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Smooth Non-Zero Rest-Mass Evolution Across Time-Like Infinity

Helmut Friedrich

Abstract. It is shown that solutions to Einstein's field equations with positive cosmological constant can include *non-zero rest-mass fields* which coexist with and travel unimpeded across a smooth conformal boundary. This is exemplified by the coupled Einstein-massive-scalar field equations for which the mass m is related to the cosmological constant λ by the relation $3\,m^2 = 2\,\lambda$. Cauchy data for the conformal field equations can in this case be prescribed on the (compact, space-like) conformal boundary $\mathcal{J}^+$. Their developments backwards in time induce a set of standard Cauchy data on space-like slices for the Einstein-massive-scalar field equations which is open in the set of all Cauchy data for this system.

1. Introduction

In his work on *conformal cyclic cosmology* [15] Roger Penrose conjectures the occurrence of concentric circles in the CMB which reflect bursts of gravitational radiation resulting from encounters of supermassive black holes in an aeon preceding the big bang of our present one. A recent analysis [10] of the CMB based on WMAP data and an independent, even more recent, study of the CMB maps observed by the Planck collaboration [1] indeed seem to identify ring like structures in the CMB sky. While these findings indicate strong support of Penrose's proposal, the theoretical reasoning which led to them still raises questions.

The underlying picture is that of a smooth, time-oriented conformal structure $\mathcal{C}$ of signature $(-,+,+,+)$ on a 4-dimensional manifold $\mathcal{M} \sim \mathbb{R} \times S$ with compact 3-manifold S, into which an infinite sequence of *aeons*, time-oriented 'physical' solutions to Einstein's field equations with cosmological constant $\lambda > 0$, are conformally embedded so that any two neighbouring aeons are separated by a *crossover* 3-*surface* $\mathcal{X} \sim S$ which is space-like with respect to the conformal structure. The aeons start with a big bang that 'touches'

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the preceding crossover surface while their future end represents an exponentially expanding space-time for which the following crossover surface defines a smooth conformal boundary in the sense proposed by Roger Penrose in [14].

Consider the Einstein equations with cosmological constant $\lambda > 0$ and vanishing energy momentum tensor and denote by $\mathcal{D}_S$ the set of Cauchy data (i.e. solutions to the constraints) for these equations on 3-manifolds diffeomorphic to S. Not all the space-times developing from these data admit a smooth conformal boundary diffeomorphic to S. Notable examples are the Schwarzschild–de Sitter solution, which includes black holes and admits only patches of a smooth conformal boundary [9], and the Nariai solution, which is geodesically complete but does not even admit a piece of a smooth conformal boundary [3]. Nevertheless, the class of solutions which do admit a smooth conformal boundary diffeomorphic to S is fairly rich. This is a consequence of a peculiar feature of Einstein's field equations. In the vacuum case, they admit representations in terms of the conformal fields, referred to as *conformal fields equations* (see Sect. 2), which induce under suitable gauge assumptions equations that are hyperbolic even where the conformal factor vanishes or becomes negative [5,8].

The conformal equations can be solved backwards in time with Cauchy data which are prescribed on the future conformal boundary $\mathcal{J}^+ \sim S$. The freedom to prescribe data on $\mathcal{J}^+$ is essentially the same as in the standard Cauchy problem, though, due to the fact that the conformal boundary is geometrically a very special hypersurface relative to the solution space-time, there are differences in the interpretation of the data [6]. Let $\mathcal{A}_S$ denote the Cauchy data pertaining to solutions obtained by such backward developments. It turns out that $\mathcal{A}_S$ is an open subset of $\mathcal{D}_S$ (if endowed with a natural Sobolev topology). This follows by the argument used in [7] to show the non-linear stability of de Sitter space. In fact, due to their hyperbolicity, the conformal field equations see in principle no difference between backward and forward evolution. The data on the conformal boundary which have been evolved backwards can thus also be evolved forwards into domains foliated by Cauchy hypersurfaces on which the conformal factor is negative. The resulting solutions to the conformal field equations extend smoothly across the conformal boundary. Cauchy stability for hyperbolic systems then implies that data in $\mathcal{D}_S$ which are sufficiently close to $\mathcal{A}_S$ also develop into domains where the conformal factor is negative and the field equations themselves then ensure that the set where the conformal factor vanishes defines a smooth conformal boundary for the vacuum space-times arising from the given data in $\mathcal{D}_S$ (which are thus seen to be in fact in $\mathcal{A}_S$).

The solution to the conformal field equation in the future of the conformal boundary again defines a solution to the Einstein equations we started with. This solution is also determined uniquely by the Cauchy data we prescribed in the past of the conformal boundary.

These results generalize to matter fields whose energy momentum tensor is trace free and which obey conformally covariant field equations. This has been exemplified in detail in [8] by the Maxwell or the Yang–Mills equations but it holds true for other such equations.[1] It follows that if a solution to these equations does admit a smooth conformal boundary in the future, this is true also for all solutions which are, in terms of Cauchy data on some given time slice, close to the given one. In all these cases, gravitational radiation or other field excitations will travel unimpeded across the conformal boundary.

There remains the question of what will happen to the prospective conformal boundary in the presence of fields with non-vanishing rest-mass, in which case the energy momentum tensor has non-vanishing trace. In [15], it has been assumed that there will be some past neighbourhood of the crossover surface in which only zero rest-mass fields will be present. While it is far from obvious how massive fields behave at the end of an unlimited spatial expansion, this certainly seems to be a strong requirement. At present, no process is known which would allow one to justify it. It is the main purpose of the following analysis to show that this restriction may not be necessary.

There is a second problem, which arises right at the crossover surfaces. The solution to the Einstein equations obtained in the future of a crossover surface $\mathcal{X}$ by extending the solution to the conformal field equations smoothly across $\mathcal{X}$ will start to contract and thus rather resemble a time-reversed version of an exponentially expanding space-time instead of a big bang solution as required by the standard scenario. It is suggested therefore in [15] that in the immediate future of $\mathcal{X}$ each aeon space-time evolves instead according to the 'isotropic cosmological singularity' model studied by Paul Tod (cf. [17], which also gives references to earlier work in this direction). In this setting, it is assumed that the space-time admits a conformal rescaling which blows up the space-time near the big bang so that the latter can be represented by a space-like hypersurface smoothly attached to the past end of the original space-time. The idea then is to identify this hypersurface and the fields obtained on it by the blow-up procedure with the preceding crossover surface and the data induced on that by the smooth conformal extension from the previous aeon.

There appears to be a basic difference, however, between the 'blow-down' procedure underlying the construction of the conformal boundary considered above and the blow-up procedure defining the isotropic cosmological singularity picture. The latter seems to admit no version of conformal field equations which induce hyperbolic evolution equations near the conformal boundary under fairly general assumptions. In fact, the freedom to prescribe initial data for the future evolution on the past boundary turns out to be rather restricted [17]. Evolving such restricted data into the future, performing a slight generic perturbation of the data induced on some Cauchy hypersurface, and then evolving backwards will, more likely than not, result in a space-time which does not admit a conformal blow-up leading to a smooth setting.

In general, it is not clear to what extent data induced on $\mathcal{X}$ from the previous aeon can be evolved further in the new setting and if they can the extension procedure will not be stable. Moreover, it is not clear which mechanism should convert on $\mathcal{X}$ the evolution law carried across $\mathcal{X}$ with the conformal

[1] Other such cases have recently been worked out in [11] and [12].

field equations into an evolution law consistent with the isotropic cosmological singularity setting. This problem will be addressed again in Sect. 5.

Required is an insight into the asymptotic behaviour at time-like infinity of fields with non-vanishing rest-mass, coupled to Einstein's equations

$$\hat{R}_{\mu\nu} - \frac{1}{2}\,\hat{R}\,\hat{g}_{\mu\nu} + \lambda\,\hat{g}_{\mu\nu} = \kappa\,\hat{T}_{\mu\nu}, \tag{1.1}$$

with cosmological constant $\lambda > 0$. The precise properties of the equations will depend very much on the specific nature of the chosen matter field and each of them will need a special analysis. We shall concentrate here on the case of a non-linear scalar field ϕ which obeys an equation of the form

$$\hat{\nabla}_\mu \hat{\nabla}^\mu \phi - (m^2\,\phi + V'(\phi)) = 0, \tag{1.2}$$

with energy momentum tensor

$$\hat{T}_{\mu\nu} = \hat{\nabla}_\mu\phi\,\hat{\nabla}_\nu\phi - \left(\frac{1}{2}\,(\hat{\nabla}_\rho\phi\,\hat{\nabla}^\rho\phi + m^2\,\phi^2) + V(\phi)\right)\hat{g}_{\mu\nu}, \tag{1.3}$$

and a potential $V(\phi) = \mu\,\phi^3 + \phi^4\,U(\phi)$ with an arbitrary real coefficient μ and a smooth real-valued function U.

The future stability of such systems has been studied under quite general assumptions by Hans Ringström [16]. His analysis will not be followed here. Employing a general wave gauge, he made a skillful choice of gauge source functions which allowed him to obtain estimates that conveniently control the long-time behaviour of the fields, which is at the focus of his work. A sharp statement about the asymptotic behaviour of the fields, however, which is our main interest here, requires an optimal control (whatever that may mean in the end) on the fields as well as on the coordinates with respect to which the behaviour of the fields is expressed. It is not easy to see whether Rinström's coordinates admit a precise description of the asymptotics of the gravitational and the matter fields.

To see precisely what may go wrong at the prospective conformal boundary we study instead in Sect. 2 the conformal field equations for general matter fields along the lines indicated in [8] and then specialize in Sect. 3 the matter model to that of a non-linear massive scalar field. As expected, it turns out that the conformal equations for the scalar field as well as those for the geometric background fields are in general strongly singular; if the equations are written so that the principal parts are well behaved independent of the sign of the conformal factor Ω, in general factors of the form Ω^{-k}, $k = 1, 2$, occur in the lower order terms. These will blow-up precisely at the set where the conformal factor Ω approaches zero, i.e. at the prospective location of the conformal boundary. One type of singularity is associated with the coefficient μ. Assuming that $\mu = 0$, we get rid of it. The remaining singularities are related to the mass m. It turns out that the singular terms occur in the scalar field equation as well as in the geometric background equations always in the form $\left(m^2 - \frac{2}{3}\,\lambda\right)\Omega^{-k}$, $k = 1, 2$. Somewhat unexpected, the complete system of conformal field equations will thus be regular if the single condition

$$(*) \qquad m^2 = \frac{2}{3}\,\lambda,$$

is imposed. We note that it can be non-trivially satisfied for real $m \neq 0$ only with the present sign of the cosmological constant. In the rest of the introduction we shall assume this condition to be satisfied.

The construction of initial data for the system (1.1), (1.2), (1.3) has been studied under fairly general assumptions in [4]. From the discussion above, it is clear, however, that not all of these data will develop into solutions that admit a smooth conformal boundary. To get an insight into the class of solutions which do admit such boundaries, we analyse in Sect. 4 the constraints induced by the conformal field equations on the set $\{\Omega = 0\}$, assuming it to be given as a smooth compact 3-manifold in the conformally extended solution space-time. As observed already in the vacuum case [6], the Hamiltonian constraint drops out. This leads to a considerable simplification; the data can be prescribed freely up to solving a linear system.

In a suitable gauge, the conformal field equations for the coupled system induce again hyperbolic evolution equations that preserve the constraints and the gauge conditions. They can be used to determine forward and backward time developments of the data on $\{\Omega = 0\}$ which provide away from $\{\Omega = 0\}$ solutions to the system (1.1), (1.2), (1.3). Denote by $\mathcal{A}_S$ the Cauchy data induced on the Cauchy hypersurfaces of these solutions and by $\mathcal{D}_S$ all Cauchy data for the system (1.1), (1.2), (1.3). It follows then as before, that the set $\mathcal{A}_S$ is open in $\mathcal{D}_S$. In fact, it follows with the observations mentioned above that the solutions, their asymptotic behaviour, as well as their smooth extensibility across the conformal boundary are not only stable under perturbations of the scalar field and the geometric background fields but the perturbations may also involve zero rest-mass fields.

That there has to be observed a specific relation between the mass m and the cosmological constant λ could give rise to worries if each of these quantities already had a specific meaning of its own independent of the other one. So far, however, the matter field and its mass have no specific interpretation and if they are given one, the relation to the cosmological may even acquire some predictive power ('a relation between the cosmological constant and the dark matter' ?). All this depends on the choice of the matter field and the role assigned to it in a space-time model. In this context, it should be emphasized that the discussion in this article has been restricted to the scalar field only for the purpose of illustration, other fields could be considered as well.

To get an idea of the order of magnitude of the mass considered here we use the value $\lambda \approx 1.7 \times 10^{-121}$ in Planck units given in [2] (ignoring the fact that the cosmological model underlying the derivation of this value is different from the one referred to in the beginning of this article). Replacing m in the equation above by $\frac{m\,c}{\hbar}$ and converting units we find the exceedingly small mass $m \approx 4 \times 10^{-33}\, eV/c^2$. It is interesting to note that in a study concerned with the recent acceleration of the universe, Leonard Parker and Alpan Raval were led to consider, by a completely different reasoning, masses of a similar order of magnitude [13].

This article focusses on the unexpected fact that the conformal equations can be regular at the conformal boundaries. This does not mean that cases of the Einstein scalar field system for which condition $(*)$ is violated cannot be of interest. On the contrary, it would be most interesting to understand the significance of condition $(*)$ with respect to the asymptotic behaviour of solutions in this set and to see whether the system (1.1), (1.2), (1.3) admits a range of masses for which the notion of crossover surface can be generalized and the conformal structure can be extended in a unique way. After all, it can be expected that transitions from exponentially expanding to big bang phases are accompanied with losses of smoothness and that the insistence on too strict smoothness requirements may obstruct modelling such transitions.

In Sect. 5, we discuss whether the evolution of massive fields across the crossover surfaces may allow us to get some insight into the problem of this 'phase transition'. To simplify matters, we set $\kappa = 1$ (or absorb it into the scalar field) and assume a scaling by a constant overall factor so that $\lambda = 3$ whence $m = \sqrt{2}$. Then we consider spatially homogeneous solutions with a linear massive scalar field so that the physical fields are of the form

$$\hat{g} = -dt^2 + f^2\, k, \quad \phi = \phi(t) \quad \text{on} \quad \mathbb{R} \times S,$$

where $f = f(t) > 0$ and k denotes a Riemannian metric with constant curvature $R_{abcd}[k] = 2\,\epsilon\, k_{a[c}\, k_{d]b}$, $\epsilon = 1, 0, -1$. In a convenient conformal and coordinate gauge the conformal metric then takes the form

$$g = -d\tau^2 + k,$$

and the conformal field equations reduce to a regular system of ODE's of second order for Ω and the rescaled matter field $\psi = \Omega^{-1}\,\phi$ and a constraint which involves Ω, ψ and their derivatives of first order. We consider solutions determined by the backward development of data on $\{\tau = 0\} = \{\Omega = 0\}$. The constraint is satisfied if $\Omega'(0) = -1$, while the data $\psi(0)$, $\psi'(0)$ can be prescribed freely.

The most interesting case $\epsilon = 1$ is discussed in some detail. If $\psi(0)$, $\psi'(0)$ are chosen to vanish, the solution is given by $\Omega_{dS} = -\sin\tau$, $\psi_{dS} = 0$. Its restriction to the interval $-\pi < \tau < 0$ is conformal to the de Sitter space-time. The stability properties of Ω_{dS}, ψ_{dS} then ensure that there exists a large set of smooth solutions Ω, $\psi \not\equiv 0$ so that $\Omega(\tau_z) = \Omega(0) = 0$ for some $\tau_z < 0$ and $\Omega > 0$ in the interval $]\tau_z, 0[$, in which it assumes its absolute maximum value Ω_m at a point τ_m. The corresponding physical solutions can be thought of as arising from asymptotic data on the 'crossover surface' $\{\tau = \tau_z\}$, developing a 'waist' of volume $\Omega_m^{-3}\, Vol(\mathbb{S}^3)$ at τ_m, and then expanding exponentially until they approach the next crossover surface at $\{\tau = 0\}$. We denote the set of these solutions by $\mathcal{B}$. All solutions in $\mathcal{B}$ are non-linearly stable under generic perturbations involving the scalar field, the geometric fields, and zero rest-mass fields.

There is a solution not belonging to $\mathcal{B}$ which is of particular interest in our context. It is given by $\Omega_* = -\tau$, $\psi_* = \sqrt{2}$. Its restriction to the domain where $\Omega > 0$ defines a physical field that is given in terms of the coordinate $t = -\log(-\tau)$ by

$$\tilde{g} = -dt^2 + e^{2t}\,k, \quad \phi = \sqrt{2}\,e^{-t}.$$

As $t \to -\infty$ the matter field ϕ grows unboundedly while it decays and the metric shows a de Sitter-type expansion behaviour as $t \to \infty$. If the solution Ω_*, ψ_* could be approximated on any given interval of the form $[z, 0]$, $z < 0$, by solutions in $\mathcal{B}$ there will exist solutions with an arbitrarily narrow waist. The restriction of such solutions to the range $]\tau_m, 0[$ would, from the point of view of observational data, hardly be distinguishable from solutions which start with a big bang and then expand exponentially. No attempt is made in this article to decide about this question because it requires a detailed analysis of the solution space.

2. The Conformal Field Equations

We consider a 4-dimensional manifold M with smooth boundary $\mathcal{J}$ and interior $\hat{M} = M \setminus \mathcal{J}$. Let $\hat{g}$ and g denote Lorentz metrics on $\hat{M}$ and M, respectively, which satisfy

$$g_{\mu\nu} = \Omega^2\,\hat{g}_{\mu\nu} \quad \text{on} \quad \hat{M},$$

with a conformal factor Ω that is given by a smooth function on M such that

$$\Omega > 0 \quad \text{on} \quad \hat{M}, \quad \Omega = 0,\; d\Omega \neq 0 \quad \text{on} \quad \mathcal{J}.$$

In the following, $\mathcal{J}$ will be thought of as being space-like with respect to g, though in the end this will be a consequence of the field equations (cf. (4.3)). It is assumed that $\hat{g}$ satisfies Einstein's field equations

$$\hat{R}_{\mu\nu} - \frac{1}{2}\,\hat{R}\,\hat{g}_{\mu\nu} + \lambda\,\hat{g}_{\mu\nu} = \kappa\,\hat{T}_{\mu\nu}, \tag{2.1}$$

with cosmological constant $\lambda > 0$. The matter fields will be specified later.

To formulate these equations in terms of the conformal metric g, the conformally transformed matter fields, and a number of fields derived from them, we note the contraction

$$-\,\hat{R} + 4\,\lambda = \kappa\,\hat{T}, \tag{2.2}$$

and use the general conformal transformation relation

$$R_{\mu\nu} + \frac{2}{\Omega}\,\nabla_\mu\,\nabla_\nu\Omega + \left(\frac{1}{\Omega}\,\nabla_\rho\,\nabla^\rho\Omega - \frac{3}{\Omega^2}\,\nabla_\rho\Omega\,\nabla^\rho\Omega\right) g_{\mu\nu} = \hat{R}_{\mu\nu}, \tag{2.3}$$

and its trace

$$R + \frac{6}{\Omega}\,\nabla_\rho\,\nabla^\rho\Omega - \frac{12}{\Omega^2}\,\nabla_\rho\Omega\,\nabla^\rho\Omega = \frac{1}{\Omega^2}\,\hat{R}, \tag{2.4}$$

where the covariant derivative operator ∇ and the index operations on the left-hand side refer to the metric g. With the definition

$$s \equiv \frac{1}{4}\,\nabla_\rho\,\nabla^\rho\Omega + \frac{1}{24}\,\Omega\,R, \tag{2.5}$$

equation (2.4) takes the form

$$6\,\Omega\,s - 3\,\nabla_\rho\Omega\,\nabla^\rho\Omega = \frac{1}{4}\,\hat{R}, \tag{2.6}$$

and equation (2.3) can be rewritten

$$\nabla_\mu \nabla_\nu \Omega = -\,\Omega\, L_{\mu\nu} + s\, g_{\mu\nu} + \Omega\, \hat{S}_{\mu\nu}, \tag{2.7}$$

where

$$L_{\mu\nu} = \frac{1}{2}\left(R_{\mu\nu} - \frac{1}{6}\, R\, g_{\mu\nu}\right) \quad \text{and} \quad \hat{S}_{\mu\nu} = \frac{1}{2}\left(\hat{R}_{\mu\nu} - \frac{1}{4}\, \hat{R}\, \hat{g}_{\mu\nu}\right),$$

denote the Schouten tensor of g and (half of) the trace-free part of the Ricci tensor of $\hat{g}$, respectively. To derive a differential equation for s we observe that the general conformal transformation relations for covariant derivatives and derived tensor fields and the Bianchi identity imply

$$g^{\rho\nu}\, \nabla_\rho\, \hat{S}_{\nu\mu} = \frac{1}{\Omega^2}\, \hat{g}^{\rho\nu}\, \hat{\nabla}_\rho\, \hat{S}_{\nu\mu} + \frac{2}{\Omega}\, \nabla^\rho\, \Omega\, \hat{S}_{\rho\mu} = \frac{1}{8\,\Omega^2}\, \hat{\nabla}_\mu\, \hat{R} + \frac{2}{\Omega}\, \nabla^\rho\, \Omega\, \hat{S}_{\rho\mu}.$$

Applying a derivative to (2.7), commuting on the left-hand side covariant derivatives, and performing a contraction then gives

$$\nabla_\mu\, s = -\,\nabla^\rho \Omega\, L_{\rho\mu} + \nabla^\rho \Omega\, \hat{S}_{\rho\nu} + \frac{1}{24\,\Omega}\, \hat{\nabla}_\mu\, \hat{R}. \tag{2.8}$$

With the decomposition

$$R_{\mu\rho\nu\lambda} = C_{\mu\rho\nu\lambda} + 2\left\{g_{\mu[\nu}\, L_{\lambda]\rho} + L_{\mu[\nu}\, g_{\lambda]\rho}\right\},$$

of the curvature tensor into the conformal Weyl tensor and the Schouten tensor the once contracted Bianchi identity for the curvature tensor of g can be written

$$\nabla_\mu C^\mu{}_{\rho\nu\lambda} = 2\, \nabla_{[\nu}\, L_{\lambda]\rho}, \tag{2.9}$$

and the analogue for $\hat{g}$ reads

$$\hat{\nabla}_\mu \hat{C}^\mu{}_{\rho\nu\lambda} = 2\, \hat{\nabla}_{[\nu}\, \hat{L}_{\lambda]\rho}. \tag{2.10}$$

With the conformal covariance relations

$$C^\mu{}_{\rho\nu\lambda} = \hat{C}^\mu{}_{\rho\nu\lambda}, \qquad \nabla_\mu(\Omega^{-1}\, C^\mu{}_{\rho\nu\lambda}) = \Omega^{-1}\, \hat{\nabla}_\mu\, \hat{C}^\mu{}_{\rho\nu\lambda},$$

and the definition

$$W^\mu{}_{\rho\nu\lambda} \equiv \Omega^{-1}\, C^\mu{}_{\rho\nu\lambda},$$

equation (2.10) can be written

$$\nabla_\mu\, W^\mu{}_{\rho\nu\lambda} = 2\, \Omega^{-1}\, \hat{\nabla}_{[\nu}\, \hat{L}_{\lambda]\rho}, \tag{2.11}$$

while (2.9) reads

$$\nabla_\nu\, L_{\lambda\rho} - \nabla_\lambda\, L_{\nu\rho} = \nabla_\mu \Omega\; W^\mu{}_{\rho\nu\lambda} + 2\, \hat{\nabla}_{[\nu}\, \hat{L}_{\lambda]\rho}. \tag{2.12}$$

Taking now into account the the field equations (2.1), we get the equations above in the form

$$6\,\Omega\, s - 3\, \nabla_\rho \Omega\, \nabla^\rho \Omega - \lambda = -\frac{\kappa}{4}\, \hat{T}, \tag{2.13}$$

$$\nabla_\mu\, \nabla_\nu \Omega + \Omega\, L_{\mu\nu} - s\, g_{\mu\nu} = \frac{\kappa}{2}\, \Omega\, T^*_{\mu\nu}, \tag{2.14}$$

$$\nabla_\mu\, s + \nabla^\rho \Omega\, L_{\rho\mu} = \frac{\kappa}{2}\, \nabla^\rho \Omega\, T^*_{\rho\mu} - \frac{\kappa}{24\,\Omega}\, \hat{\nabla}_\mu\, \hat{T}, \tag{2.15}$$

$$\nabla_\nu L_{\lambda\rho} - \nabla_\lambda L_{\nu\rho} - \nabla_\mu \Omega\, W^\mu{}_{\rho\nu\lambda} = 2\,\hat{\nabla}_{[\nu}\,\hat{L}_{\lambda]\rho}, \tag{2.16}$$

$$\nabla_\mu W^\mu{}_{\rho\nu\lambda} = 2\,\Omega^{-1}\,\hat{\nabla}_{[\nu}\,\hat{L}_{\lambda]\rho}, \tag{2.17}$$

with

$$T^*_{\rho\mu} = \hat{T}_{\rho\mu} - \frac{1}{4}\,\hat{T}\,\hat{g}_{\rho\mu}, \qquad \hat{\nabla}_\rho \hat{L}_{\mu\nu} = \frac{\kappa}{2}\,\hat{\nabla}_\rho \left(\hat{T}_{\mu\nu} - \frac{1}{3}\,\hat{T}\,\hat{g}_{\mu\nu}\right).$$

The first of these equations may be considered as a constraint which will be satisfied if the initial data are arranged accordingly: *If* (2.13) *holds at a point* p *and* (2.14), (2.15) *are satisfied on a connected neighbourhood* U *of* p, *then* (2.13) *holds on* U. In fact, a direct calculation using (2.14), (2.15) implies that

$$\nabla_\mu \left(6\,\Omega\, s - 3\,\nabla_\rho\Omega\,\nabla^\rho\Omega - \lambda + \frac{\kappa}{4}\,\hat{T}\right) = 0.$$

The equations above for the tensorial unknowns Ω, s, $L_{\mu\nu}$, $W^\mu{}_{\rho\nu\lambda}$, have to be combined with equations which determine the metric and the connection. One possibility to do this is to write the structural equations as equations for the unknowns $e^\mu{}_k$, $\Gamma_i{}^k{}_j$, where the first set of fields is the coefficients of a g-orthonormal frame field $e_k = e^\mu{}_k\,\partial_{x^\mu}$ with respect to a coordinate system x^μ so that $g(e_i, e_j) = g_{\mu\nu}\,e^\mu{}_i\,e^\nu{}_j = \eta_{ij}$ and the second set of fields is the associated connection coefficients $\Gamma_i{}^k{}_j$ defined by $\nabla_i\,e_j = \Gamma_i{}^k{}_j\,e_k$ with $\nabla_i = \nabla_{e_i}$, which satisfy $\Gamma_{ijk} = -\Gamma_{ikj}$ where $\Gamma_{ijk} = \Gamma_i{}^l{}_k\,\eta_{lj}$. In terms of these unknowns the structural equations take the form of the *torsion-free condition*

$$e^\mu{}_{i,\,\nu}\,e^\nu{}_j - e^\mu{}_{j,\,\nu}\,e^\nu{}_i = (\Gamma_j{}^k{}_i - \Gamma_i{}^k{}_j)\,e^\mu{}_k, \tag{2.18}$$

and the *Ricci identity*

$$\begin{aligned} &\Gamma_l{}^i{}_{j,\,\mu}\,e^\mu{}_k - \Gamma_k{}^i{}_{j,\,\mu}\,e^\mu{}_l + 2\,\Gamma_{[k}{}^{i\,p}\,\Gamma_{l]pj} - 2\,\Gamma_{[k}{}^p{}_{l]}\,\Gamma_p{}^i{}_j \\ &\quad = \Omega\,W^i{}_{jkl} + 2\,\{g^i{}_{[k}\,L_{l]j} + L^i{}_{[k}\,g_{l]j}\}. \end{aligned} \tag{2.19}$$

If equations (2.13) to (2.17) are expressed in terms of the frame and combined with the structural equations, they are equivalent to Einstein's vacuum equations where $\Omega \neq 0$.

The vacuum case is characterized by vanishing right-hand side of equations (2.13) to (2.17). If the resulting system is written with respect to a suitable choice of coordinates and frame field, and if the conformal factor is controlled by specifying the Ricci scalar as a function of the coordinates (which can locally be prescribed in an arbitrary way), the combined system implies equations *which are hyperbolic even where* Ω *changes sign*. Moreover, they preserve the constraints and the gauge conditions. Discussions of this fact, giving various versions of hyperbolic systems, can be found in [5–7]. The case of zero rest-mass fields for which the energy momentum tensor is trace free is similar and has been discussed in [8]. The details will not be reproduced here.

In the following, we will be interested in fields with non-vanishing rest-mass. The further analysis depends very much on the specific behaviour of the matter fields and the associated energy momentum tensor under conformal rescalings.

3. The Non-Linear Massive Scalar Field

In the following, we consider a scalar field ϕ that satisfies an equation of the form

$$\hat{\nabla}_\mu \hat{\nabla}^\mu \phi - (m^2\,\phi + V'(\phi)) = 0, \tag{3.1}$$

with energy momentum tensor

$$\hat{T}_{\mu\nu} = \hat{\nabla}_\mu \phi\, \hat{\nabla}_\nu \phi - \left(\frac{1}{2}\,(\hat{\nabla}_\rho \phi\, \hat{\nabla}^\rho \phi + m^2\,\phi^2) + V(\phi)\right) \hat{g}_{\mu\nu}, \tag{3.2}$$

and a potential of the form

$$V(\phi) = \mu\,\phi^3 + \phi^4\, U(\phi), \tag{3.3}$$

where μ is a real coefficient and U a smooth real-valued function. This form is assumed because we wish to discuss the cosmological constant, the mass term, and the constant coefficient μ separately and because $V'(0) = 0$ ensures that the coupled Einstein scalar field equations admit solutions with $\phi = 0$.

In four dimensions holds for arbitrary smooth functions ϕ the transformation law

$$\left(\Box_g - \frac{1}{6}\,R\right)[\Omega^{-1}\,\phi] = \Omega^{-3}\,\left(\Box_{\hat{g}} - \frac{1}{6}\,\hat{R}\right)[\phi]. \tag{3.4}$$

In terms of the new unknown

$$\psi = \Omega^{-1}\,\phi,$$

equation (3.1) takes then with (2.2) the form

$$\left(\Box_g - \frac{1}{6}\,R\right)[\psi] = \Omega^{-2}\left(m^2 - \frac{2}{3}\,\lambda + \frac{\kappa}{6}\,\hat{T}\right)\psi + \Omega^{-3}\,V'(\Omega\,\psi), \tag{3.5}$$

where

$$\Omega^{-3}\,V'(\Omega\,\psi) = 3\,\mu\,\Omega^{-1}\,\psi^2 + 4\,\psi^3\,U(\Omega\,\psi) + \Omega\,\psi^4\,U'(\Omega\,\psi).$$

To ensure the regularity of this term where $\Omega \to 0$ it will be assumed in the following that

$$\mu = 0.$$

The trace of the energy momentum tensor (3.2) now reads

$$\hat{T} = -\Omega^2\,\left\{\nabla_\rho(\Omega\,\psi)\,\nabla^\rho(\Omega\,\psi) + 2\,m^2\,\psi^2 + 4\,\Omega^{-2}\,V(\Omega\,\psi)\right\}, \tag{3.6}$$

so that

$$T_* \equiv \Omega^{-2}\hat{T} = -\nabla_\rho(\Omega\,\psi)\,\nabla^\rho(\Omega\,\psi) - 2\,m^2\,\psi^2 - 4\,\Omega^{-2}\,V(\Omega\,\psi), \tag{3.7}$$

is well behaved as $\Omega \to 0$. The relations (2.2) and (2.6) imply

$$\nabla_\rho\Omega\,\nabla^\rho\Omega = -\frac{\lambda}{3} + 2\,\Omega\,s + \frac{\kappa}{12}\,\Omega^2\,T_*, \tag{3.8}$$

and thus

$$\begin{aligned} T_* = &\left(\frac{\lambda}{3} - 2\,m^2\right)\psi^2 - 2\,\Omega\,(s\,\psi^2 + \psi\,\nabla_\pi\Omega\,\nabla^\pi\psi) \\ &- \Omega^2\,\left(\frac{\kappa}{12}\,\psi^2\,T_* + \nabla_\pi\psi\,\nabla^\pi\psi\right) - 4\,\Omega^{-2}\,V(\Omega\,\psi). \end{aligned} \tag{3.9}$$

From this we get

$$T_* = \left(1 + \frac{\kappa}{12}\,\psi^2\,\Omega^2\right)^{-1} \left\{ \left(\frac{\lambda}{3} - 2\,m^2\right)\psi^2 - 2\,\Omega\,(s\,\psi^2 + \psi\,\nabla_\pi\Omega\,\nabla^\pi\psi) \right.$$
$$\left. - \,\Omega^2\,\nabla_\pi\psi\,\nabla^\pi\psi - 4\,\Omega^{-2}\,V(\Omega\,\psi) \right\}, \qquad (3.10)$$

which is a smooth function for all (real) values of the unknowns ψ, $\nabla_\mu\psi$, Ω, and s. In the following calculations it will be convenient, however, to use the equation in the form (3.9). It follows

$$\square_g\psi - \frac{1}{6}\,R\,\psi = \Omega^{-2}\left(m^2 - \frac{2}{3}\,\lambda\right)\psi + \frac{\kappa}{6}\,T_*\psi + 4\,\psi^3\,U(\Omega\,\psi) + \Omega\,\psi^4\,U'(\Omega\,\psi). \qquad (3.11)$$

We note that if the background fields are given this is a semi-linear equation for ψ whose right-hand side depends via T_* also on the derivative $\nabla_\mu\psi$. Its most conspicuous feature, however, is the first term on the right-hand side, which is singular where $\Omega \to 0$.

We are in a position now to state our main result.

Theorem 3.1. *Consider the energy momentum tensor given by* (3.2)*, a potential* (3.3) *with* $\mu = 0$*, and the coupled system of equations* (2.13)*,* (2.14)*,* (2.15)*,* (2.16)*,* (2.17)*,* (2.18)*,* (2.19)*,* (3.11) *for the unknowns*

$$e^\mu{}_k, \quad \Gamma_i{}^k{}_j, \quad \Omega, \quad s, \quad L_{\mu\nu}, \quad W^\mu{}_{\rho\nu\lambda}, \quad \psi. \qquad (3.12)$$

If and only if the single condition

$$m^2 = \frac{2}{3}\,\lambda, \qquad (3.13)$$

is satisfied this system is regular in the sense that on the right-hand side of the equations no terms of the form Ω^{-k}*,* $k > 0$*, occur and the right-hand side is in fact a smooth function of the unknowns.*

Remarks. We note that the condition above can be satisfied with real m only with the de Sitter-type sign of the cosmological constant.

Because some of the equations involve derivatives of the energy momentum tensor they contain derivatives of ψ of second order. Applying a derivative to (3.11) and commuting operators one obtains a wave equation for $\nabla_\mu\psi$ and thus altogether a quasi-linear, overdetermined system of equations for the unknowns (3.12) and $\nabla_\mu\psi$. After fixing a suitable gauge one can extract from the complete set of equations a hyperbolic evolution system which preserves the gauge conditions and the constraints. Since various versions of this procedure have been discussed at length in the references given above we shall not go into the details here.

It should be noted that the compactness of the manifold S plays no role in this result.

Proof of Theorem 3.1*.* It follows immediately that (3.13) renders equation (3.11) regular. Equations (2.18) and (2.19) are obviously regular. We discuss the nature of the singularity of the remaining equations. The trace-free part of the energy momentum tensor takes the form

$$\begin{aligned} T^*_{\mu\nu} &= \nabla_\mu(\Omega\psi)\,\nabla_\nu(\Omega\,\psi) - \frac{1}{4}\,\nabla_\rho(\Omega\psi)\,\nabla^\rho(\Omega\psi)\;g_{\mu\nu}, \\ &= \psi^2\left(\nabla_\mu\Omega\,\nabla_\nu\Omega - \frac{1}{4}\,\nabla_\pi\Omega\,\nabla^\pi\Omega\,g_{\mu\nu}\right) \\ &\quad + 2\,\Omega\,\psi\,\left(\nabla_{(\mu}\Omega\,\nabla_{\nu)}\psi - \frac{1}{4}\,\nabla_\pi\Omega\,\nabla^\pi\psi\,g_{\mu\nu}\right) \\ &\quad + \Omega^2\,\left(\nabla_\mu\psi\,\nabla_\nu\psi - \frac{1}{4}\,\nabla_\pi\psi\,\nabla^\pi\psi\,g_{\mu\nu}\right), \end{aligned} \tag{3.14}$$

and is thus regular. It follows that equation (2.13), given now by (3.8) and equations (2.14), and (2.15) with $\hat{\nabla}_\mu\,\hat{T} = \nabla_\mu\,(\Omega^2\,T_*)$, are regular as $\Omega \to 0$. Critical are equations (2.16) and (2.17). With the notation above we have

$$\hat{L}_{\mu\nu} = \frac{\kappa}{2}\,T^*_{\mu\nu} - \frac{\kappa}{24}\;\hat{T}\,\hat{g}_{\mu\nu} + \frac{\lambda}{6}\,\hat{g}_{\mu\nu}.$$

While the first two terms on the right-hand side are regular since $\hat{T}\,\hat{g}_{\mu\nu} = T_*\,g_{\mu\nu}$, the last term is singular if it is expressed in terms of $g_{\mu\nu}$. However, this term is annihilated by the operator $\hat{\nabla}_\rho$ and it follows

$$\begin{aligned} \hat{\nabla}_\rho\hat{L}_{\mu\nu} &= \frac{\kappa}{2}\,\hat{\nabla}_\rho\,T^*_{\mu\nu} - \frac{\kappa}{24}\,\Omega^{-2}\,\nabla_\rho(\Omega^2\,T_*)\,g_{\mu\nu} \\ &= \frac{\kappa}{2}\,\nabla_\rho\,T^*_{\mu\nu} - \frac{\kappa}{24}\,\nabla_\rho T_*\,g_{\mu\nu} - \frac{\kappa}{12}\,\Omega^{-1}\,T_*\,\nabla_\rho\Omega\,g_{\mu\nu} \\ &\quad + \frac{\kappa}{2}\,\Omega^{-1}\Big(3\nabla_{(\rho}\,\Omega\,T^*_{\mu)\nu} + \nabla_{[\rho}\Omega\,T^*_{\mu]\nu} + \nabla_\nu\Omega\,T^*_{\rho\mu} \\ &\qquad - g_{\rho\mu}\,T^*_{\nu\delta}\nabla^\delta\Omega - g_{\nu\rho}\,T^*_{\mu\delta}\nabla^\delta\Omega\Big), \end{aligned}$$

whence

$$\begin{aligned} \hat{\nabla}_{[\rho}\hat{L}_{\mu]\nu} &= \frac{\kappa}{2}\left\{\nabla_{[\rho}\,T^*_{\mu]\nu} - \frac{1}{12}\,\nabla_{[\rho}T_*\,g_{\mu]\nu}\right. \\ &\quad \left. + \,\Omega^{-1}\left(\nabla_{[\rho}\Omega\,T^*_{\mu]\nu} + \nabla^\pi\Omega\,T^*_{\pi[\rho}\,g_{\mu]\nu} - \frac{1}{6}\,T_*\,\nabla_{[\rho}\Omega\,g_{\mu]\nu}\right)\right\}. \end{aligned} \tag{3.15}$$

Direct calculations using (3.14), (2.14) and (3.8) give

$$\begin{aligned} &\nabla_{[\rho}\,T^*_{\mu]\nu} \\ &\quad = \frac{\lambda}{6}\,\psi\,\nabla_{[\rho}\psi\,g_{\mu]\nu} - \psi\,\nabla_{[\rho}\Omega\,\nabla_{\mu]}\psi\,\nabla_\nu\Omega - \left(\frac{3}{2}\,s\,\psi^2 + \frac{1}{2}\,\psi\,\nabla_\pi\Omega\,\nabla^\pi\,\psi\right)\nabla_{[\rho}\Omega\,g_{\mu]\nu} \\ &\quad + \Omega\left\{-\frac{1}{2}\,\nabla_\pi\psi\,\nabla^\pi\psi\,\nabla_{[\rho}\Omega\,g_{\mu]\nu} - \left(2\,s\,\psi + \frac{1}{2}\,\nabla_\pi\Omega\,\nabla^\pi\psi\right)\nabla_{[\rho}\psi\,g_{\mu]\nu}\right. \\ &\quad + \psi^2\left(\nabla_{[\rho}\Omega\,L_{\mu]\nu} + \frac{1}{2}\,\nabla^\pi\Omega\,L_{\pi[\rho}\,g_{\mu]\nu}\right) \\ &\quad - \psi\left(\nabla_{[\rho}\Omega\,\nabla_{\mu]}\nabla_\nu\psi + \frac{1}{2}\,\nabla^\pi\Omega\,\nabla_\pi\nabla_{[\rho}\psi\,g_{\mu]\nu}\right) \end{aligned}$$

$$+\nabla_{[\rho}\Omega\,\nabla_{\mu]}\psi\,\nabla_\nu\psi - \frac{\kappa}{2}\psi^2\,\nabla_{[\rho}\Omega\,T^*_{\mu]\nu} - \frac{\kappa}{4}\psi^2\,\nabla^\pi\Omega\,T^*_{\pi[\rho}\,g_{\mu]\nu}\Big\}$$
$$+\,\Omega^2\left\{\psi\,\nabla_{[\rho}\psi\,L_{\mu]\nu} - \nabla_{[\rho}\psi\,\nabla_{\mu]}\nabla_\nu\psi - \frac{1}{2}\,\nabla^\pi\psi\,\nabla_\pi\nabla_{[\rho}\psi\,g_{\mu]\nu}\right.$$
$$\left.-\frac{\kappa}{24}\,\psi\,T_*\,\nabla_{[\rho}\psi\,g_{\mu]\nu} - \frac{\kappa}{2}\,\psi\,\nabla_{[\rho}\psi\,T^*_{\mu]\nu}\right\},$$

$$\begin{aligned}
-\frac{1}{12}\,\nabla_{[\rho}T_*\,g_{\mu]\nu}
&= \left(\frac{m^2}{3}-\frac{\lambda}{18}\right)\psi\,\nabla_{[\rho}\psi\,g_{\mu]\,\nu} + \frac{1}{6}\,(s\,\psi^2 + \psi\,\nabla_\pi\Omega\,\nabla^\pi\psi)\nabla_{[\rho}\Omega\,g_{\mu]\nu}\\
&\quad + \Omega\left\{\left(\frac{1}{2}\,s\,\psi + \frac{1}{6}\,\nabla_\pi\Omega\,\nabla^\pi\psi\right)\nabla_{[\rho}\psi\,g_{\mu]\nu} + \frac{1}{6}\,\nabla_\pi\psi\,\nabla^\pi\psi\,\nabla_{[\rho}\Omega\,g_{\mu]\nu}\right.\\
&\quad \left.+\frac{1}{6}\,\psi\,\nabla^\pi\Omega\,\nabla_\pi\nabla_{[\rho}\psi\,g_{\mu]\nu} - \frac{1}{6}\,\psi^2\,\nabla^\pi\Omega\,L_{\pi[\rho}\,g_{\mu]\nu} + \frac{\kappa}{12}\,\psi^2\,\nabla^\pi\Omega\,T^*_{\pi[\rho}\,g_{\mu]\nu}\right\}\\
&\quad + \Omega^2\left\{\frac{\kappa}{72}\,\psi\,T_*\,\nabla_{[\rho}\psi\,g_{\mu]\,\nu} + \frac{\kappa}{12}\,\psi\,\nabla^\pi\psi\,T^*_{\pi[\rho}\,g_{\mu]\nu}\right.\\
&\quad \left.-\frac{1}{6}\,\psi\,\nabla^\pi\psi\,L_{\pi[\rho}\,g_{\mu]\nu} + \frac{1}{6}\,\nabla^\pi\psi\,\nabla_\pi\nabla_{[\rho}\psi\,g_{\mu]\nu}\right\},\\
&\quad - \frac{1}{3}\left\{\left(2\,\Omega^{-3}\,V(\Omega\,\psi) - \Omega^{-2}\,\psi\,V'(\Omega\,\psi)\right)\nabla_{[\rho}\Omega\,g_{\mu]\nu} - \Omega^{-1}\,V'(\Omega\psi)\nabla_{[\rho}\psi\,g_{\mu]\nu}\right\}
\end{aligned}$$

$$\begin{aligned}
\nabla_{[\rho}\,\Omega\,T^*_{\mu]\nu} &= \frac{\lambda}{12}\,\psi^2\,\nabla_{[\rho}\Omega\,g_{\mu]\nu}\\
&\quad + \Omega\left\{-\frac{1}{2}(s\,\psi^2 + \psi\,\nabla_\pi\Omega\,\nabla^\pi\,\psi)\nabla_{[\rho}\Omega\,g_{\mu]\nu} + \psi\,\nabla_{[\rho}\Omega\,\nabla_{\mu]}\psi\,\nabla_\nu\Omega\right\}\\
&\quad + \Omega^2\left\{\nabla_{[\rho}\Omega\,\nabla_{\mu]}\psi\,\nabla_\nu\psi - \left(\frac{1}{4}\nabla_\pi\psi\,\nabla^\pi\psi + \frac{\kappa}{48}\,\psi^2\,T_*\right)\nabla_{[\rho}\Omega\,g_{\mu]\nu}\right\},
\end{aligned}$$

$$\begin{aligned}
\nabla^\pi\Omega\,T^*_{\pi[\rho}\,g_{\mu]\nu} &= -\frac{\lambda}{4}\,\psi^2\,\nabla_{[\rho}\Omega\,g_{\mu]\nu}\\
&\quad +\Omega\left\{\left(\frac{3}{2}\,s\,\psi^2 + \frac{1}{2}\,\psi\,\nabla_\pi\Omega\,\nabla^\pi\psi\right)\nabla_{[\rho}\Omega\,g_{\mu]\nu} - \frac{\lambda}{3}\,\psi\,\nabla_{[\rho}\psi\,g_{\mu]\nu}\right\}\\
&\quad +\Omega^2\left\{\left(2\,s\,\psi + \nabla_\pi\Omega\,\nabla^\pi\psi + \frac{\kappa}{12}\Omega\,\psi\,T_*\right)\nabla_{[\rho}\psi\,g_{\mu]\nu}\right.\\
&\quad \left.+\left(\frac{\kappa}{16}\,\psi^2\,T_* - \frac{1}{4}\,\nabla_\pi\psi\,\nabla^\pi\psi\right)\nabla_{[\rho}\Omega\,g_{\mu]\nu}\right\},
\end{aligned}$$

$$\begin{aligned}
-\frac{1}{6}\,T_*\,\nabla_{[\rho}\Omega\,g_{\mu]\nu} &= \left(\frac{m^2}{3}-\frac{\lambda}{18}\right)\psi^2\,\nabla_{[\rho}\Omega\,g_{\mu]\nu}\\
&\quad + \Omega\left(\frac{1}{3}\,s\,\psi^2 + \frac{1}{3}\,\psi\,\nabla_\pi\Omega\,\nabla^\pi\psi\right)\nabla_{[\rho}\Omega\,g_{\mu]\nu}
\end{aligned}$$

$$+\,\Omega^2\Big(\frac{1}{6}\,\nabla_\pi\psi\,\nabla^\pi\psi+\frac{\kappa}{72}\,\psi^2\,T_*\Big)\nabla_{[\rho}\Omega\,g_{\mu]\nu}$$
$$+\frac{2}{3}\,\Omega^{-2}\,V(\Omega\,\psi)\,\nabla_{[\rho}\Omega\,g_{\mu]\nu},$$

whence

$$\frac{1}{\Omega}\left(\nabla_{[\rho}\Omega\,T^*_{\mu]\nu}+\nabla^\pi\Omega\,T^*_{\pi[\rho}\,g_{\mu]\nu}-\frac{1}{6}\,T_*\,\nabla_{[\rho}\Omega\,g_{\mu]\nu}\right)$$
$$=\frac{1}{3\,\Omega}\,\left(m^2-\frac{2\lambda}{3}\right)\psi^2\,\nabla_{[\rho}\Omega\,g_{\mu]\nu}$$
$$+\left(\frac{4}{3}\,s\,\psi^2+\frac{1}{3}\,\psi\,\nabla_\pi\Omega\,\nabla^\pi\psi\right)\nabla_{[\rho}\Omega\,g_{\mu]\nu}+\psi\,\nabla_{[\rho}\Omega\,\nabla_{\mu]}\psi\,\nabla_\nu\Omega-\frac{\lambda}{3}\,\psi\,\nabla_{[\rho}\psi\,g_{\mu]\nu}$$
$$+\,\Omega\,\left\{\nabla_{[\rho}\Omega\,\nabla_{\mu]}\psi\,\nabla_\nu\psi+\left(2\,s\,\psi+\nabla_\pi\Omega\,\nabla^\pi\psi+\frac{\kappa}{12}\Omega\,\psi\,T_*\right)\nabla_{[\rho}\psi\,g_{\mu]\nu}\right.$$
$$\left.+\left(\frac{\kappa}{18}\,\psi^2\,T_*-\frac{1}{3}\,\nabla_\pi\psi\,\nabla^\pi\psi\right)\nabla_{[\rho}\Omega\,g_{\mu]\nu}\right\}$$
$$+\,\frac{2}{3}\,\Omega^{-3}\,V(\Omega\,\psi)\,\nabla_{[\rho}\Omega\,g_{\mu]\nu},$$

and thus finally

$$\hat{\nabla}_{[\rho}\hat{L}_{\mu]\nu}=\frac{\kappa}{2}\left[\frac{1}{3}\left(1+\frac{\psi}{\Omega}\right)\left(m^2-\frac{2\,\lambda}{3}\right)\psi\,\nabla_{[\rho}\psi\,g_{\mu]\,\nu}+\Omega\,K_{\rho\mu\nu}\right], \tag{3.16}$$

with the field

$$K_{\rho\mu\nu}=\left(\frac{\kappa}{18}\,\psi^2\,T_*-\frac{2}{3}\,\nabla_\pi\psi\,\nabla^\pi\psi\right)\nabla_{[\rho}\Omega\,g_{\mu]\nu}$$
$$+\left(\frac{1}{2}\,s\,\psi+\frac{2}{3}\,\nabla_\pi\Omega\,\nabla^\pi\psi+\frac{\kappa}{12}\Omega\,\psi\,T_*\right)\nabla_{[\rho}\psi\,g_{\mu]\nu}$$
$$-\psi\,\nabla_{[\rho}\Omega\,\nabla_{\mu]}\nabla_\nu\psi+\psi^2\,\nabla_{[\rho}\Omega\,L_{\mu]\nu}+\frac{1}{3}\,\psi^2\,\nabla^\pi\Omega\,L_{\pi[\rho}\,g_{\mu]\nu}$$
$$-\frac{1}{3}\,\psi\,\nabla^\pi\Omega\,\nabla_\pi\nabla_{[\rho}\psi\,g_{\mu]\nu}$$
$$-\frac{\kappa}{2}\psi^2\,\nabla_{[\rho}\Omega\,T^*_{\mu]\nu}-\frac{\kappa}{6}\psi^2\,\nabla^\pi\Omega\,T^*_{\pi[\rho}\,g_{\mu]\nu}+2\,\nabla_{[\rho}\Omega\,\nabla_{\mu]}\psi\,\nabla_\nu\psi$$
$$+\Omega\left\{-\nabla_{[\rho}\psi\,\nabla_{\mu]}\nabla_\nu\psi-\frac{1}{3}\,\nabla^\pi\psi\,\nabla_\pi\nabla_{[\rho}\psi\,g_{\mu]\nu}-\frac{\kappa}{36}\,\psi\,T_*\,\nabla_{[\rho}\psi\,g_{\mu]\nu}\right.$$
$$\left.-\frac{\kappa}{2}\,\psi\,\nabla_{[\rho}\psi\,T^*_{\mu]\nu}+\frac{\kappa}{12}\,\psi\,\nabla^\pi\psi\,T^*_{\pi[\rho}\,g_{\mu]\nu}+\psi\,\nabla_{[\rho}\psi\,L_{\mu]\nu}-\frac{1}{6}\,\psi\,\nabla^\pi\psi\,L_{\pi[\rho}\,g_{\mu]\nu}\right\}$$
$$+\frac{1}{3}\,\Omega^{-3}\,\psi\,V'(\Omega\,\psi)\,\nabla_{[\rho}\Omega\,g_{\mu]\nu}+\frac{1}{3}\,\Omega^{-2}\,V'(\Omega\psi)\,\nabla_{[\rho}\psi\,g_{\mu]\nu},$$

which is regular as $\Omega\to 0$. From this, our assertion follows immediately. □

4. The Constraints on a Hypersurface $\{\Omega = 0\}$

In [6], it has been observed that the constraints induced by the conformal vacuum field equations with positive cosmological constant simplify on a hypersurface $\mathcal{J} = \{\Omega = 0\}$. Cauchy data for the conformal field equations on such a hypersurface are referred to as *asymptotic initial data*. In the following, it will be shown that also in the case of the Einstein scalar field system satisfying $(*)$ the construction of asymptotic initial data is considerably simpler than the construction of standard Cauchy data for the Einstein scalar field system (cf. [4]).

To derive and analyse the constraints, the compact manifold $\mathcal{J}$ will be thought of as being embedded as a Cauchy hypersurface into a smooth solution to the conformal field equations. It will be convenient to assume the solution metric g to be given in terms of Gauss coordinates based on $\mathcal{J}$ so that $\Omega > 0$ in the past and $\Omega < 0$ in the future of $\mathcal{J}$ close to it. Then

$$g = -d\tau^2 + h_{ab}\, dx^a\, dx^b,$$

with $x^0 \equiv \tau = 0$ and $\partial_\tau \Omega < 0$ on $\mathcal{J}$. The Christoffel symbols are given by

$$\Gamma_a{}^0{}_b[g] = \frac{1}{2}\, h_{ab,0} = \chi_{ab}, \quad \Gamma_0{}^0{}_b[g] = 0, \quad \Gamma_a{}^0{}_0[g] = 0, \quad \Gamma_0{}^c{}_0[g] = 0,$$
$$\Gamma_a{}^c{}_0[g] = \Gamma_0{}^c{}_a[g] = h^{ce}\, \chi_{eb}, \qquad \Gamma_a{}^c{}_b[g] = \Gamma_a{}^c{}_b[h].$$

where χ_{ab} denotes the second fundamental form and h^{ab} the inverse of the metric h_{ab} induced on $\{\tau = const.\}$. With this notation, we can state the following result.

Proposition 4.1. *Assume $\lambda > 0$. On a smooth, orientable, compact 3-manifold $\mathcal{J}$ let be given a smooth Riemannian metric h_{ab} with covariant derivative operator D_a, and smooth scalar fields ψ_0, ψ_1 so that*

$$\int_S X^a\, \rho_a\, d\mu_h = 0 \quad \textit{for any conformal vector field } X \textit{ admitted by } h, \qquad (4.1)$$

where

$$\rho_a = \frac{\kappa}{3}\, \Sigma\, (\psi_0\, D_a\psi_1 - 2\, \psi_1\, D_a\psi_0) \quad \textit{with} \quad \Sigma = -\sqrt{\frac{\lambda}{3}}.$$

Furthermore, let w_{ab} be a smooth, symmetric, trace-free solution to the equation

$$D^a w_{ab} = \rho_b. \qquad (4.2)$$

Initial data (i.e. solutions to the constraints) for the conformal Einstein scalar field equations with cosmological constant λ and mass $m > 0$ satisfying the condition $(*)$ are then derived in a suitable conformal gauge from these data on $\mathcal{J}$ as follows.

- *The fields ψ_0, ψ_1 constitute the Cauchy data ψ and $\nabla_0\psi$ for the scalar field ψ on $\mathcal{J}$.*
- *The inner metric and the second fundamental form on $\mathcal{J}$ are given by*

$$h_{ab} \quad \textit{and} \quad \chi_{ab} = 0.$$

– *The conformal factor, its time derivative, and the function s are given by*

$$\Omega = 0, \quad \nabla_0 \Omega = \Sigma, \quad s = 0.$$

– *The Schouten tensor of g is given by*

$$L_{ab}[g] = L_{ab}[h], \quad L_{0a}[g] = L_{a0}[g] = 0, \quad L_{00}[g] = \frac{1}{4}\, R[h] - \frac{1}{6}\, R[g],$$

where $R[g]$ is to be considered as a smooth conformal gauge source function which can be given arbitrarily.

– *The rescaled conformal Weyl tensor $W^{\mu}{}_{\nu\lambda\rho}$ is specified in terms of its electric part with respect to $\mathcal{J}$, which is given by w_{ab}, and its magnetic part which is given by*

$$w^*_{cd} = -\frac{1}{\Sigma}\, D_a\, L_{bc}[h]\, \epsilon_d{}^{ab}.$$

Proof. Regularity of T_* and the relation $\hat{T} = \Omega^2\, T_*$ imply that

$$\hat{T} = 0, \quad \hat{\nabla}_\mu \hat{T} = 0 \quad \text{on} \quad \mathcal{J}.$$

The restriction of (2.13) to $\mathcal{J}$ thus gives

$$\nabla_0 \Omega = \Sigma = -\sqrt{\frac{\lambda}{3}}. \tag{4.3}$$

The restriction of (2.14) with $\mu = a$, $\nu = 0$ is satisfied because of the values of the Christoffel symbols and because $\nabla_0 \Omega$ is constant on $\mathcal{J}$ while it gives with $\mu = a$, $\nu = b$ the relation

$$0 = \nabla_a \nabla_b \Omega - s\, g_{ab} = -\chi_{ab}\, \Sigma - s\, h_{ab},$$

which implies that the trace-free part of χ_{ab} vanishes and its trace is given by $\chi = -\Sigma^{-1}\, s$. So far we did not make use of the conformal gauge freedom. It allows us to perform arbitrary rescalings with positive conformal factors θ and can be removed by prescribing the Ricci scalar $R[g]$ as given function near $\mathcal{J}$ and by prescribing θ and its time derivative on $\mathcal{J}$. Leaving the freedom to choose θ on $\mathcal{J}$ untouched, its time derivative can be chosen there to achieve

$$\chi = 0 \quad \text{whence} \quad s = 0 \quad \text{and} \quad \chi_{ab} = 0 \quad \text{on} \quad \mathcal{J}.$$

Observing now that $C^{\mu}{}_{\nu\rho\lambda} = \Omega\, W^{\mu}{}_{\nu\rho\lambda} = 0$ on $\mathcal{J}$ the last relation implies with Gauss' equation that

$$\frac{1}{2}\, R_{ab}[g] - \frac{1}{12}\, R[g]\, h_{ab} = L_{ab}[g] = L_{ab}[h] \equiv R_{ab}[h] - \frac{1}{4}\, R[h]\, h_{ab} \quad \text{on} \quad \mathcal{J},$$

and thus by contraction

$$R[g] + 2\, R_{00}[g] = R[h],$$

whence

$$L_{00}[g] = \frac{1}{2}\, R_{00}[g] - \frac{1}{12}\, R[g]\, g_{00} = \frac{1}{4}\, R[h] - \frac{1}{6}\, R[g].$$

It holds

$$T^*_{\mu\nu} = \psi^2 (\nabla_\mu \Omega\, \nabla_\nu \Omega - \frac{1}{4}\, \nabla_\pi \Omega\, \nabla^\pi \Omega\, g_{\mu\nu}) \quad \text{on} \quad \mathcal{J}.$$

The constraint induced by (2.15) with $\mu = a$ reduces thus to

$$L_{0a}[g] = L_{a0}[g] = 0.$$

It follows that the initial data for $L_{\mu\nu}[g]$ can be expressed completely in terms of $L_{ab}[h]$ and the gauge-dependent quantity $R[g]$ which can be prescribed arbitrarily near $\mathcal{J}$.

Because $\hat{\nabla}_{[\nu}\,\hat{L}_{\lambda]\rho} = 0$ on $\mathcal{J}$ by (3.13) and (3.16), the constraints induced by (2.16) are given by

$$\nabla_a\, L_{b\rho} - \nabla_b\, L_{a\rho} = \Sigma\; W_{\rho 0ab}.$$

The relation with $\rho = 0$ is satisfied by the fields given above and implies no condition. The remaining relation can be written in the form

$$D_a\, L_{bc}[h] - D_b\, L_{ac}[h] = \Sigma\; w^*_{cd}\, \epsilon^d{}_{ab},$$

where D_a denotes the h-covariant derivative operator and $w^*_{ab} \equiv -\frac{1}{2} W_{a0cd}\, \epsilon_b{}^{cd}$ the $\mathcal{J}$- magnetic part of $W^\mu{}_{\nu\rho\lambda}$. It is saying that the magnetic part of the rescaled conformal Weyl tensor is given on $\mathcal{J}$ by the Cotton tensor defined by h_{ab}.

One of the constraints implied by (2.17) is obtained by restricting

$$\nabla_\mu\, W^\mu{}_{0ab} = 2\,\Omega^{-1}\,\hat{\nabla}_{[a}\,\hat{L}_{b]0} = \kappa\, K_{ab0},$$

to $\mathcal{J}$. With the results above it follows that the restriction of K_{ab0} to $\mathcal{J}$ vanishes and the constraint reduces to

$$D^a\, w^*_{ab} = 0.$$

This is just the differential identity satisfied by the Cotton tensor and gives no new condition. The only remaining constraint is given by the restriction to $\mathcal{J}$ of

$$\nabla_\mu\, W^\mu{}_{0a0} = 2\,\Omega^{-1}\,\hat{\nabla}_{[a}\,\hat{L}_{0]0} = \kappa\, K_{a00},$$

which can be written with the results above

$$D^b\, w_{ba} = \rho_a \quad \text{with} \quad \rho_a = \frac{\kappa}{3}\,\Sigma\,(\psi_0\,\nabla_a\nabla_0\psi - 2\,\nabla_0\psi\,\nabla_a\psi).$$

Because

$$\nabla_a\nabla_0\psi = \partial_a\nabla_0\psi - \chi_a{}^c\, D_c\psi = \partial_a\partial_0\psi,$$

the field ρ_a can be expressed completely in terms of κ, Σ, and the data ψ_0 and $\nabla_0\psi$ for the scalar field on $\mathcal{J}$. For any smooth vector field X^a on $\mathcal{J}$ holds

$$\int_S X^a\,\rho_a\, d\mu_h = \int_S X^a\, D^b\, w_{ab}\, d\mu_h = \int_S w_{ab}\,(D^{(b}X^{a)} - \frac{1}{3}\, D_c X^c\, h^{ab})\, d\mu_h.$$

because w_{ab} is trace free. It follows that the data h, ψ, and $\nabla_0\psi$ must be given such that (4.1) holds true. If this condition is satisfied, the well known properties of the operator $\mathbb{L}_h = div \circ \mathcal{L}_h$ then imply that the equation $\mathbb{L}_h X = \rho$ is solvable and a solution to (4.2) is provided by the tensor $(\mathcal{L}_h X)_{ab}$ calculated from X. Here the divergence of a covariant, symmetric, trace free tensor field w_{ab} is the 1-form $(div\, w)_b = -D^a w_{ab}$ and the conformal Killing operator $\mathcal{L}_h$ acts on a 1-form X_a by $(\mathcal{L}_h X)_{ab} = D_{(b}X_{a)} - \frac{1}{3}\, D_c X^c h_{ab}$. □

Remarks. All possible smooth asymptotic initial data for which Ω is decreasing near $\mathcal{J}$ are obtained by the procedure described above.

The relation (4.3) (already observed in [14]) shows that the set $\{\Omega = 0\}$ is necessarily a space-like hypersurface.

The initial data for the wave equation which needs to be derived for $\nabla_\mu \psi$ from (3.5) (or from (3.11) with (3.13)) are given by $\nabla_a \psi = \partial_a \psi_0$, $\nabla_0 \psi = \psi_1$, $\nabla_0 \nabla_a \psi = \nabla_a \nabla_0 \psi = \partial_a \psi_1$, and the datum $\nabla_0 \nabla_0 \psi$, which can be read off from (3.5), is found to be

$$\nabla_0 \nabla_0 \psi = h^{ab}\, D_a D_b \psi_0 - \frac{1}{6}\, R[g]\, \psi_0 + \left(\frac{\kappa\, \lambda}{6} - 4\, U(0) \right) \psi_0^3.$$

Given an asymptotic initial data set as above and a space-time gauge in which the conformal field equations imply hyperbolic evolution equations, the latter allow us to determine a past Cauchy development of the data which provides, where $\Omega > 0$, a unique maximal, globally hyperbolic, future asymptotically simple solution $\hat{g} = \Omega^{-2}\, g$, $\phi = \Omega\, \psi$ of the system (1.1), (1.2), (1.3) with (3.13).

There exists, however, also a unique maximal, globally hyperbolic future Cauchy development of the data on which $\Omega < 0$. The conformal field equations are left invariant by the transition under which $\Omega \to -\Omega$, $W^\mu{}_{\nu\lambda\rho} \to -W^\mu{}_{\nu\lambda\rho}$, $\psi \to -\psi$ while all other unknowns remain unchanged. The fields $\hat{g} = \Omega^{-2}\, g$, $\phi = \Omega\, \psi$ obtained from the future development thus define again a solution to (1.1), (1.2), (1.3) with (3.13), which now is asymptotically simple in the past.

Because the metric h_{ab} is not subject to an analogue of the Hamiltonian constraint, neither the topology of $\mathcal{J}$ nor the conformal structure defined by h_{ab} is restricted in any way. It may appear then that the procedure offers too much freedom. This is not the case. As discussed in the proof, there remains the freedom to perform on $\mathcal{J}$ transitions of the form

$$\Omega \to \Omega^* = \theta\, \Omega, \quad g \to g^* = \theta^2\, g,$$

with positive conformal factors θ. The effect of these rescalings on the free data is

$$h_{ab} \to h^*_{ab} = \theta^2\, h_{ab}, \quad \psi_0 \to \psi^*_0 = \theta^{-1}\, \psi_0,$$

$$\psi_1 \to \psi^*_1 = \theta^{-2}\, \psi_1, \quad w_{ab} \to w^*_{ab} = \theta^{-1}\, w_{ab}.$$

These rescalings change the conformal representation of the solution to the conformal field equations but leave the associated ‘physical solution’ unchanged.

To assess the freedom to prescribe data one should observe that (4.2) leaves the freedom to add to a given solution of the inhomogeneous equation an arbitrary solution of the homogeneous equation $D^a\, w_{ab} = 0$. In spite of the condition $\chi_{ab} = 0$ on $\mathcal{J}$, which reflects the particular nature of the conformal boundary, the procedure described above thus admits essentially the same freedom to prescribe data as the standard Cauchy problem for the system (1.1), (1.2), (1.3) with (3.13). This conclusion is supported by the following observation.

Denote by S a smooth 3-manifold diffeomorphic to $\mathcal{J}$, by $\mathcal{D}_S$ the set of smooth Cauchy data on S for the system (1.1), (1.2), (1.3) satisfying $(*)$, and by $\mathcal{A}_S$ the subset of the Cauchy data $d_S \in \mathcal{D}_S$ for which the corresponding solution space-times develops so as to admit a smooth conformal boundary $\mathcal{J}^+ \sim S \sim \mathcal{J}$ in the future. S can be thought of as being embedded as a Cauchy hypersurface into this solution so that the data induced by the solution are isometric to the data d_S. The solution induces on $\mathcal{J}^+$ an asymptotic initial data set d_S^* for the conformal field equations. The past development of the data d_S^* with the conformal field equations is conformal to the solution developed from d_S. It induces on S data d'_S for the conformal field equations. As discussed above, the future evolution of these data by the conformal field equations extends smoothly beyond $\mathcal{J}^+$ into a domain which can be foliated by Cauchy hypersurfaces on which $\Omega < 0$.

With data $\bar{d}_S$ on S which are obtained by a small (non-linear) perturbation of d_S we can associate data $\bar{d}'_S$ for the conformal field equations which represent a small perturbation of d'_S so that in terms of suitable Sobolev topologies $\bar{d}'_S \to d'_S$ as $\bar{d}_S \to d_S$ and vice versa. Cauchy stability for the conformal field equation implies that data $\bar{d}'_S$ which are close enough to d'_S will then also develop into a domain foliated by Cauchy hypersurfaces on which $\Omega' < 0$ and the analogue of (4.3) for Ω', which is a consequence of the field equations where $\Omega' = 0$, ensures that the set $\{\Omega' = 0\}$ is a smooth hypersurface diffeomorphic to S. Solutions arising from data close enough to d_S will thus be asymptotically simple in the future and belong to $\mathcal{A}_S$. *The set $\mathcal{A}_S$ is thus open in the set $\mathcal{D}_S$* if the latter is endowed with a suitable Sobolev topology. From the results of [8] this statement can be generalized to include perturbations involving fields with trace-free energy momentum tensor which satisfy conformally covariant field equations.

5. Spatially Homogeneous Solutions

In the following, we study solutions to the equation considered in Theorem 3.1 for which the conformal factor Ω and the metric g are defined on $\mathbb{R} \times S$ with $S = \mathbb{S}^3$, $\mathbb{T}^3$ or $\mathbb{H}^3_*$ (a factor space of hyperbolic 3-space) and take the form

$$\Omega = \Omega(\tau), \qquad g = -d\tau^2 + l^2\, k,$$

with a function $l = l(\tau)$ and a 3-metric $k = k_{ab}\, dx^a\, dx^b = k_\epsilon$ of constant curvature $R_{abcd}[k] = 2\,\epsilon\, k_{a[c}\, k_{b]d}$, where $\epsilon = 1, 0, -1$ respectively. We write also $\tau = x^0$ and assume for simplicity

$$V = 0, \qquad \kappa = 1, \qquad \lambda = 3,$$

and thus $m^2 = 2$ to take care of (3.13). The non-vanishing Christoffel symbols and the second fundamental form χ_{ab} of the slices $\{\tau = const.\}$ are then given by

$$\chi_{ab} = \Gamma_a{}^0{}_b[g] = l\, l'\, k_{ab}, \quad \Gamma_0{}^a{}_c[g] = \Gamma_c{}^a{}_0[g] = \frac{1}{l}\, l'\, k^a{}_c, \quad \Gamma_b{}^a{}_c[g] = \Gamma_b{}^a{}_c[k],$$

where $' = \frac{d}{d\tau}$. The Ricci scalar and the Ricci tensor are given by

$$R[g] = \frac{6}{l^2}\,(\epsilon + l\,l'' + (l')^2), \qquad R_{00}[g] = -3\,\frac{l''}{l}, \qquad R_{a0}[g] = R_{0a}[g] = 0,$$
$$R_{ab}[g] = \{2\,\epsilon + l\,l'' + 2\,(l')^2\}\,k_{ab},$$

and the Schouten tensor by

$$L_{00}[g] = \frac{1}{2\,l^2}\,(\epsilon - 2\,l\,l'' + (l')^2), \quad L_{a0}[g] = L_{0a}[g] = 0, \quad L_{ab}[g] = \frac{1}{2}\,(\epsilon + (l')^2)\,k_{ab}.$$

Because the line element above is conformally flat it follows that $W^{\mu}{}_{\nu\lambda\rho} = 0$, which leads to a considerable simplification.

It will be assumed in the following that the conformal time coordinate τ vanishes on a set $\{\Omega = 0\}$ and the metric g satisfies the conformal gauge condition $R[g] = 6\,\epsilon$. Fixing R still leaves some freedom to perform rescalings. This can be used to restrict the metric and the second fundamental form on $\{\Omega = 0\}$ so that $l = 1$ and $l' = 0$ there. With these requirements and the expression for the Ricci scalar above, it follows that

$$l\,l'' + (l')^2 + \epsilon\,(1 - l^2) = 0, \quad l(0) = 1, \quad l'(0) = 0,$$

which implies that $l = 1$. Where $\Omega > 0$ the physical fields can then be given in the form

$$\hat{g} = \Omega^{-2}\,g = -dt^2 + f^2\,d\omega^2, \qquad \phi = \Omega\,\psi, \tag{5.1}$$

with

$$f(t) = \frac{1}{\Omega(\tau(t))}, \qquad \frac{dt}{d\tau} = \frac{1}{\Omega(\tau)}, \tag{5.2}$$

so that the information on the geometry is completely encoded in the conformal factor.

Because g is conformally flat we have by (2.9) and (2.10) $\nabla_{[\nu}\,L_{\lambda]\rho} = 0$ and $\hat{\nabla}_{[\nu}\,\hat{L}_{\lambda]\rho} = 0$ so that equations (2.16), (2.17) are trivially satisfied and we are left with the equations

$$2\,\Omega\,s - \nabla_{\rho}\Omega\,\nabla^{\rho}\Omega = 1 - \frac{1}{12}\,\Omega^2\,T_*, \tag{5.3}$$

$$\nabla_{\mu}\,\nabla_{\nu}\Omega = -\,\Omega\,L_{\mu\nu} + s\,g_{\mu\nu} + \frac{1}{2}\,\Omega\,T^*_{\mu\nu}, \tag{5.4}$$

$$\nabla_{\mu}\,s = -\,\nabla^{\rho}\Omega\,L_{\rho\mu} + \frac{1}{2}\,\nabla^{\rho}\Omega\,T^*_{\rho\mu} - \frac{1}{12}\,\nabla_{\mu}\Omega\,T_* - \frac{1}{24}\,\Omega\,\nabla_{\mu}T_*, \tag{5.5}$$

$$\Box_g\psi - \frac{1}{6}\,R\,\psi = \frac{1}{6}\,T_*\psi, \tag{5.6}$$

where

$$T^*_{\mu\nu} = \nabla_{\mu}(\Omega\psi)\,\nabla_{\nu}(\Omega\,\psi) - \frac{1}{4}\,\nabla_{\rho}(\Omega\psi)\,\nabla^{\rho}(\Omega\psi)\;g_{\mu\nu}, \tag{5.7}$$

$$T_* = -\nabla_{\rho}(\Omega\,\psi)\,\nabla^{\rho}(\Omega\,\psi) - 4\,\psi^2. \tag{5.8}$$

The assumed symmetry implies

$$T_* = ((\Omega\,\psi)')^2 - 4\,\psi^2, \tag{5.9}$$

$$T^*_{00} = \frac{3}{4}\,((\Omega\,\psi)')^2, \qquad T^*_{a0} = T^*_{0a} = 0, \qquad T^*_{ab} = \frac{1}{4}\,((\Omega\psi)')^2\,k_{ab}, \tag{5.10}$$

and the equations reduce to

$$2\,\Omega\,s + (\Omega')^2 = 1 - \frac{1}{12}\,\Omega^2\,T_*, \tag{5.11}$$

$$\Omega'' = -\frac{1}{2}\,\epsilon\,\Omega - s + \frac{3}{8}\,\Omega\,((\Omega\,\psi)')^2, \tag{5.12}$$

$$0 = -\frac{1}{2}\,\epsilon\,\Omega + s + \frac{1}{8}\,\Omega\,((\Omega\psi)')^2, \tag{5.13}$$

$$s' = \frac{1}{2}\,\epsilon\,\Omega' - \frac{3}{8}\,\Omega'\,((\Omega\,\psi)')^2 - \frac{1}{12}\,\Omega'\,T_* - \frac{1}{24}\,\Omega\,T_*', \tag{5.14}$$

$$-\psi'' - \epsilon\,\psi = \frac{1}{6}\,T_*\psi. \tag{5.15}$$

Obviously there is some redundancy in this system. Solving for the last term on the right-hand side of (5.13) inserting the result in the last term on the right-hand side of (5.12) gives $s = \frac{1}{4}(-\Omega'' + \epsilon\,\Omega)$, which is just (2.5). Conversely, this expression for s implies with (5.12) the relation (5.13). Solving (5.13) instead for s and using this to replace s in (5.12) gives

$$\Omega'' + \left(\epsilon - \frac{1}{2}\,((\Omega\,\psi)')^2\right)\Omega = 0. \tag{5.16}$$

Using (5.9) in (5.15) gives

$$\psi'' + \left(\epsilon + \frac{1}{6}\,((\Omega\,\psi)')^2 - \frac{2}{3}\,\psi^2\right)\psi = 0. \tag{5.17}$$

Inserting s from (5.13) in (5.11) gives

$$(\Omega')^2 = 1 + \Omega^2\left(\frac{1}{3}\,\psi^2 + \frac{1}{6}\,((\Omega\,\psi)')^2 - \epsilon\right). \tag{5.18}$$

The first two equations above provide a closed evolution system for Ω and ψ while the third equation should be read as a constraint. It will be satisfied if it holds for one value of τ and the first two equations are satisfied (not an immediate calculation). It can be shown that (5.14) follows if the other equations are satisfied. Using (5.2), equations (5.16), (5.17), (5.18) can be derived where $\Omega \neq 0$ directly from the equations implied by (2.1), (3.1), (3.2) for the functions in (5.1).

If data are prescribed on $\{\Omega = 0\} = \{\tau = 0\}$ and if is assumed that Ω is positive for $\tau < 0$ close to the crossover surface $\{\tau = 0\}$, the constraint (5.18) shows that the equations above must be solved with initial conditions

$$\Omega(0) = 0, \;\; \Omega'(0) = -1 \quad \text{and free data} \quad \psi(0), \;\; \psi'(0).$$

The vacuum solutions, obtained by setting $\psi(0) = 0$, $\psi'(0) = 0$, satisfy $\psi = 0$ and $\Omega = \Omega_\epsilon$ with

$$\Omega_1 = -\sin\tau, \;\; -\pi < \tau < 0, \quad \Omega_0 = -\tau, \;\; \tau < 0, \quad \Omega_{-1} = -\sinh\tau, \;\; \tau < 0.$$

Observing (5.2), the corresponding physical solutions $\hat{g}_\epsilon = \Omega_\epsilon^{-2}\,g_\epsilon$ are then given by the de Sitter solution

$$\hat{g}_1 = -dt^2 + \cosh^2 t\;k_1, \quad t \in \mathbb{R},$$

which expands in both time directions, and by

$$\hat{g}_0 = -dt^2 + e^{2t}\, k_0, \quad t \in \mathbb{R}, \qquad\qquad \hat{g}_{-1} = -dt^2 + \sinh^2 t\; k_{-1}, \quad t > 0.$$

When $\psi \not\equiv 0$ it follows by (5.16) in the cases $\epsilon \leq 0$ that the functions $\Omega_\epsilon(\tau)$ grow even faster backwards in time and most likely diverge for a finite value $\tau_* < 0$ of the parameter τ. The precise behaviour of the solutions near that point, or, in other words, the precise expansion behaviour of the associated physical solutions $\hat{g}_\epsilon = \Omega^{-2}\, g_\epsilon$, $\epsilon = 0, -1$, at the big bang indicated by that point, depends on the initial data for ψ and requires a detailed analysis which cannot be given here.

In the following we consider the case $\epsilon = 1$ in somewhat more detail *and drop the index* ϵ *everywhere*. The structure of the system (5.16), (5.17), (5.18) implies that for data ψ and ψ' at $\tau = 0$ which are sufficiently small the solutions $\Omega(\tau)$ will also be oscillatory and may stay close to the de Sitter solution for a long conformal time τ. It is an interesting problem to characterize the initial data $\psi(0)$ and $\psi'(0)$ for which the solutions to the system system (5.16), (5.17), (5.18) exist for all conformal times τ. That there do exist non-trivial data with this property other than oscillatory solutions is shown by the solution Ω_*, ψ_* to (5.16), (5.17), (5.18) which is given by

$$\Omega_* = -\tau, \quad \psi_* = \sqrt{2}. \tag{5.19}$$

The physical fields corresponding to the restriction of this solution to the domain $-\infty < \tau < 0$ are given in terms of the coordinate $t = -\log(-\tau)$ by

$$\tilde{g} = -dt^2 + e^{2t}\, d\omega^2, \quad \phi = \sqrt{2}\, e^{-t},$$

and are thus quite different from those in the de Sitter case. As $t \to -\infty$ the matter field diverges while it decays and the metric shows a de Sitter-type expansion behaviour as $t \to \infty$.

Because Ω_* does not approach zero a second time but $\Omega_* \to \infty$ as $\tau \to -\infty$, it is not possible to obtain a global stability result by the argument used before, but the situation may be of considerable interest in our context. Consider the backward evolution of initial data of the form

$$\Omega(0) = 0, \;\; \Omega(0) = -1, \;\; \psi(0) = \sqrt{2} - \delta, \;\; \psi'(0) = -\bar{\delta}, \;\; 0 \leq \delta < \sqrt{2}, \;\; 0 \leq \bar{\delta}.$$

Numerical calculations with data so that $0 \leq \delta$, $\bar{\delta} << 1$ show that there exist solutions Ω which stay close to Ω_* for $\tau < 0$ and $|\tau|$ small enough and which are monotonically increasing with $\Omega < -\tau$. After assuming a maximum value Ω_m at some $\tau_m < 0$, the solutions decrease until they vanish at some point $\tau_z < \tau_m$. The corresponding physical solutions on $]\tau_z, 0[$ can be thought of as arising from initial data on the 'crossover surface' $\{\tau = \tau_z\}$, of developing a 'waist' of volume $\Omega_m^{-3}\, Vol(\mathbb{S}^3)$ at τ_m, and approaching the next crossover surface at $\{\tau = 0\}$. Again these solutions would have the stability property pointed out above. First calculations show that by suitable choices of the data the value of $|\tau_m|$, the maximum value Ω_m, and the value of $|\tau_z|$ can be made to increase. This raises an interesting question, which is related to the second type of problems addressed in the introduction:

Do there exist solutions of this type which approximate for given $z < 0$ the solution Ω_* on the interval $[z, 0]$ arbitrarily well ?

A positive answer would show the existence of solutions which still have the stability property but whose waist would be arbitrarily narrow. The restriction of such solutions to the range $]\tau_m, 0[$ would, from the point of view of observational data, hardly be distinguishable from solutions which start with a big bang and then expand exponentially. Again, no attempt is made here to analyse this question.

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Helmut Friedrich
Max-Planck-Institut für Gravitationsphysik
Am Mühlenberg 1
14476 Golm, Germany
e-mail: hef@aei.mpg.de

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