymbol{u}^0 \quad ; \quad \nabla \cdot n\boldsymbol{w} = 0 \tag{41}$$

where the linear operator $\boldsymbol{L}$ contains the viscous terms, the friction terms and the $\boldsymbol{w} \times \boldsymbol{b}$-term. The normal component of $\boldsymbol{w}$ on the boundary is zero. The homogeneous system in 41 has only a trivial solution $\boldsymbol{w}_i = \boldsymbol{w}_e = 0$ since the viscous terms and the friction terms are dissipative.

$$\int (\boldsymbol{w}_i \cdot \nabla \cdot \pi_i + \boldsymbol{w}_e \cdot \nabla \cdot \pi_e)\, d^3\boldsymbol{x} + \int \left(\frac{\nu_{ee}}{\Omega_e} n^2 (\boldsymbol{w}_e - \boldsymbol{w}_i)^2 + \frac{\nu_{eo}}{\Omega_e} n \boldsymbol{w}_e^2 + \frac{\nu_{io}}{\Omega_i} n \boldsymbol{w}_i^2 \right) d^3\boldsymbol{x} \geq 0 \quad (42)$$

Based on this property it can be shown that the plasma velocities and the current density are bounded by a limit which is defined by the source terms. If the source terms s_e, s_i are zero there is no plasma current. For details see [17]. If the source terms kept small enough a self-consistent magnetic field exists. The system 41 has the same structure as the Stokes problem in hydrodynamics and therefor the same techniques of solution apply.

6 The friction model

The Stokes model presents a differential equation for the velocity $\boldsymbol{u}$ given the other terms in the force balance. Dropping the viscous forces leaves the friction forces in the equations. This model is relevant in strong magnetic fields, for weakly ionized plasmas or plasmas with a strong neutral background as in the case of Wendelstein IIA [18]. The model may be also important in the boundary region of fusion plasmas where recycling from the wall provides a large amount of neutral gas. Elastic collisions with the neutral background and charge exchange processes reduce the momentum of the charged particles and thus represent a loss term in the momentum balance. Since the neutral gas strongly interacts with the walls we may assume the neutral gas to be at rest. The equations of the friction model are

$$\begin{aligned} 0 &= -\nabla \beta_i + n\{\boldsymbol{E} + \boldsymbol{u}_i \times \boldsymbol{b}\} - \frac{\nu_{ee}}{\Omega_e} n^2 (\boldsymbol{u}_i - \boldsymbol{u}_e) - \frac{\nu_{io}}{\Omega_i} n \boldsymbol{u}_i \\ 0 &= -\nabla \beta_e - n\{\boldsymbol{E} + \boldsymbol{u}_e \times \boldsymbol{b}\} - \frac{\nu_{ee}}{\Omega_e} n^2 (\boldsymbol{u}_e - \boldsymbol{u}_i) - \frac{\nu_{eo}}{\Omega_e} n \boldsymbol{u}_e \end{aligned} \quad (43)$$

This model has been analysed in [7]and [10]. In this paper existence and uniqueness of the self-consistent magnetic field has been investigated showing a fundamental difference to the model of ideal MHD. The friction model presents an algebraic system for the vectors $\boldsymbol{u}_i$ and $\boldsymbol{u}_e$. Inverting this system with respect to $\boldsymbol{u}_i$ and $\boldsymbol{u}_e$ and inserting the result into the equations of continuity leads to an quasi-linear elliptic system for the scalar variables n and ϕ. In contrast to the ideal MHD-model boundary conditions on n and ϕ can be imposed. Let us write the system in the short form

$$\boldsymbol{g} = \boldsymbol{L}\boldsymbol{u} \quad , \quad \boldsymbol{u} = (\boldsymbol{u}_i, \boldsymbol{u}_e) \quad , \quad \boldsymbol{g} = (\nabla \beta_i + n\boldsymbol{E}, \nabla \beta_e + n\boldsymbol{E}) \quad (44)$$

where $\boldsymbol{L}$ is a matrix. $\boldsymbol{L}$ is the operator from the previous section without the viscous terms. The matrix is invertible and inserting the result into the equations of continuity yields an elliptic system of second order

$$\nabla \cdot n \boldsymbol{L}^{-1} \boldsymbol{g} = \boldsymbol{s} \quad , \quad \boldsymbol{s} = (s_i, s_e) \quad (45)$$

This system can be interpreted in different ways. Given the density and the electric field this is a linear system for β_i, β_e. A second option is to fix the temperature and to compute the density and the electric potential. To shorten the notation we introduce the parameters $\epsilon_1, ...\epsilon_3$

$$\epsilon_1 = 2n \frac{\nu_{ee}}{\Omega_e} \ , \ \epsilon_2 = \frac{1}{2}(\frac{\nu_{io}}{\Omega_i} + \frac{\nu_{eo}}{\Omega_e}) \ , \ \epsilon_3 = \frac{1}{2}(\frac{\nu_{io}}{\Omega_i} - \frac{\nu_{eo}}{\Omega_e}) \quad (46)$$

and after some manipulation the system can be modified to

$$\begin{aligned} 0 &= \nabla \cdot \frac{\boldsymbol{b} \times \nabla n}{b^2} - \nabla \cdot n\{\frac{\epsilon_2}{b^2} \nabla_\perp \phi + \frac{1}{(\epsilon_1 + \epsilon_2)} \nabla_\parallel \phi\} \\ \frac{s}{\beta_0} &= \nabla \cdot n \frac{\boldsymbol{b} \times \nabla \phi}{b^2} - \nabla \cdot \{\frac{(\epsilon_1 + \epsilon_2)}{b^2} \nabla_\perp n + \frac{1}{\epsilon_2} \nabla_\parallel n\} \end{aligned} \quad (47)$$

with $\beta_0 = \beta/n$ and $s = s_e = s_i$. The first equation is a linear Poisson-type equation for the potential ϕ, the second equation is a quasi-linear equation for the density. These properties are destroyed when we neglect the interaction with the neutral background ($\epsilon_2 \longrightarrow 0$). In that case magnetic surfaces and density surfaces coincide. The neutral background decouples plasma and magnetic surfaces and the model is also applicable to ergodic regions. Given the density the first equation is a linear elliptic equation for the electric potential. Together with boundary condition the equation has a unique solution. The density equation is quasi-linear and elliptic and can be solved iteratively. The convergence is still open, however, if a solution exists, it is Hölder-continuous and the current density is bounded in the Hölder space. The plasma source term determines the limit and if the source term is small enough a self-consistent magnetic field exists.

Since the interaction with the neutral background is small, the variation of density and potential on irrational magnetic surfaces is very small, too. An increase of the magnetic field reduces the friction term and thus improves the confinement. However, on rational surfaces with closed field lines a strong perpendicular variation can arise leading to convective cells and enhanced plasma loss. The size is determined by shear and collision frequencies. However, viscosity may also be important in convective cells and since shear viscosity decreases with growing magnetic field in can even happen that convective losses grow with growing magnetic field. Such effects have been observed in the stellarator Wendelstein IIA [18].

7 Navier-Stokes model

In the previous section we considered a slowly diffusing plasma and neglected inertial forces. If the plasma is rotating these forces, however, may be important. For the sake of simplicity we neglect the heat fluxes and approximate the friction forces by $\boldsymbol{F}_k = \sum_l \alpha_{lk} \boldsymbol{v}_l$. Momentum balance equations, equation of continuity and $\nabla \cdot \boldsymbol{j} = 0$ represent a complete set of equations for the functions $\mathbf{v}_k, p_k, \phi$.

$$\nabla \cdot (m_k n_k \boldsymbol{v}_k : \boldsymbol{v}_k + \pi_k + p_k) \;=\; q_k n_k(\boldsymbol{E} + \boldsymbol{v}_k \times \boldsymbol{B}) + \mathbf{F}_k \tag{48}$$

$$\nabla \cdot \mathbf{j} = 0 \quad ; \quad \nabla \cdot n_k \boldsymbol{v}_k = S_k \tag{49}$$

In this formulation the equations are applicable to a multifluid plasma including impurities. The one-fluid approximation has been investigated in [11]. Die ideal gas law $p_k = n_k T_k \; ; \; T_k > 0$ yields a relation between pressure and density. As boundary condition we impose $n_k = n_a, T_k = T_a$ with constant and positive values of n_a and T_a. The electric potential is assumed to be constant on the boundary which can be set to zero without loss of generality. The equations are second order, however, the inertial terms introduce non-linearities, which cannot be treated by standard methods. The constant-density model can be handled as in hydrodynamics: The existence of a weak solution in a Sobolev space can be proven. If the boundary is smooth and all coefficients are differentiable the solutions is also Hölder-continuous. In hydrodynamics viscosity comes with the Laplace operator while in collision plasma the Braginski viscosity accounts for the effect of the magnetic field. The mathematical structure, however, is the same and therefor the results of hydrodynamic theory can be extended to the Navier Stokes model in plasmas.

8 Summary and conclusions

Islands and ergodicity present a generic feature of three-dimensional magnetic surfaces of stellarators. Vacuum fields with a minimum of islands and ergodic regions can be constructed, however, plasma currents modify the topology of surfaces discontinuously and for this reason any attempt to construct ideal MHD-equilibrium in stellarator iteratively with respect to beta is bound to fail. Collisions and plasma transport decouple plasma surfaces (density and temperature) from magnetic surfaces and allow for islands and ergodicity, while plasma surfaces remain smooth and differentiable. Density and temperature profiles are the result of a second order transport equations and the Hölder space is the appropriate function space to deal with such problems. The transport equations can be solved for any magnetic field which is Hölder-continuous and the upper bound of the density and temperature is determined by the source terms of electrons and ions for any topology of magnetic surfaces. Thus the plasma currents are controlled by the source terms and the issue of a self-consistent magnetic field can be formulated as a fixed point problem in a Banach space.

This mathematical approach simulates the experimental procedure, where an external vacuum field confines a plasma at any pressure and temperature. Refuelling and heating power control the maximum density and temperature. However, the magnitude of the magnetic field and its topology affect the results, too. Any reasonable model of plasma equilibrium should be able to describe these effects. In this paper heat conduction has been neglected. Since including thermal conduction would add another second order equation the analysis becomes more complicated, however the procedure would remain the same as above.

Any plasma model which results in Hölder-continuous plasma currents fits into the scheme described above. In this paper a fluid description of the plasma has been considered, the general scheme, however, also applies to kinetic model with neoclassical viscosity. The basic requirement is that singularities in the current density be absent and the current density be bounded by external sources. Uniqueness of solutions is still an open problem. It is expected that due to non-linearities bifurcations and multiple solutions exist. In particular, the neighbourhood of rational magnetic surfaces may become unstable due to field line curvature leading to convective cells and enhanced plasma loss.

The fixed point method does not present a method for numerical computations. Computing plasma flow in a fixed magnetic field could be done by finite element methods on a grid as it is done in hydrodynamics. The iterative procedure with respect to the magnetic field, however, requires an adaptive grid to account for the fine structure of magnetic islands and convective cells.

References

[1] Kruskal M, Kulsrud R; *Phys. Fluids*, **1** (1958)

[2] Johnson J, *Comp. Phys. Reports*, **4**, 39 (1986)

[3] Grad H *Phys. Fluids* **10** (1967) 137

[4] Spitzer L jr. *Phys. Fluids* **1** (1958) 253

[5] Boozer A *Phys. Fluids* **27** , (1984) 2110

[6] Reiman A, Greenside H, Comp. Phys. Com. **43**, (1986) 147

[7] Kisslinger J, Wobig H, Europhys. Conf. Abstracts Vol. 9F Part I, 453

[8] Lortz D, Zeitschrift für Angewandte Physik und Mathematik, vol. 21 (1970) 196-211

[9] Chodura R, Schlüter A, *J. of Comp.* **41**, 68 (1981)

[10] Wobig H, *Z. f. Naturforsch.* **41a**, 1101 (1986)

[11] Spada M, Wobig H, *IPP-report 2/312* and J. Phys. A: Math. Gen. **25** 1575, (1992)

[12] Hayashi T, *Theory of Fusion Plasmas* (Chexbres) EUR 12149 EN 11,1988

[13] Gilbarg D, Trudinger N S, *Elliptic Partial Differential Equations,* Springer 1983, p. 280

[14] Ladyshenskaja O A, *Funktionalanalytische Untersuchungen der Navier-Stokeschen Gleichungen*, Akademie Verlag Berlin 1965

[15] Temam R, *Navier Stokes Equations*, North- Holland, Amsterdam, New York, Oxford 1984

[16] Braginskii S I; *Rev. of Plasma Phys.* Vol. I, p. 250, Consultants Bureau New York 1965

[17] Beidler C D, Igitkhanov Y, Wobig H, *On Electric Fields in Stellarator Equilibria*, Fusion Science and Technology Vol. 46, No.1, p.64 (2004)

[18] Grieger G, Ohlendorf W, Pacher H D, Wobig H, Wolf G H, Proc. of 4^{th} Int. Conf. on Plasma Phys. and Contr. Fusion, Madison 1971, IAEA Vienna Vol. 3, 37