Pfirsch-Schlüter impurity transport in stellarators

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Abstract

The Pfirsch-Schlüter transport of impurities is calculated for stellarator geometry. Contrary to the tokamak case, where the only contribution to the particle flux arises from friction between different species, in a stellarator there is another source of impurity transport due to pressure anisotropy. However, the pressure anisotropy term is usually smaller than the friction term, so that the impurity particle flux is comparable to that in a tokamak. The pressure anisotropy is nonetheless important as it causes the transport to be non-ambipolar and thereby determines the radial electric field. The presence of impurities therefore affects the radial electric field in a stellarator. The heat flux is also affected qualitatively by impurities.

I Introduction

The accumulation of highly charged impurity ions poses a potentially serious threat to fusion plasma performance, as was recognized already 50 years ago [1]. The problem is that collisional transport processes tend to drive such impurities into the plasma, against the bulk plasma density gradient. Theoretically, the problem is particularly severe in stellarators because of their lack of intrinsic ambipolarity. In tokamaks, the collisional transport is independent of the radial electric field to lowest order in a gyroradius expansion. In stellarators, however, this is not the case and the transport of each species is, in general, proportional to a linear combination of all the "thermodynamic forces"

$$A_{a_1} \equiv \frac{d\ln p_a}{d\psi} + \frac{e_a}{T_a}\frac{d\phi}{d\psi},$$
$$A_{a_2} \equiv \frac{d\ln T_a}{d\psi},$$

where p_a denotes the pressure of species a, T_a its temperature, ϕ the electrostatic potential, and ψ is a flux function serving as a radial coordinate. Specifically, for impurities ($a = z$) with high charge number, $Z \gg 1$, there is a term in the transport proportional to

$$A_{z_1} = \frac{d\ln p_z}{d\psi} + \frac{Ze}{T_z}\frac{d\phi}{d\psi}.$$

The radial electric field in the second term on the right normally points inward in stellarators ("ion root" operation) and since this term contains the large multiplier $Z \gg 1$, it will cause a large inward transport.

In the present paper, we calculate the stellarator impurity transport in the one case where it is possible to do so fully analytically, namely, in the limit of short mean free path, the Pfirsch-Schlüter regime. This limits the applicability of the calculation to cool edge plasmas, but has the appeal that the calculation can be carried out without any further approximations (other than the usual gyroradius expansion). For lower collisionalities, the transport needs to be calculated numerically, which is generally done with numerical codes using simplified collision operators. The analytical results of the present paper should be useful for the qualitative insight they offer and as a benchmark on the codes, enabling for instance the accuracy of simplified collision operators to be assessed.

Stellartor impurity transport has been considered in the literature before, mainly in the framework of the Hirshman-Sigmar moment formalism [2, 3]. The present calculation, which is based on the direct expansion of the drift kinetic equation by Hazeltine and Hinton [4], retains some terms of geometrical nature that are otherwise neglected, see, e.g. [5], and is used here to derive explicit transport coefficients.

II Expansion of the kinetic equation

We consider a plasma in the Pfirsch-Schlüter regime consisting of electrons, hydrogenic bulk ions and a single species of highly charged impurities with charge Z. Indices i refer to the bulk ions, whereas indices z represent impurity quantities. To take account for the geometry of the magnetic field and without employing a specific spatial coordinate system, we define a geometry-dependent quantity u in the following way. Consider the equilibrium current $\boldsymbol{j_0}$ satisfying $\boldsymbol{j_0} \times \boldsymbol{B} = \nabla p_0$, where p_0 is the equilibrium pressure. Let u be defined via

$$\begin{aligned} \boldsymbol{h} &\equiv \frac{\boldsymbol{j_0}}{p_0'} = \frac{1}{p_0'}\left(\frac{\boldsymbol{b} \times \nabla p_0}{B} + j_{0_\parallel}\boldsymbol{b}\right) \\ &\equiv \frac{\boldsymbol{b} \times \nabla\psi}{B} + u\boldsymbol{B}, \end{aligned}$$

where $\boldsymbol{b}$ is the unit vector along the magnetic field and a prime denotes derivation with respect to the radial spatial coordinate ψ. Using $\boldsymbol{\nabla} \cdot \boldsymbol{h} = 0$, we find

$$\nabla_\parallel u \quad = \quad \frac{1}{p_0'}\nabla_\parallel\left(\frac{j_{0_\parallel}}{B}\right) = \frac{2}{B^2}(\boldsymbol{b} \times \boldsymbol{\nabla}\psi) \cdot \boldsymbol{\nabla}\ln B,$$

which we will employ frequently. Here, $\nabla_\parallel = \boldsymbol{b} \cdot \boldsymbol{\nabla}$ is the gradient along the magnetic field.

The radial impurity transport is given by [6]

$$\langle \boldsymbol{\Gamma}_z \cdot \nabla\psi \rangle = \left\langle \int f_z \boldsymbol{v_d} \cdot \nabla\psi d^3v \right\rangle, \quad (1)$$

where the drift velocity $\boldsymbol{v}_d$ is

$$\boldsymbol{v}_d = \frac{\boldsymbol{b} \times \nabla\phi}{B} + \left(\frac{v_\perp^2}{2} + v_\parallel^2\right) \frac{\boldsymbol{b} \times \nabla \ln B}{\Omega},$$

and $\langle .. \rangle$ denotes an average over the flux surface, defined as the volume average between two neighboring flux surfaces. As we will show in Sec. III, this equation can be rewritten as

$$\langle \boldsymbol{\Gamma}_z \cdot \nabla\psi \rangle = \frac{1}{e_z} \left(\langle uBR_{z_\parallel} \rangle + \left\langle \left(\frac{B}{2} \nabla_\parallel u + u \nabla_\parallel B \right) (p_{z_\parallel} - p_{z_\perp}) \right\rangle \right), \quad (2)$$

where $R_{z_\parallel}$ is the impurity-ion friction force and terms of 2^{nd} order in $\delta_i \equiv \rho_i/L$ have been neglected, ρ_i being the ion Larmor radius and L the plasma dimension. The perpendicular and parallel pressures $p_\perp$ and $p_\parallel$, respectively, are defined as

$$\begin{pmatrix} p_\parallel \\ p_\perp \end{pmatrix} = \int m \begin{pmatrix} v_\parallel^2 \\ \frac{v_\perp^2}{2} \end{pmatrix} f d^3v.$$

The flow velocities are, as usual in neoclassical theory, assumed to be one order smaller than the thermal velocities, $V_a \sim \delta_a v_{th,a}$, which is the only possibility in most stellarator configurations [7]. We can thus conclude that the particle flux consists of two essentially different components: one driven by parallel friction and one by pressure anisotropy [2]. As we shall see, the former contribution dominates in the Pfirsch-Schlüter regime, but the latter is nonetheless important since it is not intrinsically ambipolar and therefore determines the radial electric field [8].

In order to calculate the friction force and the pressure anisotropy, we solve the drift kinetic equation and order as usual in δ_i to find in 0^{th} order

$$C_a(f_{a_0}) = v_\parallel \nabla_\parallel f_{a_0},$$

the solution to which is a stationary Maxwellian, and in 1^{st} order

$$C_a(f_{a_1}) = v_\parallel \nabla_\parallel f_{a_1} + \boldsymbol{v}_d \cdot \nabla f_{a_0} + \frac{e_a}{T_a} v_\parallel f_{a_0} \nabla_\parallel \phi. \quad (3)$$

The independent coordinates in velocity space are $\varepsilon_a = m_a v^2/2 + e_a\phi$ and $\mu_a = m_a v_\perp^2/(2B)$. We solve this equation by making a subsidiary expansion in the shortness of the mean-free path, $\Delta_i \equiv \lambda_{ii}/L \ll 1$, where λ_{ii} is the ion mean free path [6, 9]. The mean free paths of the electrons and impurity ions are then automatically short, too, unless their temperatures are very much higher than the ion temperature. Superscript indices correspond to this expansion whereas subscript indices express the ordering in δ_i. In lowest order,

$$C_a(f_{a_1}^{(-1)}) \quad = \quad 0. \tag{4}$$

Note that the expansion starts with -1 as the collision term in lowest order is proportional to Δ_i^{-1}. The homogeneous solution for the general linearized collision operator yields for the -1^{st} order distribution function the shifted Maxwellian

$$f_{a_1}^{(-1)} = \left(\frac{p_{a_1}^{(-1)}}{p_a} + \frac{m_a}{T_a} v_\| V_{a_\|}^{(-1)} + \left(x_a^2 - \frac{5}{2} \right) \frac{T_{a_1}^{(-1)}}{T_a} \right) f_{a_0},$$

with equal flow velocities $V_{i_\|}^{(-1)} = V_{z_\|}^{(-1)}$ and temperatures $T_{i_1}^{(-1)} = T_{z_1}^{(-1)}$. $x_a^2 \equiv m_a v^2/(2T_a)$ is the normalized velocity. The 0^{th} order equation becomes

$$C_a(f_{a_1}^{(0)}) \quad = \quad v_\| f_{a_0} \left(\frac{e_a}{T_a} \nabla_\| \phi_1^{(-1)} + \frac{\nabla_\| p_{a_1}^{(-1)}}{p_a} + \left(x_a^2 - \frac{5}{2} \right) \frac{T_{a_1}^{(-1)}}{T_a} + \frac{m_a}{T_a} \nabla_\| \left(v_\| V_{a_\|}^{(-1)} \right) \right) \tag{5}$$

Employing momentum conservation, one can show that the -1^{st} order flow velocities vanish for all particle species, which is done in the appendix. Defining the parallel thermodynamic forces as

$$A_{a_{1_\|}}^{(-1)} \equiv \frac{\nabla_\| p_{a_1}^{(-1)}}{p_a} + \frac{e_a}{T_a} \nabla_\| \phi_1^{(-1)}, \qquad A_{a_{2_\|}}^{(-1)} \equiv \frac{\nabla_\| T_{a_1}^{(-1)}}{T_a},$$

the Spitzer problem (5) becomes

$$C_a(f_{a_1}^{(0)}) \quad = \quad v_\| f_{a_0} \left(A_{a_{1_\|}}^{(-1)} + \left(x_a^2 - \frac{5}{2} \right) A_{a_{2_\|}}^{(-1)} \right). \tag{6}$$

We follow Braginskii's ansatz [10] and decompose the distribution functions in Sonine polynomials $L_j^{(m)} \equiv \frac{1}{j!} \frac{e^x}{x^m} \frac{d^j}{dx^j} \left(x^{j+m} e^{-x} \right)$ as

$$\begin{aligned} f_{z_1}^{(0)} &= \frac{m_z}{T_z} v_\| f_{z_0} \sum_k a_{z_k} L_k^{(3/2)}(x^2) \\ f_{i_1}^{(0)} &= \frac{m_i}{T_i} v_\| f_{i_0} \sum_k a_{i_k} L_k^{(3/2)}(x^2). \end{aligned} \tag{7}$$

Defining the following matrix elements

$$\begin{aligned} M_{ab}^{jk} &= \frac{\tau_{ab}}{n_a} \int v_\| L_j^{(3/2)}(x^2) C_{ab} \left[\frac{m_a}{T_a} v_\| L_k^{(3/2)}(x_a) f_{a_0}, f_{b_0} \right] d^3 v, \\ N_{ab}^{jk} &= \frac{\tau_{ab}}{n_a} \int v_\| L_j^{(3/2)}(x^2) C_{ab} \left[f_{a_0}, \frac{m_b}{T_b} v_\| L_k^{(3/2)}(x^2) f_{b_0} \right] d^3 v, \end{aligned}$$

which can be found in [6, 10], we can rewrite (6) as

$$\sum_{b,k} \frac{m_a}{T_a \tau_{ab}} \left(M_{ab}^{jk} a_{a_k} + N_{ab}^{jk} a_{b_k} \right) = A_{a_{1\|}}^{(-1)} \delta_{j0} - \frac{5}{2} A_{a_{2\|}}^{(-1)} \delta_{j1}$$

where $\tau_{ab} = 3\sqrt{\pi}/(4\hat{\nu}_{ab}) = 3\pi^{3/2}\epsilon_0^2 m_a v_{th,a}^3/(n_b e_a^2 e_b^2 \ln\Lambda)$ and $\ln\Lambda$ is the Coulomb logarithm. Due to the symmetry properties of the matrix elements, the matrix does not have full rank, and thus a solubility condition for the values of the parallel thermodynamic forces on the right-hand side exists. This condition, making the $j = 0$ equations for ions and impurities linearly dependent, reads

$$\sum_a p_a A_{a_{1\|}}^{(-1)} = 0$$

and follows directly from momentum conservation. Thus, $A_{z_{1\|}}^{(-1)}$ can be expressed as $-\frac{p_i}{p_z} A_{i_{1\|}}^{(-1)}$. Assuming equal equilibrium temperatures $T_{i_0} = T_{z_0}$, which is realistic for usual values of Z, and using Eq. (4), one can show that $A_{z_{2\|}}^{(-1)} = A_{i_{2\|}}^{(-1)}$, which leaves only two unknowns on the right-hand side of the system of equations. The coefficients can then be written as functions of the bulk ion quantities in the form

$$\begin{aligned} a_{i_0} - a_{z_0} &= \frac{T_i}{m_i \hat{\nu}_{ii}} \left(\alpha_0 A_{i_{1\|}}^{(-1)} + \beta_0 A_{2\|}^{(-1)} \right) \\ a_{a_j} &= \frac{T_i}{m_i \hat{\nu}_{ii}} \left(\alpha_{a_j} A_{i_{1\|}}^{(-1)} + \beta_{a_j} A_{2\|}^{(-1)} \right), j \geq 1. \end{aligned} \tag{8}$$

In the limit of large Z, solving the matrix problem becomes especially easy since the impurity-ion collision operator can be approximated by

$$C_{zi}(f_{z1}^{(0)}) = -\frac{\boldsymbol{R_{zi}}}{m_z n_z} \cdot \frac{\partial f_{z1}^{(0)}}{\partial \boldsymbol{v}} + \frac{m_i n_i}{m_z n_z \tau_{iz}} \frac{\partial}{\partial \boldsymbol{v}} \cdot \left((\boldsymbol{v} - \boldsymbol{V}_z^{(0)}) f_{z1}^{(0)} + \frac{T_i}{m_z} \frac{\partial f_{z1}^{(0)}}{\partial \boldsymbol{v}} \right)$$

and yields

$$C_{zi}(f_{z_1}^{(0)}) = v_\| f_{z_0} A_{z_{1\|}}^{(-1)},$$

which cancels the first expression on the right, and the remaining equation

$$C_{zz}(f_{z1}^{(0)}) = v_\| f_{z_0} \left(x^2 - \frac{5}{2} \right) A_{2\|}^{(-1)}$$

depends only on impurity parameters and can be solved analytically for a_{z_1} and a_{z_2} to yield

$$\begin{aligned} a_{z_1} &= \frac{75}{32} \sqrt{\frac{\pi}{2}} A_{z_{2\|}}^{(-1)} \frac{T_z}{m_z \hat{\nu}_{zz}} \\ a_{z_2} &= -\frac{5}{8} \sqrt{\frac{\pi}{2}} A_{z_{2\|}}^{(-1)} \frac{T_z}{m_z \hat{\nu}_{zz}}. \end{aligned}$$

The ion problem can then be solved with the impurity coefficients given.

Exploiting the orthogonality properties of the Sonine polynomials, one can relate the coefficients to the parallel flow velocities $V_{a_\parallel}^{(0)}$ and heat fluxes $q_{a_\parallel}^{(0)}$,

$$\begin{aligned} a_{a_0} &= \frac{1}{n_a}\int v_\parallel f_{a_1}^{(0)} d^3v = V_{a_\parallel}^{(0)} \\ a_{a_1} &= \frac{1}{n_a}\int v_\parallel L_1^{(3/2)} f_{a_1}^{(0)} d^3v = -\frac{2}{5}\frac{q_{a_\parallel}^{(0)}}{p_a}. \end{aligned}$$

Using these relations and neglecting terms of order n_z/n_i, the two unknown quantities can be expressed as

$$\begin{aligned} A_{i_{1\parallel}}^{(-1)} &= -\frac{m_i\hat{\nu}_{ii}}{T_i}\frac{(2/5)\beta_0\sum_a q_{a_\parallel}^{(0)}/T_a + \beta_{i_1}n_i\left(V_{i_\parallel}^{(0)} - V_{z_\parallel}^{(0)}\right)}{(\alpha_{i_1}\beta_0 - \alpha_0\beta_{i_1})n_i} \\ A_{2\parallel}^{(-1)} &= \frac{m_i\hat{\nu}_{ii}}{T_i}\frac{(2/5)\alpha_0\sum_a q_{a_\parallel}^{(0)}/T_a + \alpha_{i_1}n_i\left(V_{i_\parallel}^{(0)} - V_{z_\parallel}^{(0)}\right)}{(\alpha_{i_1}\beta_0 - \alpha_0\beta_{i_1})n_i}. \end{aligned} \tag{9}$$

The remaining task is to find expressions for the flow velocities and heat fluxes, which can be obtained from the condition of particle and energy conservation. From the 0^{th} order velocity moment of (3), we obtain

$$B\nabla_\parallel\left(\frac{n_aV_{a_\parallel}^{(0)}}{B} - \frac{p_a}{e_a}A_{a_1}u\right) = 0,$$

and from the energy moment

$$\sum_a B\nabla_\parallel\left(\frac{q_{a_\parallel}^{(0)}}{T_aB} - \frac{5}{2}\frac{p_a}{e_a}A_{a_2}u\right) = 0.$$

Due to the assumption of equal equilibrium temperatures, $A_{i_2} \approx A_{z_2}$. Thus,

$$\begin{aligned} V_{a_\parallel}^{(0)} &= \frac{T_a}{e_a}A_{a_1}uB + K_a(\psi)B, \\ \sum_a \frac{q_{a_\parallel}^{(0)}}{T_a} &= \frac{5}{2}\sum_a\frac{p_a}{e_a}A_2uB + L(\psi)B, \end{aligned} \tag{10}$$

where $K_a(\psi)$ and $L(\psi)$ are integration constants, for which we can find expressions by noting that $\langle B\nabla_\parallel M\rangle$ vanishes for all scalars M, and therefore the constraints

$$\left\langle B\sum_a\frac{q_{a_\parallel}^{(0)}}{T_a}\right\rangle = 0, \qquad \left\langle B\left(V_{i_\parallel}^{(0)} - V_{z_\parallel}^{(0)}\right)\right\rangle = 0$$

must hold. Another constraint can be obtained from the flux-surface average of B times the velocity moment of the 1st order equation, which reads

$$C_a(f^{(1)}_{a_1}) \quad = \quad v_\parallel \nabla_\parallel f^{(0)}_{a_1} + \boldsymbol{v_d}\cdot\nabla f_{a_0} + \frac{e_a}{T_a} v_\parallel f_{a_0} \nabla_\parallel \phi^{(0)}_1. \tag{11}$$

Exploiting momentum conservation, we find the condition

$$\sum_a \left\langle \nabla_\parallel B \left(p_{a_\parallel} - p_{a_\perp}\right)\right\rangle \quad = \quad \sum_a \left\langle \nabla_\parallel B \int m_a v^2 P_2(\xi) f^{(1)}_{a_1} d^3v \right\rangle = 0, \tag{12}$$

where $P_2(\xi)$ is the 2nd Legendre polynomial and $\xi = v_\parallel/v$. Due to the orthogonality properties of the Legendre polynomials, only the P_2 components of the distribution functions contribute. These first occur in 1st order in the mean-free path expansion, so Eq. (11) must be solved. This involves solving Spitzer problem of the form

$$C_a(P_2(\xi) f_{a_0} g_{a_j}) \quad = \quad (-1)^j P_2(\xi) x^2 f_{a_0} L^{(3/2)}_j(x^2), \tag{13}$$

which defines the functions g_{a_j}. These are qualitatively different from the type of Spitzer problem occurring before, which involved P_1 instead of P_2. The results (for the first few coefficients) can be found in [11]. Defining

$$\{g_{a_j}\} \quad \equiv \quad \frac{5}{2}\frac{\gamma_{a_j}}{\hat{\nu}_{aa}},$$

where $\{A\} = 8/(3\sqrt{\pi}) \int_0^\infty A x^4 \exp(-x^2) dx$ denotes an average over velocity space, and using Eqs. (12), (13) and the solutions for the coefficients a_{a_j}, we find the expressions for the integration constants to read

$$\begin{aligned}
L(\psi) &= -\frac{5}{2}\sum_a \frac{p_a}{e_a} A_{a_2} \frac{\langle uB^2\rangle}{\langle B^2\rangle} \\
K_z(\psi) &= \frac{T_i}{e_i}\left(A_{i_1} - \frac{1}{Z}A_{z_1}\right)\frac{\langle uB^2\rangle}{\langle B^2\rangle} + K_i(\psi) \\
K_i(\psi) &= -\frac{T_i}{e_i}\left(A_{i_1} G_1(\psi) + A_{i_2}\frac{\gamma_{i_1}}{\gamma_{i_0}}\left(G_1(\psi) - \frac{\langle uB^2\rangle}{\langle B^2\rangle}\right)\right. \\
&\quad \left. + \frac{\gamma_{i_2}}{\gamma_{i_0}}\left(\left(A_{i_1} - \frac{1}{Z}A_{z_1}\right)\eta_2 + A_{i_2}\eta_1\right)\left(G_1(\psi) - \frac{\langle uB^2\rangle}{\langle B^2\rangle} + \frac{1}{3}\frac{\langle \frac{B}{2}\nabla_\parallel u \nabla_\parallel B\rangle}{\langle(\nabla_\parallel B)^2\rangle}\right)\right)
\end{aligned}$$

where $G_1(\psi) \equiv \langle \nabla_\parallel B (u\nabla_\parallel B + B\nabla_\parallel u/2)\rangle / \langle (\nabla_\parallel B)^2\rangle$ and

$$\begin{aligned}
\eta_1 &= \frac{\alpha_0\beta_{i_2} - \beta_0\alpha_{i_2}}{\alpha_{i_1}\beta_0 - \alpha_0\beta_{i_1}}, \\
\eta_2 &= \frac{\alpha_{i_1}\beta_{i_2} - \beta_{i_1}\alpha_{i_2}}{\alpha_{i_1}\beta_0 - \alpha_0\beta_{i_1}}.
\end{aligned}$$

For simplicity, the expansion was truncated after two components and the impurity pressure anisotropy, being much smaller than the ion pressure anisotropy, was neglected in (12). The coefficients γ_{a_j} are given in the appendix. Since the equilibrium temperatures of the ions and impurities are assumed to be equal, the difference between the total temperatures is small (of the order δ_i), and therefore it is not necessary to distinguish between these two quantities.

III Particle transport

In this section, we calculate the radial impurity particle flux (1). Inserting the drift velocity, we find

$$\langle \Gamma_z \cdot \nabla\psi \rangle = \langle n_z (u\boldsymbol{B} - \boldsymbol{h}) \cdot \nabla\phi \rangle - \frac{1}{e_z} \left\langle \left(p_{z_\perp} + p_{z_\parallel} \right) \frac{B}{2} \nabla_\parallel u \right\rangle .$$

On the other hand, we can write

$$\begin{aligned} \left\langle e_z n_{z_0} u B \nabla_\parallel \phi \right\rangle &= \left\langle e_z u B \int \frac{m_z}{T_z} v_\parallel^2 f_{z_0} d^3 v \nabla_\parallel \phi \right\rangle \\ &= - \left\langle e_z u B \int v_\parallel^2 \nabla_\parallel f_z d^3 v \right\rangle + \left\langle u B R_{z_\parallel} \right\rangle \\ &= \left\langle u B R_{z_\parallel} \right\rangle + \left\langle e_z p_{z_\parallel} B \nabla_\parallel u \right\rangle + \left\langle e_z u B (p_{z_\parallel} - p_{z_\perp}) \nabla_\parallel \ln B \right\rangle , \end{aligned}$$

and thus Eq. (2) follows.

If we estimate the magnitude of the two terms contributing to the impurity flux relative to each other, we find that

$$\frac{\text{pressure anisotropy term}}{\text{friction term}} \sim \frac{\Delta^2}{Z^4} \ll 1.$$

The friction term thus dominates strongly, and not only in the Δ expansion, as expected from the order in which friction and pressure anisotropy occur, but also by a factor Z^4, which becomes large even for relatively small values of the impurity charge Z. Therefore, the impurity pressure anisotropy term will be neglected in the transport calculation. The friction part can be calculated to be

$$\begin{aligned} \left\langle u B R_{z_\parallel} \right\rangle &= - \left\langle u B R_{i_\parallel} \right\rangle = - \left\langle u B p_i A_{i_{1\parallel}}^{(-1)} \right\rangle \\ &= - \frac{m_i p_i \hat{\nu}_{ii}}{e_i (\alpha_{i_1} \beta_0 - \alpha_0 \beta_{i_1})} \left(\langle u^2 B^2 \rangle - \frac{\left\langle u B^2 \right\rangle^2}{\langle B^2 \rangle} \right) \left(\beta_{i_1} \left(A_{i_1} - \frac{1}{Z} A_{z_1} \right) + \beta_0 A_2 \right) . \end{aligned}$$

The entire effect of the magnetic field geometry is contained in the factor

$$\left(\langle u^2 B^2 \rangle - \frac{\left\langle u B^2 \right\rangle^2}{\langle B^2 \rangle} \right) ,$$

which is positive by the Schwartz inequality. In the limit of large Z and for $Z_{\rm eff} = 2$, this yields

$$\langle uBR_{z_\parallel}\rangle \quad = \quad -\frac{m_i p_i \hat{\nu}_{ii}}{e_i}\left(\langle u^2B^2\rangle - \frac{\langle uB^2\rangle^2}{\langle B^2\rangle}\right)\left(0.50\left(A_{i_1} - \frac{1}{Z}A_{z_1}\right) - 0.41A_2\right).$$

IV Ambipolarity

In a stellarator that is not quasisymmetric, the radial electric field is not arbitrary but is instead set by the condition that the transport should be ambipolar [8]. The important part of the transport in this respect is the pressure anisotropy term as friction is intrinsically ambipolar. Since the pressure anisotropy terms scale as $p_a/\hat{\nu}_{aa}$ and the impurities are much more collisional than the bulk ions due to their higher mass and charge, the impurity term is only a small correction to the ion pressure anisotropy and can therefore be neglected. The radial electric field is nevertheless affected by the presence of impurities, because the Equation (11) that determines the pressure anisotropy of the bulk ions contains the collision operator C_{iz}. The pressure anisotropy therefore depends on $Z_{\rm eff}$. In the limit $Z_{\rm eff} - 1 \ll 1$, the radial electric field is equal to that in a pure plasma and is given in Ref. [5]. In the opposite, Lorentz limit ($Z_{\rm eff} \gg 1$), the ion collision operator reduces to a simple pitch-angle scattering operator,

$$C_i(f_{i_1}^{(1)}) \approx C_{iz}(f_{i_1}^{(1)}) = \nu_{iz}\left(\mathcal{L}(f_{i_1}^{(1)}) + \frac{m_i}{T_i}v_\parallel V_{z_\parallel} f_{i_0}\right).$$

As the Legendre polynomials are eigenfunctions of this operator, the Spitzer problems (13) can be solved analytically in this limit, which allows comparing these results with the results in a pure plasma to estimate the effect of the impurities. The corresponding coefficients are given in Appendix A. The part of $f_{i_1}^{(1)}$ proportional to P_2 is then found to read

$$f_{i_1}^{(1)}|_{P_2} \quad = \quad -\frac{1}{3\nu_{iz}}\left(v_\parallel \nabla_\parallel f_{i_1}^{(0)} + \boldsymbol{v}_d\cdot\nabla f_{i_0}\right)|_{P_2},$$

where

$$\boldsymbol{v}_d\cdot\nabla f_{i_0}|_{P_2} \quad = \quad -\frac{m_a}{3e_a}v^2 P_2(\xi) f_{i_0}(A_{i_1} - L_1^{(3/2)}(x^2)A_{i_2})\frac{B}{2}\nabla_\parallel u$$

and $v_\parallel \nabla_\parallel f_{i_1}^{(0)}|_{P_2}$ is found from (7), truncated after $k = 2$, where a_{i_0} and a_{i_1} can be read off from (10) and a_{i_2} from (8) and (9). Furthermore, using

$$p_{i_\parallel} - p_{i_\perp} \quad = \quad \frac{3}{5}p_i\left\{\frac{f_{i_1}^{(1)}|_{P_2}}{P_2(\xi)f_{i_0}}\right\},$$

we can calculate the bulk ion pressure anisotropy contribution to the radial flux for large Z and Z_{eff}, and after some tedious algebra we obtain

$$\begin{aligned}
&\left\langle \left(\frac{B}{2}\nabla_{\|}u + u\nabla_{\|}B \right) (p_{i_{\|}} - p_{i_{\perp}}) \right\rangle \\
= \quad & -\frac{3p_iT_i}{(Z_{\text{eff}}-1)\hat{\nu}_{ii}e_i} \left[\left(0.47A_{i_1} + 1.93\frac{1}{Z}A_{z_1} + 9.90A_{i_2} \right) \cdot (G_2(\psi) - G_1(\psi)) \right. \\
& \left. + \left(1.29A_{i_2} - 0.65\left(A_{i_1} - \frac{1}{Z}A_{z_1} \right) \right) \cdot G_3(\psi) \right],
\end{aligned}$$

where

$$\begin{aligned}
G_2(\psi) &= \left\langle \left(\frac{B}{2}\nabla_{\|}u + u\nabla_{\|}B \right)^2 \right\rangle, \\
G_3(\psi) &= \left\langle \frac{B}{2}\nabla_{\|}u \left(\frac{B}{2}\nabla_{\|}u + u\nabla_{\|}B \right) \right\rangle - \left\langle \frac{B}{2}\nabla_{\|}u\nabla_{\|}B \right\rangle G_1(\psi)
\end{aligned}$$

and $G_1(\psi)$ has already been defined in Sec. II. The pressure anisotropy of a pure plasma has been calculated before [5], and, in the notation used in this paper, reads

$$\begin{aligned}
\left\langle \left(\frac{B}{2}\nabla_{\|}u + u\nabla_{\|}B \right) (p_{i_{\|}} - p_{i_{\perp}}) \right\rangle &= -\frac{3p_iT_i}{\hat{\nu}_{ii}e_i} \left[(1.80A_{i_1} + 3.21A_{i_2}) (G_2(\psi) - G_1(\psi)) \right. \\
& \left. + 0.10A_{i_2}G_3(\psi) \right].
\end{aligned}$$

Except for the quantitative change in the magnitude of the coefficients, there is no great qualitative change when impurities are present, as the contribution from A_{z_1} is proportional to $1/Z$ and thus relatively small. The only qualitative difference is the additional contribution from A_{i_1} in front of the second geometric term. The magnitude of the two geometric terms $G_2 - G_1$ and G_3 is strongly dependent on the exact geometry of the device, but the latter term tends to be slightly smaller than or comparable to the first one. Although the magnitude of the ion distribution function coefficients increases when impurities are present, the overall contribution to the pressure anisotropy decreases due to the large factor $Z_{\text{eff}} - 1$ in the denominator.

V Heat flux

The radial heat fluxes are also influenced by the presence of impurities, and are given by

$$\begin{aligned}
\langle \boldsymbol{q}_a \cdot \nabla\psi \rangle &= T_a \left\langle \int f_a \left(x^2 + \frac{e_a\phi}{T_a} - \frac{5}{2} \right) \boldsymbol{v}_d \cdot \nabla\psi d^3v \right\rangle \\
&= T_a \langle \boldsymbol{\Gamma}_a \cdot \nabla\psi \rangle - \frac{m_a}{e_a} \left\langle \frac{B}{2}\nabla_{\|}u \int f_{a_1} \left(v_{\|}^2 + \frac{v_{\perp}^2}{2} \right) \left(x^2 + \frac{e_a\phi_1}{T_a} - \frac{3}{2} \right) d^3v \right\rangle.
\end{aligned}$$

The particle flux has already been calculated in the last section, and we can rewrite the second term, using the orthogonality properties of the Sonine and the Legendre polynomials and neglecting components proportional to P_2, which first occur in 1^{st} order in the Δ expansion and are thus two orders smaller than the other terms, as

$$\frac{m_a}{e_a}\left\langle \frac{B}{2}\nabla_\| u \int f_{a_1}\left(v_\|^2+\frac{v_\perp^2}{2}\right)\left(x^2+\frac{e_a\phi_1}{T_a}-\frac{3}{2}\right)d^3v\right\rangle = -\frac{p_a}{e_a}\left\langle uB\left(\frac{5}{2}A_{2_\|}^{(-1)}+A_{a_{1\|}}^{(-1)}\right)\right\rangle .$$

Combining this term with the expression for the particle flux, the total heat flux becomes

$$\begin{aligned}\langle \boldsymbol{q}\cdot\nabla\psi\rangle &= \frac{5}{2}\sum_a \frac{T_a}{e_a}p_{a_0}\left\langle uBA_{2_\|}^{(-1)}\right\rangle \\ &= \frac{5}{2}\sum_a \frac{T_a}{e_a}p_{a_0}\frac{m_i\hat{\nu}_{ii}}{e_i}\left(\left\langle u^2B^2\right\rangle-\frac{\left\langle uB^2\right\rangle^2}{\langle B^2\rangle}\right)\frac{\alpha_{i_1}\left(A_{i_1}-\frac{1}{Z}A_{z_1}\right)+\alpha_0 A_{i_2}}{\alpha_{i_1}\beta_0-\alpha_0\beta_{i_1}}.\end{aligned}$$

VI Conclusions and Summary

We have calculated the Pfirsch-Schlüter impurity particle and heat fluxes in a stellarator. Compared with the axisymmetric case, the particle transport is qualitatively different and consists of two separate terms, one tokamak-like term that is proportional to the friction force, and one term related to the difference between parallel and perpendicular pressures, which is multiplied by a geometric factor that vanishes in axisymmetric systems. The first term is proportional to the collision frequency and the second one is inversely proportional to it. When comparing the magnitude of their contributions in the Pfirsch-Schlüter regime, one therefore finds that the friction term is considerably larger than the pressure anisotropy term, both with respect to the ordering in the shortness of the mean free path and to the ordering of the impurity charge $Z \gg 1$. Since the two terms differ by a factor of Z^4, even for relatively small impurity charges the friction term will dominate strongly. As the friction term is intrinsically ambipolar, the small pressure anisotropy term, which does not have this property, is nonetheless important for determining the radial electric field. The main contribution comes from the bulk ion pressure anisotropy, which is affected quantitatively by the presence of impurities, but the qualitative effect of the impurities is minor.

The circumstance that the neoclassical pressure anisotropy becomes small at high collisionality may have implications for the conclusion drawn in Refs. [8, 12] and [13] that the radial electric field is usually set by neoclassical rather than turbulent transport in stellarators, even if the turbulent fluxes exceed the neoclassical ones. This result was based on the observation that the particle flux produced by gyrokinetic turbulence is intrinsically ambipolar (in leading order) whereas the neoclassical transport is not. Since the neoclassical non-ambipolarity decreases with increasing collisionality, however, one would expect that turbulence could affect the electric field in cool edge plasmas if the small non-ambipolarity of the turbulence can compete with the similarly small one of the neoclassical transport.

The Pfirsch-Schlüter heat flux is unremarkable. The direct contribution from the impurities is smaller than that from the bulk ions, but the latter is affected by the presence of impurities. In a pure plasma, the heat flux exceeds the particle flux by a large factor, but they are comparable when impurities are present.

VII Appendix A: Coefficients of the distribution functions

For fixed $Z, Z_{\rm eff}$, the coefficients of the distribution functions can easily be calculated numerically. Here we give some values for different impurity charge and effective charge, assuming equal equilibrium temperatures. The values for $Z_{\rm eff} \gg 1$ are analytical limits.

Table I: Coefficients of $f_{i_1}^{(0)}$

	$Z = 6, Z_{\rm eff} = 2$	large Z, $Z_{\rm eff} = 2$	large Z, $Z_{\rm eff} \gg 1$
α_0	-1.618	-2.591	-4.51/($Z_{\rm eff}$ -1)
α_{i_1}	0.520	0.738	2.71/($Z_{\rm eff}$ -1)
α_{i_2}	-0.005	0.084	-0.39/($Z_{\rm eff}$ -1)
β_0	-1.301	-1.843	-6.77($Z_{\rm eff}$ -1)
β_{i_1}	1.673	2.205	11.28/($Z_{\rm eff}$ -1)
β_{i_2}	-0.613	-0.781	-6.77($Z_{\rm eff}$ -1)

Table II: Coefficients of $f_{z_1}^{(0)}$

	$Z = 6, Z_{\rm eff} = 2$	large Z, $Z_{\rm eff} = 2$
α_0	-1.618	-2.591
α_{z_1}	0.006	0.000
α_{z_2}	0.000	0.000
β_0	-1.301	-1.843
β_{z_1}	0.029	0.000
β_{z_2}	-0.006	0.000

Table III: Coefficients γ_{i_j}

	large Z, $Z_{\text{eff}} \gg 1$
γ_{i_0}	-2.41/(Z_{eff} -1)
γ_{i_1}	-6.02 /(Z_{eff}-1)
γ_{i_2}	-4.51 / (Z_{eff}-1)

VIII Appendix B: Proof that the -1^{st} order flow velocities vanish

We can show the -1^{st} order flow velocities to vanish: Due to particle conservation, one gets the constraint

$$\begin{aligned} 0 &= \int v_\parallel \nabla_\parallel \left(\frac{m_a}{T_a} v_\parallel V_\parallel^{(-1)} \right) \\ &= \frac{1}{3}\frac{m_a}{T_a}\left(\nabla_\parallel V_\parallel^{(-1)} - V_\parallel^{(-1)} \nabla_\parallel \ln B \right) \underbrace{\int v^2 f_{a_0} d^3v}_{\neq 0} \\ \Rightarrow V_\parallel^{(-1)} &= K_{-1}(\psi) B. \end{aligned}$$

Now consider (5) to find

$$\begin{aligned} \sum_a \left\langle B \int m_a v_\parallel C_a(f_{a_1}^{(0)}) d^3v \right\rangle &= -\sum_a \left\langle B \int m_a P_2(\xi) v^2 f_{a_1}^{(-1)} d^3v \nabla_\parallel \ln B \right\rangle \\ &= \sum_a \left\langle \left(p_{a_\parallel}^{(-1)} - p_{a_\perp}^{(-1)} \right) \nabla_\parallel B \right\rangle = 0 \end{aligned}$$

As in Sec. II, we define

$$C_a(P_2(\xi) f_{a_0} h_a) = x^2 P_2(\xi) f_{a_0},$$

which yields

$$f_{a_1}^{(-1)}|_{P_2} = \frac{m_a}{T_a} K_{-1}(\psi) P_2(\xi) f_{a_0} h_a \nabla_\parallel B.$$

Thus,

$$\sum_a \left\langle \left(p_{a_\parallel}^{(-1)} - p_{a_\perp}^{(-1)} \right) \nabla_\parallel B \right\rangle = \frac{3}{5} K_{-1}(\psi) \left\langle (\nabla_\parallel B)^2 \right\rangle \underbrace{\sum_a m_a n_a \{h_a\}}_{\neq 0},$$

and K_{-1} must vanish.

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