Universal Terms of Entanglement Entropy for 6d CFTs

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Abstract

We derive the universal terms of entanglement entropy for 6d CFTs by applying the holographic and the field theoretical approaches, respectively. Our formulas are conformal invariant and agree with the results of [34, 35]. Remarkably, we find that the holographic and the field theoretical results match exactly for the C^2 and Ck^2 terms. Here C and k denote the Weyl tensor and the extrinsic curvature, respectively. As for the k^4 terms, we meet the splitting problem of the conical metrics. The splitting problem in the bulk can be fixed by equations of motion. As for the splitting on the boundary, we assume the general forms and find that there indeed exists suitable splitting which can make the holographic and the field theoretical k^4 terms match. Since we have much more equations than the free parameters, the match for k^4 terms is non-trivial.

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Contents

1 Introduction

Entanglement entropy (EE) plays an important rule in the fields of gravity [1] and quantum many-body physics [2, 3]. It is non-local and provides a useful tool to probe the quantum correlations. It can be calculated by applying the holographic method [4, 5] and the perturbative approach [6]. For recent developments in EE, please refer to [7, 8, 9, 10, 11, 12, 13, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28]. The leading term of EE obeys the area law. However, in spacetime dimensions higher than two, it is not universal but depends on the cutoff of the system. In contrast to the leading term, the logarithmic term of EE in even spacetime dimensions is universal and thus is of great interest.

The logarithmic term of EE for CFTs in 2d is given by [29, 30]

$$S_\Sigma|_{\log} = \frac{c}{3} \log\big(\frac{L}{\pi\delta}\sin(\frac{\pi l}{L})\big) \tag{1}$$

where l and L are the length of the subsystem and total system, repesctively. δ denotes the cutoff and c is the central charge of the CFT.

The logarithmic term of EE for 4d CFTs is proposed by [31]

$$S_\Sigma|_{\log} = \log(\ell/\delta)\frac{1}{2\pi}\int_\Sigma \big[c(C^{ijkl}h_{ik}h_{jk} - trk^2 + \frac{1}{2}(trk)^2) - aR_\Sigma\big], \tag{2}$$

where C_{ijkl} is the Weyl tensor, k is the extrinsic curvature and R_Σ is the intrinsic Ricci scalar. a and c are the central charges of 4d CFTs. Eq.(2) is firstly derived by using the holographic entanglement entropy (HEE) of Einstein gravity [31]. Later, by applying Dong's formula [32], it is shown in [33] that the general higher derivative gravity $S(g, R)$ yields the same results.

So far, not much is known about the logarithmic term of EE for 6d CFTs except [34, 35]. In [34], Hung, Myers and Smolkin (HMS) obtain the logarithmic term of EE for 6d CFTs in case of zero extrinsic curvatures. Because the condition $K_{aij} = 0$ breaks the conformal invariance, their formulas are not conformal invariant. In [35], Safdi study the cases with $B_3 = \frac{B_2 - \frac{B_1}{2}}{3}$ in flat space, where B_i are the central charges of 6d CFTs. Since the 'flat-space condition' is imposed, the results of [35] are not conformal invariant either. Now let us briefly review their works.

HMS derive the universal terms of EE for CFTs as the entropy of its Weyl anomaly [34, 36]. In six dimensions, the trace anomaly takes the following form

$$\langle T^i{}_i \rangle = \sum_{n=1}^{3} B_n I_n + 2A\, E_6, \tag{3}$$

where E_6 is the Euler density and I_i are conformal invariants defined by

$$\begin{aligned} I_1 &= C_{kijl} C^{imnj} C_m{}^{kl}{}_n \,, \qquad I_2 = C_{ij}{}^{kl} C_{kl}{}^{mn} C_{mn}{}^{ij} \,, \\ I_3 &= C_{iklm} (\nabla^2 \delta^i_j + 4 R^i{}_j - \frac{6}{5} R\, \delta^i_j) C^{jklm} \,. \end{aligned} \tag{4}$$

For entangling surfaces with the rotational symmetry, only Wald entropy contributes to HEE. Thus, we have

$$S_{\rm EE} = \log(\ell/\delta) \int d^4x \sqrt{h} \left[2\pi \sum_{n=1}^{3} B_n \frac{\partial I_n}{\partial R^{ij}{}_{kl}} \tilde{\varepsilon}^{ij}\, \tilde{\varepsilon}_{kl} + 2\, A\, E_4 \right]_\Sigma \,, \tag{5}$$

where

$$\frac{\partial I_1}{\partial R^{ij}{}_{kl}} \tilde{\varepsilon}^{ij}\, \tilde{\varepsilon}_{kl} = 3 \left(C^{jmnk}\, C_m{}^{il}{}_n\, \tilde{\varepsilon}_{ij}\, \tilde{\varepsilon}_{kl} - \frac{1}{4} C^{iklm}\, C^j{}_{klm}\, \tilde{g}^\perp_{ij} + \frac{1}{20} C^{ijkl}\, C_{ijkl} \right), \tag{6}$$

$$\frac{\partial I_2}{\partial R^{ij}{}_{kl}} \tilde{\varepsilon}^{ij}\, \tilde{\varepsilon}_{kl} = 3 \left(C^{klmn}\, C_{mn}{}^{ij}\, \tilde{\varepsilon}_{ij}\, \tilde{\varepsilon}_{kl} - C^{iklm}\, C^j{}_{klm}\, \tilde{g}^\perp_{ij} + \frac{1}{5} C^{ijkl}\, C_{ijkl} \right), \tag{7}$$

$$\begin{aligned} \frac{\partial I_3}{\partial R^{ij}{}_{kl}} \tilde{\varepsilon}^{ij}\, \tilde{\varepsilon}_{kl} &= 2 \left(\Box C^{ijkl} + 4\, R^i{}_m C^{mjkl} - \frac{6}{5} R\, C^{ijkl} \right) \tilde{\varepsilon}_{ij}\, \tilde{\varepsilon}_{kl} - 4\, C^{ijkl}\, R_{ik}\, \tilde{g}^\perp_{jl} \\ &\quad + 4\, C^{iklm}\, C^j{}_{klm}\, \tilde{g}^\perp_{ij} - \frac{12}{5} C^{ijkl}\, C_{ijkl} \,. \end{aligned}$$

For entangling surfaces without the rotational symmetry but with zero extrinsic curvature, the anomaly of entropy of $C_{ijkl} \Box C^{ijkl}$ should be added to eq.(5). This contribution is used by [37] to explain the HMS mismatch [34] recently. It should be mentioned that there is another proposal for the resolution of HMS puzzle. In [38, 39], the authors suggest to use the entropy of total derivatives to explain the HMS mismatch. It is really counterintuitive that total derivatives could contribute to non-trivial entropy. If so, the logarithmic term of EE would violate the conformal invariance and depend on the approach of regularization. This strongly implies the results of [38, 39] are unreliable. Actually, by applying the LM regularization [7], it is found that the entropy of total derivatives is indeed trivial [40].

Now let us turn to the work of [35]. The universal term of EE for 6d CFTs with $B_3 = \frac{B_2 - \frac{B_1}{2}}{3}$ in flat space is given by

$$S_\Sigma|_{\log} = \log(\ell/\delta) \int_\Sigma 2AE_4 + 6\pi\big[3(B_2 - \frac{B_1}{4})J + B_3 T_3\big], \tag{8}$$

where $J = T_1 - 2T_2$ and T_i is given by

$$T_1 = (\mathrm{tr}\bar{k}^2)^2, \quad T_2 = \mathrm{tr}\bar{k}^4, \quad T_3 = (\nabla_a k)^2 - \frac{25}{16}k^4 + 11k^2 trk^2 - 6(trk^2)^2 - 16k trk^3 + 12trk^4. \tag{9}$$

Here $\bar{k}$ denotes the traceless part of the extrinsic curvature. For simplicity the extrinsic curvature in the time-like direction is set to be zero in [35]. In our notation, we have $K_{zij} = K_{\bar{z}ij} = \frac{1}{2}k_{ij}$. It should be mentioned that the 'flat-space condition breaks the conformal invariance. As a result, T_3 is not conformal invariant [35].

In this paper, we investigate the most general cases. By applying the holographic and the field theoretical methods respectively, we derive the universal terms of EE for 6d CFTs. Our formulas are conformal invariant and reduce to those of [34, 35] when imposing the conditions they use. Remarkably, we find that the holographic and the field theoretical results match for the C^2 and Ck^2 terms. As for the k^4 terms, we have to deal with the splitting problem of the conical metrics. The splitting problem appears because one can not distinguish r^2 and r^{2n} $(n \to 1)$ in the expansions of the conical metrics. We can fix the splitting problem in the bulk by applying equations of motion. As for the splitting problem on the boundary, we assume the general expressions and find that there does exist suitable splittings which can make the holographic and the field theoretical k^4 terms match.

It should be mentioned that the splitting problem does not affect the logarithmic term of EE for 4d CFTs. By using the field theoretical method, we only need the entropy of curvature-squared terms to determine the logarithmic term. It can be easily checked that the splittings do not affect the entropy of curvature-squared terms. As in the holographic approach, applying the background method [33], we can expand the action $S(g, R)$ around a background curvature $\bar{R}$. According to [33], only the squared terms $(R - \bar{R})^2$ contribute to the 4d logarithmic terms. However, as we have mentioned above, the squared terms are irrelevant to the splitting problem. Thus, the splittings do not affect the 4d logarithmic terms from both the field theoretical and the holographic viewpoints. For the 6d logarithmic terms, we need to calculate the entropy of cubic curvature terms. It turns out that the only cubic curvature term that is irrelevant to the splittings is the Love-lock term. However, the central charges of CFTs dual to Love-lock gravity and the curvature-squared gravity are not independent but constrained by $B_3 = \frac{B_2 - \frac{B_1}{2}}{3}$. Thus, to study the most general case in 6d, we have to deal with the splitting problem.

An overview of this paper is as follows: We begin with the discussions of the splitting problem for the conical metrics in Sect. 2. In Sect. 3, we derive the universal terms of EE for 6d CFTs by applying the holographic method. We firstly derive the results from a smart-constructed action and then prove that the general action produces the same results. In Sect. 4, we use the field theoretical method to calculate the universal terms of EE for 6d CFTs. We compare the field theoretical results with the holographic ones and get good agreements. We conclude with a brief discussion of our results in Sect. 5.

2 The splitting problem

In general, we have to deal with the splitting problem for the squashed cones in order to derive the holographic entanglement entropy (HEE). Let us briefly review this problem in this section. The splitting problems appear because we can not distinguish r^2 and r^{2n} in the expansions of conical metrics. That is because r^2 and r^{2n} become the same order in the limit $n \to 1$ when we calculate HEE. According to [32, 41], the general regularized squashed conical metric is

$$ds^2 = e^{2A}[dzd\bar{z} + T(\bar{z}dz - zd\bar{z})^2] + 2iV_i(\bar{z}dz - zd\bar{z})dy^i \\ +(g_{ij} + Q_{ij})dy^i dy^j, \tag{10}$$

where g_{ij} is the metric on the transverse space and is independent of $z, \bar{z}$. $A = -\frac{\epsilon}{2}\lg(z\bar{z} + a^2)$ is regularized warp factor. T, V_i, Q_{ij} are defined as [32, 37, 41]

$$\begin{aligned}
T &= \sum_{n=0}^{\infty} \sum_{m=0}^{P_{a_1...a_n}+1} e^{2mA} T_{m\ a_1...a_n} x^{a_1}...x^{a_n}, \\
V_i &= \sum_{n=0}^{\infty} \sum_{m=0}^{P_{a_1...a_n}+1} e^{2mA} V_{m\ a_1...a_n i} x^{a_1}...x^{a_n}, \\
Q_{ij} &= \sum_{n=1}^{\infty} \sum_{m=0}^{P_{a_1...a_n}} e^{2mA} Q_{m\ a_1...a_n ij} x^{a_1}...x^{a_n}.
\end{aligned} \tag{11}$$

Here $z, \bar{z}$ are denoted by x^a and $P_{a_1...a_n}$ is the number of pairs of $z, \bar{z}$ appearing in $a_1...a_n$. For example, we have $P_{zz\bar{z}} = P_{z\bar{z}z} = P_{\bar{z}zz} = 1$, $P_{z\bar{z}z\bar{z}} = 2$ and $P_{zz...z} = 0$. Expanding T, V, Q to the first few terms in Dong's notations, we have

$$\begin{aligned}
T &= T_0 + e^{2A} T_1 + O(x), \\
V_i &= U_{0\ i} + e^{2A} U_{1\ i} + O(x), \\
Q_{ij} &= 2K_{aij} x^a + Q_{0\ abij} x^a x^b + 2e^{2A} Q_{1\ z\bar{z}ij}\ z\bar{z} + O(x^3)
\end{aligned} \tag{12}$$

How to split W (W denote T, V, Q) into $\{W_0, W_1, ..., W_{P+1}\}$ is an important problem. It should be mentioned that the splitting problem is ignored in the initial works of Dong and Camps [32, 41]. However they both change their mind and realize the splitting is necessary later [1]. Recently Camps etal generalize the conical metrics to the case without Z_n symmetry, where the splitting problem appears naturally [42]. Our metric eq.(10) can be regarded as a special case of [42] that with Zn symmetry. Inspired by [7], it is expected that the splitting problem can be fixed by equations of motion. Let us take Einstein gravity in vacumm as an example. We denote the quations of motion by $E_{\mu\nu} = R_{\mu\nu} - \frac{R-2\Lambda}{2} G_{\mu\nu} = 0$. Focus on terms which are important near $x^a = 0$, we have

$$\begin{aligned}
R_{ab} &= 2K_{(a}\nabla_{b)}A - g_{ab}K^c\nabla_c A + e^{2A}[(12T_1 + 4U^2)g_{ab} - Q_{1\ abi}{}^{i}] \\
&\quad + K_{aij}K_b{}^{ij} + (12T_0 + 8U_0U_1)g_{ab} - Q_{0\ abi}{}^{i} \\
R_{ai} &= 3\varepsilon_{ba}V^b_i + D^m K_{ami} - D_i K_a,
\end{aligned}$$

[1] We thank Dong and Camps for discussions on this problem.

$$
\begin{aligned}
R_{ij} &= r_{ij} + 8U_iU_j - Q_1{}^a{}_{aij} + e^{-2A}[2K_{aim}K^{am}{}_j - K^aK_{aij} + 16U_{0\ (i}U_{1\ j)} - Q_0{}^a{}_{aij}], \\
R &= r + 16U^2 + 24T_1 - 2Q_1{}^a{}_a{}^i{}_i + e^{-2A}(3K_{aij}K^{aij} - K^aK_a + 24T_0 - 2Q_0{}^a{}_a{}^i{}_i + 32U_0U_1), \quad (13)
\end{aligned}
$$

where $A = -\frac{\epsilon}{2}\log z\bar{z}$, $\varepsilon_{z\bar{z}} = \frac{i}{2}$ and $g_{z\bar{z}} = \frac{1}{2}$. Let us firstly consider the leading term of E_{zz}, we get

$$
E_{zz} = 2K_z\nabla_z + ... = -\epsilon\frac{K_z}{z} + ... = 0. \quad (14)
$$

Requiring the above equation to be regular near the cone, we obtain the minimal surface condition $K_z = K_{\bar{z}} = 0$ [7]. To derive T_0 and Q_0, we need consider the subleading terms of $E_{z\bar{z}}, E_{ij}$ and E^μ_μ. We have

$$
\begin{aligned}
&E_{z\bar{z}} = e^{2A}(...) + [Q_{0\ z\bar{z}i}{}^{i} - 2K_{zij}K_{\bar{z}}^{ij} + K_zK_{\bar{z}} - 4U_0U_1] = 0, \\
&E_{ij} = (...) + e^{-2A}[2K_{aim}K^{am}{}_j - K^aK_{aij} + 16U_{0\ (i}U_{1\ j)} - Q_0{}^a{}_{aij} \\
&\qquad -\frac{1}{2}g_{ij}(3K_{aij}K^{aij} - K^aK_a + 24T_0 - 2Q_0{}^a{}_a{}^i{}_i + 32U_0U_1)] = 0, \\
&E^\mu{}_\mu = (...) + \frac{2-D}{2}e^{-2A}[3K_{aij}K^{aij} - K^aK_a + 24T_0 - 2Q_0{}^a{}_a{}^i{}_i + 32U_0U_1] = 0. \quad (15)
\end{aligned}
$$

Here (...) denote the leading terms which can be used to determine $T_1, U_{1i}, Q_{1z\bar{z}ij}$ and g_{ij}. From the subleading terms of the above equations, we can derive a unique solution

$$
\begin{aligned}
&T_0 = \frac{1}{24}(K_{aij}K^{aij} - K_aK^a), \\
&Q_{0z\bar{z}ij} = (K_{zim}K_{zj}{}^m - \frac{1}{2}K_zK_{\bar{z}ij} + c.c.) + 4U_{0\ (i}U_{1\ j)} \quad (16)
\end{aligned}
$$

As we shall show below, a natural choice would be $U_{0\ i} = 0$. It should be mentioned that eq.(16) are also solutions to the general higher derivative gravity if we require that the higher derivative gravity has an AdS solution. In the next section, we shall use eq.(16) to derive the universal terms of EE for 6d CFTs. Actually, we only need a weaker condition near the boundary

$$
\begin{aligned}
&T_0 = \frac{1}{24}(K_{aij}K^{aij} - x\ K_aK^a) + O(\rho^2), \\
&Q_{0z\bar{z}ij} = K_{zim}K_{zj}{}^m - y\ K_zK_{\bar{z}ij} - z\ g_{ij}K_zK_{\bar{z}} + c.c + O(\rho) \quad (17)
\end{aligned}
$$

with x, y, z are some constants which are not important. Here ρ is defined in the FG expansion eq.(78) and $\rho \to 0$ corresponds to the boundary. Actually, as we shall show in sect.3.2, eq.(17) is the necessary condition that all the higher derivative gravity in the bulk gives the same formulas of the universal terms of EE.

To end this section, let us make some comments. Besides the equations of motion, there are several other constraints which may help to fix the splitting.

1. The entropy reduces to Wald entropy in stationary spacetime.

Let us take $\nabla_\mu R_{\nu\rho\sigma\alpha}\nabla^\mu R^{\nu\rho\sigma\alpha}$ as an example. In stationary spacetime, we have $K_{aij} = Q_{zzij} = Q_{\bar{z}\bar{z}ij} = 0$. Applying the method of [37], we can derive the HEE as

$$
S_{HEE} = S_{Wald} + \int dy^{D-2}\sqrt{g}128\pi(Q_{0z\bar{z}ij}Q_{0z\bar{z}}^{ij} + 9T_0^2 + 5(U_{0\ i}U_0{}^i)^2 + \text{mixed terms of } T_0, Q_0, U_0). \quad (18)
$$

To be consistent with Wald entropy, we must have $T_0 = U_{0\ i} = Q_{0z\bar{z}ij} = 0$ in stationary spacetime. This implies that $T_0, U_{0\ i}$ and $Q_{0z\bar{z}ij}$ should be either zero or functions of the extrinsic curvatures. This is indeed the case for the splitting eqs.(16). By dimensional analysis, we note that $U_{0\ i} \sim O(K)$. However, it is impossible to express $U_{0\ i}$ in terms of the extrinsic curvature K_{aij}. Thus, a natural choice would be $U_{0\ i} = 0$.

2. The entropy of conformal invariant action is also conformal invariant.

In the bulk, we can use gravitational equations of motion to fix the splittings of conical metrics. However, we do not have dynamic gravitational fields on the boundary. Then how can we determine the splittings on the boundary? For the cases with gravity duals, in principle, we can derive the conical metric on the boundary from the one in the bulk. As for the general cases, we do not know how to fix the splittings. If we focus on the case of CFTs, the conformal symmetry can help. As we know, the universal terms of EE for CFTs are conformal invariant. Recall that we can derive the the universal terms of EE as the entropy of the Weyl anomaly [34, 31, 36]. Thus, the entropy of conformal invariants (Weyl anomaly) must be also conformal invariant. Let us call this condition as the 'conformal constraint' . Expanding the Weyl tensor in powers of e^{2A}, we have

$$\begin{aligned} C_{z\bar{z}z\bar{z}} &= e^{4A}C_{1\ z\bar{z}z\bar{z}} + e^{2A}C_{0\ z\bar{z}z\bar{z}} \\ C_{zi\bar{z}j} &= e^{2A}C_{1\ zi\bar{z}j} + C_{0\ zi\bar{z}j}, \\ C_{ikjl} &= C_{1\ ikjl} + e^{-2A}C_{0\ ikjl} \end{aligned} \tag{19}$$

The 'conformal constraint' requires that both C_1 and C_0 are conformal invariant. Assuming the general splittings in 6d spacetime

$$\begin{aligned} &T_0 = z_1 K_{amn}K^{amn} + z_2 K_a K^a \\ &Q_{0\ z\bar{z}ij} = (x_1 K_{zim}K_{\bar{z}\ \ j}^{\ m} + x_2\ g_{ij}K_{zmn}K_{\bar{z}}^{\ mn} + y_1 K_z K_{\bar{z}ij} + y_2\ g_{ij}K_z K_{\bar{z}}) + c.c. \end{aligned} \tag{20}$$

By using the 'conformal constraint', we get

$$x_1 = 1 - 2y_1,\ x_2 = \frac{1}{4} - 6z_1 - \frac{y_1}{3},\ y_2 = -\frac{1}{16} - 6z_2 - \frac{y_1}{24}. \tag{21}$$

Thus the 'conformal constraint' cannot fix the splittings on the boundary completely.

3. The splittings should yield the correct universal terms of EE for CFTs.

Another natural constraint for the splittings on the boundary is that it should give the correct universal term of EE for CFTs. By 'correct', we mean it agrees with holographic results. Remarkably, the splitting problem does not affect the universal terms of EE for 4d CFTs . From the viewpoint of CFTs, we can derive the universal terms of EE as the entropy of the Weyl anomaly. In 4d spacetime, the Weyl anomaly are curvature-squared terms whose entropy can not include T_0 and Q_0 by using Dong's formula [32]. From the viewpoint of holography, the situation is similar. For the general higher derivative gravity $S(g, R)$, it has been proved that T_0 and Q_0 does not contribute to the logarithmic terms of EE [33]. As for the 6d CFTs, the splitting problems do matter. To be consistent with the holographic results, in sect. 4, we shall derive the splittings eq.(20) with

$$x_1 = 1,\ x_2 = \frac{1}{4} - 6z_1,\ y_1 = 0,\ y_2 = -\frac{1}{16} - 6z_2. \tag{22}$$

This constraint is better than the 'conformal constraint' but still could not fix the splittings completely. It seems that we have some freedom to split the conical metrics on the boundary and this freedom does not affect the universal terms of EE.

4. The splittings does not affect the entropy of Love-lock gravity and topological invariants.

Love-lock gravity is special in several aspects. In particular, it becomes topological invariant in critical dimensions. Thus the entropy of Love-lock gravity must be also topological invariant in critical dimensions. This strong constrains the possible form of the entropy of Love-lock gravity. We know the answer is the Jacobson-Myers formula [43]. In general, we would get different entropy from the conical metrics with different splittings. Thus, we must check if the splittings affect the entropy of Love-lock gravity. It is clear that the splittings does not affect the Wald entropy. Thus, we focus on the anomaly of entropy $K_{zij}K_{\bar{z}kl}\frac{\partial^2 L}{\partial R_{zizj}\partial R_{\bar{z}k\bar{z}l}}$ [32]. Note that T_0 and Q_0 only appear in the curvatures $R_{z\bar{z}z\bar{z}}$ and $R_{zi\bar{z}j}$ but not R_{ijkl}. While only R_{ijkl} can appear in $\frac{\partial^2 L}{\partial R_{zizj}\partial R_{\bar{z}k\bar{z}l}}$ for Love-lock gravity. Thus the splittings indeed does not affect the entropy of Love-lock gravity.

3 Holographic method

In this section, we derive the universal logarithmic terms of EE for 6d CFTs by using the holographic method. We firstly derive the results from a smart-constructed bulk action and then prove that the general action produces the same results.

3.1 Logarithmic terms of EE from a smart-constructed action

For the curvature-squared gravity and Love-Lock gravity, the splitting problem does not matter. However, the central charges of the corresponding CFTs are not independent but constrained by $B_3 = \frac{B_2 - \frac{B_1}{2}}{3}$. To cover the general CFTs, we must cosider at least one cubic curvature term. Below we construct two special cubic curvature terms M_1 and M_2 which are designed to correspond to I_1 and I_2 eq.(4), respectively. We use these smart-constructed cubic curvature terms to derive universal terms of EE for 6d CFTs. It turns out that they help quite a lot to simplify the calculations.

Consider the following action

$$S = \int d^7x\sqrt{-\hat{G}}(\hat{R} + 30 + \lambda_1 M_1 + \lambda_2 M_2) \tag{23}$$

where we have set the AdS radius $l = 1$ and M_1, M_2 are constructed as

$$M_1 = \tilde{R}_{\mu\nu\rho\sigma}\tilde{R}^{\mu\alpha\beta\sigma}\tilde{R}^{\nu}{}_{\alpha\beta}{}^{\rho}, \qquad M_2 = \tilde{R}_{\mu\nu}{}^{\rho\sigma}\tilde{R}_{\rho\sigma}{}^{\alpha\beta}\tilde{R}_{\alpha\beta}{}^{\mu\nu}. \tag{24}$$

Here $\tilde{R}$ is defined by

$$\begin{aligned}
\tilde{R}_{\mu\nu\rho\sigma} &= \hat{R}_{\mu\nu\rho\sigma} + (\hat{G}_{\mu\rho}\hat{G}_{\nu\sigma} - \hat{G}_{\mu\sigma}\hat{G}_{\nu\rho}),\\
\tilde{R}_{\mu\nu} &= \hat{R}_{\mu\nu} + 6\hat{G}_{\mu\nu},\\
\tilde{R} &= \hat{R} + 42.
\end{aligned} \tag{25}$$

It should be mentioned that M_i $(i = 1, 2)$ can be regarded as the bulk counterparts to the conformal invariants I_i eq.(4). They only contribute to the holographic Weyl anomaly with respect to I_i $(i = 1, 2)$. According to [33], the holographic Weyl anomaly for the above action is

$$\langle T^i{}_i \rangle = \sum_{n=1}^{3} B_n I_n + 2A E_6, \tag{26}$$

with the central charges given by

$$\begin{aligned} A &= \pi^3, \\ B_1 &= -\frac{1}{16} + \lambda_1, \\ B_2 &= -\frac{1}{64} + \lambda_2, \\ B_3 &= \frac{1}{192}. \end{aligned} \tag{27}$$

It is expected that the universal terms of EE for 6d CTFs takes the following form

$$S_{\text{EE}} = \log(\ell/\delta) \int_\Sigma d^4x\sqrt{h_0} \left[2\pi \sum_{n=1}^{3} B_n F_n + 2\, A\, E_4 \right] \tag{28}$$

where F_n are conformal invariants need to be determined and E_4 is the Euler density. From eqs.(23,27), it is clear that we can use HEE of M_1 and M_2 to derive F_1 and F_2, respectively. Knowing F_1 and F_2, one can use HEE of Einstein gravity to obtain F_3.

3.1.1 F_1 and F_2

Now let us start to derive the universal terms of EE. We kindly suggest the readers who are not familiar with the related skills to read the Appendix. A firstly.

We firstly discuss the Wald entropy of action eq.(23). After some calculations, we get

$$\begin{aligned} S_{\text{Wald}} &= 2\pi \int d\rho d^4y\sqrt{h}\big[2 + 3\lambda_1 \epsilon^{\mu\nu}\epsilon_{\rho\sigma}\tilde{R}_\mu{}^{\alpha\beta\sigma}\tilde{R}_{\nu\alpha\beta}{}^\rho + 3\lambda_2 \epsilon^{\mu\nu}\epsilon_{\rho\sigma}\tilde{R}^{\rho\sigma\alpha\beta}\tilde{R}_{\alpha\beta\mu\nu} \\ &= 2\pi \int d\rho d^4y\sqrt{h}\big[2 + \rho^2(3\lambda_1 \tilde{\epsilon}^{ij}\tilde{\epsilon}_{kl} C_i{}^{mnl} C_{jmn}{}^k + 3\lambda_2 \tilde{\epsilon}^{ij}\tilde{\epsilon}_{kl} C^{kl}{}_{mn} C^{mn}{}_{ij}\big] + \text{irrelevant terms} \\ &= S_E + 2\pi \int d\rho d^4y \frac{\sqrt{h_0}}{2\rho}\big[3\lambda_1 \tilde{\epsilon}^{ij}\tilde{\epsilon}_{kl} C_i{}^{mnl} C_{jmn}{}^k + 3\lambda_2 \tilde{\epsilon}^{ij}\tilde{\epsilon}_{kl} C^{kl}{}_{mn} C^{mn}{}_{ij} + (4k_1 + k_2) C_{ijkl} C^{ijkl} \\ &\quad - k_2 g^\perp_{ij} C^i{}_{klm} C^{jklm}\big] + \text{irrelevant terms} \\ &= S_E + 2\pi \log(\ell/\delta) \int_\Sigma d^4x\sqrt{h_0}\big[3\lambda_1 (C^{jmnk}\, C_m{}^{il}{}_n\, \tilde{\varepsilon}_{ij}\, \tilde{\varepsilon}_{kl} - \frac{1}{4} C^{iklm}\, C^j{}_{klm}\, \tilde{g}^\perp_{ij} + \frac{1}{20} C^{ijkl}\, C_{ijkl}) \\ &\quad + 3\lambda_2 (C^{klmn}\, C_{mn}{}^{ij}\, \tilde{\varepsilon}_{ij}\, \tilde{\varepsilon}_{kl} - C^{iklm}\, C^j{}_{klm}\, \tilde{g}^\perp_{ij} + \frac{1}{5} C^{ijkl}\, C_{ijkl})\big], \end{aligned} \tag{29}$$

where S_E is the universal terms of EE for pure Einstein gravity. We leave the derivation of S_E to the next subsection. Let us discuss the above calculations briefly. The R^3 terms in action eq.(23) gives two kinds of contributions. The first kind of contributions come from their Wald entropy, such as the C^2 terms in the second and third lines of eq.(29). The second kind of contributions are due to their non-trivial corrections of $\overset{(2)}{g}{}_{ij}$ eq.(80) and $\overset{(2)}{X}{}^i$ eq.(84) in $\sqrt{h}$. The k_1, k_2 terms in the third and fourth lines of eq.(29) come from corrections of $\overset{(2)}{g}_{ij}$. Note that $\sqrt{h}$ contains only the linear term of $\overset{(2)}{X^i}$ in the relevant order $\frac{1}{\rho}$. According to equations of motion $\frac{\delta S_{\rm HEE}}{\delta X^i}=0$, the linear terms of $\overset{(2)}{X^i}$ should vanish on-shell (at least for Einstein gravity). This is indeed the case. As we shall show in the next subsection, the coefficient of $\overset{(2)}{X^i}$ vanishes on-shell in the relevant order $\frac{1}{\rho}$.

From eqs.(27,28,29), we can read out Wald-entropy-part of F_1 and F_2 as

$$\begin{aligned}
F_{W1} &= 3(C^{jmnk}\,C_{m\ \ n}^{\ \ il}\,\tilde{\varepsilon}_{ij}\,\tilde{\varepsilon}_{kl}-\frac{1}{4}\,C^{iklm}\,C^{j}_{\ \ klm}\,\tilde{g}^{\perp}_{ij}+\frac{1}{20}\,C^{ijkl}\,C_{ijkl}) \\
F_{W2} &= 3(C^{klmn}\,C_{mn}^{\ \ \ \ ij}\,\tilde{\varepsilon}_{ij}\,\tilde{\varepsilon}_{kl}-C^{iklm}\,C^{j}_{\ \ klm}\,\tilde{g}^{\perp}_{ij}+\frac{1}{5}\,C^{ijkl}\,C_{ijkl}),
\end{aligned} \tag{30}$$

which match the field theoretical results eqs.(6,7) exactly.

Now let us go on to discuss the anomaly of entropy for action eq.(23). According to eqs.(90,91), we only need to keep trK^4 and $(trK^2)^2$ among the K^4 terms. Thus, we can drop all terms including $K_{am}^{\ \ \ m}$. This helps us to simplify calculations. Note also that, as we have shown in Sect. 2, $Q_{0abij}\sim K^2, T_0\sim K^2$.

For $M_1=\tilde{R}_{\mu\nu\rho\sigma}\tilde{R}^{\mu\alpha\beta\sigma}\tilde{R}^{\nu}_{\ \ \alpha\beta}{}^{\rho}$, we derive

$$\begin{aligned}
S_{A_1} &= 24\pi K_{zij}K_{\bar{z}mn}\tilde{R}^{imjn}-12\pi K_{zij}K_{\bar{z}mn}(K_a^{in}K^{ajm}-K_a^{ij}K^{amn}) \\
&\quad -96\pi K_{zil}K_{\bar{z}j}^{\ \ l}\tilde{R}_{z\bar{z}}^{\ \ \ ij}+48\pi K_{zil}K_{\bar{z}j}^{\ \ l}(K_{zjk}K_{\bar{z}i}^{\ \ \ k}-K_{zik}K_{\bar{z}j}^{\ \ \ k}) \\
&\quad +96\pi K_{zij}K_{\bar{z}}^{\ \ ij}\tilde{R}_{z\bar{z}z\bar{z}}-48\pi K_{zij}K_{\bar{z}}^{\ \ ij}(-3T_0) \\
&= \rho^2(24\pi\bar{k}_{zij}\bar{k}_{\bar{z}mn}C^{imjn}-12\pi\bar{k}_{zij}\bar{k}_{\bar{z}mn}(\bar{k}_a^{in}\bar{k}^{ajm}-\bar{k}_a^{ij}\bar{k}^{amn}) \\
&\quad -96\pi\bar{k}_{zil}\bar{k}_{\bar{z}j}^{\ \ l}C_{z\bar{z}}^{\ \ \ ij}+48\pi\bar{k}_{zil}\bar{k}_{\bar{z}j}^{\ \ l}(\bar{k}_{zjk}\bar{k}_{\bar{z}i}^{\ \ \ k}-\bar{k}_{zik}\bar{k}_{\bar{z}j}^{\ \ \ k}) \\
&\quad +96\pi\bar{k}_{zij}\bar{k}_{\bar{z}}^{\ \ ij}C_{z\bar{z}z\bar{z}}+24\pi(\bar{k}_{zij}\bar{k}_{\bar{z}}^{\ \ ij})^2)+O(\rho^3)
\end{aligned} \tag{31}$$

where k_{aij} is the extrinsic curvature on the entangling surface Σ and $\bar{k}_{aij}$ is the traceless part of k_{aij}.

For $M_2=\tilde{R}_{\mu\nu}^{\ \ \ \rho\sigma}\tilde{R}_{\rho\sigma}^{\ \ \ \alpha\beta}\tilde{R}_{\alpha\beta}^{\ \ \ \mu\nu}$, we have

$$\begin{aligned}
S_{A_2} &= -384\pi K_{zl}^{\ \ (i}K_{\bar{z}}^{\ \ j)l}\tilde{R}_{zi\bar{z}j}+192\pi K_{zl}^{\ \ (i}K_{\bar{z}}^{\ \ j)l}(K_{zjk}K_{\bar{z}i}^{\ \ \ k}-Q_{0z\bar{z}ij}) \\
&= -384\pi\rho^2\bar{k}_{zl}^{\ \ (i}\bar{k}_{\bar{z}}^{\ \ j)l}C_{zi\bar{z}j}-192\pi\rho^2\bar{k}_{zl}^{\ \ (i}\bar{k}_{\bar{z}}^{\ \ j)l}\bar{k}_{zjk}\bar{k}_{\bar{z}i}^{\ \ \ k}+O(\rho^3)
\end{aligned} \tag{32}$$

From eq.(31,32), we can derive the 'anomay'-part of F_1 and F_2 as

$$\begin{aligned}
F_{A1} &= \frac{S_{A_1}}{2\pi}=12\bar{k}_{zij}\bar{k}_{\bar{z}mn}C^{imjn}-6\bar{k}_{zij}\bar{k}_{\bar{z}mn}(\bar{k}_a^{in}\bar{k}^{ajm}-\bar{k}_a^{ij}\bar{k}^{amn}) \\
&\quad -48\bar{k}_{zil}\bar{k}_{\bar{z}j}^{\ \ l}C_{z\bar{z}}^{\ \ \ ij}+24\bar{k}_{zl}^{\ \ i}\bar{k}_{\bar{z}}^{\ \ lj}(\bar{k}_{zjk}\bar{k}_{\bar{z}i}^{\ \ \ k}-\bar{k}_{zik}\bar{k}_{\bar{z}j}^{\ \ \ k})+48\bar{k}_{zij}\bar{k}_{\bar{z}}^{\ \ ij}C_{z\bar{z}z\bar{z}}+12(\bar{k}_{zij}\bar{k}_{\bar{z}}^{\ \ ij})^2 \\
F_{A2} &= \frac{S_{A_2}}{2\pi} \\
&= -192\bar{k}_{zl}^{\ \ (i}\bar{k}_{\bar{z}}^{\ \ j)l}C_{zi\bar{z}j}-96\bar{k}_{zl}^{\ \ (i}\bar{k}_{\bar{z}}^{\ \ j)l}\bar{k}_{zjk}\bar{k}_{\bar{z}i}^{\ \ \ k}.
\end{aligned} \tag{33}$$

Now we can obtain $F_1=F_{W1}+F_{A1}$ and $F_2=F_{W2}+F_{A2}$ from eqs.(30,33). This is one of our main results. Let us make some discussions. Firstly, we have used eqs.(17). So we require that our action has an asymptotically AdS solution. Secondly, our results eqs.(30,33) are consistent with those of [34, 35]. We have shown above that our results agree with the field theoretical results eqs.(6,7) when the extrinsic curvature vanishes[34]. As for the case of non-zero extrinsic curvature, let us compare our results with [35]. In [35], Safdi obtain the universal terms of EE for 6d CFTs with $B_3 = \frac{B_2 - \frac{B_1}{2}}{3}$ in flat space as eq.(8). For simplicity Safdi takes vanishing extrinsic curvature in the time-like direction. In our notation, we have $K_{zij} = K_{\bar{z}ij} = \frac{1}{2}k_{ij}$. Since now we do not know F_3, we set $B_3 = 0, B_1 = 2B_2$ for simplicity (We leave the derivation of F_3 to the next subsection). Note also that we have $C_{ijkl} = 0$ in flat space. Take all the above simplifications into account, we derive

$$S_\Sigma|_{\log} = \log(\ell/\delta) \int_\Sigma 2AE_4 + 9\pi B_2[(tr\bar{k}^2)^2 - 2tr\bar{k}^4] \tag{34}$$

which exactly agrees with the results of [35]. Thirdly, our $\sqrt{h_0}F_1$ and $\sqrt{h_0}F_2$ are obviously conformal invariant. That is because, similar to C_{ijkl}, $\bar{k}_{aij}$ are conformal tensors. In other words, we have $g_{ij} \to e^{2\sigma}g_{ij}, C^i_{jkl} \to C^i_{jkl}$ and $\bar{k}_{aij} \to e^{\sigma}\bar{k}_{aij}$ under conformal transformations. To end this section, we rewrite F_1 and F_2 in covariant expressions

$$\begin{aligned} F_1 \;=\;& 3(C^{jmnk}\,C_m{}^{il}{}_n\,\tilde{\varepsilon}_{ij}\,\tilde{\varepsilon}_{kl} - \frac{1}{4}C^{iklm}\,C^j{}_{klm}\,\tilde{g}^{\perp}_{ij} + \frac{1}{20}C^{ijkl}\,C_{ijkl}) \\ & +3\bar{k}^a{}_{ij}\bar{k}_{amn}C^{imjn} - \frac{3}{2}\bar{k}^b{}_{ij}\bar{k}_{bmn}(\bar{k}^{in}_a\bar{k}^{ajm} - \bar{k}^{ij}_a\bar{k}^{amn}) \\ & +3\tilde{\varepsilon}^{ab}\bar{k}_{ail}\bar{k}_{bj}^{\ l}\varepsilon^{cd}C_{cd}{}^{ij}\tilde{+}3\tilde{\varepsilon}^{ab}\bar{k}_{ail}\bar{k}_{bj}^{\ l}\tilde{\varepsilon}^{cd}\bar{k}_c{}^i{}_k\bar{k}_d{}^{kj} \\ & -\bar{k}^a{}_{mn}\bar{k}_a{}^{mn}C^{ijkl}\,\tilde{\varepsilon}_{ij}\,\tilde{\varepsilon}_{kl} + \frac{3}{4}(\bar{k}^a{}_{ij}\bar{k}_a{}^{ij})^2 \end{aligned} \tag{35}$$

$$\begin{aligned} F_2 \;=\;& 3(C^{klmn}\,C_{mn}{}^{ij}\,\tilde{\varepsilon}_{ij}\,\tilde{\varepsilon}_{kl} - C^{iklm}\,C^j{}_{klm}\,\tilde{g}^{\perp}_{ij} + \frac{1}{5}C^{ijkl}\,C_{ijkl}) \\ & -12\bar{k}^a{}_l{}^{(i}\bar{k}_a{}^{j)l}C_{minj}\tilde{g}^{\perp mn} - 6\bar{k}^a{}_l{}^{(i}\bar{k}_a{}^{j)l}\bar{k}^b{}_{jk}\bar{k}_{bi}{}^k. \end{aligned} \tag{36}$$

3.1.2 F_3

In this subsection, we derive the universal terms of EE for 6d CFTs dual to Einstein gravity. Using the results together with F_1 and F_2, we can derive F_3.

Recall that the HEE of Eintein gravity is

$$S_{HEE} = 4\pi \int d\rho d^4y\sqrt{h} \tag{37}$$

Applying the approach of [46], we have

$$\begin{aligned} h_{\rho\rho} \;=\;& \frac{1}{4\rho^2} + \frac{1}{\rho}\overset{(1)}{X}{}^j\overset{(1)}{X}{}^i\overset{(0)}{g}_{ij} + (\overset{(1)}{X}{}^i\overset{(1)}{X}{}^j\overset{(1)}{g}_{ij} + \overset{(1)}{X}{}^i\overset{(1)}{X}{}^j\overset{(1)}{X}{}^k\partial_k\overset{(0)}{g}_{ij} + 4\overset{(2)}{X}{}^i\overset{(1)}{X}{}^j\overset{(0)}{g}_{ij}) \\ =\;& \frac{1}{4\rho^2}[1 + \frac{1}{16}\rho k^i k_i + \rho^2(\frac{1}{16}k^i k^j\overset{(1)}{g}_{ij} + 2\overset{(2)}{X}{}^i k^j\overset{(0)}{g}_{ij})]. \end{aligned} \tag{38}$$

Here we have used $\overset{(1)}{X}{}^i = \frac{1}{8}k^i$ eq.(85) and the following ansatz of $\overset{(0)}{g}_{ij}$

$$\begin{aligned} \overset{(0)}{g}_{ij}dx^i dx^j = dzd\bar{z} + T(\bar{z}dz - zd\bar{z})^2 + 2iV_{\hat{i}}(\bar{z}dz - zd\bar{z})dy^{\hat{i}} \\ +(g_{\hat{i}\hat{j}} + Q_{\hat{i}\hat{j}})dy^{\hat{i}}dy^{\hat{j}}, \end{aligned} \tag{39}$$

where T, V, Q are given by

$$T = \sum_{n=0}^{\infty} T_{a_1...a_n} x^{a_1}...x^{a_n}, \quad V_{\hat{i}} = \sum_{n=0}^{\infty} V_{a_1...a_n \hat{i}} x^{a_1}...x^{a_n} = U_{\hat{i}} + ...,$$
$$Q_{\hat{i}\hat{j}} = \sum_{n=1}^{\infty} Q_{a_1...a_n ij} x^{a_1}...x^{a_n} = -2x^a k_{a\hat{i}\hat{j}} + x^a x^b Q_{ab\hat{i}\hat{j}} + ... \tag{40}$$

Here x^a denote $z, \bar{z}$ and $y^{\hat{i}}$ are coordinates on the four-dimensional entangling surface. Using eq.(39), we have $\overset{(1)}{X}{}^i \overset{(1)}{X}{}^j \overset{(1)}{X}{}^k \partial_k \overset{(0)}{g}_{ij} \sim O(x^a)$ and thus can be ignored on the entangling surface. It should be mentioned that, by choosing suitable coordinates, we can alway write the metric in the form of eq.(39) [32]. Note also that the extrinsic curvature in this subsection (Schwimmer-Theisen notation [46]) is different from the one of other sections (Dong's notation[32]) by a minus sign.

Similarly for $h_{\hat{i}\hat{j}}$, we have

$$h_{\hat{i}\hat{j}} = \frac{1}{\rho}[\overset{(0)}{h}_{\hat{i}\hat{j}} + \rho(\overset{(1)}{g}_{\hat{i}\hat{j}} - \frac{1}{4}k^a k_{a\hat{i}\hat{j}}) + \rho^2 \overset{(2)}{h}_{\hat{i}\hat{j}}], \tag{41}$$

with $\overset{(2)}{h}_{\hat{i}\hat{j}}$ given by

$$\begin{aligned}
\overset{(2)}{h}_{\hat{i}\hat{j}} &= \partial_{\hat{i}} \overset{(1)}{X}{}^m \partial_{\hat{j}} \overset{(1)}{X}{}^n \overset{(0)}{g}_{mn} + \partial_{\hat{i}} \overset{(2)}{X}{}^m \partial_{\hat{j}} \overset{(0)}{X}{}^n \overset{(0)}{g}_{mn} + \partial_{\hat{i}} \overset{(0)}{X}{}^m \partial_{\hat{j}} \overset{(2)}{X}{}^n \overset{(0)}{g}_{mn} \\
&\quad + (\partial_{\hat{i}} \overset{(1)}{X}{}^m \partial_{\hat{j}} \overset{(0)}{X}{}^n + \partial_{\hat{i}} \overset{(0)}{X}{}^m \partial_{\hat{j}} \overset{(1)}{X}{}^n)(\overset{(1)}{g}_{mn} + \overset{(1)}{X}{}^k \partial_k \overset{(0)}{g}_{mn}) \\
&\quad + \partial_{\hat{i}} \overset{(0)}{X}{}^m \partial_{\hat{j}} \overset{(0)}{X}{}^n (\overset{(2)}{g}_{mn} + \overset{(1)}{X}{}^k \partial_k \overset{(1)}{g}_{mn} + \frac{\overset{(1)}{X}{}^k \overset{(1)}{X}{}^l}{2} \partial_k \partial_l \overset{(0)}{g}_{mn} + \overset{(2)}{X}{}^k \partial_k \overset{(0)}{g}_{mn}) \\
&= (\frac{1}{64} \partial_{\hat{i}} k^m \partial_{\hat{j}} k^n + \partial_{\hat{i}} \overset{(2)}{X}{}^m \partial_{\hat{j}} \overset{(0)}{X}{}^n + \partial_{\hat{i}} \overset{(0)}{X}{}^m \partial_{\hat{j}} \overset{(2)}{X}{}^n) \overset{(0)}{g}_{mn} \\
&\quad + \frac{1}{8}(\partial_{\hat{i}} k^m \overset{(1)}{g}_{m\hat{j}} + \partial_{\hat{j}} k^m \overset{(1)}{g}_{m\hat{i}}) + \frac{1}{32} \epsilon_{mn} (\partial_{\hat{i}} k^m k^n U_{\hat{j}} + \partial_{\hat{i}} k^m k^n U_{\hat{i}}) \\
&\quad + \overset{(2)}{g}_{\hat{i}\hat{j}} + \frac{1}{8} k^a \partial_a \overset{(1)}{g}_{\hat{i}\hat{j}} + \frac{1}{64} k^a k^b Q_{ab\hat{i}\hat{j}} + \overset{(2)}{X}{}^k \partial_k \overset{(0)}{g}_{mn}
\end{aligned} \tag{42}$$

Let us try to simplify the above formula. Focus on the $\overset{(2)}{X}{}^m$ terms which are relevant to the logarithmic terms of EE, we have

$$\begin{aligned}
S_{\overset{(2)}{X}} &= 4\pi \log(\ell/\delta) \int_\Sigma d^4y \sqrt{h_0} [\overset{(2)}{X}{}^i k^j \overset{(0)}{g}_{ij} + \overset{(0)}{h}{}^{\hat{i}\hat{j}} \partial_{\hat{i}} \overset{(0)}{X}{}^m \partial_{\hat{j}} \overset{(2)}{X}{}^n \overset{(0)}{g}_{mn} + \frac{1}{2} \overset{(2)}{X}{}^k \partial_k \overset{(0)}{g}_{mn}] \\
&= 4\pi \log(\ell/\delta) \int_\Sigma d^4y \sqrt{h_0} [\overset{(2)}{X}{}^i \overset{(0)}{g}_{ij} \overset{(0)}{h}{}^{\hat{m}\hat{n}} (k^j_{\hat{m}\hat{n}} - \partial_{\hat{m}} \partial_{\hat{n}} \overset{(0)}{X}{}^j + \overset{(0)}{\gamma}{}^{\hat{l}}{}_{\hat{m}\hat{n}} \partial_{\hat{l}} \overset{(0)}{X}{}^j - \overset{(0)}{\Gamma}{}^j{}_{kl} \partial_{\hat{m}} \overset{(0)}{X}{}^k \partial_{\hat{n}} \overset{(0)}{X}{}^l] \\
&\quad + 4\pi \log(\ell/\delta) \int_\Sigma d^4y \partial_{\hat{j}} (\sqrt{h_0} \overset{(0)}{h}{}^{\hat{i}\hat{j}} \partial_{\hat{i}} \overset{(0)}{X}{}^m \overset{(2)}{X}{}^n \overset{(0)}{g}_{mn}) \\
&= 4\pi \log(\ell/\delta) \int_\Sigma d^4y \sqrt{h_0} D_{\hat{i}} \overset{(2)}{X}{}^{\hat{i}}
\end{aligned} \tag{43}$$

where $\gamma^{\hat{l}}_{\hat{m}\hat{n}}$ and $D_{\hat{i}}$ are the Levi-Civita connection and covariant derivatives on the entangling surface Σ, respectively. In the above derivations, we have used the definition of the extrinsic curvature

$$k^j_{\hat{m}\hat{n}} = \partial_{\hat{m}} \partial_{\hat{n}} \overset{(0)}{X}{}^j - \overset{(0)}{\gamma}{}^{\hat{l}}{}_{\hat{m}\hat{n}} \partial_{\hat{l}} \overset{(0)}{X}{}^j + \overset{(0)}{\Gamma}{}^j{}_{kl} \partial_{\hat{m}} \overset{(0)}{X}{}^k \partial_{\hat{n}} \overset{(0)}{X}{}^l. \tag{44}$$

Now it is clear that we can drop $\overset{(2)}{X}$ safely on closed entangling surfaces. Thus eq.(42) can be simplified as

$$\begin{aligned}
\overset{(2)}{h}_{\hat{i}\hat{j}} &= \frac{1}{64}\partial_{\hat{i}}k^m\partial_{\hat{j}}k^n\overset{(0)}{g}_{mn}+\frac{1}{8}(\partial_{\hat{i}}k^m\overset{(1)}{g}_{m\hat{j}}+\partial_{\hat{j}}k^m\overset{(1)}{g}_{m\hat{i}})+\frac{1}{32}\epsilon_{mn}(\partial_{\hat{i}}k^mk^nU_{\hat{j}}+\partial_{\hat{i}}k^mk^nU_{\hat{i}}) \\
&\quad +\overset{(2)}{g}_{\hat{i}\hat{j}}+\frac{1}{8}k^a\partial_a\overset{(1)}{g}_{\hat{i}\hat{j}}+\frac{1}{64}k^ak^bQ_{ab\hat{i}\hat{j}} \\
&= \frac{1}{64}(\nabla_{\hat{i}}k^m\nabla_{\hat{j}}k^n\overset{(0)}{g}_{mn}-k^mk^nR_{m\hat{i}n\hat{j}})+\frac{1}{8}(\nabla_{\hat{i}}k^m\overset{(1)}{g}_{m\hat{j}}+\nabla_{\hat{j}}k^m\overset{(1)}{g}_{m\hat{i}}+k^m\nabla_m\overset{(1)}{g}_{\hat{i}\hat{j}})+\overset{(2)}{g}_{\hat{i}\hat{j}},
\end{aligned} \tag{45}$$

where ∇_i are the covariant derivatives with respect to $\overset{(0)}{g}_{mn}$. From eqs.(38,45), we can derive the logarithm term of EE for Einstein gravity as

$$\begin{aligned}
S_E &= \pi\log(\ell/\delta)\int_\Sigma d^4y\sqrt{h_0}[2\overset{(2)}{h}{}^{\hat{i}}{}_{\hat{i}}-\overset{(1)}{g}_{\hat{i}\hat{j}}\overset{(1)}{g}{}^{\hat{i}\hat{j}}+\frac{1}{2}(\overset{(1)}{g}{}^{\hat{i}}{}_{\hat{i}})^2+\frac{1}{2}k^ak_{a\hat{i}\hat{j}}\overset{(1)}{g}{}^{\hat{i}\hat{j}}-\frac{3}{16}k^ak_a\overset{(1)}{g}{}^{\hat{i}}{}_{\hat{i}} \\
&\qquad +\frac{1}{8}k^ak^b\overset{(1)}{g}_{ab}-\frac{1}{16}k^ak_{a\hat{i}\hat{j}}k_bk^{b\hat{i}\hat{j}}+\frac{7}{512}(k^ak_a)^2].
\end{aligned} \tag{46}$$

The definitions of $\overset{(1)}{g}, \overset{(2)}{g}$ can be found in the Appendix.A with $k_1 = k_2 = 0$. After some complicated calculations, we find that eq.(46) is conformal invariant up to some total derivatives. This can be regarded as a check of eq.(46). Please refer to Appendix B for the proof of the conformal invariance of eq.(46). Using eq.(46) together with F_1 and F_2 of sect. 2.2.1, we can derive F_3.

$$\begin{aligned}
F_3 &= -192\pi^2E_4+12F_1+3F_2 \\
&\quad +\overset{(2)}{h}{}^{\hat{i}}{}_{\hat{i}}-\frac{1}{2}\overset{(1)}{g}_{\hat{i}\hat{j}}\overset{(1)}{g}{}^{\hat{i}\hat{j}}+\frac{1}{4}(\overset{(1)}{g}{}^{\hat{i}}{}_{\hat{i}})^2+\frac{1}{4}k^ak_{a\hat{i}\hat{j}}\overset{(1)}{g}{}^{\hat{i}\hat{j}}-\frac{3}{32}k^ak_a\overset{(1)}{g}{}^{\hat{i}}{}_{\hat{i}} \\
&\quad +\frac{1}{16}k^ak^b\overset{(1)}{g}_{ab}-\frac{1}{32}k^ak_{a\hat{i}\hat{j}}k_bk^{b\hat{i}\hat{j}}+\frac{7}{1024}(k^ak_a)^2
\end{aligned} \tag{47}$$

This is one of our main results. Now let us consider some special cases below.

Case I: $k_{aij} = 0$,

$$\begin{aligned}
S_E &= \pi\log(\ell/\delta)\int_\Sigma d^4y\sqrt{h_0}[2\overset{(2)}{g}{}^{\hat{i}}{}_{\hat{i}}-\overset{(1)}{g}_{\hat{i}\hat{j}}\overset{(1)}{g}{}^{\hat{i}\hat{j}}+\frac{1}{2}(\overset{(1)}{g}{}^{\hat{i}}{}_{\hat{i}})^2] \\
&= \log(\ell/\delta)\int_\Sigma d^4y\sqrt{h_0}[2\pi\sum_{n=1}^{3}B_nF_{W_n}+2AE_4+B_3\Delta S]
\end{aligned} \tag{48}$$

where $F_{W_n} = \frac{\partial I_n}{\partial R^{ij}{}_{kl}}\tilde{\varepsilon}^{ij}\,\tilde{\varepsilon}_{kl}$ denote the Wald entropy eqs.(6,7,8). B_n and A are the central charges of CFTs dual to Einstein gravity, which can be found in eq.(27) with $\lambda = 0$. ΔS is the famous HMS mismatch [34], which was firstly found by Hung, Myers and Smolkin that the holographic universal terms of EE does not match the CFT ones even for entangling surface with zero extrinsic curvature. Recently, the authors of [37] find that HMS have ignored the anomaly of entropy of I_3. Taking into account such contributions, the holographic and CFT results indeed match. After some tedious calculations, we derive ΔS as

$$\Delta S = -4\pi(\qquad C_{mn}{}^{rs}C^{mnkl}\tilde{g}^{\perp}_{sl}\tilde{g}^{\perp}_{rk}-C_{mnr}{}^{s}C^{mnrl}\tilde{g}^{\perp}_{sl}$$

$$
\begin{aligned}
&+ \quad 2C_m{}^n{}_r{}^s C^{mkrl}\tilde{g}^{\perp}_{ns}\tilde{g}^{\perp}_{kl} - 2C_m{}^n{}_r{}^s C^{mkrl}\tilde{g}^{\perp}_{nl}\tilde{g}^{\perp}_{ks} \\
&+ \quad \frac{4}{3}\tilde{g}^{\perp}_{ij}\tilde{g}^{\perp}_{kl}\tilde{g}^{\perp}_{mn}\tilde{g}^{\perp}_{rs}C^{ikmr}C^{jlns} - \frac{4}{3}\tilde{g}^{\perp}_{ij}\tilde{g}^{\perp}_{kl}\tilde{g}^{\perp}_{mn}C^{ikm}{}_s C^{jlns}) \qquad (49)
\end{aligned}
$$

Note that the first two lines of eq.(49) was derived by HMS [34] under the conditions $k_{aij} = 0$ and $R_{abci} = 3\epsilon_{ab}V_{ci} = 0$. If we drop the second condition, we get some new terms in the last line of eq.(49). Actually, these new terms are proportional to $R_{abci}R^{abci}$.

Case II: flat $\overset{(0)}{g}_{ij}$ and zero $\overset{(1)}{g}_{ij} = \overset{(2)}{g}_{ij} = 0$. Note that this means the bulk spacetime is pure AdS.

$$
S_E = \frac{\pi}{512}\log(\ell/\delta)\int_{\Sigma} d^4y\sqrt{h_0}[16\partial_{\hat{i}}k^m\partial^{\hat{i}}k^n \overset{(0)}{g}_{mn} + 7(k^a k_a)^2 - 16k^a k_{a\hat{i}\hat{j}}k_b k^{b\hat{i}\hat{j}}] \qquad (50)
$$

In the above derivations, we have used the flat condition $R_{aibj} = 0$. For simplicity, we set $U_i = 0$. This is also the case studied in [35]. Compare eq.(50) with

$$
S_{\Sigma}|_{\log} = \log(\ell/\delta)\int_{\Sigma} d^4x\sqrt{h_0}\left[2\,A\,E_4 + 2\pi\sum_{n=1}^{3}B_n F_n\right], \qquad (51)
$$

we can derive F_3 as

$$
F_3 = \frac{3}{16}(16\partial_{\hat{i}}k^m\partial^{\hat{i}}k^n \overset{(0)}{g}_{mn} + 7(k^a k_a)^2 - 16k^a k_{a\hat{i}\hat{j}}k_b k^{b\hat{i}\hat{j}}) - 192\pi^2 E_4 + 12F_1 + 3F_2 \qquad (52)
$$

with E_4 and F_n given by

$$
\begin{aligned}
E_4 &= \frac{1}{128\pi^2}\delta^{i_1i_2i_3i_4}_{j_1j_2j_3j_4}R_{\Sigma}^{j_1j_2}{}_{i_1i_2}R_{\Sigma}^{j_3j_4}{}_{i_3i_4} = \frac{1}{32\pi^2}\delta^{i_1i_2i_3i_4}_{j_1j_2j_3j_4}k^{aj_1}{}_{i_1}k_a^{j_2}{}_{i_2}k^{bj_3}{}_{i_3}k_b^{j_4}{}_{i_4} \\
F_1 &= -\frac{3}{2}\bar{k}_{bij}\bar{k}^b{}_{mn}(\bar{k}_a^{in}\bar{k}^{ajm} - \bar{k}_a^{ij}\bar{k}^{amn}) + 3\epsilon^{ab}\bar{k}_{ail}\bar{k}_{bj}{}^l\epsilon^{cd}\bar{k}_{ck}{}^i\bar{k}_d{}^{jk} + \frac{3}{4}(\bar{k}_{aij}\bar{k}^{aij})^2 \\
F_2 &= -6\bar{k}_{al}{}^i\bar{k}^{alj}\bar{k}_{bjk}\bar{k}^b{}_i{}^k \qquad (53)
\end{aligned}
$$

To derive E_4 in the above equation, we have used the 'flat-space condition' $R^{\|}_{ijkl} = R_{\Sigma ijkl} - (k_{aik}k^a{}_{jl} - k_{ail}k^a{}_{jk}) = 0$. F_1 and F_2 are obtained from eqs.(35,36) with $C_{ijkl} = 0$. Eqs.(52,53) apply to the case with flat space-time on the boundary. This is also the case studied in [35]. Recall that the author of [35] makes two further assumptions [35]. The first one is $B_3 = \frac{B_2 - \frac{B_1}{2}}{3}$. And the second assumption is zero extrinsic curvature in the time-like direction. So we can drop the indices (a,b,c,d) in eqs.(52,53). We get

$$
S_{\Sigma}|_{\log} = \log(\ell/\delta)\int_{\Sigma} d^4x\sqrt{h_0}\left[2\,A\,E_4 + 3\pi B_1(\frac{3}{2}T_1 - T_2) - 12\pi B_2(T_2) + 6\pi B_3(T_3 + 9T_1 - 12T_2)\right] \quad (54)
$$

where the definitions of T_n can be found in eq.(9). Note that eq.(54) reduces to the result of [35] eq.(8) when $B_3 = \frac{B_2 - \frac{B_1}{2}}{3}$. This is a non-trivial check of our results.

3.2 Logarithmic terms of EE from a general action

In this subsection, we investigate the universal terms of EE by using the general higher derivative gravity. We prove that it yields the same results as the above section. Our main method is the background-field approach developed in [33]. For simplicity, we focus on the action without the derivatives of the curvature $S(g_{\mu\nu}, R_{\mu\nu\sigma\rho})$. We assume this action has an asymptotically AdS solution.

We firstly expand the action around a referenced curvature $\bar{R}_{\mu\nu\rho\sigma} = -(\hat{G}_{\mu\rho}\hat{G}_{\nu\sigma} - \hat{G}_{\mu\sigma}\hat{G}_{\nu\rho})$. According to [33], only the first few terms are relevant to the holographic Weyl anomaly and the logarithmic term of EE. We have

$$
\begin{aligned}
S(g_{\mu\nu}, R_{\mu\nu\sigma\rho}) &= \int d^7x\sqrt{-\hat{G}}[\ \sum_{n=0}^{3}\sum_{i=1}^{m_n} c_i^n \tilde{K}_i^n + O(\rho^4)\ ] \\
&= \int d^7x\sqrt{-\hat{G}}[-\frac{c_1^1}{12}(\hat{R}+30) + c_1^2\tilde{R}_{\mu\nu\rho\sigma}\tilde{R}^{\mu\nu\rho\sigma} + c_3^3\tilde{R}\tilde{R}_{\mu\nu\rho\sigma}\tilde{R}^{\mu\nu\rho\sigma} + c_6^3\tilde{R}_{\mu\nu}\tilde{R}^{\mu}{}_{\rho\sigma\beta}\tilde{R}^{\nu\rho\sigma\beta} \\
&\qquad + c_7^3 M_2 + c_8^3 M_1 + O(\rho^4)\ ]
\end{aligned}
\tag{55}
$$

where c_i^n are constants determined by the action and m_n is the number of independent scalars constructed from appropriate contractions of n curvature tensors. For example, $m_1 = 1, m_2 = 3, m_3 = 8$. $\tilde{K}_i^n = K_i^n|_{[\hat{R}\to(\hat{R}-\bar{R})]}$ with K_i^n the independent scalars constructed from n curvature tensors. For example, we have

$$
\begin{aligned}
&K_1^1 = \hat{R}, \\
&K_i^2 = (\hat{R}_{\mu\nu\rho\sigma}\hat{R}^{\mu\nu\rho\sigma},\ \hat{R}_{\mu\nu}\hat{R}^{\mu\nu},\ \hat{R}^2), \\
&K_i^3 = (\hat{R}^3, \hat{R}\hat{R}_{\mu\nu}\hat{R}^{\mu\nu}, \hat{R}\hat{R}_{\mu\nu\rho\sigma}\hat{R}^{\mu\nu\rho\sigma}, \hat{R}_{\mu}^{\nu}\hat{R}_{\nu}^{\rho}\hat{R}_{\rho}^{\mu}, \hat{R}^{\mu\nu}\hat{R}^{\rho\sigma}\hat{R}_{\mu\rho\sigma\nu}, \hat{R}_{\mu\nu}\hat{R}^{\mu\rho\sigma\lambda}\hat{R}^{\nu}{}_{\rho\sigma\lambda}, \\
&\qquad \hat{R}_{\mu\nu\rho\sigma}\hat{R}^{\mu\nu\lambda\chi}\hat{R}^{\rho\sigma}{}_{\lambda\chi}, \hat{R}_{\nu\nu\rho\sigma}\hat{R}^{\nu\lambda\chi\sigma}\hat{R}^{\nu}{}_{\lambda\chi}{}^{\rho}), \\
&...
\end{aligned}
\tag{56}
$$

For simplicity, we focus on the case with $c_1^2 = 0$ in this paper. Without loss of generality, we set $c_1^1 = -12, c_3^3 = \lambda_3, c_6^3 = \lambda_4, c_7^3 = \lambda_2, c_8^3 = \lambda_1$. Then the general action becomes

$$
S = \int d^7x\sqrt{-\hat{G}}[\hat{R} + 30 + \lambda_1 M_1 + \lambda_2 M_2 + \lambda_3\tilde{R}\tilde{R}_{\mu\nu\rho\sigma}\tilde{R}^{\mu\nu\rho\sigma} + \lambda_4\tilde{R}_{\mu\nu}\tilde{R}^{\mu}{}_{\rho\sigma\beta}\tilde{R}^{\nu\rho\sigma\beta} + O(\rho^4)] \tag{57}
$$

Please refer to eq.(24) and eq.(25) for the defination of M_n and $\tilde{R}$, respectively. According to [33], the Weyl anomaly of dual CFTs is $\langle T^i{}_i\rangle = \sum_{n=1}^{3} B_n\, I_n + 2A\, E_6$ with central charges given by eq.(27)

$$
\begin{aligned}
A &= \pi^3, \\
B_1 &= -\frac{1}{16} + \lambda_1, \\
B_2 &= -\frac{1}{64} + \lambda_2, \\
B_3 &= \frac{1}{192}.
\end{aligned}
\tag{58}
$$

Remarkably, the CFTs dual to actions eq.(23) and eq.(57) have the same central charges. This means that they must have the same universal terms of entanglement entropy too. Thus $\tilde{R}\tilde{R}_{\mu\nu\rho\sigma}\tilde{R}^{\mu\nu\rho\sigma}$ and $\tilde{R}_{\mu\nu}\tilde{R}^{\mu}{}_{\rho\sigma\beta}\tilde{R}^{\nu\rho\sigma\beta}$ can not contribute to universal terms of EE in order to be consistent with the results of the above section.

Following the approach of sect. 2.2, we find that the Wald entropy of $\tilde{R}\tilde{R}_{\mu\nu\rho\sigma}\tilde{R}^{\mu\nu\rho\sigma}$ and $\tilde{R}_{\mu\nu}\tilde{R}^{\mu}{}_{\rho\sigma\beta}\tilde{R}^{\nu\rho\sigma\beta}$ is indeed irrelevant to the universal terms of EE. However, mismatches may come from the anomaly of entropy if we choose $Q_{0z\bar{z}ij}$ and T_0 freely. Focus on the relevant terms, we get the anomaly of entropy as

$$\begin{aligned} S_{\text{Anomaly}} &= -\int d\rho d^4y\sqrt{h}\big[\lambda_3 K_{zij}K_{\bar{z}}{}^{ij}(3K_{amn}K^{amn} - K^a K_a - 2Q_{0am}^{a\ \ m} + 24T_0) \\ &+ \lambda_4 K_{zij}K_{\bar{z}}{}^{ij}(K_{zmn}K_{\bar{z}}{}^{mn} - Q_{0\ z\bar{z}m}{}^{m} + 6T_0) + \frac{\lambda_4}{2}K_{zil}K_{\bar{z}}{}^{l}{}_{j}(2K^{aim}K_{am}{}^{j} - K_a K^{aij} - Q_a^{aij})\big] + ... \\ &= 0 + ... \end{aligned} \tag{59}$$

where '...' denotes terms irrelevant to the logarithmic terms of EE. In the above derivations, we have used eq.(17) and the fact that only the trK^4 and $(trK^2)^2$ terms contribute to the universal term of EE for 6d CFTs. Now it is clear that $\tilde{R}\tilde{R}_{\mu\nu\rho\sigma}\tilde{R}^{\mu\nu\rho\sigma}$ and $\tilde{R}_{\mu\nu}\tilde{R}^{\mu}{}_{\rho\sigma\beta}\tilde{R}^{\nu\rho\sigma\beta}$ indeed do not contribute to the logarithmic terms. So the higher derivative gravity with $c_1^2 = 0$ gives the same results as those of sect. 2.1. While for the case with c_1^2 non-zero, the calculation is a little complicated. But there is no reason this case would gives a different result. We leave it as an exercise for the readers. Finally, it should be mentioned that we can regard eq.(59) as another derivation of eq.(17). That is because different higher derivative gravity must give the same formula of universal terms of EE. Thus eq.(59) must be zero.

4 Field theoretical method

In this section, we compute the universal terms of EE by using the field theoretical method and then compare with the holographic results. Similar to the bulk case, we meet the splitting problem. Since now we do not know how to fix the splitting problem on the boundary, we assume the most general expressions. We find that there indeed exists suitable splittings which could make the holographic and the field theoretical results match.

4.1 F_1 and F_2

Let us firstly study the case of F_1 and F_2. We find that the field theoretical results exactly match the holographic ones for the C^2 and Ck^2 terms. As for the k^4 terms, one meet with the splitting problem for $q_{0z\bar{z}ij}$ and t_0. Since now we do not know how to fix the splitting for t, q on the boundary, we assume the following general expressions

$$\begin{aligned} &t_0 = z_1 k_{amn}k^{amn} + z_2 k_a k^a \\ &q_{0\ z\bar{z}ij} = (x_1 k_{zim}k_{\bar{z}}{}^{m}{}_{j} + x_2\ g_{ij}k_{zmn}k_{\bar{z}}{}^{mn} + y_1 k_z k_{\bar{z}ij} + y_2\ g_{ij}k_z k_{\bar{z}}) + c.c. \end{aligned} \tag{60}$$

Recall that, in sect.3.1, we have already proved that the field theoretical results match the holographic ones for Wald entropy (C^2 terms), so we focus on the anomaly of entropy below.

For I_1 we get the anomaly of entropy as

$$\begin{aligned} S_1 &= 24\pi\bar{k}_{zij}\bar{k}_{\bar{z}mn}C^{imjn} - 12\pi\bar{k}_{zij}\bar{k}_{\bar{z}mn}C_0^{imjn} \\ &\quad -96\pi\bar{k}_{zi}^{m}\bar{k}_{\bar{z}mj}C_{z\bar{z}}^{ij} + 48\pi\bar{k}_{zi}^{m}\bar{k}_{\bar{z}mj}C_{0\ z\bar{z}}^{ij} \end{aligned}$$

$$96\pi\bar{k}_{zmn}\bar{k}_{\bar{z}}{}^{mn}C_{z\bar{z}z\bar{z}}-48\pi\bar{k}_{zmn}\bar{k}_{\bar{z}}{}^{mn}C_{0\ z\bar{z}z\bar{z}} \tag{61}$$

where $\bar{k}_{aij}$ is the traceless part of the extrinsic curvature and C_0 is defined in the Appendix. C. Comparing eq.(61) with eq.(33), we find that the Ck^2 terms match exactly. If we require that the k^4 terms also match, we get a unique solution to eq.(60)

$$x_1=1,\ x_2=\frac{1}{4}-6z_1,\ y_1=0,\ y_2=-\frac{1}{16}-6z_2 \tag{62}$$

It is interesting to check if the field theoretical and holographic results for F_2 also match under this condition (62). This is can be regarded as a self-consistent testing. As we shall show below, this is indeed the case.

Let us go on to compute the anomaly of entropy for I_2

$$S_2 \ = \ -384\pi\bar{k}_{z}^{(i}{}_{m}\bar{k}_{\bar{z}}{}^{j)m}C_{zi\bar{z}j}+192\pi\bar{k}_{z}^{(i}{}_{m}\bar{k}_{\bar{z}}{}^{j)m}C_{0\ zi\bar{z}j}, \tag{63}$$

where C_0 is given by eqs.(98). Similarly to the case of I_1, the Ck^2 terms of eq.(33) and eq.(63) match exactly. The k^4 terms also match if we impose the condition eq.(62). So our results have passed the self-consistent testing. Note that comparing the holographic results and the field theoretical results for F_1 and F_2 does not fix z_1, z_2.

To end this section, we show some details of the derivation of eq.(62). For simplicity, we focus on the case of vanishing extrinsic curvature in the time-like direction (the general case gives the same results). Then we can replace k_{aij} by $\frac{1}{2}k_{ij}$. From eqs.(61,63,102), we can derive the k^4 terms as

$$\begin{aligned}
&B_1S_1+B_2S_2\\
=\ &3\pi[B_1(x_1-2)-4B_2x_1]trk^4-\frac{3\pi}{2}[B_1(x_1-2y_1-3)-4B_2(1+x_1-2y_2)]ktrk^3\\
&+\frac{3}{20}\pi[B_1(21+2x_1+28x_2+168z_1)+4B_2(1+2x_1-12x_2-72z_1)](trk^2)^2\\
&+\frac{3}{160}\pi[B_1(19+6y_1-56y_2-336z_2)+4B_2(9-14y_1+24y_2+144z_2)]k^4\\
&+\frac{3}{80}\pi[B_1(-79+3x_1-28x_2-32y_1+112y_2-168z_1+672z_2)\\
&\qquad -4B_2(29+7x_1-12x_2-48y_1+48y_2-72z_1+288z_2)]k^2trk^2
\end{aligned} \tag{64}$$

For 6d CFTs with $B_3=0$, the holographic k^4 terms eq.(54) becomes

$$\begin{aligned}
S_{\Sigma}|_{\log} \ &= \ \log(\ell/\delta)\int_{\Sigma}d^4x\sqrt{h_0}\left[3\pi B_1(\frac{3}{2}T_1-T_2)-12\pi B_2(T_2)\right]\\
&= \ \log(\ell/\delta)\int_{\Sigma}3\pi\big[-(B_1+4B_2)trk^4+(B_1+4B_2)ktrk^3+\frac{3}{2}B_1(trk^2)^2-\frac{3}{8}(3B_1+4B_2)k^2trk^2\\
&\qquad +\frac{3}{64}(3B_1+4B_2)k^4\big].
\end{aligned} \tag{65}$$

Compare eq.(64) with eq.(65), we find a unique solution

$$x_1=1,\ x_2=\frac{1}{4}-6z_1,\ y_1=0,\ y_2=-\frac{1}{16}-6z_2 \tag{66}$$

Note that B_1 and B_2 are independent central charges, so there are ten equations (64) for six unkown parameters. Thus it is really non-trivial that we have consistent solutions.

4.2 F_3

Now let us go on to study the F_3 term. In sect. 3.1.2, we have discussed the holographic F_3 for two interesting cases. In this first case we set $k_{aij} = 0$ and derive the C^2 terms of F_3 eq.(48). While in the second case, we focus on the flat boundary spacetime and obtain the k^4 terms of F_3 eqs.(48,54). In this section, we calculate the corresponding field theoretical results and compare with the holographic ones. We find that the C^2 terms of F_3 indeed match. This can be regarded as a resolution of the HMS puzzle [34, 37]. While for the k^4 terms, we have to deal with the splitting problem. We assume eqs.(60,62) and check if this assumption of splitting could pass the F_3 test or not.

Case **I**: $k_{aij} = 0$

Let us firstly investigate the case with zero extrinsic curvature. It is found by HMS [34] that there are mismatches between the holographic and the field theoretical universal terms of EE even for the entangling surfaces with zero extrinsic curvature. Recently, the authors of [**?**] find that HMS have ignored the anomaly of entropy from the Weyl anomaly I_3. After taking into account this contribution, the holographic and CFT universal terms of EE indeed match [37]. For simplicity [34, 37] both focus on the cases with $k_{aij} = 0$ and $R_{abci} = 3\epsilon_{ab}V_{ci} = 0$. Here we drop the second constraint $R_{abci} = 3\epsilon_{ab}V_{ci} = 0$ and check if the holographic and the field theoretical results still match. We only need to compare ΔS eq.(49) with the anomaly of entropy from I_3. That is because the anomaly of entropy of I_1 and I_2 vanishes for $k_{aij} = 0$. Note further that the anomaly of entropy of I_3 only comes form $C_{ijkl}\Box C^{ijkl} \cong -\nabla_m C_{ijkl}\nabla^m C^{ijkl}$ for the case of zero extrinsic curvature.

When the extrinsic curvature vanishes, the splitting problem disappears and the anomaly of entropy for the gravitational action with one derivative of the curvature is given by [37]

$$\begin{aligned} S_{Anomaly} &= 2\pi \int d^d y \sqrt{g} \Big[64 \Big(\frac{\partial^2 L}{\partial \nabla_z R_{zizl} \partial \nabla_{\bar z} R_{\bar z k \bar z l}} \Big)_{\alpha_1} \frac{Q_{zzij} Q_{\bar z \bar z kl}}{\beta_{\alpha_1}} \\ &+ 96 i \Big(\frac{\partial^2 L}{\partial \nabla_z R_{zizl} \partial \nabla_{\bar z} R_{\bar z z \bar z k}} \Big)_{\alpha_1} \frac{Q_{zzij} V_{\bar z k}}{\beta_{\alpha_1}} + c.c \\ &+ 144 \Big(\frac{\partial^2 L}{\partial \nabla_z R_{z \bar z z l} \partial \nabla_{\bar z} R_{\bar z z \bar z k}} \Big)_{\alpha_1} \frac{V_{zl} V_{\bar z k}}{\beta_{\alpha_1}} \Big], \end{aligned} \tag{67}$$

where Q, V are defined in the conical metric

$$\begin{aligned} ds^2 = e^{2A}[dz d\bar z + e^{2A} T (\bar z dz - z d\bar z)^2] + 2 i e^{2A} V_i (\bar z dz - z d\bar z) dy^i \\ + (g_{ij} + Q_{ij}) dy^i dy^j. \end{aligned} \tag{68}$$

Here $A = -\frac{\epsilon}{2} \lg(z\bar z + a^2)$ is regularized warp factor and V_i, Q_{ij} are defined as

$$\begin{aligned} &V_i = U_i + z V_{zi} + \bar z V_{\bar z i} + O(z^2), \\ &Q_{ij} = z^2 Q_{zzij} + \bar z^2 Q_{\bar z \bar z ij} + 2 z \bar z e^{2A} Q_{z \bar z ij} + O(z^3) \end{aligned} \tag{69}$$

Applying the formula eq.(67), we derive the anomaly of entropy of $C_{ijkl}\Box C^{ijkl} \cong -\nabla_m C_{ijkl}\nabla^m C^{ijkl}$ as

$$S_A = \int d^4 y \sqrt{h} \big[128\pi \bar Q_{zzij} \bar Q_{\bar z \bar z}{}^{ij} + 432\pi V_{zi} V_{\bar z}{}^{i} \big]. \tag{70}$$

It should be mentioned that the total entropy of $\Box C_{ijkl}C^{ijkl}$ vanishes by using the approach of [32, 37].

Substituting the conical metric eq.(68) with $A=0$ into ΔS eq.(49), we get

$$\Delta S = \left[128\pi \bar{Q}_{zzij}\bar{Q}_{\bar{z}\bar{z}}{}^{ij} + 432\pi V_{zi}V_{\bar{z}}{}^{i}\right]. \tag{71}$$

which is exactly the same as eq.(70). Thus the holographic and the field theoretical results match for the C^2 terms of F_3.

Case **II**: flat $\overset{(0)}{g}_{ij}$

Now let us go on to study the second case with flat spacetime on the boundary. The holographic F_3 term is given by eq.(54)

$$S_\Sigma|_{\log} = \log(\ell/\delta)\int_\Sigma d^4y\sqrt{h_0}\,\left[6\pi B_3(T_3+9T_1-12T_2)\right] \tag{72}$$

with

$$T_1=(\mathrm{tr}\bar{k}^2)^2,\quad T_2=\mathrm{tr}\bar{k}^4,\quad T_3=(\nabla_a k)^2-\frac{25}{16}k^4+11k^2trk^2-6(trk^2)^2-16ktrk^3+12trk^4. \tag{73}$$

Applying the method developed in [32, 37], we can derive $2\pi F_3$ as the entropy of Weyl anomaly I_3. We list the results below.

I For $ds^2=dzd\bar{z}+(1+\frac{z+\bar{z}}{2})^2dy_1^2+dy_2^2+dy_3^2+dy_4^2$, we obtain the entropy of I_3 as

$$S_I|_{\log}=\frac{27\pi}{8} \tag{74}$$

which agrees with the holographic result eq.(72). Here and below we drop the factor from the integral dy^4.

II For $ds^2=dzd\bar{z}+(1+\frac{z+\bar{z}}{2})^2(dy_1^2+\sin^2 y_1dy_2^2)+dy_3^2+dy_4^2$, we derive the entropy of I_3 as

$$S_{II}|_{\log}=30\pi, \tag{75}$$

which matches the holographic result eq.(72).

III For $ds^2=dzd\bar{z}+(1+\frac{z+\bar{z}}{2})^2(dy_1^2+\sin^2 y_1dy_2^2+\sin^2 y_1\sin^2 y_2dy_3^2)+dy_4^2$, we get the entropy of I_3 as

$$S_{III}|_{\log}=\frac{459\pi}{8}, \tag{76}$$

which is consistent with the holographic result eq.(72).

IV For $ds^2=dzd\bar{z}+(1+\frac{z+\bar{z}}{2})^2(dy_1^2+\sin^2 y_1dy_2^2+\sin^2 y_1\sin^2 y_2dy_3^2+\sin^2 y_1\sin^2 y_2\sin^2 y_3^2dy_4^2)$, we have the entropy of I_3

$$S_{IV}|_{\log}=0, \tag{77}$$

which also agrees with the holographic result eq.(72).

Now it is clear that the splittings eq.(60,62) have passed the F_3 test. Remarkably, we cannot fix the splittings completely by comparing the holographic and field theoretical universal terms of EE. It seems that we have more than one way to split the conical metrics on the boundary and such freedom does not affect the universal terms of EE.

5 Conclusions

We have investigated the universal terms of EE for 6d CFTs by applying holographic and the field theoretical methods, respectively. Our results agree with those of [34, 35]. We find the holographic and the field theoretical results match for the C^2 and Ck^2 terms. While for the k^4 terms, we meet the splitting problem for the conical metrics. We fix the splitting problem in the bulk by using two different methods. The first one is by using equations of motion and second one is requiring that all the higher derivative theories of gravity yield the same logarithmic terms of EE. These two methods give consistent results for the splitting in the bulk. As for the splitting on the boundary, we assume the general forms and find there indeed exists suitable splitting which can make the holographic and CFT k^4 terms match. Since we have much more equations than the free parameters, this match is non-trivial. Remarkably, we can not fix the splitting on the boundary completely by comparing the holographic and field theoretical results. It seems that we have some freedom to split the conical metrics on the boundary and such freedom does not affect the universal terms of EE for CFTs. That is not surprising. That is because the terms (Weyl anmoly) we studied are quite special. Actually, for Love-lock gravity, arbitrary splitting would not affect the entropy. How to fix the splitting problem on the boundary is an interesting problem. For the cases with gravity duals, we could obtain the conical metrics on the boundary from the one in the bulk. While for the general cases, now it is not clear to us how to fix this problem. We hope to address this problem in future. Finally, we want to point out how much our holographic results F_i eqs.(35,36,47) depend on the splittings. It is clear that the combinations $(F_3 - 3F_2 - 12F_1)$ and $(2F_1 + F_2)$ are independent of the splittings. That is because they can be derived from the holographic entanglement entropy of Einstein gravity and Love-lock gravity which are irrelevant to the splittings. Without loss of generality, we choose F_2 as the third independent combination of F_i. As mentioned above, the splitting problem does not affect the C^2 and Ck^2 terms. Thus, only the k^4 terms of F_2 are relevant to the splitting problem.

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A FG expansion and Schwimmer-Theisen approach

A.1 FG expansion

For asymptotically AdS space-time, we can expand the bulk metric in the Fefferman-Graham gauge

$$ds^2 = \hat{G}_{\mu\nu} dx^\mu dx^\nu = \frac{1}{4\rho^2} d\rho^2 + \frac{1}{\rho} g_{ij} dx^i dx^j , \tag{78}$$

where $g_{ij} = \overset{(0)}{g}_{ij} + \rho \overset{(1)}{g}_{ij} + ... + \rho^{\frac{d}{2}}(\overset{(\frac{d}{2})}{g}_{ij} + \overset{(\frac{d}{2})}{h}_{ij} \log \rho) + ...$ Interestingly,

$$\overset{(1)}{g}_{ij} = -\frac{1}{d-2}(\overset{(0)}{R}_{ij} - \frac{\overset{(0)}{R}}{2(d-1)} \overset{(0)}{g}_{ij}), \tag{79}$$

can be determined completely by PBH transformation [44, 45] and thus is independent of equations of motion. However, the higher order terms $\overset{(2)}{g}_{ij}, \overset{(3)}{g}_{ij} ...$ are indeed constrained by equations of motion. We have

$$\begin{aligned} \overset{(2)}{g}_{ij} \quad = \quad & k_1 \, C_{mnkl} C^{mnkl} \; \overset{(0)}{g}_{ij} + k_2 \, C_{iklm} C_j{}^{\; klm} \\ & + \frac{1}{d-4} \Bigg[\frac{1}{8(d-1)} \nabla_i \nabla_j R - \frac{1}{4(d-2)} \Box R_{ij} + \frac{1}{8(d-1)(d-2)} \Box R \; \overset{(0)}{g}_{ij} \\ & - \frac{1}{2(d-2)} R^{kl} R_{ikjl} + \frac{d-4}{2(d-2)^2} R_i{}^k R_{jk} + \frac{1}{(d-1)(d-2)^2} R R_{ij} \\ & + \frac{1}{4(d-2)^2} R^{kl} R_{kl} \; \overset{(0)}{g}_{ij} - \frac{3d}{16(d-1)^2 (d-2)^2} R^2 \; \overset{(0)}{g}_{ij} \Bigg], \end{aligned} \tag{80}$$

For action eq.(23), we have

$$k_1 = \frac{3}{80}(5\lambda_1 + 14\lambda_2), \quad k_2 = \frac{3}{4}(\lambda_1 - 4\lambda_2) \tag{81}$$

The following formulas are useful [33]

$$\begin{aligned} & \tilde{R} \sim o(\rho^2), \quad \tilde{R}_{ij} \sim o(\rho), \quad \tilde{R}_{i\rho} \sim o(\rho), \quad \tilde{R}_{\rho\rho} \sim o(1) \\ & \tilde{R}_{i\rho j\rho} \sim o(\frac{1}{\rho}), \quad \tilde{R}_{\rho ijk} \sim o(\frac{1}{\rho}) \\ & \tilde{R}_{ijkl} = \frac{C_{ijkl}}{\rho}. \end{aligned} \tag{82}$$

A.2 Schwimmer-Theisen approach

Denote the transverse space of the squashed cone by m. The embedding of the 5-dimensional submanifold m into 7-dimensional bulk is described by $X^\mu = X^\mu(\sigma^\alpha)$, where $X^\mu = \{x^i, \rho\}$ are bulk coordinates and $\sigma^\alpha = \{y^a, \tau\}$ are coordinates on m. We choose a gauge

$$\tau = \rho, \quad h_{a\tau} = 0, \tag{83}$$

where $h_{\alpha\beta}$ is the induced metric on m. Let us expand the embedding functions as

$$X^i(\tau, y^i) = \overset{(0)}{X^i}(y^a) + \overset{(1)}{X^i}(y^a)\tau + \overset{(2)}{X^i}(y^a)\tau^2 + ... \tag{84}$$

Diffeomorphism preserving the FG gauge (78) and above gauge (83) uniquely fixes a transformation rule of the embedding functions $X^\mu(y^a, \tau)$ [46]. From this transformation rule, we can identity $\overset{(1)}{X^i}(y^a)$ with $\frac{1}{8}k^i(y^a)$

$$\overset{(1)}{X^i}(y^a) = \frac{1}{8}k^i(y^a), \tag{85}$$

where k^i is the trace of the extrinsic curvature of the entangling surface Σ in the boundary where CFTs live. From eq.(84), we can derive the induced metric on m as

$$h_{\tau\tau} = \frac{1}{4\tau^2}\Big(1 + \frac{1}{4}\, k^i k^j \overset{(0)}{g}_{ij}\, \tau + \cdots\Big), \qquad h_{ab} = \frac{1}{\tau}\Big(\overset{(0)}{h}_{ab} + \overset{(1)}{h}_{ab}\, \tau + ...\Big), \tag{86}$$

with

$$\overset{(0)}{h}_{ab} = \partial_a \overset{(0)}{X^i} \partial_b \overset{(0)}{X^j}\ \overset{(0)}{g}_{ij}, \qquad \overset{(1)}{h}_{ab} = \overset{(1)}{g}_{ab} - \frac{1}{2} k^i k^j_{ab} \overset{(0)}{g}_{ij}. \tag{87}$$

Thus, we have

$$\sqrt{h} = \sqrt{\overset{(0)}{h}}\, \frac{1}{2\rho^3} + ... \tag{88}$$

Using eq.(84), we can also derive the extrinsic curvature K of m as

$$K^i_{ab} = (k^i_{ab} - \frac{k^i}{4} \overset{(0)}{h}_{ab}) + ... \tag{89}$$

Note that all the other components of $K^\mu_{\alpha\beta}$ are higher order terms which do not contribute to the logarithmic terms.

In Dong's notation, we list some useful formulas.

$$K_{aij} = \frac{\bar{k}_{aij}}{\sqrt{\rho}} + ...,\ K_a \sim \rho^{3/2}, \quad K_{aij}K^{aij} = \rho \bar{k}_{aij}\bar{k}^{aij} + ... \tag{90}$$

From eqs.(88,90), we notice that only the following terms could contribute to the universal term of EE for 6d CFTs

$$\sqrt{h} trK^4, \quad \sqrt{h}(trK^2)^2 \tag{91}$$

B The conformal invariance of F_3

In this section, we prove that the logarithmic terms of EE for Einstein gravity S_E eq.(46) are conformal invariant. Recall that F_3 is a combination of S_E and the conformal invariants F_1, F_2, E_4. Thus, equivalently, we shall prove F_3 is conformal invariant. For simplicity, we focus on the infinitesimal conformal transformations. According to [44], we have

$$\begin{aligned}
\delta \overset{(0)}{g}_{ij} &= 2\sigma \overset{(0)}{g}_{ij}\\
\delta \overset{(1)}{g}_{ij} &= \nabla_i \nabla_j \sigma\\
\delta \overset{(2)}{g}_{ij} &= -2\sigma \overset{(2)}{g}_{ij} + \frac{1}{2}\nabla^k\sigma\nabla_k \overset{(1)}{g}_{ij} - \frac{1}{2}\nabla^m\sigma\nabla_{(i} \overset{(1)}{g}_{j)m} + \frac{1}{2} \overset{(1)}{g}{}_{(i}{}^{k} \nabla_{j)}\nabla_k\sigma.
\end{aligned} \tag{92}$$

and

$$\begin{aligned}
\delta k^m_{ij} &= -h_{ij}\tilde{g}^{\perp mn}\nabla_n\sigma\\
\delta k^m &= -2\sigma k^m - 4\tilde{g}^{\perp mn}\nabla_n\sigma
\end{aligned}$$

$$\delta\Gamma^m_{ij} = \delta^m_i \nabla_j\sigma + \delta^m_j \nabla_i\sigma - \overset{(0)}{g}_{ij}\nabla^m\sigma$$
$$\delta R_{ijkl} = 2\sigma R_{ijkl} + \overset{(0)}{g}_{il}\nabla_j\nabla_k\sigma - \overset{(0)}{g}_{ik}\nabla_j\nabla_l\sigma + \overset{(0)}{g}_{jk}\nabla_i\nabla_l\sigma - \overset{(0)}{g}_{jl}\nabla_i\nabla_k\sigma \tag{93}$$

Substituting eqs.(92,93) into eq.(46), we get

$$\begin{aligned}
\delta_\sigma S_E \quad = \quad & \pi\log(\ell/\delta)\int_\Sigma d^4y\sqrt{h_0}[\overset{(1)}{g}{}^{ij} h_{ij}k^m\nabla_m\sigma + \frac{1}{4}h^{ij}k^m\nabla_m\nabla_i\nabla_j\sigma + \frac{1}{4}\tilde{g}^{\perp mn}h^{ij}k^l R_{limj}\nabla_n\sigma \\
& -2\overset{(1)}{g}{}^{ij}k^m_{ij}\nabla_m\sigma + h^{ij}\nabla^m\overset{(1)}{g}_{ij}\nabla_m\sigma - \tilde{g}^{\perp mn}h^{ij}\nabla_m\sigma\nabla_n\overset{(1)}{g}_{ij} - \frac{3}{32}k^m k_m k^n\nabla_n\sigma \\
& +\frac{1}{4}k^m k^n\nabla_m\nabla_n\sigma - \overset{(1)}{g}_{ij}k_i\nabla_j\sigma + \frac{1}{2}k^m k_{mij}k^{nij}\nabla_n\sigma - \overset{(1)}{g}{}^{i}{}_{m}h^{mj}\nabla_i\nabla_j\sigma + \overset{(1)}{g}{}^{ij}h_{ij}h^{mn}\nabla_m\nabla_n\sigma \\
& -h^{ij}\nabla^k\nabla_i\overset{(1)}{g}_{jk} + \frac{1}{16}h^{ij}k^m\nabla_i k_j\nabla_m\sigma - \frac{1}{16}h^{ij}k^m\nabla_i k_m\nabla_j\sigma + \frac{1}{2}h^{ij}\nabla_i k^m\nabla_j\nabla_m\sigma \\
& -\frac{5}{32}k^m k_m h^{ij}\nabla_i\nabla_j\sigma + \frac{1}{2}k_m k^{mij}\nabla_i\nabla_j\sigma - 2\overset{(1)}{g}{}^{ij}h_{ik}\nabla^k\tilde{g}^{\perp}{}_{jl}\nabla^l\sigma - \frac{1}{4}h^{ij}\nabla_i k^m\nabla_j\tilde{g}^{\perp}{}_{mn}\nabla_n\sigma \\
& -\frac{1}{4}\tilde{g}^{\perp ij}h^{kl}\nabla_k k_i\nabla_j\nabla_l\sigma].
\end{aligned} \tag{94}$$

Let us try to simplify the above complicated results. The trick is to replace the covariant derivative ∇_i with respect to $\overset{(0)}{g}_{ij}$ by the intrinsic covariant derivative D_i with respect to h_{ij} as much as possible. Besides, we find the following formulas are useful:

$$\begin{aligned}
& h^m_i h^n_j\nabla_m V_n = D_i(h^n_j V_n) - k^m_{ij}V_m \\
& h^m_i h^n_j\nabla_m\nabla_n\sigma = D_i D_j\sigma - k^m_{ij}\nabla_m\sigma \\
& h^m_n\nabla_m h_{ij} = k_{inj} + k_{jni} \\
& k_m\nabla_n h^{mn} = k^m k_m \\
& k^m h^p_i h^q_j h^l_k R_{mpql} = k^m(\nabla_j k_{mik} - \nabla_k k_{mij}) \\
& k^m h^{ik}h^{jl}R_{mijk}\nabla_l\sigma = \frac{1}{2}D_i\sigma D^i(k^m k_m) - D^i\sigma D^j(k^m k_{mij}) + \nabla_i k_m k^{mij}\nabla_j\sigma \\
& h^{ij}k^m\nabla_i\nabla_m\nabla_j\sigma = D^i(h^j_i k^m\nabla_m\nabla_j\sigma) - k^m k^n\nabla_i\nabla_j\sigma - h^{ij}\nabla_i k^m\nabla_m\nabla_j\sigma
\end{aligned} \tag{95}$$

Applying the above formulas, we can simplify $\delta_\sigma S_E$ as

$$\begin{aligned}
\delta_\sigma S_E \quad = \quad & \pi\log(\ell/\delta)\int_\Sigma d^4y\sqrt{h_0}D^i[\frac{1}{4}h^j_i k^m\nabla_m\nabla_j\sigma + \overset{(1)}{g}_{mn}h^{mn}D_i\sigma - h_{ij}\overset{(1)}{g}{}^{jm}\nabla_m\sigma \\
& -\frac{1}{8}k^m k_m D_i\sigma + \frac{1}{4}k^m k_{mij}D^j\sigma].
\end{aligned} \tag{96}$$

which are just total derivatives. Now it is clear that S_E eq(46) and thus F_3 eq.(47) are conformal invariant up to some total derivatives.

C Weyl tensor

The Weyl tensor in D-dimensional spacetime is defined as

$$C_{\mu\nu\rho\sigma} = R_{\mu\nu\rho\sigma} - \frac{2}{D-2}(g_{\mu[\rho}R_{\sigma]\nu} - g_{\nu[\rho}R_{\sigma]\mu}) + \frac{2}{(D-1)(D-2)}R\ g_{\mu[\rho}g_{\sigma]\nu}. \tag{97}$$

Here we list some useful formulas.

$$C_{z\bar{z}z\bar{z}} \quad = \quad e^{4A}C_{1\ z\bar{z}z\bar{z}} + e^{2A}C_{0\ z\bar{z}z\bar{z}},$$

$$
\begin{aligned}
C_{0\ z\bar{z}z\bar{z}} &= -3T_0 + \frac{1}{D-2}\left[K_{zmn}K_{\bar{z}}^{mn} - Q_{0\ z\bar{z}\ m}^{\ \ \ m} + 6T_0\right] \\
&\quad -\frac{1}{4(D-1)(D-2)}(3K_{cmn}K^{cmn} - K_cK^c - 2Q_{0c\ m}^{\ \ c\ m} + 24T_0) \qquad (98)\\
C_{zi\bar{z}j} &= e^{2A}C_{1\ zi\bar{z}j} + C_{0\ zi\bar{z}j}, \\
C_{0\ zi\bar{z}j} &= K_{zjl}K_{\bar{z}\ i}^{\ l} - Q_{0\ z\bar{z}ij} \\
&\quad -\frac{1}{D-2}\left[K_{cjl}K^{cl}_{i} - \frac{1}{2}K^cK_{cij} - \frac{1}{2}Q_0{}^{c}{}_{cij} + g_{ij}(K_{zmn}K_{\bar{z}}{}^{mn} - Q_{0\ z\bar{z}m}{}^{m} + 6T_0)\right] \\
&\quad +\frac{1}{2(D-1)(D-2)}g_{ij}(3K_{cmn}K^{cmn} - K_cK^c - 2Q_{0c\ m}^{\ \ c\ m} + 24T_0) \qquad (99)\\
C_{ikjl} &= C_{1\ ikjl} + e^{-2A}C_{0\ ikjl}, \\
C_{0\ ikjl} &= K_{ail}K^{a}{}_{jk} - K_{aij}K^{a}{}_{kl} \\
&\quad -\frac{2}{D-2}\left[g_{i[j}R_{0\ l]k} - g_{k[j}R_{0\ l]i}\right] \\
&\quad +\frac{2}{(D-1)(D-2)}g_{i[j}g_{l]k}(3K_{cmn}K^{cmn} - K_cK^c - 2Q_{0c\ m}^{\ \ c\ m} + 24T_0) \qquad (100)\\
R_{0\ ij} &= 2K_{aim}K^{am}{}_{j} - K^aK_{aij} - Q^a_{0aij} \qquad (101)
\end{aligned}
$$

Let us focus on the case of [35] with $K_{aij} = \frac{1}{2}k_{ij}, Q_{0\ z\bar{z}ij} = \frac{1}{4}q_{ij}$ and $D = 6$. We have

$$
\begin{aligned}
C_{0\ z\bar{z}z\bar{z}} &= \frac{1}{80}(2k_{mn}k^{mn} + k^2 - 3q) - \frac{9}{5}t_0 \\
C_{0\ zi\bar{z}j} &= -\frac{1}{8}q_{ij} + \frac{1}{8}kk_{ij} + \frac{1}{80}g_{ij}(k_{mn}k^{mn} - 2k^2 + q) - \frac{9}{10}t_0g_{ij} \\
C_{0\ ikjl} &= (k_{il}k_{jk} - k_{ij}k_{kl}) - \frac{1}{2}(g_{i[j}r_{0\ l]k} - g_{k[j}r_{0\ l]i}) + \frac{1}{10}g_{i[j}g_{l]k}(3k_{mn}k^{mn} - k^2 - 2q + 24t_0) \qquad (102)\\
r_{0\ ij} &= 2k_{im}k^{m}{}_{j} - kk_{ij} - q_{ij} \qquad (103)
\end{aligned}
$$

References

[1] T. Nishioka, S. Ryu and T. Takayanagi, J. Phys. A 42: 504008 (2009).

[2] L. Amico, R. Fazio, A. Osterloh and V. Vedral, Rev. Mod. Phys. 80, 517 (2008).

[3] J. Eisert, M. Cramer, and M. Plenio, Rev. Mod. Phys. 82, 277 (2010).

[4] S. Ryu and T. Takayanagi, Phys. Rev. Lett. **96**, 181602 (2006) [arXiv:hep-th/0603001].

[5] S. Ryu and T. Takayanagi, JHEP **0608**, 045 (2006) [arXiv:hep-th/0605073[hep-th]].

[6] V. Rosenhaus and M. Smolkin, JHEP 1412 (2014) 179 .

[7] A. Lewkowycz and J. Maldacena, Generalized gravitational entropy, JHEP 1308 (2013) 090, [arXiv:1304.4926].

[8] A. Almheiri, X. Dong and D. Harlow, "Bulk Locality and Quantum Error Correction in AdS/CFT", [arXiv:1411.7041[hep-th]].

[9] E. Mintun, J. Polchinski and V. Rosenhaus, "Bulk-Boundary Duality, Gauge Invariance, and Quantum Error Correction", [arXiv:1501.06577[hep-th]].

[10] D. V. Fursaev, A. Patrushev and S. N. Solodukhin, Phys.Rev. D88 (2013) 4, 044054. [arXiv:1306.4000 [hep-th]].

[11] D. V. Fursaev, Proof of the Holographic Formula for Entanglement Entropy, JHEP 0609 (2006) 018.

[12] H. Casini, M. Huerta and R. C. Myers, Towards a derivation of holographic entanglement entropy, JHEP05(2011)036 [arXiv:1102.0440 [hep-th]].

[13] M. Nozaki, T. Numasawa and T. Takayanagi, "Quantum Entanglement of Local Operators in Conformal Field Theories," Phys. Rev. Lett. **112**, 111602 (2014) [arXiv:1401.0539 [hep-th]].

[14] S. He, T. Numasawa, T. Takayanagi and K. Watanabe, "Quantum Dimension as Entanglement Entropy in 2D CFTs," Phys. Rev. D **90**, 041701 (2014) [arXiv:1403.0702 [hep-th]].

[15] R. Bousso, H. Casini, Z. Fisher and J. Maldacena, Phys. Rev. D **90**, 044002 (2014) [arXiv:1404.5635 [hep-th]].

[16] R. Bousso, H. Casini, Z. Fisher and J. Maldacena, arXiv:1406.4545 [hep-th].

[17] V. Rosenhaus and M. Smolkin, "Entanglement Entropy Flow and the Ward Identity," arXiv:1406.2716 [hep-th].

[18] A. Bhattacharyya, A. Kaviraj and A. Sinha, JHEP 1308, 012 (2013) [arXiv:1305.6694 [hep-th]];

[19] A. Bhattacharyya, M. Sharma and A. Sinha, JHEP 1401, 021 (2014) [arXiv:1308.5748 [hep-th]].

[20] A. Bhattacharyya, M. Sharma, JHEP 1410 (2014) 130 [arXiv:1405.3511 [hep-th]].

[21] V. Balasubramanian, B. Czech, B. D. Chowdhury and J. de Boer, JHEP 1310 (2013) 220 [arXiv:1305.0856[hep-th]].

[22] V. Balasubramanian, B. D. Chowdhury, B. Czech, J. de Boer and M. P. Heller, Phys. Rev. D 89, 086004 (2014) [arXiv:1310.4204[hep-th]].

[23] R. C. Myers, J. Rao and S. Sugishita, JHEP 1406 (2014) 044 [arXiv:1403.3416[hep-th]].

[24] M. Headrick, R. C. Myers and J. Wien, JHEP 10: 149 (2014) [arXiv:1408.4770[hep-th]].

[25] S. Cremonini and X Dong, Phys. Rev. D 89, 065041 (2014) [arXiv:1311.3307[hep-th]].

[26] D. Fursaev, Bulletin MSRU, v.1 (2014) 1-9 [arXiv:1406.5635[gr-qc]].

[27] J. Erdmenger, M. Flory and C. Sleight, JHEP06(2014)104 [arXiv:1401.5075 [hep-th]].

[28] K. Sen and A. Sinha, JHEP 1407 (2014) 098 [arXiv:1405.7862 [hep-th]].

[29] P. Calabrese and J. Cardy, Entanglement entropy and quantum field theory, J. Stat. Mech. 0406, P002 (2004) [arXiv:hep-th/0405152].

[30] C. Holzhey, F. Larsen and F. Wilczek, Geometric and renormalized entropy in conformal field theory, Nucl. Phys. B 424, 443 (1994) [arXiv:hep-th/9403108].

[31] S. N. Solodukhin, Phys. Lett. B **665** (2008) 305 [arXiv:0802.3117].

[32] X. Dong, JHEP01(2014)044 [arXiv:1310.5713[hep-th]].

[33] R. X. Miao, Class. Quant. Grav. **31** (2014) 065009.

[34] L. Y. Hung, R. C. Myers and M. Smolkin, JHEP 1104 (2011) 025 [arXiv:1101.5813].

[35] B. R. Safdi, JHEP12(2012)005 [arXiv:1206.5025 [hep-th]].

[36] R. C. Myers and A. Sinha, JHEP **1101**, 125 (2011) [arXiv:1011.5819].

[37] R. X. Miao and W. z. Guo, [arXiv:1411.5579[hep-th]].

[38] A. Astaneh, A Patrushev and S. N. Solodukhin, [arXiv:1411.0926[hep-th]].

[39] A. Astaneh, A Patrushev and S. N. Solodukhin, [arXiv:1412.0452[hep-th]].

[40] X. Dong and R. X. Miao, "Generalized Gravitational Entropy of Total Derivatives", to appear.

[41] J. Camps, JHEP 1403 (2014) 070 [arXiv:1310.6659[hep-th]].

[42] J. Camps and W. R. Kelly, JHEP03(2015)061 [arXiv:1412.4093[hep-th]].

[43] T. Jacobson and R. C. Myers, Phys. Rev. Lett. **70**, 3684 (1993) [arXiv:hep-th/9305016].

[44] C. Imbimbo, A. Schwimmer, S. Theisen and S. Yankielowicz, Class. Quant. Grav. **17** (2000) 1129 [arXiv:hep-th/9910267].

[45] A. Schwimmer and S. Theisen, JHEP 1405 (2014) 012 [arXiv:1311.4746 [hep-th]].

[46] A. Schwimmer and S. Theisen, Nucl.Phys. B **801** (2008) 1-24 [arXiv:0802.1017 [hep-th]].