

A note on the resolution of the entropy discrepancy

Yue Huang^{*†}, Rong-Xin Miao[‡]

**State Key Laboratory of Theoretical Physics, Institute of Theoretical Physics,
Chinese Academy of Sciences, Beijing, 100190*

*†Kavli Institute for Theoretical Physics China,
Chinese Academy of Sciences, Beijing, 100190*

*‡Max Planck Institute for Gravitational Physics (Albert Einstein Institute)
Am Mühlenberg 1, 14476 Golm, Germany*

Abstract

It is found by Hung, Myers and Smolkin that there is entropy discrepancy between the field theoretical and the holographic results for the CFTs in 6d spacetime. Recently, there appears two different proposals for the resolution of this puzzle. One proposes to use the anomaly of entropy and the generalized Wald entropy to resolve the HMS puzzle. While the other one suggests to use the entropy of total derivatives to explain the HMS mismatch. We investigate these two proposals carefully in this note. By studying the example of Einstein gravity, we find that it is the proposal of [2] rather than the one of [3, 4] that can solve the HMS puzzle. Besides, we find that there is arbitrariness in the derivations of Wald entropy. And only the total entropy is well-defined.

^{*}Email: huangyue@itp.ac.cn

[‡]Email: rong-xin.miao@aei.mpg.de

Contents

1	Introduction	1
2	The HMS mismatch	2
3	The APS proposal	3
4	The MG proposal	4
5	Further support	5
6	The arbitrariness of Wald entropy	6
7	Conclusion	8
A	Detailed calculations	8

1 Introduction

Hung, Myers and Smolkin (HMS) find that the logarithmic term of entanglement entropy derived from the field theoretical approach does not match the holographic result for 6d CFTs [1]. Recently, two different approaches are proposed to resolve this puzzle. One proposes to use the anomaly of entropy and the generalized Wald entropy from the Weyl anomaly to solve this puzzle [2]. While the other one suggests to use the entropy from total derivatives in the Weyl anomaly to explain the HMS mismatch [3, 4]. However, the results of [3, 4] are highly based on the FPS regularization [5]. By applying the LM regularization [6] or Dong's regularization [7] instead, it is found that the entropy of covariant total derivatives is indeed trivial [8]. This implies that the proposal of [3, 4] is unreliable. In this note, we provide further evidence that the HMS puzzle can not be solved by the approach of [3, 4].

It is really counterintuitive that total derivatives could contribute to non-trivial entropy. If so, the logarithmic term of entanglement entropy (EE) for CFTs would lose conformal invariance and depend on the approaches of regularization [3, 4]. That is completely unacceptable. Entropy is physical and thus should be independent of the regularizations one choose. The authors of [3, 4] argue that this is not a problem for 4d CFTs. That is because no total derivatives appear in the holographic Weyl anomaly in 4d spacetime [9]. However, total derivatives do appear in the holographic Weyl anomaly in 6d spacetime. In the holographic analysis, these total derivatives are fixed. The authors of [3, 4] propose to use the entropy from these total derivatives to explain the HMS mismatch. They did not consider all of the total derivatives but only choose some of them to explain the HMS mismatch [4].

In this note, we apply the method of [3, 4] to investigate the logarithmic term of EE for 6d CFTs dual to Einstein gravity. We consider all of the total derivatives in the holographic Weyl anomaly and

find that the field theoretical result does not match the holographic one. Thus, the proposal of [3, 4] does not resolve the HMS puzzle [1].

The paper is organized as follows. In Sect. 2, we briefly review the HMS mismatch [1] and two possible resolutions [2, 3, 4]. In Sect. 3, we use the methods of [3, 4] to calculate the logarithmic term of EE for 6d CFTs dual to Einstein gravity. It turns out their proposal can not explain the HMS mismatch. In Sect. 4, we apply the methods of [2] to study the logarithmic term of EE for 6d CFTs dual to Einstein gravity. We find the proposal of [2] can indeed resolve the HMS puzzle. In Sect.5, we provide further evidence that it is the proposal of [2] rather than those of [3, 4] that can resolve the HMS puzzle. In sect.6, we show that there is arbitrariness in the derivations of Wald entropy. And only the total entropy is well-defined. We conclude in Sect. 7.

2 The HMS mismatch

In this section, we briefly review the HMS mismatch [1]. It was found by Hung, Myers and Smolkin find that the logarithmic term of entanglement entropy derived from the field theoretical approach does not match the holographic result for 6d CFTs [1]. For simplicity, HMS focus on the cases with zero extrinsic curvature.

In the field theoretical approach, the logarithmic term of EE can be derived as the entropy of the Weyl anomaly [1, 5]. In six dimensions, the trace anomaly takes the following form

$$\langle T^i{}_i \rangle = \sum_{n=1}^3 B_n I_n + 2A E_6 + \nabla_i \hat{J}^i, \quad (1)$$

where E_6 is the Euler density, $\nabla_i J^i$ are total derivatives and I_i are conformal invariants defined by

$$\begin{aligned} I_1 &= C_{kijl} C^{imnj} C_m{}^{kl}{}_n, & I_2 &= C_{ij}{}^{kl} C_{kl}{}^{mn} C_{mn}{}^{ij}, \\ I_3 &= C_{iklm} (\nabla^2 \delta_j^i + 4R^i{}_j - \frac{6}{5} R \delta_j^i) C^{jklm}. \end{aligned} \quad (2)$$

For entangling surfaces with the rotational symmetry, only Wald entropy contributes to holographic entanglement entropy. Thus, we have

$$S_{\text{FEE}} = \log(\ell/\delta) \int d^4x \sqrt{h} \left[2\pi \sum_{n=1}^3 B_n \frac{\partial I_n}{\partial R^{ij}{}_{kl}} \tilde{\varepsilon}^{ij} \tilde{\varepsilon}_{kl} + 2A E_4 \right]_{\Sigma}, \quad (3)$$

where

$$\frac{\partial I_1}{\partial R^{ij}{}_{kl}} \tilde{\varepsilon}^{ij} \tilde{\varepsilon}_{kl} = 3 \left(C^{jmnk} C_m{}^{il}{}_n \tilde{\varepsilon}_{ij} \tilde{\varepsilon}_{kl} - \frac{1}{4} C^{iklm} C^j{}_{klm} \tilde{g}_{ij}^{\perp} + \frac{1}{20} C^{ijkl} C_{ijkl} \right), \quad (4)$$

$$\frac{\partial I_2}{\partial R^{ij}{}_{kl}} \tilde{\varepsilon}^{ij} \tilde{\varepsilon}_{kl} = 3 \left(C^{klmn} C_{mn}{}^{ij} \tilde{\varepsilon}_{ij} \tilde{\varepsilon}_{kl} - C^{iklm} C^j{}_{klm} \tilde{g}_{ij}^{\perp} + \frac{1}{5} C^{ijkl} C_{ijkl} \right), \quad (5)$$

$$\begin{aligned} \frac{\partial I_3}{\partial R^{ij}{}_{kl}} \tilde{\varepsilon}^{ij} \tilde{\varepsilon}_{kl} &= 2 \left(\square C^{ijkl} + 4R^i{}_m C^{mjkl} - \frac{6}{5} R C^{ijkl} \right) \tilde{\varepsilon}_{ij} \tilde{\varepsilon}_{kl} - 4 C^{ijkl} R_{ik} \tilde{g}_{jl}^{\perp} \\ &\quad + 4 C^{iklm} C^j{}_{klm} \tilde{g}_{ij}^{\perp} - \frac{12}{5} C^{ijkl} C_{ijkl}. \end{aligned}$$

The logarithmic term of EE can also be derived from the holographic entanglement entropy. We call this method as the holographic approach. Take Einstein gravity as an example. The logarithmic term of EE is given by [1]

$$S_{\text{HEE}} = 2\pi \log(\ell/\delta) \frac{\tilde{L}^5}{\ell_{\text{P}}^5} \int_{\Sigma} d^4x \sqrt{h} \left[\frac{1}{2} h^{ij} \overset{(2)}{g}_{ij} + \frac{1}{8} (h^{ij} \overset{(1)}{g}_{ij})^2 - \frac{1}{4} \overset{(1)}{g}_{ij} h^{jk} \overset{(1)}{g}_{kl} h^{li} \right] \quad (6)$$

The mismatch between holographic result eq.(6) and field theoretical result eq.(3) becomes

$$\begin{aligned} \Delta S = -4\pi B_3 \log(\ell/\delta) \int_{\Sigma} d^4x \sqrt{h} \quad & (C_{mn}{}^{rs} C^{mnkl} \tilde{g}_{sl}^{\perp} \tilde{g}_{rk}^{\perp} - C_{mnr}{}^s C^{mnr} \tilde{g}_{sl}^{\perp} \\ & + 2C_m{}^n{}_r{}^s C^{mkr} \tilde{g}_{ns}^{\perp} \tilde{g}_{kl}^{\perp} - 2C_m{}^n{}_r{}^s C^{mkr} \tilde{g}_{nl}^{\perp} \tilde{g}_{ks}^{\perp}), \end{aligned} \quad (7)$$

This is the famous HMS mismatch. Note that the above equations are derived in the case of zero extrinsic curvatures.

It is proposed to use the anomaly of entropy and the generalized Wald entropy to explain the HMS mismatch in [2]. When the extrinsic curvatures vanish, only $C_{ijkl}^2 C^{ijkl} \simeq -\nabla_m C_{ijkl} \nabla^m C^{ijkl}$ in I_3 contributes to non-zero anomaly of entropy. Take into account these contributions, the field theoretical and the holographic results match exactly. Note that the entropy of total derivatives vanishes by applying the LM regularization [6] or Dong's regularization [8]. However, the authors of [3, 4] claim that, in addition to $-\nabla_m C_{ijkl} \nabla^m C^{ijkl}$, the total derivatives $B_3 \nabla_m (C_{ijkl} \nabla^m C^{ijkl}) + \nabla_i \hat{J}^i$ also contribute to the logarithmic terms of EE. They find that the entropy of total derivatives is non-zero by applying the FPS regularization [5]. And they propose to use the entropy from total derivatives to explain the HMS puzzle [3, 4].

3 The APS proposal

In this section, we use the method of [3, 4] to calculate the entropy from total derivatives carefully. It turns out that the field theoretical result does not match the holographic one. Thus, the proposal of [3, 4] does not solve the HMS puzzle.

In the field theoretical approach, the logarithmic term of EE can be derived from the entropy of the Weyl anomaly [1, 5]. For Einstein gravity, the holographic Weyl anomaly is given by [10, 11].

$$\langle T^i{}_i \rangle = 2\pi^3 E_6 - \frac{1}{16} I_1 - \frac{1}{64} I_2 + \frac{1}{192} (I_3 - C_5) + \nabla_i J^i \quad (8)$$

where we have set the Newton's constant $G = \frac{1}{16\pi}$ and the AdS radius $l = 1$. $C_5 = \frac{1}{2} \square C_{ijkl} C^{ijkl}$ and the total derivative is given by [11]

$$\nabla_i J^i = \frac{1}{960} (15C_3 - 18C_4 - 3C_6 + 20C_7) \quad (9)$$

with C_k defined as

$$\begin{aligned} C_3 &= \nabla_i [R_{mn} \nabla^i R^{mn} - \frac{1}{6} R \nabla^i R] \\ C_4 &= \nabla_i [R_{mn} \nabla^m R^{in} - \frac{1}{3} R^{im} \nabla^n R_{mn} - \frac{1}{18} R \nabla^i R] \end{aligned}$$

$$\begin{aligned}
C_6 &= \nabla_i \left[\frac{1}{2} R^{im} \nabla_m R - R_{mn} \nabla^m R^{in} \right] \\
C_7 &= \nabla_i \left[R^{kmni} \nabla_k R_{nm} + \frac{1}{4} R_{mnkl} \nabla^i R^{mnkl} + \frac{1}{8} R_{im} \nabla_m R - \frac{1}{4} R_{mn} \nabla^m R^{in} \right]. \tag{10}
\end{aligned}$$

In the case of zero extrinsic curvatures, the entropy of E_6, I_1, I_2 reduces to Wald entropy. Thus, the HMS mismatch can only come from $(I_3 - C_5)$ and $\nabla_i J^i$. Interestingly, although [2] and [4] use different approaches of regularizations, they both find that the total entropy minus the Wald entropy of $(I_3 - C_5)$ can explain the HMS mismatch. The LM regularization [6] or Dong's regularization [7] is used in [2]. As a result the entropy of total derivatives is trivial in the approach of [2]. While a different regularization [5] is used in [4]. It turns out that the entropy of total derivatives is non-zero in their approach [4]. To resolve the HMS puzzle in the approach of [4], it is necessary to prove that the entropy from the total derivative $\nabla_i J^i$ eq.(9) vanishes. This is, however, not the case.

Now let us focus on the FPS regularization [5]. For simplicity, we use the following regularized conical metric

$$ds^2 = f_n(r) dr^2 + r^2 d\tau^2 + (\delta_{ij} + 2\tilde{H}_{ij} r^{2n} \cos t \sin t) dy^i dy^j, \tag{11}$$

where $f_n = \frac{r^2 + b^2 n^2}{r^2 + b^2}$ and $\tau \sim \tau + 2n\pi$. Following the approaches of [3, 4], we obtain

$$\begin{aligned}
& \int_0^{2\pi n} d\tau \int_0^{r_0} dr \int dy^{D-2} \sqrt{g} \nabla_i J^i = \int_0^{2\pi n} d\tau \int dy^{D-2} \sqrt{g} J^r \Big|_{r=0}^{r=r_0} \\
&= \int dy^{D-2} \frac{\pi}{40} (n-1) (bx)^{4(n-1)} \frac{x^8 (tr\tilde{H})^2 + c_1 x^6 + c_2 x^4 + c_3 x^2 + c_4}{(1+x^2)^4} \Big|_{x=0}^{x=\infty} + O(n-1)^2 \\
&= \int dy^{D-2} \frac{\pi}{40} (n-1) (tr\tilde{H})^2 + O(n-1)^2 \tag{12}
\end{aligned}$$

where we have replaced r by bx in the above derivations. Note that [3, 4] choose to drop the contribution at $r = 0$ ($x = 0$). Thus c_k are irrelevant to the final results. From eq.(12), we can derive the entropy of eq.(9) as

$$S_{\text{APSTD}} = - \lim_{n \rightarrow 1} \partial_n \left(\int dx^D \sqrt{g} \nabla_i J^i \right) = - \frac{\pi}{40} \int dy^{D-2} (tr\tilde{H})^2 \tag{13}$$

which is non-zero. Note that eq.(13) works in the Lorentzian signature, which differs from its Euclidean form by a minus sign. Now it is clear that the proposal of [3, 4] can not solve the HMS puzzle [1]. In other words, the field theoretical and the holographic results of the logarithmic term of EE does not match in the approach of [3, 4].

4 The MG proposal

In this section, we prove that the entropy of the total derivative eq.(9) indeed vanishes in the approach of [2]. Thus, the proposal of [2] does resolve the HMS puzzle [1]. We apply Dong's regularization [7] instead of the FPS regularization [5] in this section.

Let us focus the following regularized conical metric

$$ds^2 = \frac{1}{(r^2 + b^2)^{1-\frac{1}{n}}} (dr^2 + r^2 d\tau^2) + (\delta_{ij} + 2\tilde{H}_{ij} r^2 \cos t \sin t) dy^i dy^j \tag{14}$$

with $\tau \sim \tau + 2\pi$. For the total derivative eq.(9), we firstly expand it in powers of \tilde{H} and then do the τ integral. It turns out that only the \tilde{H}^2 terms contribute to the entropy. The other terms are either in higher order $O(n-1)^2$ or vanishing in the limit $b \rightarrow 0$. Focus on the \tilde{H}^2 terms, we have

$$\begin{aligned} \int dx^D \sqrt{g} \nabla_i J^i &= b^{4-\frac{4}{n}} \int_0^\infty dx \int dy^4 \frac{\pi x}{60(x^2+1)^{\frac{2}{n}+4}} [(n-1) \sum_{k=0}^4 q_{2k} x^{2k} \\ &+ (n-1)^2 [(40tr\tilde{H}^2 - 10(tr\tilde{H})^2)x^{10} + \sum_{k=0}^4 p_{2k} x^{2k}] + O(n-1)^3] \end{aligned} \quad (15)$$

where $r = bx$ and q_{2k} are given by

$$\begin{aligned} q_0 &= 28tr\tilde{H}^2 - (tr\tilde{H})^2, \quad q_2 = 12(10tr\tilde{H}^2 - 3(tr\tilde{H})^2), \\ q_4 &= 2(78tr\tilde{H}^2 - 33(tr\tilde{H})^2), \quad q_6 = 4(16tr\tilde{H}^2 - 7(tr\tilde{H})^2), \\ q_8 &= 3(tr\tilde{H})^2 \end{aligned} \quad (16)$$

p_{2k} are irrelevant to the entropy. We find the following formulas are useful ¹

$$\begin{aligned} \int_0^\infty \frac{xdx}{(1+x^2)^{4+\frac{2}{n}}} &= \int_0^\infty \frac{x^9 dx}{(1+x^2)^{4+\frac{2}{n}}} = \frac{1}{10} + O(n-1) \\ \int_0^\infty \frac{x^3 dx}{(1+x^2)^{4+\frac{2}{n}}} &= \int_0^\infty \frac{x^7 dx}{(1+x^2)^{4+\frac{2}{n}}} = \frac{1}{40} + O(n-1) \\ \int_0^\infty \frac{x^5 dx}{(1+x^2)^{4+\frac{2}{n}}} &= \frac{1}{60} + O(n-1) \\ \int_0^\infty \frac{x^{11} dx}{(1+x^2)^{4+\frac{2}{n}}} &= \frac{60\Gamma(\frac{2}{n}-2)}{\Gamma(4+\frac{2}{n})} = -\frac{1}{4(n-1)} + O(n-1)^0 \end{aligned} \quad (17)$$

From the above equations together with $b^{4-\frac{4}{n}} = 1 + O(n-1)$, we can derive the entropy of the total derivative eq.(9) as

$$S_{\text{MGTD}} = \lim_{n \rightarrow 1} \partial_n \left(\int dx^D \sqrt{g} \nabla_i J^i \right) = 0, \quad (18)$$

which indeed vanishes. Thus the proposal of [2] does solve the HMS puzzle [1].

5 Further support

In this section, we provide further support that it is the proposal of [2] rather than those of [3, 4] that can resolve the HMS puzzle. We calculate the entropy for all of the terms in the Weyl anomaly eq.(8) by using the methods of [2] and [3, 4], respectively. It turns out that only the methods of [2] can yield consistent results with the holographic ones.

¹In principle, one should firstly integrate x from 0 to (r_0/b) and then subtract off the contributions from the singular cone with $b = 0$. The detailed approach can be found in [8]. In general, there are non-universal terms which depend on r_0 and the universal terms in the integral. Only the universal terms survive once we subtract off the contributions from the singular cone. Here we use a simpler method. We integrate x from 0 to ∞ for some suitable range of n , and then do the analytic continuation for n . It turns out that only the universal terms appear in the results. Thus the method here produces the same results as the one of [8].

In the holographic approach, the universal terms of EE for 6d CFTs dual to Einstein gravity is given by [1]

$$S_{\text{HEE}} = \pi \log(\ell/\delta) \int_{\Sigma} d^4y \sqrt{h} [2g_{\hat{i}\hat{i}}^{(2)} - g_{\hat{i}\hat{j}}^{(1)} g^{\hat{j}\hat{j}} + \frac{1}{2} (g_{\hat{i}\hat{i}}^{(1)})^2] \quad (19)$$

The above formula applies to the case with zero extrinsic curvatures. For the general case, please see [12]. For the conical metrics eqs.(11,14) with $b = 0$ and $n = 1$, the above equation becomes

$$S_{\text{HEE}} = -\frac{\pi}{40} \log(\ell/\delta) \int_{\Sigma} d^4y \sqrt{h} [8\text{tr}\tilde{H}^2 - (\text{tr}\tilde{H})^2] \quad (20)$$

Let us rewrite the holographic Weyl anomaly eq.(21) in the initial form of [10]

$$\langle T_i^i \rangle = \frac{1}{32} \left(-\frac{1}{2} R R^{ij} R_{ij} + \frac{3}{50} R^3 + R^{ij} R^{kl} R_{ikjl} - \frac{1}{5} R^{ij} \nabla_i \nabla_j R + \frac{1}{2} R^{ij} \square R_{ij} - \frac{1}{20} R \square R \right) \quad (21)$$

Note that the curvature in our notation is different from the one of [10] by a minus sign.

Using the methods of [3, 4, 5] together with the metric eq.(11), we obtain the total entropy of eq.(21) in the Lorentzian signature as

$$S_{\text{APS}} = -\frac{\pi}{5} \int_{\Sigma} d^4y \sqrt{h} [\text{tr}\tilde{H}^2], \quad (22)$$

which does not match the holographic result eq.(20) at all.

Applying the methods of [2, 7] with the conical metric eq.(14), we derive the total entropy of eq.(21) in the Lorentzian signature as

$$S_{\text{MG}} = -\frac{\pi}{40} \int_{\Sigma} d^4y \sqrt{h} [8\text{tr}\tilde{H}^2 - (\text{tr}\tilde{H})^2] \quad (23)$$

which exactly agrees with the holographic result eq.(20). Please refer to the Appendix for the derivations of eqs.(22,23). Recall that the entropy of E_6, I_1, I_2 and $(I_3 - C_5)$ is the same in the approaches of [2] and [3, 4] when the extrinsic curvatures vanish. And the only difference of the entropy in these two approaches comes from the total derivatives $\nabla_i J^i$. Thus it is expected that we have $S_{\text{MGTD}} - S_{\text{APSTD}} = S_{\text{MG}} - S_{\text{APS}}$. From eqs.(13,18,22,23), we find that this is indeed the case. This can be regarded as a check of our calculations.

Now it is clear that it is the proposal of [2] rather than those of [3, 4] that can solve the HMS puzzle.

6 The arbitrariness of Wald entropy

In this section, we show that Wald entropy is not a well-defined physical quantity. In general, it is inconsistent with the Bianchi identities. This is not surprising. In addition to Wald entropy, the anomaly of entropy [5, 7, 13] and the generalized Wald entropy [2] also contribute to the total entropy. It does not matter as long as the total entropy is well-defined. This is indeed the case. It should be mentioned that the arbitrariness of Wald entropy does not affect our above discussions, since we always focus on the total entropy in this note.

Let us take an example to illustrate the arbitrariness of Wald entropy

$$S_{\text{Wald}} = -2\pi \int_{\Sigma} dx^{D-2} \sqrt{h} \frac{\delta L}{\delta R_{ijkl}} \epsilon_{ij} \epsilon_{kl}. \quad (24)$$

We work in Euclidean signature in this section. Thus we have $\epsilon_{ij} \epsilon^{ij} = 2$, $\epsilon^{im} \epsilon^j_m = \tilde{g}^{\perp ij}$. From the Bianchi identities, we have

$$\frac{1}{4} \nabla_i R \nabla^i R = \nabla_i R^{im} \nabla_j R^j_m \quad (25)$$

The Wald entropy of the left hand side of eq.(25) is given by

$$2\pi \int_{\Sigma} dx^{D-2} \sqrt{h} \square R \quad (26)$$

And the Wald entropy of the right hand side of eq.(25) is given by

$$2\pi \int_{\Sigma} dx^{D-2} \sqrt{h} \tilde{g}^{\perp ij} \nabla_i \nabla_j R = 2\pi \int_{\Sigma} dx^{D-2} \sqrt{h} [\square R - D_i D^i R + k^a \nabla_a R] \quad (27)$$

where D_i are the intrinsic covariant derivatives and $k^a = k^a_{ij} g^{ij}$ are the traces of the extrinsic curvatures. Clearly, eq.(26) and eq.(27) are different for the cases with non-zero extrinsic curvatures. This implies that, in general, the Wald entropy is not a well-defined physical quantity. It should be mentioned Wald entropy works well for entangling surfaces Σ with the rotational symmetry. Thus nothing goes wrong in the initial work of Wald [14]. For entangling surfaces Σ with the rotational symmetry, Wald entropy becomes the total entropy and thus must be well-defined.

The total entropy of left hand side and the right hand side of eq.(25) can be calculated by using the methods of the appendix. Clearly, both sides give the same results. That is because eq.(25) is an identity. Thus the left hand side and the right hand side of eq.(25) make no differences in the approach of the appendix. This implies only the total entropy is well-defined. On the other hand, there is arbitrariness in the derivations of the Wald entropy. The Wald entropy changes when one rewrite the action into an equivalent form by using the Bianchi identities.

Let us consider another example. Let us rewrite the total derivative C_6 eq.(10) into two equivalent expressions. The first one is

$$\bar{C}_6 = \frac{1}{4} \nabla_i R \nabla^i R - \nabla_i R_{mn} \nabla^m R^{in} + R^{ij} \nabla_i \nabla_j R - 2R^{ij} \nabla_{(i} \nabla_{k)} R^k_j \quad (28)$$

and the second one is [15]

$$\hat{C}_6 = \frac{1}{4} \nabla_i R \nabla^i R - \nabla_i R_{mn} \nabla^m R^{in} + R_{ij} R_{kl} R^{ikjl} - R^i_j R^j_k R^k_i \quad (29)$$

After some calculations, we derive the Wald entropy of \bar{C}_6 and \hat{C}_6 as

$$\bar{S}_{\text{Wald}} = 0, \quad (30)$$

$$\hat{S}_{\text{Wald}} = 2\pi \int_{\Sigma} dx^{D-2} \sqrt{h} [\square R - \tilde{g}^{\perp ij} \nabla_i \nabla_j R + \tilde{g}^{\perp ij} R_{im} R^m_j - R_{ij} R_{kl} (\tilde{g}^{\perp ij} \tilde{g}^{\perp kl} - \tilde{g}^{\perp il} \tilde{g}^{\perp kj})] \quad (31)$$

Remarkably, although the total derivatives \bar{C}_6 and \hat{C}_6 are equivalent, they give different Wald entropy². This clearly shows that Wald entropy is not a well-defined physical quantity. There is too much arbitrariness in its derivations. On the other hand, the total entropy is indeed well defined. One can check that the total entropy of \hat{C}_6 and \bar{C}_6 is both zero by using the MG approach [2]. By applying the APS approach [3, 4] instead, the total entropy of \hat{C}_6 and \bar{C}_6 is non-zero, but still the same.

In conclusion, the Wald entropy itself makes no sense. There is too much arbitrariness in its derivations. Instead, only the total entropy consisted of Wald entropy[14], the generalized Wald entropy [2] and the anomaly of entropy [5, 7, 13] is well-defined.

7 Conclusion

Recently, there appears two different proposals for the resolution of HMS puzzle. One proposes to use the entropy of total derivatives to explain the HMS mismatch. While the other one proposes to use the anomaly of entropy and the generalized Wald entropy to resolve the HMS puzzle. We investigate these two proposals carefully in this note. By studying the example of Einstein gravity, we find that it is the proposal of [2] rather than those of [3, 4] that can resolve the HMS puzzle. This means that it is Dong's regularization [7] rather than the FPS regularization that yields the correct results for the entropy. It is a strong support of the work [8] that the covariant total derivatives do not contribute to non-trivial entropy. Finally, we find that there is arbitrariness in the derivations of Wald entropy. Thus, Wald entropy itself is not well-defined. It turns out that only the total entropy is well-defined. It should be mentioned that Wald entropy becomes the total entropy and thus is well-defined in stationary spacetime. Thus nothing goes wrong in the initial work of Wald [14].

Acknowledgements

R. X. Miao is supported by Sino-Germann (CSC-DAAD) Postdoc Scholarship Program. Y. Huang and R. X. Miao thank School of Astronomy and Space Science at Sun Yat-Sen University for hospitality.

A Detailed calculations

In this appendix, we provide some details for the derivations of eqs.(22,23).

By using the FPS regularization eq.(11), we can derive eq. (22). Firstly, we expand the holographic Weyl anomaly eq.(21) in powers of \tilde{H} and then do the τ integral. Here we take n as an integer. Secondly, we do the analytic continuation for n and expand the results around $n = 1$. We keep terms up to the order $O(n - 1)^2$. Finally, we do the r integral and select the terms in order $O(n - 1)$. It turns out that only the \tilde{H}^2 terms contribute to the entropy. The other terms are either in higher

²Eq.(31) is derived independently in a recent work [15]. However, they do not realize that there is arbitrariness in the derivations of Wald entropy.

order $O(n-1)^2$ or vanishing in the limit $b \rightarrow 0$. Focus on the \tilde{H}^2 terms, we have

$$\begin{aligned}
\int dr d\tau dy^4 \sqrt{g} \langle T^i_i \rangle &= b^{4n-4} \int_0^\infty dx \int dy^4 \frac{\pi (x^{4n-3})}{20 (x^2+1)^6} \left[(n-1) \sum_{k=0}^4 d_{2k} x^{2k} \right. \\
&\quad \left. + (n-1)^2 \left[\sum_{k=0}^4 c_{2k} x^{2k} + 2(3(\text{tr}\tilde{H})^2 - 10\text{tr}\tilde{H}^2)x^{10} \right] \right. \\
&\quad \left. + O(n-1)^3 \right]
\end{aligned} \tag{32}$$

where $r = bx$ and d_{2k} are given by

$$\begin{aligned}
d_0 &= 4\text{tr}\tilde{H}^2, \quad d_2 = 6(\text{tr}\tilde{H})^2 + 16\text{tr}\tilde{H}^2, \\
d_4 &= 9((\text{tr}\tilde{H})^2 + 4\text{tr}\tilde{H}^2), \quad d_6 = 40\text{tr}\tilde{H}^2, \\
d_8 &= -3(\text{tr}\tilde{H})^2 + 16\text{tr}\tilde{H}^2
\end{aligned} \tag{33}$$

Note that c_{2k} are irrelevant to the final result, so we do not list them. The first line of eq.(32) contribute to the Wald-like entropy. The second line of eq.(32) are the would-be logarithmic terms. Naively, second line of eq.(32) is in order $O(n-1)^2$. It seems to be irrelevant to the entropy. However, it becomes in order $O(n-1)$ after the integral. The magic happens because the would-be logarithmic divergence gets a $\frac{1}{n-1}$ enhancement.

In general, we have two kind of would-be logarithmic terms. One is at $x \rightarrow 0$ and the other one is at $x \rightarrow \infty$.

$$\begin{aligned}
\int_0^\infty \frac{x^{4n-5} dx}{(1+x^2)^6} &= -\frac{1}{60} \pi (n-3)(n-2)(2n-7)(2n-5)(2n-3) \csc(2\pi n), \quad 1 < \Re(n) < 4 \\
&= \frac{1}{4(n-1)} + O(n-1)^0
\end{aligned} \tag{34}$$

$$\begin{aligned}
\int_0^\infty \frac{x^{4n+7} dx}{(1+x^2)^6} &= -\frac{1}{60} \pi n(n+1)(2n-1)(2n+1)(2n+3) \csc(2\pi n), \quad -2 < \Re(n) < 1 \\
&= \frac{-1}{4(n-1)} + O(n-1)^0,
\end{aligned} \tag{35}$$

It seems that the above two integrals could not be well-defined at the same time. Thus, the authors of [3, 4] choose to drop the would-be logarithmic term at infinity eq.(35). However, as pointed out in [8], we actually do not need the condition $n < 1$ to derive eq.(35). Note also that the results after analytic continuation are both well defined for $n < 1$ and $n > 1$. So there is no reason to drop such term. However, since we are using the methods of [3, 4], we adopt their choice in this paper. Note that the would-be logarithmic term at $x \rightarrow 0$ vanishes in our case. In addition to eq.(34), we find the following formulas are useful

$$\begin{aligned}
\int_0^\infty \frac{x^{4n-3} dx}{(1+x^2)^6} &= \int_0^\infty \frac{x^{4n+5} dx}{(1+x^2)^6} = \frac{1}{10} + O(n-1) \\
\int_0^\infty \frac{x^{4n-1} dx}{(1+x^2)^6} &= \int_0^\infty \frac{x^{4n+3} dx}{(1+x^2)^6} = \frac{1}{40} + O(n-1) \\
\int_0^\infty \frac{x^{4n+1} dx}{(1+x^2)^6} &= \frac{1}{60} + O(n-1)
\end{aligned} \tag{36}$$

Using eqs.(34,36) together with $b^{4n-4} = 1 + O(n-1)$, we can derive

$$\int drd\tau dy^4 \sqrt{g} \langle T_i^i \rangle = \frac{(n-1)\pi}{5} \int dy^4 [tr\tilde{H}^2] + O(n-1)^2 \quad (37)$$

Now we get the entropy eq.(22) in Lorentzian signature

$$S_{\text{APS}} = -\lim_{n \rightarrow 1} \partial_n \int drd\tau dy^4 \sqrt{g} \langle T_i^i \rangle = -\frac{\pi}{5} \int_{\Sigma} d^4y \sqrt{h_0} [tr\tilde{H}^2]. \quad (38)$$

Note that the entropy in Lorentzian signature differs from its Euclidean form by a minus sign. In the above derivations we have dropped the would-be logarithmic term at $x \rightarrow \infty$ as [3, 4]. Even if we recover this kind of term, the field theoretical result still does not match the holographic one.

Now let us turn to derivation of eq.(23). The calculation is very similar to the above one. The only difference is that now we use Dong's regularization for the conical metric eq.(14). We obtain

$$\int drd\tau dy^4 \sqrt{g} \langle T_i^i \rangle = b^{4-\frac{4}{n}} \int_0^\infty dx \int dy^4 \frac{\pi x}{100(x^2+1)^{\frac{2}{n}+4}} [(n-1) \sum_{k=0}^4 f_{2k} x^{2k} + O(n-1)^2] \quad (39)$$

where $r = bx$ and f_{2k} are given by

$$\begin{aligned} f_0 &= 5(3(tr\tilde{H})^2 - 4tr\tilde{H}^2), & f_2 &= -80tr\tilde{H}^2, \\ f_4 &= -30((tr\tilde{H})^2 + 6tr\tilde{H}^2), & f_6 &= -200tr\tilde{H}^2, \\ f_8 &= 15(tr\tilde{H})^2 - 80tr\tilde{H}^2 \end{aligned} \quad (40)$$

The following formulas are useful

$$\begin{aligned} \int_0^\infty \frac{xdx}{(1+x^2)^{4+\frac{2}{n}}} &= \int_0^\infty \frac{x^9 dx}{(1+x^2)^{4+\frac{2}{n}}} = \frac{1}{10} + O(n-1) \\ \int_0^\infty \frac{x^3 dx}{(1+x^2)^{4+\frac{2}{n}}} &= \int_0^\infty \frac{x^7 dx}{(1+x^2)^{4+\frac{2}{n}}} = \frac{1}{40} + O(n-1) \\ \int_0^\infty \frac{x^5 dx}{(1+x^2)^{4+\frac{2}{n}}} &= \frac{1}{60} + O(n-1) \end{aligned} \quad (41)$$

From eqs.(39,40,41), we can derive

$$\int drd\tau dy^4 \sqrt{g} \langle T_i^i \rangle = -\frac{(n-1)\pi}{40} \int d^4y [8tr\tilde{H}^2 - (tr\tilde{H})^2] + O(n-1)^2 \quad (42)$$

Now we obtain the entropy eq.(23) in the Lorentzian signature

$$S_{\text{MG}} = \lim_{n \rightarrow 1} \partial_n \int drd\tau dy^4 \sqrt{g} \langle T_i^i \rangle = -\frac{\pi}{40} \int_{\Sigma} d^4y \sqrt{h_0} [8tr\tilde{H}^2 - (tr\tilde{H})^2]. \quad (43)$$

Note that Dong's formula of entropy (the first equality of eq.(43)) [7] differs from the one of FPS (the first equality of eq.(38)) [5] by a minus sign.

References

- [1] L. Y. Hung, R. C. Myers and M. Smolkin, JHEP 1104 (2011) 025 [arXiv:1101.5813].

- [2] R. X. Miao and W. Z. Guo, [arXiv:1411.5579 [hep-th]].
- [3] A. F. Aстанeh, A Patrushev and S. N. Solodukhin, [arXiv:1411.0926[hep-th]].
- [4] A. F. Aстанeh, A Patrushev and S. N. Solodukhin, [arXiv:1412.0452[hep-th]].
- [5] D. V. Fursaev, A. Patrushev and S. N. Solodukhin, Phys.Rev. D88 (2013) 4, 044054. [arXiv:1306.4000 [hep-th]].
- [6] A. Lewkowycz and J. Maldacena, JHEP 1308 (2013) 090 [arXiv:1304.4926].
- [7] X. Dong, JHEP 1401 (2014) 044 [arXiv:1310.5713[hep-th]].
- [8] X. Dong and R. X. Miao, “Generalized Gravitational Entropy from Total Derivatives”, to appear.
- [9] A. Bhattacharyya, M. Sharma and A. Sinha, On generalized gravitational entropy, squashed cones and holography, JHEP 1401, 021 (2014) [arXiv:1308.5748 [hep-th]]. A. Bhattacharyya and M. Sharma, On entanglement entropy functionals in higher derivative gravity theories, JHEP 1410, 130 (2014) [arXiv:1405.3511 [hep-th]].
- [10] M. Henningson and K. Skenderis, JHEP 9807 (1998) 023.
- [11] F. Bastianelli, S. Frolov and A. A. Tseytlin, Conformal anomaly of (2,0) tensor multiplet in six-dimensions and AdS / CFT correspondence, JHEP 0002 (2000) 013 [hep- th/0001041].
- [12] R. X. Miao, ‘Universal Terms of Entanglement Entropy for 6d CFTs’, [arXiv:1503.05538[hep-th]].
- [13] J. Camps, JHEP 1403 (2014) 070 [arXiv:1310.6659[hep-th]].
- [14] R. M. Wald, Phys. Rev. D **48**, 3427 (1993).
- [15] A. F. Aстанeh and S. N. Solodukhin, [arXiv:1504.01653[hep-th]].