# Reconnection in semi-collisional, low $-\beta$ plasmas

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**Abstract**

Reconnection of semi-collisional, low-$\beta$ plasmas is studied numerically for two model problems using a two-field description of the plasma including electron pressure effects (and hence kinetic Alfvén-wave dynamics). The tearing unstable Harris-sheet, with the global parameters of the GEM-challenge case shows a linear growth of the peak reconnection rate with the drift parameter $\rho_s$ when this scale is significantly larger than the resistive skin depth, and the island is smaller than the Harris sheet current layer width. As exemplary for a driven, rather than a spontaneous reconnection situation we study as second model system two coalescing islands, starting from a non-equilibrium situation. The peak reconnection rate again increases initially linearly with $\rho_s$ but saturates and becomes $\rho_s$ independent for larger values. In this saturated regime, no flux pile-up occurs, and the reconnection is limited by the rate of approach of the two coalescing islands. The qualitative differences between spontaneous and driven reconnection cases, and their scaling behaviour are best understood by considering the reconnection rate as a triple product of outflow Mach number, outflow to inflow channel width ratio, and magnetic energy density at a height $\rho_s$ above the X point.

## I. Introduction

Reconnection is a ubiquitous problem of plasma physics and central to the understanding of the conversion of magnetic into thermal and kinetic energy in astrophysics, space science and magnetic confinement research. Generally the observed rates of reconnection cannot be explained in terms of a simple resistive MHD model. Starting in the 90ies it has been shown by several authors (e.g. [1-4]) how two-fluid effects can significantly accelerate reconnection, by mitigating the constraints on the plasma flow in the proximity of the reconnection line (X-point). The basic topological problems, the constraints arising from purely resistive reconnection and the effects of a more generalized Ohm´s law can be understood and quantified by 2-D simulations, although 3-D effects and turbulence may still play a major role in the immediate neighbourhood of the reconnection line.

Two-fluid effects, dominating in various plasma regimes are the Hall effect, electron pressure gradient effects and electron inertia, introducing, respectively, the ion inertial length $d_i = c/\omega_{pi} = C_A/\Omega_i$, the drift scale $\rho_s = \sqrt{T_e/m_i}/\Omega_i$ and the electron inertial skin depth $d_e = c/\omega_{pe}$. Here and henceforward $\omega_{pi}, \omega_{pe}, \Omega_i$ denote ion and electron plasma and the ion cyclotron frequency, and $C_A$ the Alfvén speed. We focus in the following on plasma conditions relevant to standard aspect-ratio tokamaks, where the Hall effect is unimportant, and on phenomena proceeding at a slow enough rate $\gamma$ so that the resistive skin depth $\delta_{res} = \sqrt{2\eta/\mu_o\gamma}$ (with $\eta$ the resistivity) is larger than the electron inertial one. This is in contrast to most other reconnection studies, notably the coordinated benchmark effort GEM [1], which were motivated by space and astrophysical contexts and concentrated on high $\beta$ plasmas where the Hall effect is dominant.

Reconnection phenomena have been idealized into two extreme situations, where the reconnection either itself determines the rate and geometry of the macroscopic developments (spontaneous reconnection) or where the rate of flux change is externally imposed (forced reconnection). Sample realisations of these two situations are, on one hand the tearing instability of the Harris sheet [1], and the Taylor problem [2] on the other. Ultimate aim of simulations of idealized situations is the quest for universally valid scalings of the dependent parameters of the problem on (mainly) $\eta, d_i, \rho_s$. In the case of Hall-MHD several, apparently contradictory conclusions have been drawn in this regard [2-4] which, however, have recently been argued to be motivated by the differing formulations of the problem [5].

To try to shed light onto effects that can possibly lead to a variation of the apparent scaling with the dispersion-inducing parameter ($d_i$ or $\rho_s$, respectively), we have conducted simulations for a coalescing island case, where we are able to show a continuous variation of the dependence of the reconnection rate on $\rho_s$ when changing from a situation where plasma flow and reconnection in the X-point vicinity are rate- determining, to one where the finiteness of the drive leads to a saturation of this dependence.

To contrast this behaviour to one of essentially spontaneous reconnection, and to connect to the results of other groups for a much more extensively studied situation, we also show first the results of our model and code for the Harris-sheet geometry used in the GEM-challenge simulations (Ref. [1] and papers summarized therein).

## II. Physics Model and Governing Equations

A 2-D slab-simulation of reconnection capturing features of a tokamak plasma with a sheared magnetic field is conveniently carried out in a coordinate system with one axis (*z* in our notation) aligned with the unperturbed magnetic field at the resonant surface. At low $\beta$ the strong parallel field then justifies an incompressible plasma description, with $|\vec{B}| \approx \vec{B}\cdot\nabla z = B_{zo}$. A set of equations used for treating such plasmas in the semi-collisional regime are those of the four-field fluid-drift model derived in Refs. [6] and [7] for a cold-ion, isothermal electron plasma ignoring electron inertia. It consists of four time-dependent partial differential equations for the magnetic flux function $\psi$ , the z-component of vorticity $\omega = \vec{e}_z \cdot \nabla \times \vec{u}_\perp = \nabla^2\phi / B_{zo}$ (with $\phi$ the electrostatic potential), the electron pressure $p_e$ and the parallel flow velocity $u_{//}$. In the absence of a background density gradient, the effects of parallel sound dynamics are also negligible [8], allowing further to neglect $u_{//}$ and to set $\nabla_{//} p_e = \rho_s^2 e n_e \nabla_{//} \nabla^2 \phi$. The resulting set of 2-field equations is thus given by

$$\frac{d}{dt}\psi - \rho_s^2 \vec{B}\cdot\nabla\omega = \frac{\eta}{\mu_o}\nabla^2\psi \qquad (1)$$

$$\frac{d}{dt}\omega - \frac{1}{n_o m_i}\vec{B}\cdot\nabla j_z = \mu\nabla^2\omega \qquad (2)$$

together with

$$\nabla^2\psi = -\mu_o j_z \qquad (3)$$

and

$$\nabla^2\phi = B_{zo}\omega \qquad (4)$$

using $d/dt = \partial/\partial t + \vec{u}\cdot\nabla$, the ansatz $\vec{B} = B_{zo}\nabla z - \nabla z \times \nabla\psi$ and $\vec{u} = \frac{1}{B_{zo}}\nabla z \times \nabla\phi$.

The set is equivalent to the one used in Ref. 9.

Our dimensionless units are based on one characteristic length of the computational regime $L_0$, a magnetic field value in the poloidal plane $B_o$ and the constant mass density $n_o m_i$, giving dimensionless units (here designated by an asterisk, which we drop, however, after this section) as

$t = t^*(L_o / C_{Ao}), \vec{r} = \vec{r}^* L_o, \vec{u} = \vec{u}^* C_{Ao}, \vec{B} = \vec{B}^* B_o, \psi = \psi^*(B_o L_o), \phi = \phi^*(L_o C_{Ao} B_{zo}), \rho_s = \rho_s^* L_o$, $\eta = \eta^*(\mu_o L_o C_{Ao})$, $\mu = \mu^*(n_o m_i L_o C_{Ao})$, with $C_{Ao} = B_o / \sqrt{\mu_o n_o m_i}$. The choice of reference magnetic field is motivated by the fact that $B_{zo}$ enters the dimensionless equations only through $\rho_s$, so that a normalisation of $\vec{B}$ in terms of a poloidal component, and, in particular, of the Alfvén speed in terms of this (poloidal) $B_o$ gives more physical insight. $B_{zo}$ would enter only in the conversion of our dimensionless electrostatic potentials $\phi^*$ into dimensional ones.

Mainly for numerical reasons, we include both resistivity and viscosity as dissipative terms, chosing in general the magnetic Prandtl number $\mu^*/\eta^*$ equal to one. Scans against the resistivity are thus generally also scans against $Ha^{-2}$, with the Hartmann number given by $Ha = B_o L_o / \sqrt{\eta\mu}$.

**III. Computational Method**

The above set of equations consists of diffusive and convective terms. For the diffusive terms (resistivity and viscosity) we use an unconditionally stable Alternating Direction Implicit scheme, whereas the convective terms are treated explicitly by a leapfrog scheme alternating between the vorticity and the magnetic flux equations. The time-step is therefore constrained by a Courant Levy condition: $\Delta t < \Delta x / (|u| + \rho_s |B_x| / \Delta x)$ which takes into account the dispersive properties of the waves. To avoid grid oscillations in regions of steep gradients it was found necessary to use an Arakawa scheme for the Poisson brackets that is fourth order accurate in space and conserves rigorously energy and squared vorticity [10].

Production runs were carried out with a grid of 400x200 points using mesh refinement in the X-point proximity with homomorphic grid functions: for the cases reported here, we used minimum step sizes $\Delta y = 2 \cdot 10^{-4}$ and (for large $\rho_s$) $\Delta x = 1.25 \cdot 10^{-3}$, respectively. For low values of the dissipation coefficients, 4$^{th}$ order derivative terms in the form of hyper-resistivity and hyper-viscosity, respectively, had to be added to the R.H.S. of Eqs (1) and (2) to keep gradients to the scale of the grid [11]. We demonstrated (and describe below) that the resulting reconnection rates and the macroscopic flows and fields were independent of the numerical coefficients used for these hyper-resistivity and –viscosity terms.

## IV. Results of Simulations

The most important results of our simulations are the geometrical patterns of magnetic and flow fields, and the reconnection rates as function of time or reconnected flux. The latter representation allows particularly well to separate the effects of reconnection from those of drive, as for self-similar geometries (and, obviously, in the linear case) equal reconnected flux would imply equal forces exerted onto the plasma perpendicular to any reference flux surface taken as boundary of the reconnection zone. It is therefore particularly indicated for simulations starting from an unstable near-equilibrium state, like our (or the GEM) Harris sheet cases.

### IV.A. The Harris Sheet

To connect to the extensive information gained from the GEM benchmark exercise, we have studied, as a first problem, the Harris sheet magnetic configuration, albeit assuming – to be consistent with our ordering – a strong guide field, and consequently a constant density. We also based dimensionless quantities on a macroscopic length $L_o$ , rather than on an intrinsic plasma parameter ( $d_i$ in the GEM-convention), leading to conversion factors

$$\eta^{GEM} = \eta^{SGL}\left(L/d_i\right),\ \psi^{GEM} = \psi^{SGL}\left(L/d_i\right),\ t^{GEM} = t^{SGL}\left(L/d_i\right),\ \dot{\psi}^{GEM} = \dot{\psi}^{SGL}$$

between the dimensionless parameters of the GEM case and the present paper (*SGL*). In our units and coordinates, the unperturbed Harris-sheet field configuration is given by $B_x(y) = \tanh(y/0.039)$, with an initial seed perturbation specified as $\psi(x, y, t = 0) = 0.0078\cos(\pi x)\cos(\pi y)$. For the GEM standard parameter set $\left(L/d_i\right)^{GEM} = 12.8$. With different dispersive terms in our two-fluid formulations, only the purely resistive MHD case can be used for a direct comparison. We chose, for this reason our reference values of $\eta$ so as to correspond to a case studied in Ref. [13]. The results, both for the reconnected flux as function of time and for the topology of the flux function agree very well, over the whole period described in Ref. [13] ( $t^* \le 15.6$ in our units of Alfvén transits through the periodicity length in y-direction). As noted by the authors of [13] , the early growth rate of the perturbation agrees also well with the prediction of classical [14] linear tearing mode theory, taking the value of $\Delta'$ of the Harris sheet for vanishing island width, although the requirements for its validity ( $w_I < \delta_{tear} < \delta_{res}$ , with $w_I, \delta_{tear} = \dfrac{\eta\Delta'}{\gamma}, \delta_{res} = \sqrt{\eta/\gamma}$ the island, the tearing-layer and the resistive-layer width) are not met by the parameters and the initial conditions ( $w_I = 0.078, \delta_{tear} = 0.035, \delta_{res} = 0.035$). One would, after the adjustment of the simulations to the "inconsistent" initial conditions rather expect a Rutherford-regime [15] like behaviour with the energy flux into the island region balanced by the Ohmic dissipation. Due to the large initial island size (comparable to the current layer width of the Harris sheet), however, simulations are from the very beginning in the saturation regime. The reconnection rate should therefore rather be compared with the expression $\dot{\psi}_1 = (1.22/8) w_0 \eta \Delta'(w_0) \psi_0''(w_0)$ , which, taking the analytic formula of Biskamp [16] for $\Delta'(w_I(t=0)) = 102$ and $\psi_0''(w_I(t=0))$=11.2, gives a rate by a factor of 2.5 smaller than observed.

Figs.1.a –c show the poloidal magnetic flux contours during three characteristic phases, at times between the initial state and $t$= 17.5. (Calculations were continued till $t$=83.7 but with no further qualitative changes.) During early phases, the neutral sheet along the $y$ = 0 line extends in the x-direction between two Y-type reconnection points, which coalesce forming an X-point at later times. Tables 1 and 2 give some quantitative results of our Harris-sheet simulations, and among others, also the ratio of the current sheet width (taken at half the maximum value of $j_z$) in y $(\delta_j)$ and x-direction $(L_j)$ at the two instances of Fig.1b,c corresponding to reconnected fluxes of 0.038 and 0.066, respectively. (Note that our definition of reconnected flux $\psi_{rec}(t) = \psi(0,0,t) - \psi(\mathrm{O-point}, t)$ includes already the flux implied as reconnected by the initial conditions.) The ratio $\delta_j / L_j$ is very small, with $\delta_j$ microscopic and of the order of the resistive layer width, and $L_j$ macroscopic and of the order of the geometrical dimensions. In the scans with constant magnetic Prandtl number, the maximum of the reconnection rate $\dot{\psi}_{rec}$ scales, to very good approximation, like $1/\sqrt{\eta}$ (Fig.2).

Starting from this reference case we introduced the electron pressure, and hence a finite value of the drift parameter $\rho_s$. The latter was chosen, for the cases discussed below as 0.018, 0.036, 0.072, respectively. The largest $\rho_s$ case corresponds to the same ratio of microscopic ($\rho_s$ or $d_i$, respectively) to macroscopic dimensions (lateral and vertical elongation of box $L_x, L_y$ and current sheet width $\delta_{curr}$) as the GEM reference case, although the effect mitigating reconnection is a different one [17] (kinetic Alfvén in our vs. Whistler waves in the GEM-case), and the characteristic length appearing in the GEM-case $d_i = c/\omega_{pi}$ is the ion gyroradius based on the Alfvén (rather than the ion acoustic) speed. Tables 1 and 2 give a list of characteristic quantities derived from the calculations for the different cases for the two values of the reconnected flux: $\psi_{rec} = 0.038$ and 0.066, respectively. In case of a one-to-one correspondence between reconnected flux and island shape and width, equal reconnected flux $\psi_{rec}$ would correspond to equal energy input into the tearing zone $\delta W_{mag}$ per unit reconnected flux $\delta\psi_{rec}$ , and remains at least a well-defined reference situation in the general case.

Introduction of electron pressure effects has been found in Refs. [9,18] to allow, like the Hall effect, for a decoupling of plasma flows and field lines over a spatial scale of the order $\rho_s$. This can be seen from the fact that the term $\vec{u}\cdot\nabla\psi - \rho_s^2\vec{B}\cdot\nabla\omega = \vec{u}\cdot\nabla\psi - \rho_s^2(\vec{e}_z\times\nabla\omega)\cdot\nabla\psi$ can be re-written in the form $\vec{v}\cdot\nabla\psi$, with

$$\vec{v} = \vec{u} - \rho_s^2(\vec{e}_z\times\nabla\omega) = -\vec{e}_z\times\nabla\left(\phi - \rho_s^2\nabla^2\phi\right), \tag{6}$$

with $\vec{v}$ the field line convection velocity, and $\phi - \rho_s^2\nabla^2\phi$ its stream function. Whereas the reconnection rate for the spontaneous (self-regulating) Harris sheet case in resistive MHD is controlled by resistivity, for large enough $\rho_s$ it becomes independent of $\eta$ (or Hartmann number), as is shown in Fig. 2 for the intermediate case ($\rho_s = 0.036$). As requirement for this insensitivity, we found that the resistive skin depths $\delta_{res} = \sqrt{\eta/\gamma}$, at the actual $\gamma = \dot{\psi}_{rec}/\psi_{rec}$ determined by the simulations, should be small compared to $\rho_s$. This condition is well satisfied for $\rho_s = 0.036, 0.072$ (see Tables 1 and 2) but not for $\rho_s = 0.018$.

Fig. 3 shows the reconnection rate $\dot{\psi}_{rec}$ as a function of time and reconnected flux for the cases of Tables 1 and 2. As found in the GEM-studies, the reconnection rate is strongly increased by the decoupling of plasma flow and field line convection. This is the case already from the very beginning of each simulation, but $\dot{\psi}_{rec}$ shows a further strong increase at an instant in time occurring later for smaller $\rho_s$. The time of this increase is not related simply to the amount of reconnected flux, but rather to a characteristic change in field line geometry. Early in the calculations, all cases show two Y-type reconnection regions, merging later into a single X-point. This behaviour is qualitatively quite similar to the one described in Ref. [18]. Fig. 4 shows the flux pattern for the three different $\rho_s$- cases at a total reconnected flux $\psi_{rec} = 0.038$ and $\psi_{rec} = 0.066$. The “mixed” case $\rho_s$ = 0.018 (with $\delta_{res} \approx \rho_s$) at the earlier time exhibits still two distinct Y-type reconnection regions and only a moderately enhanced reconnection rate $\dot{\psi}_{rec}$. Reconnection is already strongly enhanced for the larger $\rho_s$-values. In particular, however, flux surfaces in these latter cases have already developed X-type reconnection geometry. This X-type flux surface geometry extends however only over a local region with an approximate height $\rho_s$, linking, on larger scales, to the remnants of the original Y-type flux surface structure. At later times, this two-region feature of the geometry disappears, as can be seen from the flux patterns at $\psi_{rec}$ = 0.066. Remnants of the two-region structure are still visible at this $\psi_{rec}$-value in the reconnected regions of the $\rho_s$ = 0.072, as the reconnected flux value is reached earlier at the higher $\rho_s$. The sudden strong increase of the reconnection rate $\dot{\psi}_{rec}$ in the finite $\rho_s$-cases appears to be linked to the expansion of the X-point geometry over a vertical height $\rho_s$ corresponding to the region of decoupling of plasma flow and field convection.

Tables 1 and 2 list some further parameters quantitatively characterizing the geometry of the reconnection region at the time-instances corresponding to the two reconnected flux values of Fig. 4: the time at which this reconnected flux value is reached, the local field line angle $\alpha$ at the X-point, $\delta_j / L_j$, and the ratio of the plasma flow channel width in *y*- ($\delta_u$) and *x*–direction ($L_u$). For characterizing this flow channel we take the stream-line tangent to *y* = $\rho_s$ and use its closest approach to *y* = 0 and *x* = 0, respectively, as definition of $\delta_u$ and $L_u$. (This definition can obviously not be used for purely resistive-MHD: as, close to the line of stagnation, the ratio $\delta_u / L_u$ depends, however, little on the actual stream-line chosen, we take in this case simply the innermost stream-line plotted).

With increasing $\rho_s$ (and concomitantly increasing reconnection rate) the current sheet at the X-point contracts dramatically in vertical, but in particular also in x-direction. At high values of $\rho_s$, hyper-resistivity and hyper-viscosity are required to suppress grid oscillations, but the reconnection rate becomes independent of their numerical values, if they are chosen low enough. In general we use a hyper-viscosity coefficient equal to that of hyper-resistivity, but we have verified the independence of the reconnection rate by varying the two coefficients also separately. When the reconnection rate becomes independent of resistivity, the current layer assumes an apparently universal aspect ratio ($\delta_j / L_j \approx 0.1$), also found in our simulations regarding coalescing islands and for the same geometry in Ref. [9]. Over a distance $|x| \le (L_u/\delta_u)\rho_s$, $|y| \le \rho_s$ the electron pressure term $-\rho_s^2 \vec{B}\cdot\nabla\omega$ dominates over convection by the plasma flow $\vec{u}\cdot\nabla\psi$ among the different contributions to the electric field $\dot{\psi}$, and determines it in the axis vicinity, in a similar form as in simulations of corresponding Hall-mitigated reconnection described in Ref. [11].

The field line angle, at the X-point in the outflow direction, after its formation, continues to open up till the "nose-like" structure extending over a vertical region $\approx \rho_s$ has disappeared. A given angle $\alpha$ is reached at a lower reconnected flux value, and in particular much earlier, for the larger $\rho_s$-values, although even the resistive MHD at later times shows a similar flux pattern. Our case of Tables 1,2 with $\rho_s = 0$ was continued to $t = 84$, when $\alpha$ reached $46^o$.

The decoupling of field lines and plasma flow over a progressively larger region is evident in the comparison of the stream function $\phi$ for the plasma flow with that for the field line convection: $\phi - \rho_s^2\nabla^2\phi$ (Fig. 5) for the case of largest $\rho_s$, near the reconnection rate maximum (at $\psi_{rec}$=0.066). The finite electron pressure term eliminates the need for the plasma streamlines to converge near the X-point to conserve the convected flux in spite of the reduction of $|B|$ in its proximity [19]. As we discuss further in the conclusions, once an X-point is formed, the ratio of in- to outflow velocity $u_{in}/u_{out}$ - equal to $\delta_u/L_u$ in Tables 1 and 2 – is closely linked to the field line angle of the separatrix. For such cases both $\delta_u$ and $L_u$ are roughly proportional to $\rho_s$, as found by Shay et al. [3]. The outflow velocity is however significantly smaller than the reference Alfvén velocity, as other forces besides inertia (notably the hydrostatic pressure difference between the O-point and the upstream region present already in the equilibrium phase) balance the major part of the upstream magnetic pressure.

The most important macroscopic result of these simulations is the observed dependence of reconnection rate $\dot{\psi}_{rec}$ on $\rho_s$. The reconnection rate has a broad maximum as function of the reconnected flux, which for all three finite $\rho_s$-cases occurs approximately for the reconnected flux value of 0.066. The maximum reconnection rate increases about linearly with $\rho_s$ between the cases with $\rho_s$ = 0.018 and 0.036 and weaker up to 0.072. A uniform scaling across the whole tested range is, however, not to be expected, as the half-width of the island is approximately equal to the largest value of $\rho_s$. A quantitative analysis of the magnetic flux and the plasma flow patterns reveals that a major contribution to this scaling comes from the shift in the location of plasma flow/field decoupling away from the X-point. As can be seen from the stream-lines in Fig. 5 the plasma in-flow along the x=0 line is nearly constant over a significant range of y, and varies also significantly less than the reconnection rate between the three cases (from $u_y$ = 0.02 for $\rho_s$ = 0.018 to 0.045 for $\rho_s$ = 0.072). The value of $\rho_s$ determines the location of plasma-field decoupling, as can be seen particularly well from the stream-lines of the advection flow (Fig. 5) which start converging towards the X-point at a height increasing with $\rho_s$. At this location they take over the magnetic flux frozen so far into the plasma flow and carry it further to the X-point: $u_y B_x(\rho_s) \approx \dot{\psi}_{rec}$.

### IV.B. Coalescent Islands Case

To approach the situation of driven reconnection, where the rate of energy input into the reconnection zone is externally imposed, we simulated as second problem, the coalescent island case in the form studied in Ref. [9]. Although also this problem concerns an energetically closed system (in the sense that the boundary conditions correspond to vanishing Poynting flux and perpendicular plasma flow), it starts far from force equilibrium.

The first phase of development is therefore characterized by the conversion of magnetic energy into kinetic energy, and is relatively independent of reconnection (a situation prevailing also, to a much lesser extent, and only over a short period, in the Harris sheet case, when we start the unstable development with a relatively large amplitude). In our normalisation, the duration of this phase is about $t \le 1$, and for sufficiently low resistivity and viscosity conserves the sum of kinetic and magnetic energy (Fig. 6). At the onset of reconnection therefore not only magnetic energy, but a significant amount of accumulated kinetic energy is available for its drive.

For our simulations we initiate calculations with a current distribution given by

$j_z(x,y) = j_o \cdot \left(r_m{}^2 - \left(x^2 + (y-d)^2\right)\right)^2$ , for $r_m{}^2 > \left(x^2 + (y-d)^2\right)$ and

$j_z(x,y) = 0$ , for $r_m{}^2 \le \left(x^2 + (y-d)^2\right)$

and the flux distribution obtained by solving Ampere´s law with the boundary conditions $\psi(\pm 1, y) = 0,\ \psi(x,1) = 0,\ \partial\psi(x,0)/\partial y = 0$. The plasma is initially assumed to be at rest, and the stream function of its flow satisfies at all times $\phi(\pm 1, y, t) = 0, \phi(x, \pm 1, t) = 0$. The intial conditions are thus symmetric about both the x=0 and the y= 0 lines. To allow, at later times, spontaneously arising, asymmetric tearing of the current sheets along the x-axis, we solve the equations in the region $(-1 \le x \le 1, 0 \le y \le 1)$. The current density is normalized so that (initially) $\max|B| = B_o$. For all cases reported here, $d$ = 0.3 and the initial flux value at the O- and X- points are 0.32, and 0.213, respectively, corresponding to a total flux available at $t$ =0 for further reconnection of 0.107.

Due to the initial non-equilibrium and the large kinetic energy acquired by the plasma prior to reconnection, the dynamics of the system, at later stages is multifaceted. Fig. 7 gives flux contours at different times, for the case with $\eta = \mu = 10^{-4}$ and $\rho_s$ =0.032. Even after complete reconnection is achieved, at $t \approx 1$ , the stored kinetic energy can lead to a tearing of the plasmoid and, again, the formation of separate island regions along the x-axis, coalescing again later. For this reason, we limit quantitative comparisons to phases where less than half of the available flux is reconnected. Fig. 8 shows the reconnection rate $\dot{\psi}_{rec}$ as a function of time and of reconnected flux, for different values of $\rho_s$, at constant $\eta = \mu = 10^{-4}$. For these and smaller values of the dissipation coefficients we have verified that the reconnection rate becomes independent of $\eta$ and $\mu$. Larger dissipation coefficients – for constant magnetic Prandtl number – at finite $\rho_s$ were actually found to reduce the reconnection rate, as viscosity damps the velocity shear which otherwise allows the decoupling of plasma flow and field convection in the X-point vicinity. Reconnection rates of Fig. 8 are much higher than in the corresponding cases of the Harris sheet (Fig. 3) showing a much stronger drive. For the same reason also the plasma outflow velocity is much closer to the Alfvén speed.

Fig. 9 shows flux contours, and overlaid, colour-coded, the magnitude of the plasma flow velocity $|u|$ at a reconnected flux $\psi_{rec}(t) = 0.253$, for some of the cases of Fig. 8. Increasing $\rho_s$ results in a widening of the outflow zone, with steep gradients of $|u|$ across the separatrix flux surface. Quantitative information on characteristic parameters for these cases is given in Table 3. The reconnection rate grows linearly with $\rho_s$ between 0.0033 and 0.01 and then weaker than linearly up to 0.02. Beyond that value it saturates. In the region of linear growth, $\dot{\psi}_{rec}$ agrees well with the trend that would be expected from a Sweet Parker like argument, namely proportional to $\delta_u / L_u \propto u_{in} / u_{out}$ . For these values of $\rho_s$, the outflow velocity equals the reference Alfvén velocity (taking $B_{ref} = B_x(0, \rho_s)$) For larger $\rho_s$, the ratio of the flow sheets $\delta_u / L_u$ continues to increase with $\rho_s$, but accompanied by a decrease in the outflow velocity. Beyond $\rho_s = 0.02$ these two trends compensate: the outflow velocity adjusts to the channel width and $B_{ref}$ so as to maintain the reconnection rate invariant. In this situation we have obviously arrived at conditions of truly "forced" reconnection, where external conditions (in this case produced by the acceleration of the two islands towards each other during the frozen-in-field phase) impose the reconnection rate. The plasma flow velocity and the flow channel geometry have then to adjust in order to accommodate the imposed, nearly constant reconnection rate in spite of the variation of $\rho_s$.

## V. Discussion

The two model problems dealt with in our simulations exhibit a range of different scaling trends with the parameter governing the plasma/magnetic field decoupling (in our case the drift parameter $\rho_s$). Starting point for any unified interpretation of the quasistationary behaviour, is the relation $E_z(0,0) = \dot{\psi}_{rec} \approx B_x(0, y_{ref}) \cdot u_y(0, y_{ref}) = B_{ref} \cdot u_{in}$ describing advection of frozen-in flux by the plasma flow, in the upstream region, prior to the plasma/magnetic field decoupling. Incompressibility relates this inflow to the outflow velocity $u_y(0, y_{ref}) = u_{in} = \delta(\phi_{ref}) / L(\phi_{ref}) \cdot u_x(x_{ref}, 0) = \delta(\phi_{ref}) / L(\phi_{ref}) \cdot u_{out}$ where $\delta(\phi_{ref}) / L(\phi_{ref}) = \delta_u / L_u$ is the ratio of outflow to inflow width of the channel formed by a stream line $\phi = \phi_{ref}$ passing close to the X-point. Following Ref. [3] we refer $u_{out}$ to the Alfvén-speed, based on the reference field $B_{ref}$, to write

$$\dot{\psi}_{rec} = M_A (\delta_u / L_u) B_{ref}^2 , \tag{7}$$

and to discuss the results in terms of the three contributing factors $(\delta_u / L_u), M_A, B_{ref}^2$.

For finite $\rho_s$, once an X-point is formed, the maximum velocity shear is across the separatrix, and, with increasing $\rho_s$, the point of effective decoupling moves upwards, leading to a concomitant increase in $\delta_u$ and $L_u$. The ratio $\delta_u / L_u$ is thus linked to the separatrix geometry near the X-point, and the variation of $\delta_u / L_u$ is strongly linked to the global field structure. In the Harris-sheet case, the sudden increase in reconnection rate evident in Fig. 3 happens simultaneously with the opening up of the separatrix over a height of the order $\rho_s$. If the geometry is strongly constrained by global properties - like in the Harris-sheet case - the ratio $\delta_u / L_u$ tends to a constant, like also in the double tearing mode - cases reported in Ref. [3]. For the $\rho_s$ = 0.072 case (corresponding to the

GEM standard data-set convention) reconnection proceeds so fast, that the original (rather large amplitude) perturbation has not been damped out by the time of Fig. 4, and the X-point angle overshoots this general trend (Table 2).

The reference value $B_{ref}$ should be linked to the location of decoupling of the plasma flow and the field line convection (e.g. $v_x(0, y_{ref}) \approx u_x(0, y_{ref})$, as proposed by Ref. [3]). The upward shift of this location, to increasing values of $B_x(0, y)$, in fact explains a significant part of the observed $\rho_s$-scaling in our Harris sheet case, as can be understood also from the magnetic field profiles shown in Fig. 10a.

The relation of the reconnection rate to an expression like Eq. (7) is more robust regarding the linear factor $B_{ref}$ derived from the frozen-in condition, than the second one, derived from the variation of the Alfvén-velocity. The variation of the outflow velocity (respectively of $M_A$, once we have chosen a definition of reference field $B_{ref}$) is dominated by the global dynamics of the problem. In the case of an "embedded" reconnection this manifests itself in a variation of the hydrostatic pressure difference between the in- and outflow region, which is not explicitly computed in incompressible calculations, but is, of course, nevertheless present and has to be included (together with the sometimes significant contribution of magnetic tension $-\int (\vec{B} \cdot \nabla)\vec{B} \cdot d\vec{u} / |u|$ to the integral of the momentum equation along a stream-line) in the Sweet-Parker pressure balance. This is evident, of course, in spontaneous reconnection, starting from an equilibrium with (in slab geometry) $|B^2| / (2\mu_o) + p = \mathrm{const.}$ In our Harris sheet case, in fact, even at later times, the Alfvén-Mach number referred to either the asymptotic field $B_x(x,1)$ or a reference field $B_x(0, \rho_s)$, remains below 0.3 for all instances in our simulations. The situation is very different in our coalescent island simulations, where the starting state is far from equilibrium, leading initially to a strong acceleration of the plasma. Flux pile-up occurs therefore in front of the reconnection zone, on a spatial scale commensurable with $\rho_s$ (Fig.10b). The outflow Mach number – referenced to $B_x(0, \rho_s)$ - is of the order 1, and the reconnection rate - for the large $\rho_s$-values approaches a self-regulated state, in which separatrix geometry, field-strength in the pre-reconnection region, and Mach-number adjust to give a reconnection rate imposed by the dynamics (i.e. the approach velocity) of the two converging plasma blobs, nearly independently of $\rho_s$.

The scaling of reconnection rate with $\rho_s$ is therefore a complex one, with differences between the spontaneous and driven situation, but varying also with $\rho_s$ (and its relation to macroscopic scales) itself. We characterize, in the following, this behaviour by the contributions to the three factors $(\delta_u/L_u), M_A, B_{ref}^2$ , and distinguish, for each problem three regimes, corresponding to increasing values of $\rho_s$: (1) the transition from purely resistive to drift parameter dominated regime, (2) an intermediate regime, where a scaling of $(d\psi_{rec}/dt) \sim \rho_s$ is observed, and (3) a regime of saturation.

In the Harris sheet case (Tables 1 and 2), $\delta_u / L_u$ makes a sudden jump in the transition from the resistive to drift parameter dominated case, but varies then only slowly (as discussed above, the high $\rho_s$-case is an exception, as the magnetic field structure still shows remnants of the starting perturbation – see Fig. 4f). The outflow Mach number referenced to the field $B_{ref}$ at the nominal decoupling point $(0, \rho_s)$ varies also relatively little, so that the dominating contribution in Eq. (7) comes from the variation of $B_{ref}$ , which is primarily due the upwards shift of the reference point (see Fig. 10a).

In the coalescent island case (Table 3) , $\delta_u / L_u$ (and $\dot{\psi}_{rec}$) show a strong increase from $\rho$ =0.0033 to 0.01, and a smaller rate of increase further on. The outflow Mach number, based on $B_{ref}$ is initially constant and close to 1, but decreases slightly when the $\rho_s$-dependence of $\dot{\psi}_{rec}$ saturates. The profiles of $B_x(0, y)$ in Fig. 10b show that the spatial maximum of the upstream magnetic field shifts away from the X-point with increasing $\rho_s$ and decreases in magnitude. Due to the strong dynamics of this system, selecting a particular instance in time (or reconnected flux) is difficult and possibly misleading: Fig. 8b shows, however, quite clearly the saturation of $\dot{\psi}_{rec}(\psi_{rec}, \rho_s)$ with $\rho_s$ over an extensive range in reconnected flux.

The actually observed reconnection rates, for each of the two model problems correlate quite well with the expression Eq. (7) (see last columns of Tables 2 and 3), but the ratio $\dot{\psi}_{rec} / \left(M_A (\delta_u / L_u) B_{ref}^2\right)$ varies significantly between the Harris sheet and the coalescing island case, and differs, in particular in the former case, significantly from 1. The two main reasons for this are, that at the reference point $(0, \rho_s)$ the electron pressure term makes already a large contribution to the magnetic field convection, and that the assumption of quasi-stationarity ($\dot{\psi}(x, y, t) = \dot{\psi}_{rec}(t)$) is only approximately satisfied.

**The question of the existence of a universal reconnection rate (as debated in Refs. [1-5]) can be discussed at hand of Eq. (7), and in the light of our results for the two model problems**

**considered. If one assumes a strong drive ($M_A \approx 1$) and normalizes by $B_{ref}{}^2$ as suggested in [3], the reconnection rate will still scales with the in/out flow ratio $\delta_u / L_u$. Only if the magnetic geometry is very robust and does not change with $\rho_s$ in the vicinity of the X point - like for the example discussed in [3] and for the GEM challenge - this ratio, and hence the reconnection rate normalized by $B_{ref}{}^2$ will remain independent of $\rho_s$. In the case of truly forced reconnection, where the rate is externally imposed by the boundary conditions [5] and kept fixed in a variation of $\rho_s$, all three factors can adjust to match the imposed rate: our coalescent island case approaches this limit for the largest two $\rho_s$-values.**

Although the parameter ordering for our simulations $(d_e < \delta_{res} < \rho_s)$ is relevant for certain tokamak applications, the results do not have a direct applicability to problems of this device, but are intended rather to contribute to the general understanding. Reconnection problems in tokamaks span a broad spectrum ranging from both spontaneous to forced reconnection. In the case of classical tearing modes, the energy inflow into the tearing layer per unit reconnected flux is given by $dW_{recon}/d\psi_{rec} = \Delta' \psi_{rec}$ , and is imposed by the field changes in the ideal plasma region, whereas the rate of energy input (the divergence of the Poynting vector $dW_{recon}/dt = \Delta' \psi_{rec}^2 \cdot d\psi_{re}/(\psi_{rec} dt)$) is then determined by the reconnection rate itself. On the other hand the growth of secondary islands at rational surfaces $q = (n+1)/n$ – forming the possible seed for neoclassical tearing modes (NTMs) – by MHD activities connected to sawteeth, or the response to suddenly applied resonant perturbations are cases of a driven reconnection, where the primary drive (e.g. the growth of the $m/n = 1/1$ mode) is hardly affected by the back-reaction. The latter and similar problems, for rotating plasmas will be examined in the future with the code described in Ref. [20], with a similar two-fluid physics model in cylindrical geometry, but including density gradients and hence diamagnetic drifts.

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TABLE 1: Characteristic results of Harris sheet simulations with $\eta = \mu = 2 \cdot 10^{-4}$ and different values of drift parameters for $\psi_{rec} = 0.038$: time of attainment ($t$), instantaneous reconnection rate ($d\psi/dt$), island width ($w_I$), field line angle at X-point ($\alpha$), resistive layer width ($\delta_{res} = \sqrt{\eta/\gamma}$), ratio of current sheet widths ($\delta_j / L_j$), ratio of outflow to inflow channel width $(\delta_u / L_u)$.

| $\rho_s$ [$10^{-2}$] | $t$ | $d\Psi_{rec}/dt$ | $w_I$ | $\alpha$ [°] | $\delta_{res}$ | $\delta_j/L_j$ | $\delta_u/L_u$ |
|---|---|---|---|---|---|---|---|
| 0 | 3.8 | 0.006 | 0.19 | Y-g. | 0.035 | 0.03 | 0.08 |
| 1.8 | 2.1 | 0.014 | 0.21 | Y-g. | 0.020 | 0.03 | 0.09 |
| 3.6 | 1.3 | 0.074 | 0.21 | 20 | 0.011 | 0.12 | 0.18 |
| 7.2 | 0.6 | 0.092 | 0.18 | 34 | 0.009 | 0.10 | 0.30 |

TABLE 2: As Table 1, but $\psi_{rec} = 0.066$. In addition, values for the outflow velocity ($u_{out}$), the reference magnetic field strength ($B_{ref}$) and the fraction of Alfvén velocity ($M_A$) are given together with the normalised reconnection rate $\dot{\psi}_{rec}$ /($B_{ref}^2$ $\delta_u/L_u$ $M_A$).

| $\rho_s$ [$10^{-2}$] | $t$ | $d\Psi_{rec}/dt$ | $w_I$ | $\alpha$ [°] | $\delta_{res}$ | $\delta_j/L_j$ | $\delta_u/L_u$ | $u_{out}$ | $B_{ref}$ | $M_A$ | $\frac{\dot{\psi}_{rec}}{B_{ref}^2 \delta_u / L_u M_A}$ |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 0 | 7.2 | 0.006 | 0.29 | Y-g. | 0.040 | 0.03 | 0.1 | 0.07 | - | - | - |
| 1.8 | 3.8 | 0.036 | 0.25 | 20 | 0.019 | 0.03 | 0.19 | 0.10 | 0.55 | 0.18 | 3.5 |
| 3.6 | 1.65 | 0.084 | 0.25 | 34 | 0.012 | 0.10 | 0.21 | 0.18 | 0.67 | 0.26 | 3.4 |
| 7.2 | 0.95 | 0.134 | 0.25 | 42 | 0.010 | 0.10 | 0.30 | 0.15 | 0.7 | 0.21 | 4.3 |

TABLE 3: Characteristic results of coalescent island simulations with $\eta = \mu = 10^{-4}$ and different values of drift parameters for $\psi_{rec} = 0.253$: time of attainment ($t$), instantaneous reconnection rate ($\dot{\psi}_{rec}$), ratio of outflow to inflow channel width $\left(\delta_u / L_u\right)$, (spatial) maximum of outflow velocity ), the reference magnetic field strength ($B_{ref}$), the fraction of Alfvén velocity ($M_A$) and the normalised reconnection rate $\dot{\psi}_{rec}$ /($\mathrm{B}_{\mathrm{ref}}^2$ $\delta_u/L_u$ $M_A$).

| $\rho_s$ [$10^{-2}$] | $t$ | $\dot{\psi}_{rec}$ | $\delta_u/L_u$ | $u_{out,\,max}$ | $B_{ref}$ | $M_A$ | $\dfrac{\dot{\psi}_{rec}}{\mathrm{B}_{\mathrm{ref}}^2 \delta_u / L_u M_A}$ |
|---|---|---|---|---|---|---|---|
| 0 | 2.0 | 0.05 | 0.06 | 1.0 | - | - | - |
| 0.33 | 1.55 | 0.08 | 0.075 | 1.0 | 1.1 | 0.9 | 0.98 |
| 1 | 1.2 | 0.2 | 0.36 | 0.77 | 0.77 | 1.0 | 0.94 |
| 2 | 0.92 | 0.25 | 0.43 | 0.63 | 0.86 | 0.73 | 1.08 |
| 3.3 | 0.85 | 0.25 | 0.53 | 0.52 | 0.84 | 0.62 | 1.08 |

**Figure Caption**

**Fig. 1**: Flux contours during the Harris sheet reconnection for the resistive MHD case ($\eta = \mu = 2 \cdot 10^{-4}$) at three characteristic times, corresponding to reconnected fluxes of (a) $\psi_{rec} = 0.038$ $(t = 3.8)$ , (b) $\psi_{rec} = 0.066$ $(t = 7.2)$ , (c) $\psi_{rec} = 0.102$ $(t = 17.5)$

**Fig. 2**: Maximum reconnection rate as function of resistivity, at constant magnetic Prandtl $\mu/\eta = 1$ for vanishing and finite ( $\rho_s = 0.036$ ) drift parameter.

**Fig. 3** (Color online): Reconnection rate $d\psi_{rec}/dt$ (equal to the z-component of the electric field at the X-point) as function of time (a) and of the reconnection rate (b) for the Harris sheet case, at different drift parameters $\rho_s$, for $\eta = \mu = 2 \cdot 10^{-4}$ .

**Fig. 4** (Color online): Flux surface for the Harris sheet case, for different, finite values of $\rho_s$ (a,d: 0.018, b,e: 0.036, c,f: 0.072) at equal amount of reconnected flux (left: $\psi_{rec}$ = 0.038, right: $\psi_{rec}$ = 0.066). The vertical bar gives the extension of $\rho_s$ .

**Fig. 5** (Color online): Iso-contours of the flow stream function (or electrostatic potential) $\phi$ (a) and of the stream function $\phi - \rho_s^2 \nabla^2 \phi$ for the flux advection velocity $\vec{v}$ (b) for the Harris sheet case, for $\rho_s = 0.072$ at $\psi_{rec}$ = 0.066.

**Fig. 6** (Color online): Sum of magnetic and kinetic energy for the coalescent island case, for the resistive MHD case and magnetic Prandtl number $\mu/\eta = 1$, and different dissipation rates.

**Fig. 7**: Magnetic flux at different times during the coalescent island simulations, for $\eta = \mu = 10^{-4}$ , $\rho_s = 0.032$.

**Fig. 8** (Color online): Reconnection rate $d\psi_{rec}/dt$ (equal to the z-component of the electric field at the X-point) as function of time (a) and of reconnected flux (b) for the coalescent island case, at different drift parameters $\rho_s$ , for $\eta = \mu = 10^{-4}$ .

**Fig. 9** (Color): Magnetic flux (iso-contours) and magnitude of plasma flow velocity $|\vec{u}|$ (colour coded) for the coalescent island case at fixed reconnected flux $\psi_{rec} = 0.253$, at different drift parameters $\rho_s$ , for $\eta = \mu = 10^{-4}$ .

**Fig. 10** (Color online): Magnetic field strength along the y-axis $B_x$(x=0,y) for the Harris sheet (a) and the coalescent island case (b) for different values of the drift parameter $\rho_s$ .