Relativistic neoclassical radial fluxes in the $1/\nu$ regime

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Abstract. The radial neoclassical fluxes of electrons in the $1/\nu-$regime are calculated with relativistic effects taken into account and compared with those in the non-relativistic approach. The treatment is based on the relativistic drift-kinetic equation with the thermodynamic equilibrium given by the relativistic Maxwell-Jüttner distribution function. It is found that for the range of fusion temperatures, $T_e < 100$ keV, the relativistic effects produce a reduction of the radial fluxes which does not exceed 10%. This rather small effect is a consequence of the non-monotonic temperature dependence of the relativistic correction caused by two counteracting factors: a reduction of the contribution from the bulk and a significant broadening with the temperature growth of the energy range of electrons contributing to transport.

The relativistic formulation for the radial fluxes given in this paper is expressed in terms of a set of relativistic thermodynamic forces which is not identical to the canonical set since it contains an additional relativistic correction term dependent on the temperature. At the same time, this formulation allows application of the non-relativistic solvers currently used for calculation of mono-energetic transport coefficients.

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1. Introduction

The role of relativistic effects in hot plasmas has been recognized as important not only in astrophysics [1, 2] but also in fusion, in particular, for the population of highly energetic runaway electrons in tokamaks [3]. However, the relativistic effects do not necessarily require an extremely high temperatures since they can be non-negligible even if T_e is only on the order of tens of keV, i.e. $T_e \ll m_{e0}c^2$. These effects appear due to the macroscopic features of the relativistic thermodynamic equilibrium given by the Maxwell-Jüttner distribution function [1,4]. An example of such effects provided by the Maxwell-Jüttner distribution function is given in a recent paper [5], where the stability criterion for collisional heat transfer from hot electrons to ions with respect to the Coulomb decoupling is studied and it is found that relativistic effects lead to qualitative changes in stability criteria. While in non-relativistic plasmas criterion is given by $T_e/T_i < 3$, relativistic effects makes it temperature-dependent and for $T_{e,i} > 75$ keV the collisional coupling between electrons and ions becomes absolutely stable.

Relativistic effects in fusion are surely not important for the ions, but the transport physics for electrons needs to be examined carefully for fusion reactor projects such as ITER [6–8] and DEMO [9,10], in which the expected electron temperature is sufficiently high, $T_e \simeq 20 - 50$ keV, and for future aneutronic fusion reactors with D–^{3}He and may be p–^{11}B reactions, which require temperatures of up to $70 - 100$ keV [11–13]. However, all transport codes (see, for example [14]) developed to date and applied for simulations of reactor scenarios are based on the non-relativistic approach. Furthermore, there is no quantitative definition of an applicability range for the non-relativistic transport models so far.

Relativistic kinetics and MHD in plasmas are usually treated in the covariant formulation [1, 2]. For neoclassical transport, however, the covariant formulation is not necessary since Lorentz invariance is of minor importance with respect to the characteristic drift velocity, $V_{\rm dr}/c \ll 1$. For this purpose, one can directly apply the relativistic drift-kinetic equation [15] with the relativistic Coulomb operator [4].

In this paper, the relativistic effects in the radial fluxes in the $1/\nu$-regime, which might be the most dangerous regime for future burning plasmas in stellarators and where the radial electric field plays no significant role, are estimated. This case was chosen for investigation because the role of the highly energetic tail of the distribution function in transport processes in this regime is expected to be the largest in comparison with other regimes. Indeed, the diffusion coefficient in the $1/\nu$-regime scales roughly as $V_{\rm dr}^2/\nu_e \propto v^7$, while in the tokamak banana-regime it scales as $\rho_{ce}^2\nu_e \propto v^{-1}$ (here, $V_{\rm dr}$ is the radial drift-velocity, ρ_{ce} is the Larmor radius and ν_e is the collision frequency).

In Sec. 2, the relativistic drift-kinetic equation (rDKE) in the mono-energetic approach with a set of thermodynamic forces which differs from the canonical one is formulated. Only radial gradients are taken into account while the parallel electric field is excluded from consideration. In Sec. 3, rigorous expressions for the radial electron fluxes and transport coefficients in the $1/\nu-$regime are derived. In particular, the expression for the relativistic radial heat flux is obtained. As a guideline, the paper [16] was used, where the same was calculated in the non-relativistic approach. In Sec. 4, the numerical comparison of the relativistic and non-relativistic transport coefficients and radial fluxes is performed, and in Sec. 5 a brief discussion of the results is given.

2. Mono-energetic drift-kinetic equation for relativistic electrons

The electron radial fluxes in toroidal plasmas (except the Ware pinch) can be calculated from the relativistic drift-kinetic equation (rDKE) for the first-order distribution function f_{e1} in the mono-energetic approach [17–19]. Using on a magnetic surface, with flux-sufrace label ρ, the set of variables (s, u, λ), where s is the coordinate along the field-line, $u = p/m_{e0} = \gamma v$ is the momentum per unit mass, $\gamma = \sqrt{1 + u^2/c^2}$ is the Lorentz-factor, $\lambda = (1 - \xi^2)/b$ is the normalized magnetic moment, where $\xi = u_\parallel/u$ is the pitch and $b = B/B_0$ is the normalized magnetic field with the reference field B_0, the mono-energetic rDKE can be written as

$$\mathcal{V}(f_{e1}) - \nu_D(u)\mathcal{L}(f_{e1}) = -\left(\mathbf{V}_{\rm dr} \cdot \nabla\rho\right) \frac{\partial F_{eMJ}}{\partial \rho}. \tag{1}$$

The first term in Eq. (1) is the mono-energetic Vlasov operator, $\mathcal{V} = (v_\parallel \mathbf{h} + \mathbf{V}_{\rm dr}) \cdot \nabla_s$, where $\mathbf{h} = \mathbf{B}/B$ and ∇_s is the gradient within the magnetic surface (here, $\dot{\lambda} = 0$). The second term is the pitch-angle scattering operator with the deflection frequency $\nu_D(u) = \nu_D^{ee}(u) + \nu_D^{ei}(u)$ (the complete expressions for relativistic ν_D^{ee} and ν_D^{ei} are given in Appendix A) and the Lorentz operator is

$$\mathcal{L} = \frac{2\xi}{b}\frac{\partial}{\partial\lambda}\left(\lambda\xi\frac{\partial}{\partial\lambda}\right). \tag{2}$$

The relativistic drift velocity can be written as

$$\mathbf{V}_{\rm dr} = \frac{c}{B^2}\mathbf{E}\times\mathbf{B} - \frac{m_{e0}cu^2(1+\xi^2)}{2e\gamma B^3}\mathbf{B}\times\nabla B \tag{3}$$

with $\mathbf{E} = -\nabla\Phi = -\Phi'\nabla\rho$ and $\Phi' \equiv d\Phi/d\rho$, where Φ is the plasma potential (here and below, $e = |e|$). One can see that only the last term in Eq. (3) contributes to $\dot{\rho} \equiv \mathbf{V}_{\rm dr} \cdot \nabla\rho$ on the right-hand side (RHS) of Eq. (1). Since our treatment is limited to the $1/\nu$-regime, only such values of E for which electrons with large E/vB make no significant contribution to transport are considered. In this case, the $\mathbf{E}\times\mathbf{B}$ drift term can be omitted in the Vlasov operator, i.e $\mathcal{V} \simeq v_\parallel \mathbf{h}\cdot\nabla_s$. (In the more general case, this term must be included to obtain the $\sqrt{\nu}$-regime which is more complex for analytical treatment and is not considered here.)

Thermodynamic equilibrium for relativistic electrons is given by the Jüttner distribution function [1] also known as the relativistic Maxwellian [4], which may be conveniently represented as

$$f_{eMJ}(u,\rho) = \frac{n_e}{\pi^{3/2}u_{te}^3}C_{MJ}(\mu_r)\mathrm{e}^{-\mu_r(\gamma-1)}, \tag{4}$$

where $u_{te} = p_{te}/m_{e0}$ is the thermal momentum per unit mass with $p_{te} = \sqrt{2m_{e0}T_e}$ and $\mu_r = m_{e0}c^2/T_e$. The Maxwell-Jüttner distribution function is normalized by density, $n_e = \int d^3u\, f_{eMJ}$, and the normalization factor is

$$C_{MJ}(\mu_r) = \sqrt{\frac{\pi}{2\mu_r}}\frac{\mathrm{e}^{-\mu_r}}{K_2(\mu_r)} \simeq 1 - \frac{15}{8\mu_r} + \mathcal{O}(1/\mu_r^2), \qquad \mu_r \gg 1, \tag{5}$$

where $K_n(x)$ is the modified Bessel function of n-th order. For convenience, the Maxwell-Jüttner distribution function is used in Eq. (1) with the Boltzmann-factor included:

$$F_{eMJ} = \mathrm{e}^{-e\Phi/T_e} f_{eMJ}. \tag{6}$$

Since plasma parameters such as density and temperature only depend on the flux-surface label, ρ, the derivative in the right-hand-side of Eq. (1) can be expressed in terms of the thermodynamic forces,

$$\frac{\partial F_{eMJ}}{\partial \rho} = [A_1(\rho) + \kappa\, A_2(\rho)]\, F_{eMJ}, \tag{7}$$

where $\kappa = \mu_r(\gamma - 1)$ is the relativistic kinetic energy normalized by T_e, and the thermodynamic forces A_1 and A_2 are defined as

$$A_1(\rho) = \frac{n_e'}{n_e} - \left(\frac{3}{2} + \mathcal{R}\right)\frac{T_e'}{T_e} - \frac{e\Phi'}{T_e}, \tag{8a}$$

$$A_2(\rho) = \frac{T_e'}{T_e}, \tag{8b}$$

with $n_e' \equiv dn_e/d\rho$, $T_e' \equiv dT_e/d\rho$, and the relativistic correction-term

$$\mathcal{R}(\mu_r) = \mu_r\left(\frac{K_3}{K_2} - 1\right) - \frac{5}{2} \simeq \frac{15}{8\mu_r} + \mathcal{O}(1/\mu_r^2), \qquad \mu_r \gg 1. \tag{9}$$

Note that in contrast to the "canonical" set of the thermodynamic forces [17–19], which depend only on the normalized gradients of density and temperature (n_e'/n_e and T_e'/T_e, respectively), and not on the absolute values of these plasma parameters, the first thermodynamic force $A_1(\rho)$ in the relativistic set Eq. (8) contains an additional temperature-dependent term.

Finally, the reduced mono-energetic rDKE can be represented as follows:

$$(\mathbf{h}\cdot\nabla_s)f_{e1} - \frac{\gamma\nu_D(u)}{u\xi}\,\mathcal{L}(f_{e1}) = -\frac{\gamma}{u\xi}\,\dot{\rho}\,[A_1(\rho) + \kappa\, A_2(\rho)]\, F_{eMJ}. \tag{10}$$

Note that similar to the non-relativistic formulation, the energy enters in Eq. (10) only as a parameter in $\gamma\nu_D(u)/u$ and the solution of Eq. (10) describes only the pitch- and spatial behavior of the distribution function f_{e1}, which is the same for both relativistic and non-relativistic approaches. With the proper choice of parameters and right-hand-side of Eq. (10), the solution from such solvers as DKES [19] and NEO-2 [20], which solve the non-relativistic DKE directly, can be interpreted as a solution of the mono-energetic relativistic DKE.

3. Relativistic radial fluxes

In this chapter, the radial fluxes of particles and energy in the $1/\nu-$regime are calculated following Ref. [16] with the mono-energetic DKE treated in the relativistic approach.

Equation (10) can be solved by integration along the field-line. Here, only the trapped electrons, $B_0/B_{\max} < \lambda < B_0/B_{\min}$, are considered ($B_{\max}$ and $B_{\min}$ are the absolute maximum and minimum of B on the given magnetic surface, respectively). Enumerating the local minima of B along the magnetic field-line by k and integrating Eq. (10) over the bounce trajectory (assumed to be closed), one can obtain

$$\frac{2\gamma\nu_D(u)}{u}\frac{\partial}{\partial\lambda}\left(\lambda I^{(k)}\frac{\partial f_{e1}^{(k)}}{\partial\lambda}\right) = \delta\rho^{(k)}\frac{\partial F_{eMJ}}{\partial\rho} \tag{11}$$

with

$$I^{(k)} = \oint_{(k)} \frac{ds}{b}\xi \ \text{ and } \ \delta\rho^{(k)} = \frac{\gamma}{u}\oint_{(k)} \frac{ds}{\xi}\dot{\rho}, \tag{12}$$

where $\delta\rho^{(k)}$ is the radial displacement of an electron due to the magnetic drift after one bounce period. To solve Eq. (11), the following trick was used [16]. Applying the explicit expression for $\mathbf{V}_{\rm dr}$ given by Eq. (3) to $\dot{\rho} = \mathbf{V}_{\rm dr}\cdot\nabla\rho$ and using the fact that powers of $\xi = \sigma\sqrt{1-\lambda b}$ with $\sigma = \pm 1$ can be expressed as

$$\xi^m = -\frac{2}{(m+2)b}\frac{\partial}{\partial\lambda}\xi^{m+2}, \tag{13}$$

one can represent the integrand for $\delta\rho^{(k)}$ in Eq. (12) as follows:

$$\frac{\dot{\rho}}{\xi} = -\frac{u^2}{\gamma}\frac{|\nabla\rho|k_G}{b^2\omega_{c0}}\frac{\partial}{\partial\lambda}\left(\xi + \frac{\xi^3}{3}\right), \tag{14}$$

where $\omega_{c0} = eB_0/(m_{e0}c)$ is the cyclotron frequency, $k_G = \mathbf{n}_\rho\cdot[\mathbf{h}\times(\mathbf{h}\cdot\nabla)\mathbf{h}]$ is the geodesic curvature of the magnetic field line and $\mathbf{n}_\rho = \nabla\rho/|\nabla\rho|$ is the unit vector normal to the magnetic surface. Then

$$\delta\rho^{(k)} = -\frac{u}{3}\frac{\partial H^{(k)}}{\partial\lambda} \ \text{ with } \ H^{(k)} = \oint_{(k)} ds\, \xi(3+\xi^2)\frac{|\nabla\rho|k_G}{b^2\omega_{c0}}. \tag{15}$$

Using this relation and the fact that $I^{(k)} = H^{(k)} = 0$ at the bottom of the magnetic wells (when $\lambda = B_0/B_{\min}$), the order of Eq. (11) can be reduced,

$$\frac{\partial f_{e1}^{(k)}}{\partial\lambda} = -\frac{H^{(k)}}{6\lambda I^{(k)}}\frac{u^2}{\gamma\nu_D(u)}\frac{\partial F_{eMJ}}{\partial\rho}. \tag{16}$$

The radial components of the particle and energy fluxes are given by

$$\Gamma_e^\rho = \langle\mathbf{\Gamma}_e\cdot\nabla\rho\rangle = \left\langle\int d^3u\,\dot{\rho}\,f_{e1}\right\rangle, \tag{17a}$$

$$Q_e^\rho = \langle\mathbf{Q}_e\cdot\nabla\rho\rangle = \left\langle\int d^3u\, m_{e0}c^2(\gamma-1)\,\dot{\rho}\,f_{e1}\right\rangle, \tag{17b}$$

where f_{e1} is the solution of the relativistic drift-kinetic equation Eq. (10) and $\langle\ldots\rangle$ means averaging on the magnetic surface.

The conductive heat flux for relativistic electrons requires special attention. According to its physical definition [17,18], the radial conductive heat flux can be found by extracting the advective and mechanical contributions from the radial component of the energy flux,

$$q_e^\rho = Q_e^\rho - T_e \Gamma_e^\rho - W_e V^\rho, \tag{18}$$

where

$$W_e = \int d^3u \, m_{e0} c^2 (\gamma - 1) \, f_{eMJ} = \left(\frac{3}{2} + \mathcal{R} \right) n_e T_e \tag{19}$$

is the energy density related to the Maxwell-Jüttner distribution function, and $V^\rho = \Gamma_e^\rho / n_e$ is the flow velocity. Finally, the radial heat flux can be written as

$$q_e^\rho = Q_e^\rho - \left(\frac{5}{2} + \mathcal{R} \right) T_e \Gamma_e^\rho. \tag{20}$$

Please note that this definition differs from the non-relativistic expression accepted in the neoclassical theory [17,18] by the additional correction term $\mathcal{R}$.

It is convenient to use common notations for both the particle and energy fluxes of the form $J_i = \left\langle \int h_i \dot{\rho} f_{e1} d^3u \right\rangle$, where $J_1 \equiv \Gamma_e^\rho$ and $J_2 \equiv Q_e^\rho / T_e$ with $h_1 = 1$ and $h_2 = \kappa \equiv \mu_r (\gamma - 1)$, respectively. Using in $\int d^3u$ the variable κ instead of u and performing the integration over λ instead of pitch, $\int d\xi = b/2 \sum_\sigma \int d\lambda / |\xi|$, one can obtain from Eq. (17) the following:

$$J_i = \frac{\pi}{2} u_{te}^3 \int_0^\infty d\kappa \sqrt{\kappa} \, h_i \gamma \left(\frac{\gamma + 1}{2} \right)^{1/2} \left\langle b \sum_{\sigma = \pm 1} \int_{1/b_{\max}^{(k)}}^{1/b_{\min}^{(k)}} d\lambda \, f_{e1}^{(k)} \frac{\dot{\rho}}{|\xi|} \right\rangle. \tag{21}$$

Then, substituting Eq. (14) into Eq. (21), an integration by parts over λ can be performed. Finally, considering the averaging over the flux-surface as the limit of integration along the field-line and applying Eq. (16), the desired expression for the radial fluxes can be obtained,

$$J_i = -n_e G_0 C_{MJ} \int_0^\infty d\kappa \, \frac{\mathrm{e}^{-\kappa} \kappa^{5/2}}{\gamma \hat{\nu}_D(u)} \left(\frac{\gamma + 1}{2} \right)^{5/2} h_i \, \frac{\partial \ln F_{eMJ}}{\partial \rho}, \tag{22}$$

with $\hat{\nu}_D(u) \equiv \nu_D(u) / \nu_{e0}$. One can check that the non-relativistic limit considered in Ref. [16] is recovered.

The coefficient G_0 in Eq. (22), identical for both relativistic and non-relativistic formulations, contains all parameters for plasmas and magnetic configuration which are specific for the considered $1/\nu-$regime,

$$G_0 = \frac{4\sqrt{2}}{9\pi^{3/2}} \, \frac{u_{te}^4}{R^2 \omega_{e0}^2 \nu_{e0}} \, \langle |\nabla \rho| \rangle^2 \, \epsilon_{\text{eff}}^{3/2}, \tag{23}$$

where R is the major radius and ϵ_{eff} is the effective ripple amplitude (not shown here; for details see [16]). The expression for the radial fluxes in the $1/\nu$ collisional regime calculated in the relativistic approach Eq. (22) is, actually, the main result of this paper.

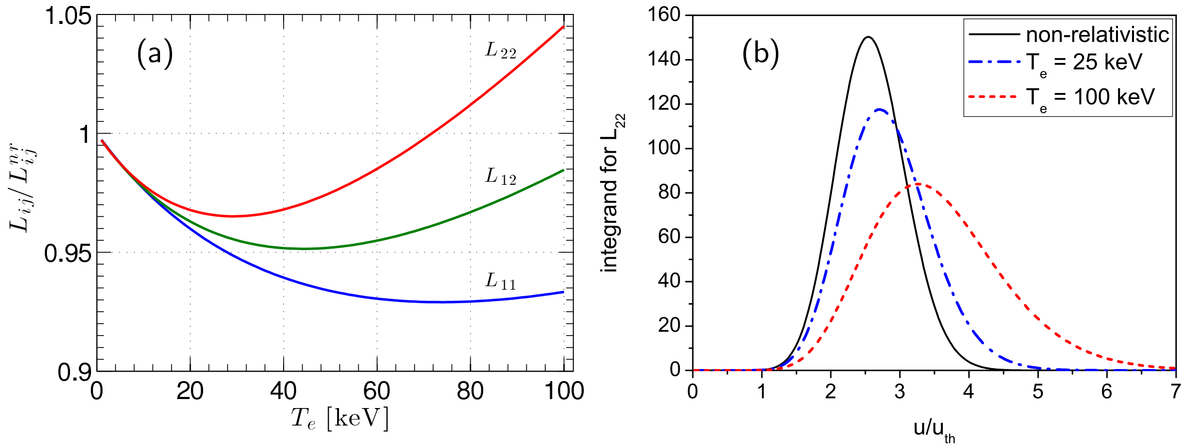

Figure 1. [Color online] a) Relativistic transport coefficients L_{ij} divided by corresponding non-relativistic values L_{ij}^{nr} are shown as the function of electron temperature. b) Integrand of L_{22} (see Eq. (25)) is plotted for the different temperatures. The value $T_e = 1$ eV is taken as the non-relativistic limit.

Substituting in Eq. (22) the derivative of the Maxwell-Jüttner distribution function Eq. (7), the radial fluxes can be expressed as

$$J_i = -n_e \sum_{j=1,2} L_{ij} A_j, \tag{24}$$

where the thermodynamic forces A_1 and A_2 are defined in Eq. (8) and the transport coefficients are equal to

$$L_{ij} = G_0 C_{MJ} \int_0^\infty d\kappa \, \frac{\mathrm{e}^{-\kappa} \kappa^{5/2}}{\gamma \hat{\nu}_D(u)} \left(\frac{\gamma+1}{2} \right)^{5/2} h_i h_j, \tag{25}$$

with $i, j = 1, 2$. This definition satisfies Onsager symmetry.

4. Comparison of relativistic and non-relativistic radial fluxes

In this chapter, the role of relativistic effects in the radial neoclassical transport is examined. Using relativistic expression for $\nu_D(u)$ (see Appendix A), direct numerical integration in Eq. (25) can be done. For comparison, the expression for non-relativistic transport is appropriate. The latter can be obtained from Eq. (25) by letting $C_{MJ} = \gamma = 1$, $\kappa = v^2/v_{te}^2$, and with the non-relativistic expression for $\nu_D(v)$. Since the geometrical part of the transport coefficients is the same for both non-relativistic and relativistic approaches, the ratio of these quantities is a pure indicator of the relativistic effects.

In Fig. 1(a), the ratios $L_{ij}/L_{ij}^{\mathrm{nr}}$ are shown as a functions of T_e for $i, j = 1, 2$ (here and below, the label "nr" indicates the non-relativistic quantities). One can see that the correction provided by the relativistic effects is not very strong (less than 7% for this range of temperature). However, a non-monotonic temperature dependence is not intuitively expected and requires an interpretation.

In Fig. 1(b), the integrand for L_{22} is plotted as a function of u/u_{te} for different temperatures. One can see that a non-monotonic temperature dependence of $L_{ij}/L_{ij}^{\rm nr}$ can be explained by superposition of two counteracting relativistic effects. The first one appears due to a reduction of the contribution from the bulk of the distribution function and prevails in the low-temperature range, $T_e < 10$ keV, leading to a decrease of transport coefficients (note that the slope in Fig. 1(a) is almost the same for all transport coefficients in this temperature range). A decrease of the bulk contribution is caused by the specific feature of the Maxwell-Jüttner distribution function and can be estimated from C_{MJ} Eq. (5). The second effect is caused by a broadening of the energy-range of contributing electrons and the shift of the maximum of the integrand into higher energies, and this leads to an increase of the transport coefficients with temperature. The latter effect appears to be important at higher temperatures. As one can see from Fig. 1(b), in the non-relativistic limit ($T_e = 1$ eV) the major contribution is coming from the electrons with $u/u_{te} \sim 1-4$, while at higher temperatures this range becomes broader and for $T_e = 100$ keV the corresponding range is $u/u_{te} \sim 1.5 - 6.5$. This effect is weaker for L_{11} than for L_{12} and L_{22} due to the lower power of κ in the integrand in Eq. (25).

Unlike the non-relativistic case, the transport coefficients do not fully characterize the transport properties of a confined plasma (because of the relativistic factor $\mathcal{R}$ in A_1) and, consequently, the comparison of particle and energy fluxes is necessary as well. In order to make a comparison with the non-relativistic limit possible, let us consider two special cases: (a) $n_e' = \Phi' = 0$, and (b) $T_e' = 0$, and the corresponding fluxes can be written, respectively, as

$$J_i^{(a)} = -n_e \left[-\left(\frac{3}{2} + \mathcal{R}\right) L_{i,1} + L_{i,2} \right] \frac{T_e'}{T_e}, \tag{26a}$$

$$J_i^{(b)} = -n_e L_{i,1} \left(\frac{n_e'}{n_e} - \frac{e\Phi'}{T_e} \right). \tag{26b}$$

In both cases, the ratio $J_i/J_i^{\rm nr}$ no longer contains the gradients and can be easily calculated. Note that in the case (b), $\Gamma_e^\rho/\Gamma_e^{\rho,{\rm nr}} = L_{11}/L_{11}^{\rm nr}$, i.e. the relativistic correction for Γ_e^ρ is identical to L_{11} which is shown in Fig. 1(a).

Following Eqs. (20) and (26), the heat flux can also be represented in a similar manner:

$$q_e^{\rho,(a)}/T_e = -n_e \left[-\left(4 + 2\mathcal{R}\right) L_{12} + L_{22} + \left(\frac{15}{4} + 4\mathcal{R} + \mathcal{R}^2\right) L_{11} \right] \frac{T_e'}{T_e}, \tag{27a}$$

$$q_e^{\rho,(b)}/T_e = -n_e \left[L_{12} - \left(\frac{5}{2} + \mathcal{R}\right) L_{11} \right] \left(\frac{n_e'}{n_e} - \frac{e\Phi'}{T_e} \right). \tag{27b}$$

In Eq. (27), the Onsager symmetry, $L_{12} = L_{21}$, was used.

In Fig. 2, the ratios of $\Gamma_e^\rho/\Gamma_e^{\rho,{\rm nr}}$, $Q_e^\rho/Q_e^{\rho,{\rm nr}}$ and $q_e^\rho/q_e^{\rho,{\rm nr}}$ for both cases are shown. The same non-monotonic dependence as in the case of the transport coefficients is clearly indicated, and relativistic correction for the $1/\nu$ radial fluxes is found to be less then 10% for the temperature range checked.

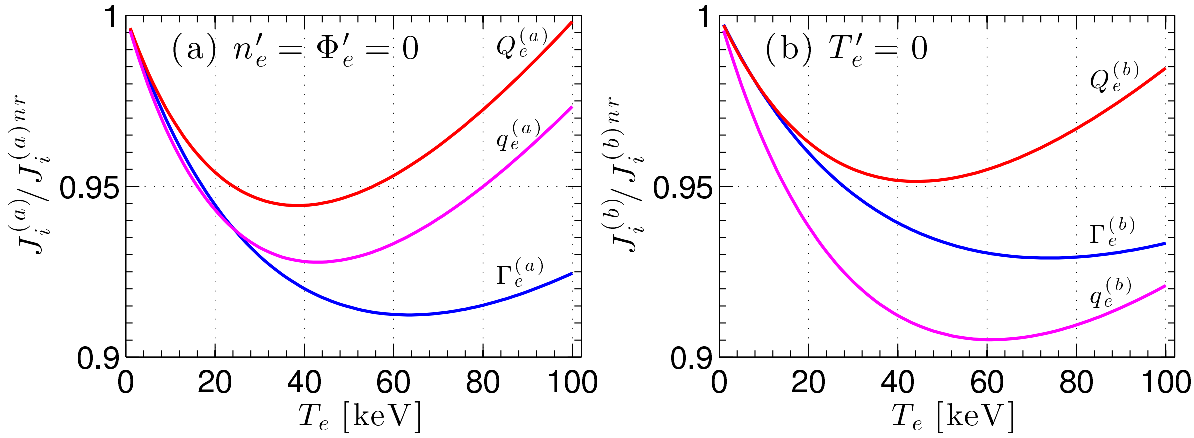

Figure 2. [Color online] The temperature dependence of the ratios J_i/J_i^{nr} for two cases, (a) and (b), respectively, is shown.

5. Summary

In this paper, the neoclassical radial fluxes for hot electrons in the $1/\nu$ regime, which is specific to stellarators, has been calculated in the relativistic approach. The choice of $1/\nu$ regime was motivated by a rather intuitive expectation that the role of relativistic effects in this regime should be most pronounced since the contribution to the radial transport from the tail of the distribution function is the largest (the integrand in the transport coefficients for the non-relativistic electrons in this regime scales in lowest order as $\propto v^7$).

For calculations, the reduced mono-energetic relativistic DKE was derived. Apart from the radial particle and energy fluxes, also the expression for the relativistic conductive heat flux was obtained. The definition for the radial fluxes in the relativistic approach has an important feature: the relativistic effects enter in the fluxes not only through the distribution function, but also through an additional temperature-dependent term in the first thermodynamic force. This relativistic term depends only on the temperature, in contrast to the canonical set of radial thermodynamic forces in which the logarithmic gradients of plasma parameters appear. Nevertheless, use of the proposed formulation has a big advantage: the transport coefficients with the relativistic effects taken into account can be calculated by the same numerical solvers which solve the non-relativistic DKE directly.

Following Ref. [16], the radial fluxes were calculated from the relativistic mono-energetic DKE and the results obtained were compared with the corresponding non-relativistic quantities. It was found that the relativistic effects for hot electrons produce a modest, but systematic reduction of the radial transport (up to 10% within the temperature range relevant for fusion). However, a non-monotonic temperature dependence of the transport coefficients is somewhat surprising. This behavior is the result of two counteracting factors present for relativistic kinetics. The first factor is related to a reduction in the relativistic Maxwellians of the weight of bulk electrons with an increase of the temperature. The second factor is caused by a broadening of the energy-range and a shift of the maximum contribution to higher energies.

This initial investigation confirms the intuitive expectation of an absence of strong relativistic effects in the radial transport in stellarator fusion plasmas. At the same time, this conclusion is not general and a similar check must also be made for the banana-regime in tokamaks. Apart from this, maybe the most important task is the calculation of the parallel electron fluxes with the relativistic effects taken into account. Based on the results provided in this paper, one may expect that within the non-relativistic neoclassical treatment both the electron radial fluxes in the banana regime and the electron bootstrap current in hot plasmas are somewhat overestimated.

Acknowledgments

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Appendix A. Relativistic expressions for deflection frequencies

Expressing the deflection frequency for the test-particle a immersed in the background b through the diffusion coefficient for pitch-angle scattering, $\nu_D^{ab}(u) = (2/u^2)D_{\theta\theta}^{ab}(u)$, and taking the general relativistic definition for $D_{\theta\theta}^{ab}(u)$ from Ref. [4], the general expression for $\nu_D^{ee}(u)$ with the relativistic Maxwellian can be written as follows:

$$\begin{aligned}\nu_D^{ee}(u) &= \nu_{e0} C_{MJ}(\mu_r) \frac{4}{\sqrt{\pi}} \times \\ &\left(\frac{\gamma}{u^3} \int\limits_0^u \left[\gamma' - 2\left(\frac{c^2}{u^2} + \frac{1}{\gamma^2}\right) j'_{0[2]02} + \frac{8}{\gamma^2}\frac{c^2}{u^2} j'_{0[3]022} \right] \frac{u'^2}{\gamma'} \mathrm{e}^{-\mu_r(\gamma'-1)} du' + \right. \\ &\left. \frac{\gamma}{u^2} \int\limits_u^\infty \left[\frac{\gamma'^2}{\gamma} - 2\left(\frac{c^2}{u^2} + \frac{u'^2}{u^2\gamma^2}\right) j_{0[2]02} + \frac{8}{\gamma^2}\frac{c^2}{u^2} j_{0[3]022} \right] \frac{u'}{\gamma'} \mathrm{e}^{-\mu_r(\gamma'-1)} du' \right).\end{aligned} \tag{A.1}$$

The specific functions $j_{l[k]*}(z)$ [4] are given by:

$$\begin{aligned} j_{0[2]02}(z) &= (z\gamma - \sigma)/4z, \\ j_{0[3]022}(z) &= [-3z\gamma + (3 + 2z^2)\sigma]/32z, \end{aligned} \tag{A.2}$$

where $\sigma(z) = \ln(z + \gamma)$ with $\gamma = \sqrt{1 + z^2}$ and $z = u/c$. Since the leading order for $z \ll 1$ is $j_{0[2]02} \simeq z^2/6$ and $j_{0[3]022} \simeq z^4/120$, the non-relativistic limit, $c \to \infty$, can be easily obtained [17, 18],

$$\nu_D^{ee}(v) = \nu_{e0} \frac{1}{x^3} \left[\left(1 - \frac{1}{2x^2}\right) \mathrm{erf}(x) + \frac{\mathrm{erf}'(x)}{2x} \right], \tag{A.3}$$

where $x = v/v_{te}$.

For ν_D^{ei}, the ion background can be taken as a non-relativistic Maxwellian. For calculations, it is sufficient to apply the high-speed-limit:

$$\nu_D^{ei}(u) = \nu_{e0} Z_{\rm eff} \frac{\gamma u_{te}^3}{u^3}. \tag{A.4}$$

One can see that the non-relativistic limit for this expression also exactly coincides with the classical approach.

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