Probability distributions of turbulent energy

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Probability density functions (PDFs) of scale-dependent energy fluctuations, $P[\delta E(\ell)]$, are studied in high-resolution direct numerical simulations of Navier-Stokes and incompressible magnetohydrodynamic (MHD) turbulence. MHD flows with and without a strong mean magnetic field are considered. For all three systems it is found that the PDFs of inertial range energy fluctuations exhibit self-similarity and monoscaling in agreement with recent solar-wind measurements [B. Hnat et al., Geophys. Res. Lett. 29(10), 86-1 (2002)]. Furthermore, the energy PDFs exhibit similarity over all scales of the turbulent system showing no substantial qualitative change of shape as the scale of the fluctuations varies. This is in contrast to the well-known behavior of PDFs of turbulent velocity fluctuations. In all three cases under consideration the $P[\delta E(\ell)]$ resemble Lévy-type gamma distributions $\sim \Delta^{-1}\exp(-|\delta E|/\Delta)|\delta E|^{-\gamma}$ The observed gamma distributions exhibit a scale-dependent width $\Delta(\ell)$ and a system-dependent γ. The monoscaling property reflects the inertial-range scaling of the Elsässer-field fluctuations due to lacking Galilei invariance of δE. The appearance of Lévy distributions is made plausible by a simple model of energy transfer.

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Turbulence in electrically conducting magnetofluids is, apart from its importance for laboratory plasmas (see, for example [1]), a key ingredient in the dynamics of, e.g., the earth's liquid core and the solar wind (see e.g. [2]). A simple description of such plasmas is the framework of incompressible magnetohydrodynamics (MHD), a fluid approximation neglecting kinetic processes occuring on microscopic scales. This approach is appropriate if the main interest is focused on the nonlinear dynamics and the inherent statistical properties of fluid turbulence. To this end, two-point increments of a turbulent field component, say f, in the direction of a fixed unit vector $\hat{\mathbf{e}}$, $\delta f(\ell) = f(\boldsymbol{r} + \hat{\mathbf{e}}\ell) - f(\boldsymbol{r})$ are analyzed, since they yield a comprehensive and scale-dependent characterization of the statistical properties of turbulent fluctuations via the associated probability density function (PDF) [3].

PDFs of temporal fluctuations [16] in the solar wind, e.g. of total (magnetic + kinetic) energy density, as measured by the WIND spacecraft are self-similar over all observed scales, exhibit monoscaling, and closely resemble gamma distributions. In contrast the PDFs of velocity and magnetic field are found to display well-known multifractal characteristics, i.e. the associated PDFs change from Gaussian at large scales to leptocurtic (fat-tailed) at small scales [4–6]. The solar wind plasma is a complex and inhomogeneous mixture of mutually interacting regions with different physical characteristics and dynamically important kinetic processes [7, 8]. Thus, it is not clear if the abovementioned solar-wind observations are caused by turbulence or some other physical phenomenon. This paper reports an investigation of turbulent PDFs based on high-resolution direct numerical simulations of physically 'simpler' homogeneous incompressible MHD and Navier-Stokes turbulence to elucidate this question. Monoscaling of the two-point PDFs of energy is found in the inertial range of macroscopically isotropic MHD turbulence, anisotropic MHD turbulence with an imposed mean magnetic field as well as in turbulent Navier-Stokes flow. The respective PDFs resemble leptocurtic gamma laws on all spatial scales in agreement with the solar-wind measurements. The monoscaling property is shown to be a consequence of lacking Galilei invariance of the energy fluctuations in combination with turbulent inertial-range scaling. The appearance of Lévy-type gamma distributions apparently results from nonlinear turbulent tranfer as suggested by similar findings in all three investigated systems and a simple reaction-rate model.

The dimensionless equations of incompressible MHD, formulated in Elsässer variables $\boldsymbol{z}^{\pm} = \boldsymbol{v} \pm \boldsymbol{b}$ with the fluid velocity $\boldsymbol{v}$ and the magnetic field $\boldsymbol{b}$ which is given in Alfvén-speed units [9], read

$$\nabla \cdot \boldsymbol{z}^{\pm} = 0 \tag{1}$$

$$\partial_t \boldsymbol{z}^{\pm} = -\boldsymbol{z}^{\mp} \cdot \nabla \boldsymbol{z}^{\pm} - \nabla P + \eta_+ \Delta \boldsymbol{z}^{\pm} + \eta_- \Delta \boldsymbol{z}^{\mp} \tag{2}$$

with the total pressure $P = p + \frac{1}{2}b^2$. The dimensionless kinematic viscosity μ and magnetic diffusivity η appear in $\eta_\pm = 1/2(\mu \pm \eta)$.

The data used in this work stems from pseudospectral high-resolution direct numerical simulations [10] based on a set of equations equivalent to Eqs. (1) and (2). It describes homogeneous fully-developed turbulent MHD and Navier-Stokes ($\boldsymbol{b} \equiv 0$) flows in a cubic box of linear size 2π with periodic boundary conditions. The initial conditions for the decaying simulation run consist of random fluctuations with total energy equal to unity. In the MHD cases total kinetic and magnetic energy are approximately equal. The initial spectral energy distri-

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bution is peaked at small wavenumbers around $k = 4$ and decreases like a Gaussian towards small scales. In the MHD setups magnetic and cross helicity are small implying $\boldsymbol{z}^+ \simeq \boldsymbol{z}^-$. The driven turbulence simulations were run towards quasi-stationary states whose energetic and helicity characteristics as mentioned above are roughly equal to the decaying run. The MHD magnetic Prandtl number $\mathsf{Pr_m} = \mu/\eta$ is unity. The Reynolds numbers of all configurations are of order 10^3.

Three cases are considered. Setup (a) represents decaying macroscopically isotropic 3D MHD turbulence. The dataset contains 9 states of fully developed turbulence each comprising 1024^3 Fourier modes. The samples are taken equidistantly in time over a period of about 3 large eddy turnover times. The angle-integrated energy spectrum of this system exhibits a Kolmogorov-like scaling law [11] in the inertial range, i.e. $E_k \sim k^{-5/3}$. The second dataset (b) contains simulation data of a driven quasi-stationary macroscopically anisotropic MHD flow with a strong constant mean magnetic field. The driving is accomplished by freezing the largest Fourier modes of the system ($k \leq 2$). The data comprises 1024^2 Fourier modes perpendicular to the direction of the mean field and 256 modes parallel to it. This dataset covers about 2 large eddy turnover times of quasi-stationary turbulence with 8 samples taken equidistantly over that period. The perpendicular energy spectrum shows Iroshnikov-Kraichnan-like behavior $E_{k_\perp} \sim k_\perp^{-3/2}$ [12, 13]. Note that this is neither claim nor clear evidence for the validity of the Iroshnikov-Kraichnan picture in this configuration. For further details of the simulations and additional references see [10]. The third simulation (c) represents a turbulent statistically isotropic Navier-Stokes flow with resolution 1024^3 which is kept stationary by the same driving method as in case b) and exhibits Kolmogorov-scaling $E_k \sim k^{-5/3}$ of the turbulent energy spectrum.

For all turbulent systems the statistical properties of $\delta f(\ell)$ which is computed over varying scale ℓ are investigated. In the present work f stands for the component of $\boldsymbol{z}^+$ in the increment direction $\hat{\mathbf{e}}$ or the fluctuation energy defined here as $E \equiv (z^+)^2$. For the macroscopically isotropic setups (a) and (c) $\hat{\mathbf{e}} = \hat{\mathbf{e}}_z$. In case (b) the unit vector points in a fixed arbitrary direction perpendicular to the mean magnetic field. Under the assumption of statistical isotropy the statistical properties of $\delta f(\ell)$ depend solely on ℓ. This is the case for setup (a), (c), and for system (b) in planes perpendicular to the mean magnetic field. The assumption also holds approximately for δE if contributions by eddies on larger-scales which are convolved into this non-Galileian-invariant quantity can be regarded as quasi-constant on scale ℓ (see below). The quantity $\langle \delta f(\ell) \rangle$ scales self-similarly with the scaling parameter α ($\alpha \geq 0$), if $\langle \delta f(\lambda \ell) \rangle = \lambda^\alpha \langle f(\ell) \rangle$ for every λ. For the associated cumulative probability distribution follows $\wp(\delta f(\ell) \leq \rho) = \wp(\lambda^{-\alpha} \delta f(\lambda \ell) \leq \rho)$ for any real ρ. This implies for the probability density P

$$P[\delta f(\ell)] = \lambda^{-\alpha} P_s[\lambda^{-\alpha} \delta f_s] \qquad (3)$$

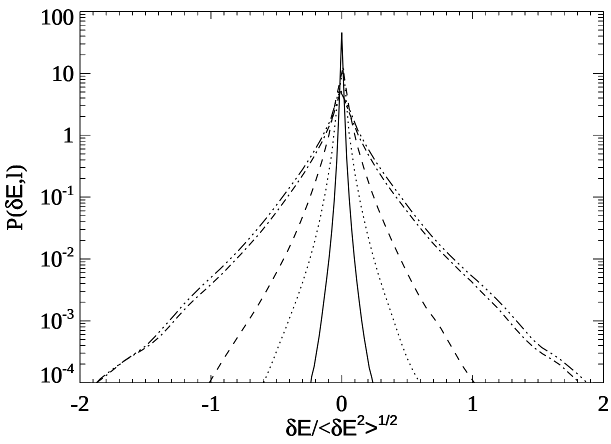

FIG. 1: The PDFs of total energy fluctuations δE on five different scales $\ell = \pi/n$ with $n = 511$ (solid), $n = 130$ (dotted), $n = 46$ (dashed), $n = 4$ (dot-dashed), $n = 1$ (3-dot-dashed).

introducing the master PDF P_s with $\delta f_s = \delta f(\lambda \ell)$. According to Eq. (3), there is a family of PDFs that can be collapsed to a single curve P_s, if α is independent of ℓ. This is known as monoscaling in contrast to multifractal scaling observed, e.g., for two-point increments of a turbulent velocity field.

To test if the abovementioned observations in the solar wind are a phenomenon related to inherent properties of turbulence time- and space-averaged increment series $\delta z^+(\ell)$ and $\delta E(\ell)$ for different ℓ, ranging between $\pi/512$ up to π are computed. In system (a) the increments are normalized using $(E^{\mathrm{T}})^{1/2}$ with $E^{\mathrm{T}} = 1/4 \int_V \mathrm{d}V[(z^+)^2 + (z^-)^2]$ to compensate for the decaying amplitude of the turbulent fluctuations. The PDFs are generated as normalized histograms of the respective increments taken over all positions in the 2π-periodic box which contains the real space fields, $\boldsymbol{v}(\boldsymbol{r})$ and $\boldsymbol{b}(\boldsymbol{r})$, computed from the available Fourier-coefficients. Fig. 1 shows $P[\delta E(\ell)]$ for various ℓ in the isotropic case (a). The non-Gaussian nature of the PDFs over all scales is evident. Similar behavior is found in the anisotropic case (b) where the increments are taken perpendicularly to the direction of the mean field as well as in the Navier-Stokes simulation (c). The PDFs are highly symmetric and become increasingly broader with growing ℓ reflecting the increase of turbulent energy towards largest scales. Interestingly, the PDFs at *all* scales have the same leptokurtic shape resembling Lévy laws. In particular, away from the center, $\delta E = 0$, the PDFs are close to gamma distributions $\sim \exp(-|\delta E|/\Delta)|\delta E|^{-\gamma}$ of different widths Δ. The exponent γ of the best fits is constant in the inertial range and amounts approximately to 3.4 (a), 4.2 (b), and 3.1 (c). In the solar wind a similar finding however with $\gamma \approx 2.5$ was reported [4].

The similarity of the $P[\delta E(\ell)]$ on different scales ℓ suggests the possibility of monoscaling. The monoscaling

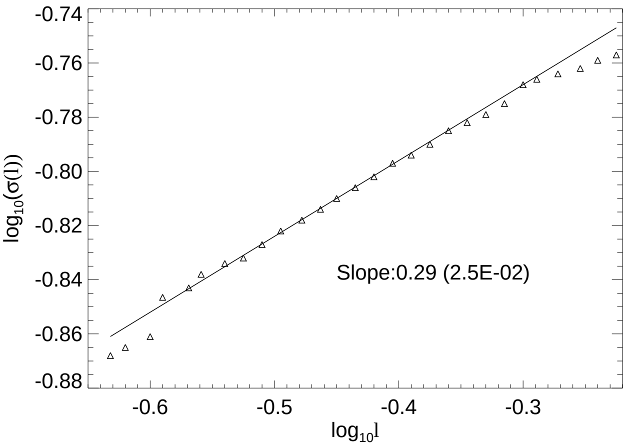

FIG. 2: Standard deviation of total energy increments within the inertial range in case (a) (triangles) with linear least-squares fit (solid line).

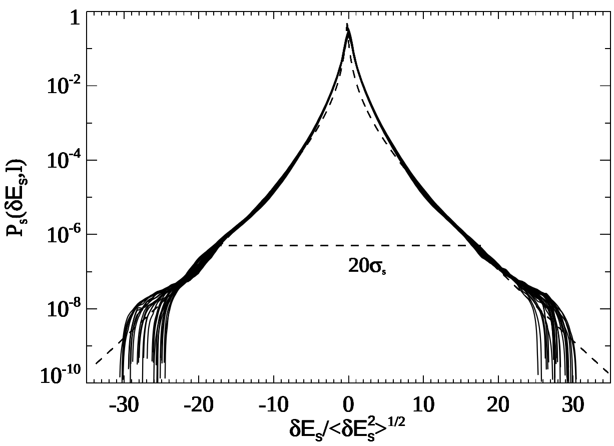

FIG. 3: Rescaled PDFs of total energy fluctuations in the inertial range of the isotropic case (a). The gamma law $10^{-3}\exp(-|\delta E|/0.35)|\delta E|^{-3.1}$ is represented by the dashed curve.

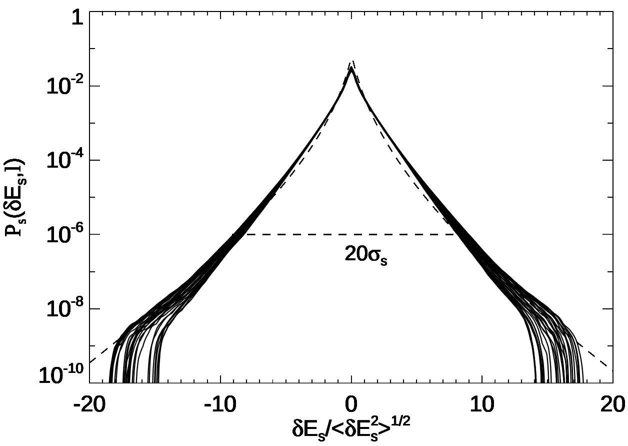

FIG. 4: Rescaled PDFs of total energy fluctuations in the inertial range of the Navier-Stokes case (c). The gamma law $10^{-3}\exp(-|\delta E|/0.4)|\delta E|^{-3.4}$ is represented by the dashed curve.

exponent is expected to be scale-independent in the inertial range only since the energy increments are not Galilei invariant. Therefore, small-scale δE also comprise contributions by larger eddies which advect the small-scale fluctuations. A linearization of δE with respect to the largest-scale contribution $(z_0^+)^2 \gg (\delta z^+)^2$ yields to lowest order $\delta E \approx (z_0^+ + \delta z^+)^2 \sim z_0^+ \delta z^+$. As a consequence, the energy increments reflect the inertial-range scaling of the turbulent Elsässer fields, i.e. $\delta E \sim \delta z^+ \sim \ell^\alpha$. To apply the rescaling procedure given by Eq. (3) (cf. also [4]) the exponent α is extracted from the PDFs by two independent techniques.

Firstly, the standard deviation is considered which is defined as $\sigma(\ell) = [\langle \delta E(\ell)^2 \rangle]^{1/2}$. In the inertial range σ exhibits power-law behavior with respect to the increment distance, $\sigma(\ell) \sim \ell^\alpha$, Fig. 2 shows the standard deviation of total energy fluctuations in the inertial range for the isotropic case (a) in double logarithmic presentation. A linear least-squares fit is carried out to obtain α. The characteristic exponents deduced in this way are $\alpha = 0.29 \pm 0.025$ for the isotropic case (a), $\alpha = 0.23 \pm 0.025$ for the anisotropic case (b), and $\alpha = 0.28 \pm 0.03$ for the Navier-Stokes flow (c). As expected these values are close to the non-intermittent scaling exponents observed for the turbulent field fluctuations, i.e. $\alpha_{\mathrm{K41}} = 1/3$ for cases (a) and (c) while $\alpha_{\mathrm{IK}} = 1/4$ for case (b).

Secondly, in the inertial range the characteristic exponents can be obtained via the amplitude of $P(0, \ell) \sim \ell^{-\alpha}$ profiting from the fact that the peaks of the PDFs are statistically the least noisy part of the distributions. The scaling exponent obtained by using this method is in good agreement with the value of α obtained via the PDF variance. Fig. 3 shows the rescaled PDFs according to Eq. (3) for MHD case (a) (similar for (b), not shown) while Fig. 4 displays the rescaled PDFs obtained from the Navier-Stokes simulation (c). The corresponding increment distances ℓ are all lying in the respective inertial range. Evidently the PDFs are self-similar and collapse for up to 20σ with weak scattering on the master PDF, P_s, when using the characteristic exponents given above. The dashed lines in both figure display the best fitting gamma laws.

The PDFs of the Elsässer field fluctuations, $\mathrm{P}[\delta z^+(\ell)]$, in system (a) (systems (b) and (c) likewise) display a different and well-known behavior as can be seen from Fig. 5. The distributions lose their small-scale leptocurtic character as ℓ increases. Due to the lacking correlation of distant turbulent fluctuations the associated distributions become approximately Gaussian at large scales. Because of the resulting multifractal scaling of

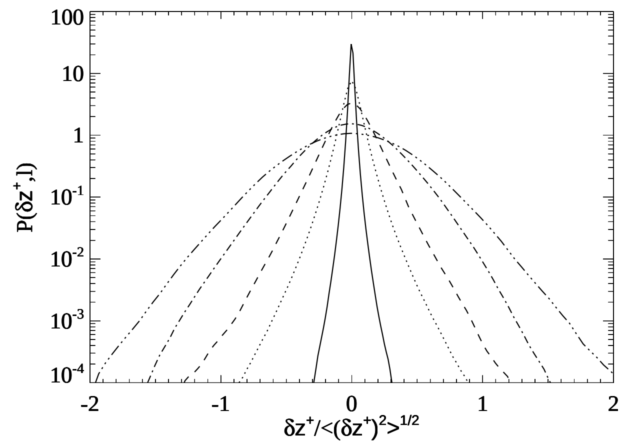

FIG. 5: The PDFs of Elsässer field fluctuations δz^+ for the same five different scales as in Fig. 1.

the PDFs which is a signature of the intermittent small-scale structure of turbulence it is obvious that they can not be collapsed onto a single curve even in the inertial range. However, one can infer the nonintermittent characteristic scaling exponent by regarding the function $P(0,\ell)$ (not shown). For example, in system (a) this function exhibits clear inertial-range scaling $\sim \ell^{-a}$ with $a = 0.33 \pm 1.5 \times 10^{-2}$ in very good agreement with α_{K41}.

The occurrence of gamma PDFs is made plausible by a simple reaction-rate ansatz [14, 15]: Consider the 'intensity' $n(e)$ of turbulent fluctuations with energy $e = |\delta E|$ such that $n(e)$ is the fraction the total turbulent energy associated with these fluctuations and the larger eddies in which they are embedded. The evolution of this function is assumed to obey the following linear rate equation:

$$\partial_t n(e) = -n(e)/\tau_-(e) + \int_e^\infty \mathrm{d}e' n(e')/\tau_+(e', e) \qquad (4)$$

where $\tau_+(e', e)$ is the time characteristic of the creation of fluctuations with energy e as a result of turbulent transfer from fluctuations with energy e' while $\tau_-(e)$ is the respective characteristic decay time. Normalization of $n(e)$ by $\int_0^\infty \mathrm{d}e' n(e')$ yields the corresponding PDF $P(e)$. In a statistically stationary state Eq. (4) then gives

$$P(e) = C_1 \int_e^\infty \mathrm{d}e' P(e') \frac{\tau_-(e)}{\tau_+(e', e)} \qquad (5)$$

where C_1 is a normalization constant. For $\tau_-(e)/\tau_+(e', e) \sim (e'/e)^\gamma$ this integral equation has the solution $P(e) = C_2 e^{-\gamma} \exp(-e/\Delta)$. Thus, the model (4) which mimics in combination with the abovementioned assumptions a direct spectral transfer process yields the observed gamma distributions. Note that the lower bound of the integral in Eq. (5) implies that energy flows from higher to lower levels where for technical simplicity very large differences between e and e' are allowed. A finite upper bound of the integral in Eq. (5) does, however, not change the result fundamentally. This suggests that the observed gamma distributions are an indication of turbulent spectral transfer.

In summary it has been shown by high-resolution direct numerical simulations of incompressible turbulent magnetohydrodynamic and Navier-Stokes flows that the monoscaling of energy fluctutation PDFs observed in the solar wind is the consequence of lacking Galilei invariance of energy increments in combination with self-similar scaling of the underlying turbulent fields. The closeness of the PDFs to Lévy-type gamma distributions is made plausible by a simple model mimicking nonlinear spectral transfer.

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