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On a moving boundary model of anomalous heat transport in a tokamak plasma

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Abstract: One of the main problems in fusion research is to understand the dynamics of heat transport in a tokamak plasma. In certain scenarios the heat flux suddenly is much larger than predicted by classical theory, 'anomalously' large. In this paper we investigate a mathematical model for the onset of 'anomalous transport' as suggested by measurements in tokamaks.

We consider a quasilinear heat equation with a heat conduction coefficient that depends piecewise linearly on the gradient of the temperature. The local non-differentiability of the coefficient gives rise to a moving front. Assuming a solution given, we investigate its smoothness and the properties of the front. Also, an ODE for the velocity of the front is derived, which leads to a front tracking technique. Then we prove existence of a unique solution, under assumptions suggested by the investigation of the front. We also give two families of parameter dependent exact solutions of the equation.

Keywords: quasilinear heat equation, moving free boundary, front tracking, plasma physics

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1 Introduction

One of the main problems in fusion research is to understand the dynamics governing the heat transport in tokamak plasmas. A tokamak is a torus-shaped device for confining plasmas by magnetic fields [16]. In cylinder coordinates (r, ϕ, z), the dominant magnetic field is the axisymmetric *toroidal* one, i.e. the one in ϕ-direction. It is produced by external coils. This field alone, however, cannot confine a plasma. An additional magnetic field in (r, z) is necessary for equilibrium. This additional magnetic field is mostly produced by a large toroidal current in the plasma, i.e. by a flow of electrons and ions in ϕ-direction. The combination of these fields results in helical magnetic field lines around the torus. Most of them are everywhere dense on torus-shaped nested surfaces, the so called *magnetic surfaces.*

Charged particles in magnetic fields cannot move freely, they have to gyrate along field lines. Since the field lines in tokamaks have complicated helical structures themselves, particle trajectories can be quite complicated. In addition, particles collide with each other, and the collisions cause displacements and change the particle trajectories. These displacements are random. Thus particles also diffuse across magnetic surfaces. Since the particles take their energy with them, this causes a diffusive transfer of heat across magnetic surfaces. This is essentially a one-dimensional process.

In certain scenarios, heat fluxes measured in tokamak experiments lead to transport coefficients which are much larger than the ones expected from classical theory, especially for electrons. Certain parameter scenarios lead to 'stiff' temperature profiles [11]: If the electron temperature gradient exceeds a critical threshold value, the heat transport increases in such a dramatic way that the then onsetting transport is called 'anomalous' by plasma physicists. A simple mathematical model for the onset of this anomalous transport suggested by the measurements [11] is the following:

Mathematical model: Problem (P)

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(\chi\left(\frac{\partial u}{\partial x}\right)\frac{\partial u}{\partial x}\right) + S(x), \tag{1}$$
$$\text{for} \quad (t,x) \in \Omega_T = (0,T)\times(0,1) \subset I\!R^2, \text{ with}$$

$$\chi(u_x) = D_0 + D_1 H(|u_x| - \eta)\ (|u_x| - \eta) \tag{2}$$

$$u(0,x) = u_0(x), \quad x \in [0,1], \tag{3}$$
$$u_x(t,0) = 0, \qquad u(t,1) = 0, \qquad t \in [0,T], \tag{4}$$

where

u represents typically the temperature,

x replaced the radius r: we slightly simplified the elliptic operator to eliminate unimportant complications;

The Heaviside function H is defined as usually,

$$H(|u_x|-\eta) := \begin{cases} 0, & |u_x| \le \eta \\ 1, & |u_x| > \eta. \end{cases}$$

$\eta > 0$ is a parameter, the threshold value for u_x, i.e. for the gradient of u. η is assumed to be constant. By $\eta \neq 0$ we exclude the degenerate case that the threshold value is reached at the left boundary for all times.

$D_0 > 0,\ D_1 > 0$ are constants,

$S(x) \ge 0$ is a source function. Especially meaningful for the anomalous heat transport problem is

$$S(x) = S_0 e^{-\frac{(x-x_0)^2}{\delta^2}}, \qquad x_0 \in [0,1], \quad \delta > 0, \quad S_0 = Const \ge 0; \tag{5}$$

Unless otherwise stated, the functions u_0 and S are assumed to be such that the solutions of Problem (P) are as smooth as possible (for instance they will have to satisfy the compatibility conditions (19) and (20)).

The heat flux is defined as

$$q(t,x) := \chi(u_x)\ u_x = (D_0 + D_1 H(|u_x|-\eta)(|u_x|-\eta))u_x, \tag{6}$$

and thus satisfies

$$q(t,x) = \begin{cases} D_0 u_x, & |u_x| \le \eta \\ D_0 u_x + D_1(|u_x|-\eta)u_x, & |u_x| > \eta. \end{cases}$$

It is easy to see that χ and q are piecewise linear and lipschitz-continuous as functions of u_x, and that $\partial q/\partial x$ depends continuously on x. As will be shown below, u_{xx} typically does jump at those (t,x) where $|u_x(t,x)| = \eta$. Thus Problem (P) should not be expected to have classical solutions. In this paper we will focus on the non-smoothness introduced by the corner in χ, assuming all other quantities to have 'adequate' smoothness, i.e. to be as smooth as possible. As will be shown below, it is adequate to treat Problem (P) as a moving-free-boundary problem.

As far as we could see, problems of type Problem (P) are not treated in the mathematical literature - though there is a rich literature on free and moving boundary problems, see [2, 3, 8, 10] and others.

In the classical book by Ladyzhenskaia et al [6], for instance, the following nonlinear version of the Stefan problem is considered:
determine the temperature $u : \Omega_T \to I\!R^+, \quad \Omega_T = (0,T) \times D, \quad D \subset I\!R^n,$ such that

$$\alpha(u)\ u_t = \nabla(\kappa(u)\nabla u)$$

in those $(t,x) \in \Omega_T$ where $u(t,x) \neq u_k,\ \ k = 1,\cdots,m,\ \ (u_0 := 0 < u_1 < \cdots < u_m)$. Here $\alpha, \kappa : I\!R^+ \to I\!R^+$ are smooth on each interval $[u_{k-1}, u_k]$ and may have a jump discontinuity at $u_k,\ k = 1,\cdots,m$. On the interfaces $S^{(k)} := \{(t,x) \in \Omega_T :\ u(t,x) = u_k\}$ the following two conditions hold

i. $[u\,]|_{S^{(k)}} := \lim\limits_{(t,x)\in S^{(k)}+} u \; - \lim\limits_{(t,x)\in S^{(k)}-} u = 0,$

ii. $b_k \cdot n + [\kappa(u)\nabla u] \cdot n|_{S^{(k)}} = 0$, where $b_k \in I\!R^n$ is a positive vector and n is the normal to $S^{(k)}$ along ∇u;

and u satisfies some initial and boundary conditions

$$u(0,x) = \psi_0(x), \quad x \in \bar{D}, \qquad u\,(t,x)|_{x\in\partial D} = 0, \quad t \in [0,T]. \tag{7}$$

Typical for the multi-phase Stefan problem as well as for some other free boundary problems [3, 8, 10] are two conditions on the free boundary (on the interface) which are sufficient to determine the motion of the boundary. The first condition usually imposes constraints on the function values of the solution (like condition i.) while the second one (condition ii., 'Stefan condition', 'energy balance') usually defines the motion of the free boundary. In the anomalous heat transport problem, however, we only know the threshold value of the temperature gradient. There is no a priori condition on an inner boundary or on its motion.

In the dissertation [1], Problem (P) was treated under several different viewpoints: mathematical properties of the equations were investigated; exact solutions were calculated; a front tracking algorithm for Problem (P) was developed, numerically analysed, thoroughly tested on examples and compared to other numerical methods; and the resulting numerical code was used to perform numerical simulations.
The developed basic numerical method AIM employs a method of lines [12, 15]: discretization by finite elements transforms quasilinear parabolic equations/systems to a system of ODEs; this system then is solved with a special adaptive time stepping. This method proved to work well for classical quasi-linear parabolic equations. The newly developed error estimates and the new strategy for the adaptive time stepping proved to be very adequate: on the chosen test cases it is as good or even better than the time-stepping based on the Kraaijevanger estimate [4], see [1, sections 4.1, 4.2].
When a free boundary (a non-degenerate front point x_F with $|u_x(t,x_F)| = \eta$, see below) is detected, a newly developed explicit *front tracking technique (FTT)* is empoyed: the FEM-discretization is refined in a small neighborhood of $x_F(t)$ and Problem (P) is split at $x_F(t)$ into two subproblems (P1) and (P2). On each side of the interface the AIM approach is applied. In addition, an ODE is solved to track and update the position of the front. This whole numerical method as developed and described in [1] proved to be especially efficient on typical anomalous transport problems [1, section 4.3].

In this paper we give an enlarged version of the mathematical analysis of Problem (P) and of the theoretical foundations of the front tracking technique developed for anomalous transport. In section 2 we give the basic definitions: required smoothness of a weak solution of Problem (P) , non-degenerate and degenerate front points. In Theorem 3.1 we assume that a solution is given, with a non-degenerate front point $x_{F,0}$ in the initial function. We derive an ODE for $x_F(t)$, $\;x_F(0) = x_{F,0}$ and show the existence of a C^1-function $x_F(t)$ in some non-empty time interval. In Theorem 4.1 we prove existence of a unique solution under assumptions suggested by Theorem 3.1. Finally, in section 5, we give two parameter-dependent families of exact solutions of eq. (33). Note that these exact solutions satisfy *only some, not all* theoretical results of the foregoing sections because they satisfy eqs. (1), (2), but not the initial and boundary conditions (4).

2 Definitions

According to Ladyzhenskaia et al [6, Th.6.7, Ch.V], eqs. (1), (2) with initial-boundary conditions (7) have at least one weak solution $u(t,x) \in \overset{\circ}{V}{}^{0,1}_{3,2}([0,T]\times D)$, the Banach space obtained by completing the linear space of smooth functions

$$u : [0,T]\times D \to \mathbb{R}, \qquad u(t,x)|_{\partial D} = 0 \ \text{ for } t\in[0,T], \qquad ||u||_{\overset{\circ}{V}{}^{0,1}_{3,2}} < \infty,$$

under the norm [6, p. 465, p. 2ff]

$$||u||_{\overset{\circ}{V}{}^{0,1}_{3,2}} = \max_{0\le t\le T}\left(\int_D |u(t,x)|^2dx\right)^{1/2} + \left(\int_0^T\left(\int_D |u_x(t,x)|^3dx\right)^{3/3} dt\right)^{1/3}.$$

Because of the nonlinear heat conductivity coefficient and the discontinuity of its first derivative w.r.t. u_x at $|u_x| = \eta$. This result may be generalized to the mixed boundary conditions of Problem (P) .

Having in mind that the heat conductivity coefficient for Problem (P) is a well defined smooth function away from $|u_x| = \eta$, we will require more smoothness for the solutions of Problem (P) in the following.

Definition 2.1. *We say that a function* $u : B \to \mathbb{R}$, $\bar{\Omega}_T \subset B \subset \mathbb{R}^2$, *is a* **solution of Problem (P)** *if:* $u \in C^{1+\alpha/2,1+\alpha}(\bar{\Omega}_T)$ *for some* $\alpha \in (0,1)$, *u satisfies Problem (P) a.e., and* u_{xx} *is defined and piece-wise continuous in* $\bar{\Omega}_T$.

Remark 2.1. *If u is a solution of Problem (P) then* $u_x \in C^{1+\alpha/2,\alpha}(\bar{\Omega}_T)$, *but in addition* u_{xx} *is piece-wise continuous in* $\bar{\Omega}_T$. *Therefore* u_x *is even Lipschitz continuous in* $\bar{\Omega}_T$ *with respect to x.*

Definition 2.2. *Let* $u = u(t,x)$ *be a solution of Problem (P) ,* $\eta > 0$ *given.*
$x_F \in (0,1)$ *is called* **(non-degenerate) front point at** *t if* $|u_x(t,x_F)| = \eta$ *and if both* $\lim_{x\to x_F^-} u_{xx}(t,x) \neq 0$ *and* $\lim_{x\to x_F^+} u_{xx}(t,x) \neq 0$. *A point* x_F *is called* **degenerate front point** *if* $|u_x(t,x_F)| = \eta$ *and* $\lim_{x\to x_F^-} u_{xx}(t,x) = 0$ *and/or* $\lim_{x\to x_F^+} u_{xx}(t,x) = 0$.

Remark 2.2. *There are two possible cases for front points* $x_F \in (0,1)$*:*
$u_x(t,x_F) = \eta$ *or* $u_x(t,x_F) = -\eta$.

Remark 2.3 (Non-degenerate front points). *At non-degenerate front points* x_F, $|u_x|$ *crosses the line* η *monotonically and* $\lim\limits_{x \to x_F^-} u_{xx}(t,x) \neq \lim\limits_{x \to x_F^+} u_{xx}(t,x)$. *The size of the jump of* u_{xx} *will be given in eqs. (13). Do we allow sign-changing jumps, as occurring for instance for* $v(x,t) := \eta(x - x_F) + (x - x_F)\ |x - x_F|$ *for* $|\eta| < 2$*? As turns out in the proof of Theorem 3.1, sign-changing jumps of* u_{xx} *cannot occur at non-degenerate front points of exact or accurately computed solutions, see eqs. (13). Thus there is no need to take care of sign-changing jumps in Definition 2.2.*

Remark 2.4 (Degenerate front points). *At a degenerate front point in anomalous transport problems* $|u_x|$ *might cross the line* η *at a saddle point or it might touch the line* η *in a local minimum or maximum.*

- *The case that* $|u_x|$ *crosses the line* η *at a saddle point was never observed in our anomalous transport studies. It thus has not been investigated and is not considered here.*
- *The case of touching of the line* η *at* $\bar{x}$ *at a local maximum or minimum, without crossing, is possible and does occur in anomalous transport problems [1, item 'Multiple front points', p. 99ff]. It is important only if anomalous transport sets in or ceases to happen at* $\bar{x}$*. In the first case it gives rise to two additional front points for larger* t*, in the other case a pair of front points disappears. Both cases are shown to happen in the example leading to [1, Fig.4.12, p. 100]. The two points of type* $\bar{x}$ *themselves do not require any special action since there is no anomalous transport at such points. In the numerical simulations, points and short intervals where* $|u_x| = \eta$ *but* $|u_x|$ *does not cross the line* η *(i.e. points in a small neighborhood of an extremum) are treated as non-front points. Numerical treatment of two non-degenerate front points which are about 4 grid points apart is discussed in [1, p. 101].*
- *What about* $|u_x| = \eta$ *in a closed subinterval* $[\bar{x}^I, \bar{x}^{II}] \subset (0,1)$ *with or without crossing of the line* η *before and afterwards? In this case* $u_{xx}(t,x) \equiv 0$ *in* $[\bar{x}^I, \bar{x}^{II}]$ *and Problem (P) reduces locally to the ordinary initial value problem*

  $$\frac{du}{dt} = S(x) \quad \text{in } [\bar{x}^I, \bar{x}^{II}], \quad u(0,x) = v_0(x), \tag{8}$$

  depending on a parameter x*. It can be integrated analytically as long as an* x*-interval with* $u_{xx}(t,x) \equiv 0$ *exists. Special sources* $S(x)$ *and initial conditions* $v_0(x)$ *will allow such solutions. The sources relevant to the anomalous transport problem, however, will not allow such* x*-intervals to persist. Though the case of* Turing bifurcations *[7] is mathematically interesting, we will not enter this field here since it is irrelevant to anomalous transport.*

In the proof of Theorem 2.1 we will apply the **Generalized Implicit Function Theorem of Clarke**. We thus introduce it here, together with related definitions.

Let $f : I\!R^m \to I\!R^n$ be locally Lipschitz continuous in a neighborhood of some $x \in I\!R^m$. Then f is almost everywhere differentiable near x (Theorem of Rademacher). Let $D_f \subset I\!R^m$ be the set where f is differentiable. Then its **generalized Jacobian in the sense of Clarke** at the point $x \in I\!R^m$ is given by

$$\partial f(x) := \mathrm{conv}\left\{ A \in I\!R^{n\times m} : \ A = \lim_{x^k \to x} Df(x^k), \ x^k \in D_f \right\} \tag{9}$$

where $Df(x^k)$ is a classical Jacobian at $x^k \in D_f$ and $\mathrm{conv}(B)$ denotes the convex hull of the set B. Note that the generalized Jacobian is a set.

Now let $H : I\!R^m \times I\!R^n \to I\!R^n$, $(y,x) \mapsto H(y,x)$, be locally Lipschitz and let $\pi_x \partial H(y,x)$ be 'the projection of $\partial H(y,x)$ onto the x-space', i.e.

$$\begin{aligned} \pi_x \partial H(y,x) \ &:= \ \big\{ M \in I\!R^{n\times n} : \ \text{matrix } [N,M] \in \partial H(y,x) \subset I\!R^{n\times(m+n)} \\ &\qquad \text{for some } N \in I\!R^{n\times m} \ \big\}. \end{aligned} \tag{10}$$

Let $\pi_y \partial H(y,x)$ be such that $[\pi_y \partial H(y,x), \pi_x \partial H(y,x)] = \partial H(y,x)$.

Then the implicit function theorem due to Clarke is:

Theorem 2.1. [14, Th.1.1]:
Suppose that $H : I\!R^m \times I\!R^n \to I\!R^n$ is a locally Lipschitz function in a neighbourhood of $(\bar{y}, \bar{x})$, and that $(\bar{y}, \bar{x})$ solves $H(\bar{y}, \bar{x}) = 0$. If $\pi_x \partial H(\bar{y}, \bar{x})$ is of maximal rank, then there exist an open neighbourhood Y of $\bar{y}$ and a function $G(.) : Y \to I\!R^n$ such that G is locally Lipschitz in Y , $G(\bar{y}) = \bar{x}$ and for every $y \in Y$, $H(y, G(y)) = 0$.

3 Properties of the front for a given solution

Theorem 3.1. *Let u be a solution of Problem (P) on $[0,T]\times[0,1]$ and let $\partial/\partial t \int\limits_0^1 u_t(t,\xi)d\xi$ be bounded. If there exists a non-degenerate front point $x_{F,0}$ at $t_0 = 0$ with $|u_x(0, x_{F,0})| = \eta$, then there are an interval $[0,T^*) \subset [0,T]$ and a C^1-function $x_F(t)$ on $[0,T^*)$ such that $x_F(t)$ is a non-degenerate front point for every $t \in [0,T^*)$, satisfying*

$$u_x(t, x_F(t)) = \eta \ sgn(u_x(t, x_{F,0})) \qquad and \qquad x_F(0) = x_{F,0}. \tag{11}$$

Moreover, define $s_t(t,x) := \partial/\partial t \left(\int\limits_0^x S(\xi) - u_t(t,\xi)d\xi \right)$. Then

$$\lim_{x \to x_F^-} u_{xt}(t,x) = -\frac{s_t(t,x_F)}{D_0}, \qquad \lim_{x \to x_F^+} u_{xt}(t,x) = -\frac{s_t(t,x_F)}{D_0 + D_1\eta} \tag{12}$$

and

$$\begin{aligned} \lim_{x \to x_F^-} u_{xx}(t,x) \ &= \ -\frac{u_t(t,x_F) - S(x_F)}{D_0}, \\ \lim_{x \to x_F^+} u_{xx}(t,x) \ &= \ -\frac{u_t(t,x_F) - S(x_F)}{D_0 + D_1\eta}. \end{aligned} \tag{13}$$

The velocity of the front point is given by

$$\dot{x}_F(t) = \frac{-s_t(t,x_F)}{u_t(t,x_F) - S(x_F)} = -\frac{u_{xt}(t,x_F)}{u_{xx}(t,x_F)}. \tag{14}$$

Proof: Assume that there exists a non-degenerate front point $x_{F,0}$ at $t_0 = 0$ with $|u_x(0,x_{F,0})| = \eta$. Without loss of generality we consider the case

$$u_x(0,x_{F,0}) = \eta. \tag{15}$$

From our assumptions follows $\lim_{x\to x_F^-} u_{xx}(t,x) \neq 0$ and $\lim_{x\to x_F^+} u_{xx}(t,x) \neq 0$ and $u_x \in C^{1+\alpha/2,\alpha}(\bar{\Omega}_T)$. It also follows that u_x is Lipschitz continuous w.r.t. x (Remark 2.1) and thus that u_x is differentiable a.e. w.r.t. x (Theorem of Rademacher [14]). We cannot apply the classical implicit function theorem [13, p. 658] which would require u_x to be continuously differentiable w.r.t. all variables, but we can apply its generalization to a.e. differentiable functions: Clarke's Theorem [14], Theorem 2.1. The generalized Jacobian (9), for our particular case, has the form

$$\partial f := \mathrm{conv}\{(u_{xt}(t,x_F^-), u_{xx}(t,x_F^-))^t, (u_{xt}(t,x_F^+), u_{xx}(t,x_F^+))^t\}.$$

Then $\pi_x\partial f$ consists of all $\beta \in I\!R$ such that for some $\gamma \in I\!R$ the vector $(\gamma,\beta)^t \in \partial f$. Since we assumed $\lim\limits_{x\to x_F^-} u_{xx}(t,x) \neq 0$ and $\lim\limits_{x\to x_F^+} u_{xx}(t,x) \neq 0$ the condition '$\pi_x\partial f$ has maximal rank' is satisfied and we can apply Theorem 2.1. Thus there exists a one-sided open neighbourhood $[0,T^*)$ of 0 and a function $x_F : [0,T^*) \to I\!R$ such that x_F is locally Lipschitz in $[0,T^*)$, $x_F(0) = x_{F,0}$ and $u_x(t,x_F(t)) = \eta$.

In order to avoid working with the implicit equation (11) for $x_F(t)$ we derive an equation for the velocity of the front point. To this end we compute the flux, defined by (6), at the front point

$$q(t,x_F) = D_0 u_x(t,x_F) = D_0\ \eta \cdot sgn(u_x(t,x_F))$$

and take the derivative with respect to the time. We get

$$\dot{x}_F(t) = -\frac{q_t(t,x_F)}{q_x(t,x_F)} = -\frac{u_{xt}(t,x_F)}{u_{xx}(t,x_F)}. \tag{16}$$

Integrating Problem (P) with respect to x in the interval $[0,x]$ we obtain

$$D_0 u_x + D_1 H(|u_x| - \eta)(|u_x| - \eta)u_x + s \ = \ 0,$$

where $s(t,x) := \int\limits_0^x S(\xi) - u_t(t,\xi)d\xi$. Differentiating with respect to the time t we get

$$(D_0 + D_1 H(|u_x| - \eta)(2|u_x| - \eta))u_{xt} + s_t = 0. \tag{17}$$

The values of u_{xt} at the front point are given by

$$\lim_{x\to x_F^-} u_{xt}(t,x) = -\frac{s_t(t,x_F)}{D_0}, \qquad \lim_{x\to x_F^+} u_{xt}(t,x) = -\frac{s_t(t,x_F)}{D_0 + D_1\eta}.$$

Taking into account the values of u_{xx} at x_F,

$$\lim_{x \to x_F^-} u_{xx}(t,x) = -\frac{u_t(t,x_F) - S(x_F)}{D_0}, \qquad \lim_{x \to x_F^+} u_{xx}(t,x) = -\frac{u_t(t,x_F) - S(x_F)}{D_0 + D_1\eta},$$

we finally get that

$$\dot{x}_F = \frac{-s_t(t,x_F)}{u_t(t,x_F) - S(x_F)},$$

which implies that $\dot{x}_F$ is continuous.

T^*, the duration of existence of the solution of (16), depends on the maximum of $\dot{x}_F$ (Peano's existence theorem [5, pp. 10]: $T^* = \min(t, \frac{1}{\max|\dot{x}_F|})$). ☐

4 Existence of a solution for given data

In Theorem 3.1 we investigated the properties of the front $x_F(t)$ for a given solution. Now we will investigate existence of a solution of Problem (P) for given data. We make the following assumptions:

Assumptions 4.1. :

- *the initial function u_0 belongs piecewise to $C^{2+\alpha}$, i.e. for $0 \le x \le x_{F,0}$ and for $x_{F,0} \le x \le 1$; u_0 has exactly one non-degenerate front point $x_{F,0} \in (0,1)$, i.e. it satisfies $|u_{0,x}(x_{F,0})| = \eta$ and the jump condition (13),*

$$D_0 \lim_{x \to x_{F,0}^-} u_{xx}(0,x) = (D_0 + D_1\ \eta) \lim_{x \to x_{F,0}^+} u_{xx}(0,x) \neq 0; \tag{18}$$

- *the source S belongs to $C^{2+\alpha}([0,1])$;*
- *at $x = 0$ and $x = 1$ u_0 and S satisfy the compatibility conditions of zeroth order*

$$u_0'(0) = 0, \qquad u_0(1) = 0, \tag{19}$$

and of first order

$$\frac{\partial}{\partial x}((D_0 + D_1 H(|u_0'| - \eta)(|u_0'| - \eta))u_0')\Big|_{x=1} + S(1) = 0 \tag{20}$$

$$\frac{\partial}{\partial x}\left(\frac{\partial}{\partial x}((D_0 + D_1 H(|u_0'| - \eta)(|u_0'| - \eta))u_0')\right)\Big|_{x=0} + S'(0) = 0;$$

- *the source function S satisfies*

$$\left|\frac{dS}{dx}\right| \le (\varepsilon + P(|u_x|))(1 + |u_x|)^4,$$

where $P(\rho) \ge 0$ is continuous, $P(\rho) \overset{\rho\to\infty}{\to} 0$, and $\varepsilon > 0$ is sufficiently small, $\varepsilon = \varepsilon(M, \nu, \mu, \mu_1, \max_{\rho \ge 0} P(\rho))$.

If these assumptions are not satisfied for $t = 0$ but for $t = t_0$ we simply transform $t' = t - t_0$.

We split Problem (P) into two subproblems (P1) and (P2) defined as

$$(P1): \begin{cases} \frac{\partial u}{\partial t} = \frac{\partial q}{\partial x} + S(x) = a(|u_x|)u_{xx} + S(x), \quad 0 < x < x_F(t),\ t > 0, \\ u(0,x) = u_0(x), \quad 0 \le x \le x_{F,0}, \\ u_x(t,0) = 0, \quad t \ge 0, \\ u_x(t, x_F(t)) = u_{0,x}(x_{F,0}),\ |u_{0,x}(x_{F,0})| = \eta, \quad t \ge 0, \end{cases}$$

and

$$(P2): \begin{cases} \frac{\partial u}{\partial t} = \frac{\partial q}{\partial x} + S(x) = a(|u_x|)u_{xx} + S(x), \quad x_F(t) < x < 1,\ t > 0 \\ u(0,x) = u_0(x), \quad x_{F,0} \le x \le 1, \\ u_x(t, x_F(t)) = u_{0,x}(x_{F,0}),\ |u_{0,x}(x_{F,0})| = \eta, \quad t \ge 0, \\ u(t,1) = 0, \quad t \ge 0, \end{cases}$$

with flux defined by

$$q(t,x) = D(t,x,u_x)u_x = \begin{cases} D_0 u_x, & |u_x| \le \eta, \\ (D_0 + D_1(|u_x| - \eta))u_x, & |u_x| \ge \eta, \end{cases} \tag{21}$$

or for the non-divergence representation

$$a(|u_x|) = \begin{cases} D_0, & |u_x| \le \eta, \\ D_0 + D_1(2|u_x| - \eta), & |u_x| \ge \eta. \end{cases} \tag{22}$$

Theorem 4.1. *Let Problem (P) be given with data satisfying Assumptions 4.1.*
1. Let $x_F(t) \in C^1([0,T])$ *be any function satisfying* $x_F(0) = x_{F,0}$ *and* $0 < x_F(t) < 1$. *Then (P1) and (P2) possess in* $[0,T]$ *unique classical solutions* $u^-(t,x)$ *and* $u^+(t,x)$, *respectively.*
2. Let $x_F(t) > 0$, $u^-(t,x)$ *and* $u^+(t,x)$ *solve the nonlinear system (P1), (P2) and*

$$\dot{x}_F(t) = -\frac{q_t(t, x_F(t))}{q_x(t, x_F(t))}, \qquad x_F(0) = x_{F,0}, \tag{23}$$

for $t \in [0,T]$. *Then*

$$u(t,x) := \begin{cases} u^-(t,x), & x \in [0, x_F], \\ u^+(t,x), & x \in [x_F, 1], \end{cases} \tag{24}$$

is the unique solution of Problem (P) in $[0,T]$.

A related theorem was proved in [1] under the additional assumption that the front $x_F(t) \in C^1[0,T]$ is known a priori.

In the numerical code accompanying [1], first (P1) and (P2) are advanced in time, then eq. (23), and then the grid is adjusted (grid refinement in a neighborhood of $x_F(t_{j+1})$). Numerical details are given in [1, p. 61ff]. Note that eq. (23) is equivalent to eq. (14), but more convenient in computations. This approach is supported by Theorem 4.1.

Proof: 1. Assume that $x_F(t)$ is any function with the mentioned properties. Then problems (P1) and (P2) possess classical solutions. This is shown in [1] by applying classical results from chapters *IV* and *VI* in [6]. The details of the proof are not repeated here because it is standard. A full text may be found in [1, Chapter 3].

2. Now assume that x_F solves (23), u^- solves (P1), and u^+ solves (P2). We have to show that $u^-(t, x_F(t)) = u^+(t, x_F(t))$ and that $x_F(t)$ and thus $u(t,x)$ are unique. Let $\varepsilon > 0$ and consider

$$u_t^\varepsilon(t,x) \;=\; \tilde{a}(|u_x|,\varepsilon)u_{xx}^\varepsilon(t,x) + S(x), \tag{25}$$

where

$$\tilde{a}(v,\varepsilon) = D_0 + D_1(\frac{1}{2} + \frac{1}{\pi}\arctan\frac{v-\eta}{\varepsilon})(2v-\eta).$$

Then $\tilde{a}(v,\varepsilon) \to a(v)$ for $\varepsilon \to 0$: We can represent $\tilde{a}$ as $\tilde{a}(v,\varepsilon) = a(v) + f(v,\varepsilon)$, where

$$f(v,\varepsilon) = \begin{cases} D_1(2v-\eta)(\frac{1}{2} + \frac{1}{\pi}\arctan\frac{v-\eta}{\varepsilon}), & v < \eta, \\ D_1(2v-\eta)(-\frac{1}{2} + \frac{1}{\pi}\arctan\frac{v-\eta}{\varepsilon}), & v > \eta, \end{cases}$$

and

$$f_v'(v,\varepsilon) = \begin{cases} D_1(1 + \frac{2}{\pi}\arctan\frac{v-\eta}{\varepsilon}) + \frac{D_1(2v-\eta)}{\pi}\frac{\varepsilon^2}{\varepsilon^2+(v-\eta)^2} & v < \eta \\ D_1(-1 + \frac{2}{\pi}\arctan\frac{v-\eta}{\varepsilon}) + \frac{D_1(2v-\eta)}{\pi}\frac{\varepsilon^2}{\varepsilon^2+(v-\eta)^2} & v > \eta. \end{cases}$$

Both f and f_v' go to zero for $\varepsilon \to 0$ and $v \neq \eta$. Note that the function f is uniformly continuous in ε since it is defined and continuous for any ε, including large ε; and

$$f(v,\varepsilon) \overset{\varepsilon\to\pm\infty}{\to} \pm\frac{D_1(2v-\eta)}{2},$$

the sign depending on v. We solve

$$\begin{aligned} u_t^\varepsilon &= \tilde{a}(|u_x|,\varepsilon)u_{xx}^\varepsilon + S(x), \quad 0 < x < x_F,\ t > 0 \\ u^\varepsilon(0,x) &= u_0(x), \qquad 0 \le x \le x_F, \\ u_x^\varepsilon(t,0) &= 0, \qquad t \ge 0, \\ |u_x^\varepsilon(t,x_F)| &= \eta, \qquad t \ge 0, \end{aligned} \tag{26}$$

and

$$\begin{aligned} u_t^\varepsilon &= \tilde{a}(|u_x|,\varepsilon)u_{xx}^\varepsilon + S(x), \quad x_F < x < 1,\ t > 0 \\ u^\varepsilon(0,x) &= u_0(x), \qquad x_F \le x \le 1, \\ u^\varepsilon(t,1) &= 0, \qquad t \ge 0, \\ |u_x^\varepsilon(t,x_F)| &= \eta, \qquad t \ge 0. \end{aligned} \tag{27}$$

Each of these problems can be transformed such that $x \in [0,1]$ (we have done this in more details, later on in the proof, for equations (28) and (30)). In this way, the function x_F enters in the main equation. The coefficient $\tilde{a}(v,\varepsilon)$ is Hölder continuous in v with a constant α, and according to [6, Ch.IV, Th.5.3] problem (26) has a unique solution in the class $C^{1+\alpha/2,2+\alpha}([0,T] \times [0,x_F])$. Problem (27) has mixed boundary conditions and Theorem 5.3 in [6, Ch.IV] is not directly applicable. However, Theorem 5.1 [6, p.170] combined with Theorem 12.1 [6, p.223] assure that the mixed boundary problem (27) has a unique solution in $C^{1+\alpha/2,2+\alpha}([0,T]\times[x_F,1])$, for $0<\alpha<1$, provided that $\tilde{a}, S \in C^{\alpha/2,\alpha}(\Omega_T)$.

Now, let us consider the difference between the solutions of (26) and (P1), $w^- := u^\varepsilon - u^-$, and the corresponding differential equation satisfied by it,

$$\begin{aligned} w^-_t &= a(|u_x|)w^-_{xx} + f(|u_x|,\varepsilon)u^\varepsilon_{xx}, \qquad 0<x<x_F(t), \\ w^-(0,x) &= 0, \qquad w^-_x(t,0) = |w^-_x(t,x_F)| = 0. \end{aligned} \tag{28}$$

We map the interval $[0,x_F]$ to $[0,1]$ through $x \mapsto \xi = \frac{x}{x_F}$. In terms of this new variable the problem reads

$$w^-_t = \frac{1}{x_F(t)^2}\left(D_0 w^-_{\xi\xi} + f(\frac{|u_\xi|}{x_F},\varepsilon)u^\varepsilon_{\xi\xi}\right), \qquad 0<\xi<1,\ t>0 \tag{29}$$

$$w^-(0,\xi) = 0, \qquad w^-_\xi(t,0) = |w^-_\xi(t,1)| = 0.$$

Again according to Ladyzhenskaia et al [6], problem (29) possesses a unique solution if the coefficients making up the problem belong to the class $C^{\alpha/2,\alpha}$. For the coefficient in front of $w^-_{\xi\xi}$ this is true because of the continuity of $x_F(t)$. In order to prove that $\frac{1}{x_F(t)^2} f(\frac{|u_\xi|}{x_F},\varepsilon)u^\varepsilon_{\xi\xi}$ belongs to the class $C^{\alpha/2,\alpha}$, we need that $u^\varepsilon_{\xi\xi\xi}$ and $u^\varepsilon_{\xi\xi t}$ exist and are continuous. To argue for this we use the fact that the solution of (26) belongs to the class $C^{(3+\alpha)/2,3+\alpha}$ since the coefficients making up the equation possess a greater smoothness.

Because problem (29) has a unique solution and $f(v,\varepsilon) \overset{\varepsilon\to 0}{\to} 0$ uniformly, it follows that the solution of (29) goes to the zero solution for $\varepsilon \to 0$, i.e. $u^\varepsilon(t,x) \overset{\varepsilon\to 0}{\to} u^-(t,x)$.

Similarly, we proceed with the interval $[x_F,1]$. We define a function $w^+(t,x)$ in $[0,T]\times[x_F,1]$, such that $w^+ = u^\varepsilon(t,x) - u^+(t,x)$ and it satisfies the problem

$$\begin{aligned} w^+_t &= a(|u_x|)w^+_{xx} + f(|u_x|,\varepsilon)u^\varepsilon_{xx}, \qquad x_F<x<1, \\ w^+(0,x) = 0, &\qquad w^+_x(t,x_F) = w^+(t,1) = 0. \end{aligned} \tag{30}$$

We transform this problem into $[0,1]$, through $\xi = \frac{x-x_F}{1-x_F}$, and obtain a linear parabolic problem

$$\begin{aligned} w^+_t &= \frac{1}{(1-x_F)^2}\left(a(|\frac{u_\xi}{(1-x_F)}|)w^+_{\xi\xi} + f(|\frac{u_\xi}{1-x_F}|,\varepsilon)u^\varepsilon_{\xi\xi}\right), \quad 0<\xi<1 \\ w^+(0,\xi) &= 0, \qquad w^+_\xi(t,0) = w^+(t,1) = 0. \end{aligned} \tag{31}$$

We use similar arguments as in the previous case. According to Ladyzhenskaia et al, problem (31) possesses a unique solution if $\frac{1}{(1-x_F)^2}a(|\frac{u_\xi}{(1-x_F)}|)$ and $\frac{1}{(1-x_F)^2}f(|\frac{u_\xi}{1-x_F}|,\varepsilon)u^\varepsilon_{\xi\xi}$

belong to $C^{\alpha/2,\alpha}$. For the latter we use the same arguments as in the previous case. The Hölder continuity of the term in front of $w^+_{\xi\xi}$ follows from the boundedness of $u_{\xi\xi}$, $u_{\xi t}$, and a_v. In this way we get that problem (30) possesses a unique solution and $w^+(t,x) \overset{\varepsilon\to 0}{\to} 0$.

Now, let $g : [0,T] \to I\!R$ and

$$g(t) = u^\varepsilon(t, x_F(t)^-) - u^\varepsilon(t, x_F(t)^+) \tag{32}$$

By taking the derivative of (32) with respect to t we obtain

$$\frac{dg(t)}{dt} = u^\varepsilon_t(t, x_F(t)^-) + \eta \dot{x}^-_F - u^\varepsilon_t(t, x_F(t)^+) - \eta \dot{x}^+_F.$$

Because of $\dot{x}^-_F = \dot{x}^+_F$ and the continuity of u^ε_t on the interface we get

$$\frac{dg(t)}{dt} = 0.$$

In addition, $g(0) = 0$ leads to $g(t) \equiv 0$, i.e, $u^\varepsilon(t, x_F(t)^-) = u^\varepsilon(t, x_F(t)^+)$. The same is true for $u(t, x_F(t)^-) = u(t, x_F(t)^+)$, for $t \in [0,T]$.

We now show that the function u defined by (24) is a solution of Problem (P) according to Definition 2.1. The functions u and u_x are continuous since $u^- \in C^{1+\alpha/2,2+\alpha}([0,T] \times [0, x_F))$ and $u^+ \in C^{1+\alpha/2,2+\alpha}([0,T] \times (x_F, 1])$ and for every fixed $t \in [0,T]$ it holds that $u^-(t, x_F(t)) = u^+(t, x_F(t)) = u(t, x_F(t))$ and $u^-_x(t, x_F(t)) = u^+_x(t, x_F(t)) = u_x(t, x_F(t)) = u_{0,x}(x_F(0))$. Furthermore, u_{xx} is continuous everywhere except at the front point $x_F(t)$. Now all we need in addition is to show that u satisfies Problem (P) . Indeed, that is the case, because for every fixed $t \in [0,T]$, $u(t,x) \equiv u^-(t,x)$ for $x \in [0, x_F(t)]$ and $u^-(t,x)$ is a solution of (P1), respectively (P) in that interval. Similarly, in $[x_F(t), 1]$ it holds for every fixed $t \in [0,T]$ that $u(t,x) \equiv u^+(t,x)$ and $u^+(t,x)$ is a solution of Problem (P2), respectively Problem (P) in the corresponding interval.

This shows existence and uniqueness of the solution $x_F(t), u(t,x)$ of Problem (P) for given solutions of the system (P1), (P2) and (23). Assume that the system (P1), (P2) and (23) has a second solution for the same initial and boundary data as $x_F(t), u(t,x)$. Then the above proof leads to a second solution $y_F(t), v(t,x)$ of Problem (P) . Note that $u(t,x) = v(t,x)$ iff $x_F(t) = y_F(t)$. Let us assume that $x_F(t) \neq y_F(t)$. Then there is a smallest $t \in [0,T]$ such that $\dot{x}_F(t) \neq \dot{y}_F(t)$. Without restriction we may assume that this happens for $t = 0$. But this is impossible because $\dot{x}_F(0)$ and $\dot{y}_F(0)$ are both completely determined by the same initial and boundary conditions and the same differential equation. $\square$

5 Parameter dependent families of exact solutions

In this section we describe two families of exact solutions of

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left((1 + D_1 H(u_x - \eta)(u_x - \eta)) \frac{\partial u}{\partial x} \right), \qquad 0 < x < 1, \quad t > 0. \tag{33}$$

This is eq. (1) with $S(x) \equiv 0$ and $D_0 = 1$. In addition we assume D_1 and η to be given. Assuming that the initial function $u_0(x)$ in eq. (3) is such that

$$\frac{\partial u_0}{\partial x}(x_F(0)) = \eta, \tag{34}$$

we use the front tracking idea and decompose eq. (33) into two sub-equations $(\tilde{P}1)$ and $(\tilde{P}2)$:

$$u_t = \begin{cases} u_{xx}, & u_x \le \eta \qquad (\tilde{P}1) \\ (1 + 2D_1 u_x - D_1\eta) u_{xx}, & u_x \ge \eta. \qquad (\tilde{P}2) \end{cases}$$

This allows us to derive exact solutions of (33). We found two families of parameter-dependent exact solutions for given D_1 and η. Note that additional families of solutions may be found by varying D_0, D_1 and η as well.

Lemma 5.1. *Let $D_1 > 0$ and η be given. Let A and $C > 0$ be parameters satisfying*

$$A + 2C^2 D_1 t \le \eta \le C + A + 2C^2 D_1 t \qquad \textit{for} \quad 0 \le t \le t_1 \tag{35}$$

for some $t_1 > 0$. Then

$$u(t,x) := \begin{cases} \frac{D_1\eta - 1}{2D_1}\left(x + \frac{A-\eta}{C}\right) + \frac{D_1\eta+1}{4CD_1^2}\left(e^{2D_1(Cx+2D_1C^2t+A-\eta)} - 1\right) + \frac{\eta^2 - A^2}{2C}, & u_x \le \eta \\ \frac{C}{2}x^2 + 2C^3 D_1 t^2 + 2C^2 D_1 x t + Ax + C(1 - D_1\eta + 2AD_1)t, & u_x \ge \eta \end{cases} \tag{36}$$

defines a family of solutions of equation (33).

Proof: A simple calculation shows that $u(t,x)$ solves eq. (33) and that the length t_1 of the time interval depends on the relative size of the parameter C. These solutions were obtained by matching the solution of $(\tilde{P}1)$ at the front point $x_F \in [0,1]$ to a polynomial in x and t that solves $(\tilde{P}2)$. Finding these solutions was by far not as easy as verifying them. Details are given in [1, p. 52f]. ☐

Example 5.1. *Choose $D_1 = 1$, $\eta = 3$, $A = 2$ and $C = 1$. Then eq. (35) is satisfied for $0 \le t \le 1$, and eq. (36) simplifies to*

$$u(t,x) = \begin{cases} \exp^{2x+4t-2} + x + \frac{1}{2}, & u_x \le \eta, \\ \frac{x^2}{2} + 2t^2 + 2tx + 2x + 2t, & u_x \ge \eta. \end{cases} \tag{37}$$

A front point x_F exists in $(0,1]$ for $t_F \in [0,1/2)$ and satisfies $2x_F + 4t_F - 2 = 0$. A numerical approximation to this solution[1] *and to its gradient are shown in Fig.1 for three different times $t_j < 1/2$. The magenta curves correspond to the solution of $(\tilde{P}1)$ and the blue curves to the solution of $(\tilde{P}2)$, resp.*

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[1] After calculating the initial and boundary values of u from (37), $u(t,x)$ was obtained numerically with the code described and analysed in [1].

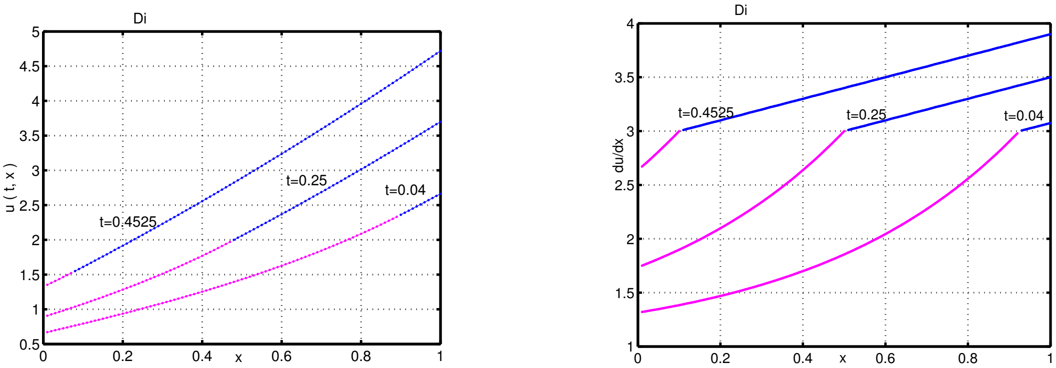

Figure 1: Solution (37) (left) and its gradient (right) for times $t_j = 0.04, 0.25, 0.45$; [1, Fig.4.3, p.90].

Lemma 5.2. *Let $D_1 > 0$ and η be given. Let $K > 0$ and $\alpha > 0$ be parameters satisfying*

$$\eta \leq \frac{1-6(K-t)}{6D_1(K-t)} \qquad \textit{for} \quad t_0 \leq t \leq t_1, \tag{38}$$

for some t_0, t_1 with $t_1 > t_0$. Then

$$u(t,x) := \begin{cases} (K-t)^\alpha f(\frac{x^2}{K-t}), & u_x \leq \eta \\ \frac{x^3}{36D_1(K-t)} + \frac{x(D_1\eta-1)}{2D_1}, & u_x \geq \eta, \end{cases} \tag{39}$$

defines a family of solutions of eq. (33). Here $f(.)$ is a solution of the Confluent Hypergeometric Equation. It is defined by

$$f(\xi) = b_1\ {}_1F_1(-\alpha, 1/2, \xi) + b_2 U(-\alpha, 1/2, \xi), \qquad \xi = \frac{x^2}{K-t},\ \xi_F = \frac{x_F^2}{K-t}$$

where

$${}_1F_1(a,c,\xi) = \frac{\Gamma(c)}{\Gamma(c-a)\Gamma(a)} \int_0^1 e^{\xi t} t^{a-1}(1-t)^{c-a-1} dt \tag{40}$$

and

$$U(a,c,\xi) = \frac{1}{\Gamma(a)} \int_0^\infty e^{-\xi t} t^{a-1}(1+t)^{c-a-1} dt. \tag{41}$$

The coefficients b_1 and b_2 are defined by the conditions

$$\begin{aligned} f(\xi_F) &= (K-t)^{1/2-\alpha} \frac{(2D_1\eta-1)\sqrt{6(D_1\eta+1)}}{3D_1}, \\ f'(\xi_F) &= \frac{\eta}{2(K-t)^{\alpha-1/2}\sqrt{6(D_1\eta+1)}}. \end{aligned}$$

Proof: A calculation shows that $u(t,x)$ solves eq. (33). This second family of solutions was obtained by matching the self-similar solution of the heat equation ($\tilde{P}1$) to a solution of eq. ($\tilde{P}2$) using an ansatz $u(t,x) = f_0(t) + x f_1(t) + \frac{x^2}{2} f_2(t) + \frac{x^3}{3} f_3(t)$. This leads to differential equations for f and the f_i, $i = 0, \ldots, 3$. Details are given in [1, p. 54f]. ☐

Example 5.2. *Choose* $D_1 = 10,\ \eta = 0.1,\ K = 1$ *and* $\alpha = 1$. *Then eq. (38) is satisfied for* $t_0 = \frac{11}{12} \leq t < 1,\ \frac{11}{12} \approx 0.91666$. *The corresponding solution is plotted in Figure 2 at times* $t_j = 0.938,\ 0.97,\ 0.99$.

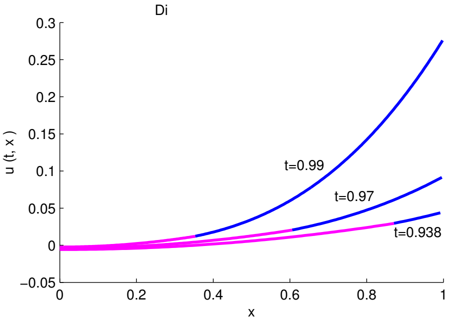

Figure 2: The exact solution (39) for $D_1 = 10,\ \eta = 0.1,\ K = 1$ and $\alpha = 1$ at times $t_j = 0.938,\ 0.97,\ 0.99$

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