

Resonant dynamics and the instability of anti-de Sitter spacetime

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We consider spherically symmetric Einstein-massless-scalar field equations with negative cosmological constant in five dimensions and analyze evolution of small perturbations of anti-de Sitter spacetime using the recently proposed resonant approximation. We show that for typical initial data the solution of the resonant system develops an oscillatory singularity in finite time. This result hints at a possible route to establishing instability of AdS under arbitrarily small perturbations.

Introduction. A few years ago two of us gave numerical evidence that anti-de Sitter (AdS) spacetime in four dimensions is unstable against black hole formation for a large class of arbitrarily small perturbations [1]. More precisely, we showed that for a perturbation with amplitude ε a black hole forms on the timescale $\mathcal{O}(\varepsilon^{-2})$. Using nonlinear perturbation analysis we conjectured that the instability is due to the turbulent cascade of energy from low to high frequencies. This conjecture was extended to higher dimensions in [2].

Since the computational cost of numerical simulations rapidly increases with decreasing ε , our conjecture was based on extrapolation of the observed scaling behavior of solutions for small (but not excessively so) amplitudes, which left some room for doubts whether the instability will persist to arbitrarily small values of ε (see e.g. [3]). To resolve these doubts, in this paper we validate and reinforce the above extrapolation with the help of a recently proposed resonant approximation [4–6]. The key feature of this approximation is that the underlying infinite dynamical system (hereafter referred to as the resonant system) is scale invariant: if its solution with amplitude 1 does something at time t , then the corresponding solution with amplitude ε does the same thing at time t/ε^2 . Moreover, the latter solution remains close to the true solution (starting with the same initial data) for times $\lesssim \varepsilon^{-2}$ (provided that the errors due to omission of higher order terms do not pile up too rapidly). Thus, by solving the resonant system we can probe the regime of arbitrarily small perturbations (whose outcome of evolution is beyond the possibility of numerical verification).

For concreteness, in this paper we focus our attention on AdS₅ (the most interesting case from the viewpoint of AdS/CFT correspondence); an extension to other dimensions is straightforward and will be presented elsewhere.

Model. For the reader's convenience, let us recall from [1, 2] the general framework for studying the spherically symmetric scalar perturbations of the AdS spacetime. The five dimensional asymptotically AdS spacetimes are parametrized by the coordinates $(t, x, \omega) \in (-\infty, \infty) \times [0, \pi/2) \times \mathbb{S}^3$ and the metric

$$ds^2 = \frac{\ell^2}{\cos^2 x} (-Ae^{-2\delta} dt^2 + A^{-1} dx^2 + \sin^2 x d\omega^2), \quad (1)$$

where $\ell^2 = -6/\Lambda$, $d\omega^2$ is the round metric on \mathbb{S}^3 , and A, δ are functions of (t, x) . For this ansatz the evolution of a self-gravitating massless scalar field $\phi(t, x)$ is governed by the following system (using units in which $8\pi G = 3$)

$$\partial_t \Phi = \partial_x (Ae^{-\delta} \Pi), \quad \partial_t \Pi = \frac{1}{\tan^3 x} \partial_x (\tan^3 x Ae^{-\delta} \Phi), \quad (2)$$

$$\partial_x A = \frac{2 + 2 \sin^2 x}{\sin x \cos x} (1 - A) - \sin x \cos x A (\Phi^2 + \Pi^2), \quad (3)$$

$$\partial_x \delta = -\sin x \cos x (\Phi^2 + \Pi^2), \quad (4)$$

where $\Phi = \partial_x \phi$ and $\Pi = A^{-1} e^\delta \partial_t \phi$. To ensure smoothness at spatial infinity and finiteness of the total mass M we impose the boundary conditions (using $\rho = \pi/2 - x$)

$$\begin{aligned} \phi(t, x) &= f_\infty(t) \rho^4 + \mathcal{O}(\rho^6), & \delta(t, x) &= \delta_\infty(t) + \mathcal{O}(\rho^8), \\ A(t, x) &= 1 - M\rho^4 + \mathcal{O}(\rho^6), \end{aligned} \quad (5)$$

where the power series expansions are uniquely determined by M and the functions $f_\infty(t), \delta_\infty(t)$. We use the normalization $\delta(t, 0) = 0$, hence t is the proper time at the center. We will solve this system for small smooth perturbations of the AdS solution $\phi = 0, A = 1, \delta = 0$.

Resonant approximation. As follows from equation (2), linearized perturbations of AdS₅ are governed by the operator $L = -\tan^{-3} x \partial_x (\tan^3 x \partial_x)$. This operator is essentially self-adjoint with respect to the inner product $(f, g) := \int_0^{\pi/2} f(x)g(x) \tan^3 x dx$. The eigenvalues and orthonormal eigenfunctions of L are $\omega_n^2 = (2n + 4)^2$ ($n = 0, 1, \dots$) and

$$e_n(x) = 2\sqrt{\frac{(n+2)(n+3)}{n+1}} \cos^4 x P_n^{(1,2)}(\cos(2x)), \quad (6)$$

where $P_n^{(1,2)}(x)$ is the Jacobi polynomial of order n .

After these preliminaries we are prepared to introduce the resonant approximation. To avoid technicalities, let us first illustrate this approach in the case of the cubic

wave equation on the *fixed* AdS₅ background¹

$$\partial_{tt}\phi + L\phi + \sec^2 x \phi^3 = 0. \quad (7)$$

Inserting the mode expansion $\phi(t, x) = \sum_n c_n(t) e_n(x)$ into (7) we get an infinite system of coupled oscillators

$$\frac{d^2 c_n}{dt^2} + \omega_n^2 c_n = \sum_{jkl} I_{jkl n} c_j c_k c_l, \quad (8)$$

where the coefficients $I_{jkl n} = -(e_j e_k e_l \sec^2 x, e_n)$ determine interactions between the modes. To factor out fast linear oscillations in (8), we change variables using the variation of constants (a.k.a. “interaction picture”)

$$c_n = \beta_n e^{i\omega_n t} + \bar{\beta}_n e^{-i\omega_n t}, \quad (9)$$

$$\frac{dc_n}{dt} = i\omega_n (\beta_n e^{i\omega_n t} - \bar{\beta}_n e^{-i\omega_n t}). \quad (10)$$

This transforms the system (8) into

$$2i\omega_n \frac{d\beta_n}{dt} = \sum_{jkl} I_{jkl n} c_j c_k c_l e^{-i\omega_n t}, \quad (11)$$

where each c_j in the sum is given by (9), thus each term in the sum has a factor $e^{-i\Omega t}$, where $\Omega = \omega_n \pm \omega_j \pm \omega_k \pm \omega_l$. The terms with $\Omega = 0$ correspond to resonant interactions, while those with $\Omega \neq 0$ are non-resonant.

Passing to slow time $\tau = \varepsilon^2 t$ and rescaling $\beta_n(t) = \varepsilon \alpha_n(\tau)$, we see that for ε going to zero the non-resonant terms $\propto e^{-i\Omega\tau/\varepsilon^2}$ are highly oscillatory and therefore negligible (at least for some time). Keeping only the resonant terms in (11) (which is equivalent to time-averaging), we obtain the infinite autonomous dynamical system (which we shall call the *resonant system*)

$$2i\omega_n \frac{d\alpha_n}{d\tau} = \sum_{jkl} I_{jkl n} \alpha_j \alpha_k \bar{\alpha}_l, \quad (12)$$

where the summation runs over the set of indices $\{j, k, l\}$ for which $\Omega = 0$ and $I_{jkl n} \neq 0$ (due to the fully resonant non-dispersive spectrum of L and vanishing of some coefficients $I_{jkl n}$, this set reduces to $\{jkl \mid j + k - l = n\}$; see footnote 3 in [6]). Note that the system (12) is invariant under the scaling $\alpha_n(\tau) \rightarrow \varepsilon^{-1} \alpha_n(\tau/\varepsilon^2)$. It is routine to show that the solutions of (11) starting from small initial data of size ε are well approximated by the solutions of (12) on the timescale $\mathcal{O}(\varepsilon^{-2})^2$. In other words,

on this timescale the dynamics of solutions of the cubic wave equation (7) is dominated by resonant interactions.

For the system (2-4) the derivation of the resonant system is similar but technically more intricate because, to begin with, one has to integrate out the constraints which (at the lowest order) results in nonlocal cubic nonlinearities in the derivatives of ϕ . Despite these complications, the resonant system has the same form as (12), namely

$$2i\omega_n \frac{d\alpha_n}{d\tau} = \sum_{j+k-l=n} C_{jkl n} \alpha_j \alpha_k \bar{\alpha}_l, \quad (13)$$

except that now the interaction coefficients $C_{jkl n}$ are given by much more complicated expressions involving integrals of products of eigenfunctions (6) and their derivatives. The system (13) was first derived in [4] and [5] using the multiscale perturbation methods³ and soon afterwards in [6] using the averaging method. It is simpler than the full system, yet still too difficult to be handled by purely analytic means, hence in what follows we analyze it using numerical and asymptotic methods.

Results. We solved in parallel the full Einstein equations (2-4) and the resonant system (13) for a variety of the same small initial data. To illustrate the results (which we believe are universal), we present them for the two-mode initial data with energy (almost) equally distributed among the modes

$$\phi(0, x) = \varepsilon \left(\frac{1}{4} e_0(x) + \frac{1}{6} e_1(x) \right), \quad \Pi(0, x) = 0. \quad (14)$$

Let us point out that, since small one-mode data and their perturbations enjoy quasiperiodic evolutions [1, 4, 9], the two-mode data are in a sense minimal in what it takes to initiate the turbulent cascade (cf.[10]).

The numerical simulations of the full Einstein equations (2-4) were previously reported in [2] in the case of Gaussian initial data. The evolution of the two-mode data (14) looks similar. For all considered small values of ε we observe collapse to a black hole in time $t_H(\varepsilon) \sim \varepsilon^{-2}$ (see Fig. 1). This scaling suggests that the instability should be seen in the resonant approximation. In what follows, we confirm this expectation and thereby give support to the conjecture that the instability is present for arbitrarily small perturbations.

For the numerical computation, the resonant system (13) must be truncated at some (possibly large) index N . As a compromise between the accuracy⁴ and the

¹ The study of this equation has been argued (see [7] and references therein) to provide insight into the problem of stability of AdS. We do not share this view (as explained in [8]) and use equation (7) solely for illustrative purposes, *not* as a toy model.

² More precisely, if $\beta_n(t)$ and $\alpha_n(\tau)$ are solutions of (11) and (12), respectively, and $\beta_n(0) = \varepsilon \alpha_n(0)$ for each n , then $|\beta_n(t) - \varepsilon \alpha_n(\varepsilon^2 t)| \lesssim \varepsilon$ for $t \lesssim \varepsilon^{-2}$.

³ In the multiscale approach, the system (13) follows from elimination of secular terms due to resonances at the third order of perturbation expansion. The fact that all secular terms can be removed in this way has been sometimes misunderstood as evidence for stability of AdS.

⁴ See [11] for the analysis of the combined error due to truncation and averaging for semilinear wave equations.

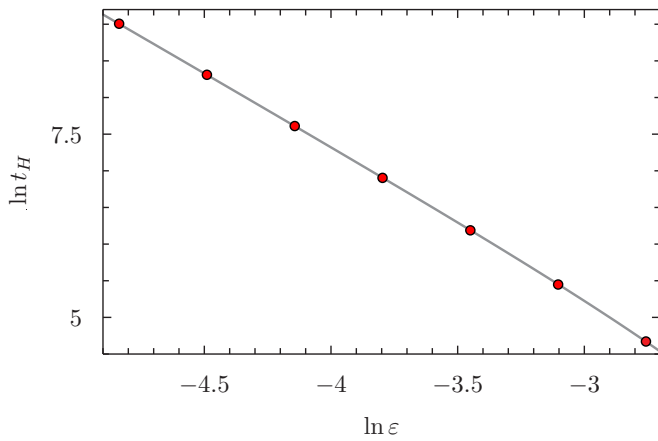


FIG. 1: Time of horizon formation vs amplitude in the evolution of initial data (14). The solid line depicts the fit of the function $-2 \ln \varepsilon + a + b\varepsilon^2$ to the numerical data $\ln t_H(\varepsilon)$. From this fit we obtain $\tau_H := \lim_{\varepsilon \rightarrow 0} \varepsilon^{-2} t_H(\varepsilon) = e^a \approx 0.514$.

computational cost, we choose here $N = 172$. To solve the truncated resonant system (TRS) numerically we use the 6th-order Gauss-Runge-Kutta method⁵. The initial data for the TRS corresponding to (14) are

$$\alpha_0(0) = 1/8, \quad \alpha_1(0) = 1/12, \quad (\alpha_n(0))_{n \geq 2} = 0. \quad (15)$$

To describe and analyze the behavior of solutions, it is convenient to use the amplitude-phase representation $\alpha_n = A_n e^{iB_n}$ in terms of which the resonant system (13) takes the following form [6]

$$2\omega_n \frac{dA_n}{d\tau} = \sum_{\substack{j+k-l=n \\ j \neq n, k \neq n}} S_{jkl n} A_j A_k A_l \sin(B_n + B_l - B_j - B_k) \quad (16)$$

$$2\omega_n \frac{dB_n}{d\tau} = T_n A_n^2 + \sum_{j \neq n} R_{jn} A_j^2 + A_n^{-1} \sum_{\substack{j+k-l=n \\ j \neq n, k \neq n}} S_{jkl n} A_j A_k A_l \cos(B_n + B_l - B_j - B_k). \quad (17)$$

where $T_n = C_{nnnn}$, $R_{jn} = C_{njjn}$ if $j \neq n$, and $S_{jkl n} = C_{jkl n}$ if all four indices are different. Explicit expressions for these coefficients are given in the appendix A in [6].

Evolving the data (15), we find that higher modes are quickly excited (see Fig. 2). For early times the amplitudes grow at the polynomial rate $A_n(\tau) \sim \tau^{n-1}$ while the phases evolve approximately linearly. At a later time

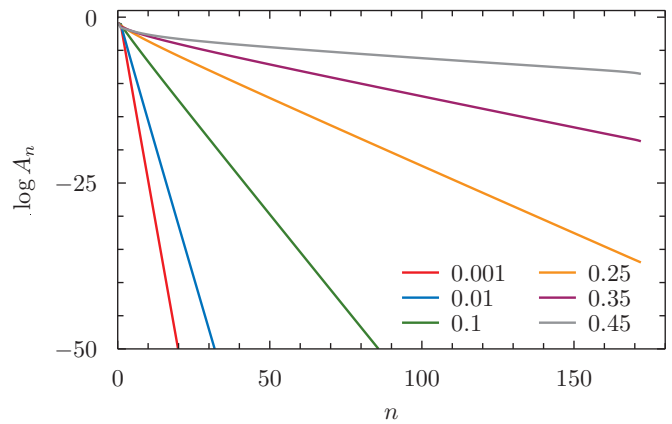


FIG. 2: The amplitude spectra (for several times τ).

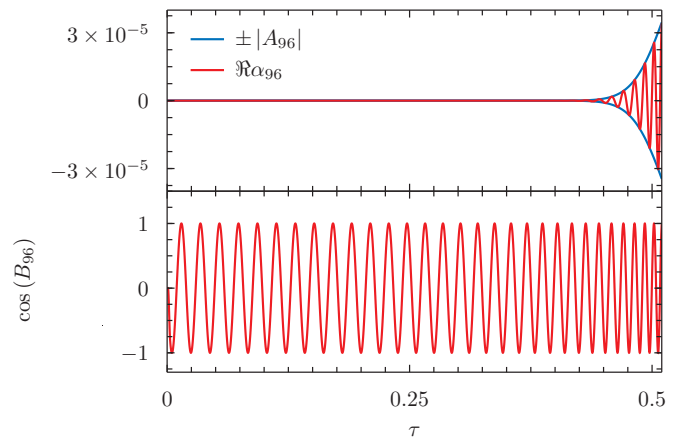


FIG. 3: Evolution of a sample high mode.

the frequencies of oscillations begin to grow rapidly (see Fig. 3). This highly oscillatory behavior accumulates at a finite time causing numerical difficulties. We find that the time-step of numerical integration, for which the algorithm is convergent, tends to zero as the cutoff N increases. This suggests that the solution of the resonant system ($N = \infty$) develops an oscillatory singularity in some finite time τ_* ⁶.

To give better evidence for blowup and analyze its character we proceed in the spirit of the analyticity strip method [12, 13], namely we make the following asymptotic ansatz for the amplitudes

$$A_n(\tau) \sim n^{-\gamma(\tau)} e^{-\rho(\tau)n} \quad \text{for } n \gg 1. \quad (18)$$

Fitting this formula to the numerical data we obtain the time dependence of the exponent $\gamma(\tau)$ and the ‘analyticity radius’ $\rho(\tau)$. As shown in Fig. 4, it appears that $\rho(\tau)$

⁵ Due to the oscillatory character of solutions, explicit schemes turn out to be unstable. More details of the numerical method and its validation are given in the supplementary material accompanying this note.

⁶ For any finite N the solution of TRS cannot blow up and, with sufficient resolution, can be numerically continued past τ_* , however this ‘afterlife’ is an artifact of truncation.

tends to zero in a finite time τ_* (with $\rho_0 = -\rho'(\tau_*) > 0$) confirming that the solution of the resonant system (13) becomes singular at τ_* . The fit also reveals that the asymptotic power-law amplitude spectrum has the exponent $\lim_{\tau \rightarrow \tau_*} \gamma(\tau) = 2$.

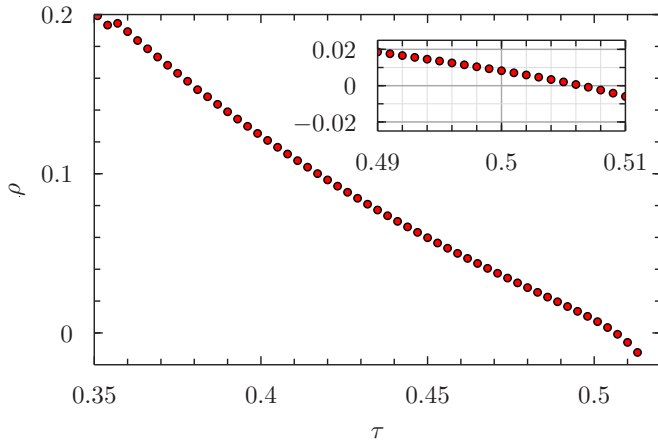


FIG. 4: The radius of analyticity $\rho(\tau)$ obtained by fitting the formula (18) to the amplitude spectrum. The point of this (notoriously poor) fit is to show that $\rho(\tau)$ hits zero in some finite time τ_* , not to determine τ_* precisely (cf. footnote 6).

Guided by these numerical findings we will now construct an asymptotic solution of the resonant system that becomes singular in finite time. We assume that for large n and $\tau \rightarrow \tau_*$

$$A_n(\tau) \sim n^{-2} e^{-\rho_0(\tau_* - \tau)n}. \quad (19)$$

To solve for the phases, we note the following asymptotic behavior of the interaction coefficients

$$T_n \sim n^5, \quad R_{jn} \sim n^2 j^3, \quad S_{\lambda_j, \lambda_k, \lambda_l, \lambda_n} \sim \lambda^4 S_{jkl n}. \quad (20)$$

Notice that the latter implies that

$$\sum_{\substack{j+k-l=n \\ j \neq n, k \neq n}} S_{jkl n} (jkl)^{-2} = \mathcal{O}(1), \quad (21)$$

that is the sum does not depend of n . Plugging (19) into Eq.(17) and using (20) and (21), we see that for $\tau \rightarrow \tau_*$ the r.h.s. of Eq.(17) is dominated by the term

$$\sum R_{jn} A_j^2 \sim n^2 \sum j^{-1} e^{-2\rho_0(\tau_* - \tau)j} \sim n^2 \ln(\tau_* - \tau), \quad (22)$$

thus the derivatives $\frac{dB_n}{d\tau}$ blow up logarithmically. Moreover, it follows from the above that for large n the phases B_n behave linearly with n , hence $B_n + B_l - B_j - B_k \approx 0$ for the resonant quartets. This implies that both sides of Eq.(16) are (approximately) independent of n , reassuring that the ansatz (19) is self-consistent. Numerical simulations indicate that the asymptotics of blowup just described is in fact universal; this is illustrated in Fig. 5 in the case of two-mode data (15).

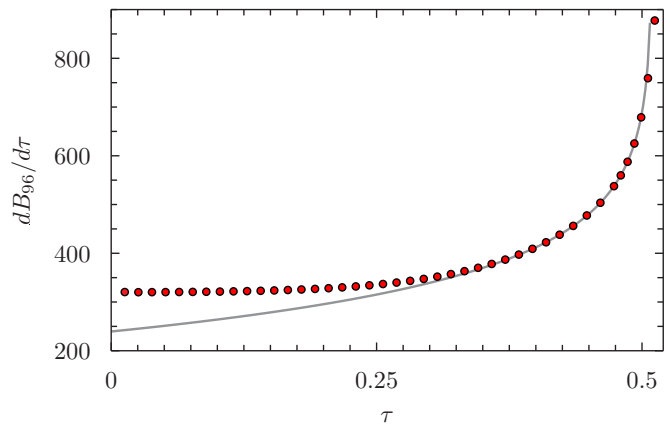


FIG. 5: Evidence for the logarithmic blowup. The solid line represents the fit of the theoretical prediction $a_n \ln(\tau_* - \tau) + b_n$ to the numerical data $\frac{dB_n}{d\tau}$ for $n = 96$. Performing this fit for all $n > 20$ we find, in accord with the asymptotic analysis, that the coefficients a_n and b_n vary linearly with n , while $\tau_* \approx 0.509$ does not depend on n . Note that τ_* agrees very well with the time of collapse τ_H given in the caption of Fig. 1.

Conclusion. To summarize, we have constructed the asymptotic solution of the resonant system that becomes singular in finite time and gave numerical evidence that this solution acts as a universal attractor for blowup. The key question is how to transfer this blowup result from the resonant system to the full system. On one hand, we see that the resonant approximation reproduces the amplitudes of true solutions remarkably well almost all the way to collapse. This is illustrated in Figs. 6 and 7 where we compare the energy spectrum and the growth of the Ricci scalar at the origin computed in parallel using the full Einstein equations and the resonant system. On the other hand, the resonant approximation does not work so well for the phases⁷. For this reason (and because of possible breakdown of the cubic approximation near collapse), it is not clear to us what (if any) is the physical interpretation of the oscillatory singularity for the resonant system⁸. Nonetheless, the fact that solutions of the resonant system blow up in finite time (for typical initial data) strongly indicates that the corresponding solutions of the full system collapse on the timescale $\mathcal{O}(\varepsilon^{-2})$.

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⁷ Note that the highly oscillatory behavior of solutions is in tension with the idea of time averaging.

⁸ Since the resonant system has no scale, it is tempting to speculate that there is a relationship between the oscillatory singularity and Choquet’s discretely self-similar solution [14].

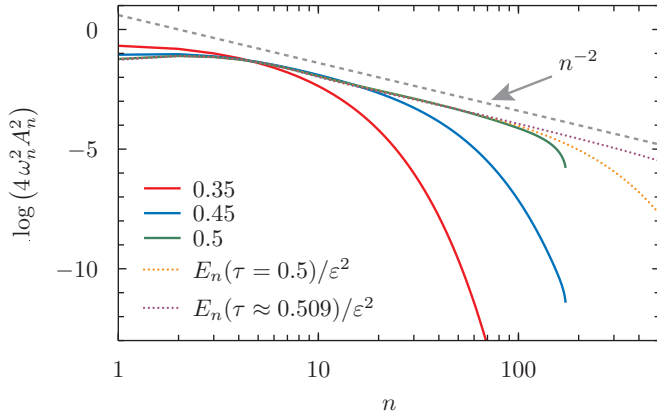


FIG. 6: Energy spectra for the full system (dotted lines) and TRS (solid lines) at sample late times (cf. Fig. 2). The power-law spectrum n^{-2} unfolds as the solutions collapse/blow up. Here $\varepsilon \approx 0.0079$ (the smallest amplitude used in simulations).

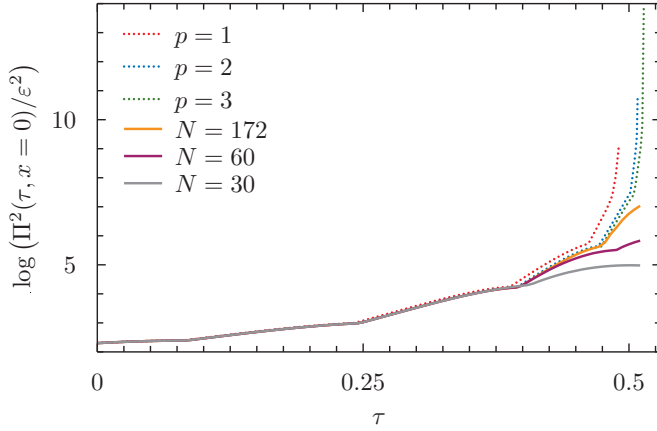


FIG. 7: Upper envelope of $\Pi^2(t, 0)$ in the evolution of initial data (14) with $\varepsilon = (2\pi)^{-3/2}2^{-p}$ for $p = 1, 2, 3$ (dotted lines). As ε decreases, the rescaled quantities $\varepsilon^{-2}\Pi^2(\varepsilon^2t, 0)$ approach a limiting curve. The corresponding solutions of TRS (solid lines) appear to approach the same curve as N increases.

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- [1] P. Bizoń, A. Rostworowski, Phys. Rev. Lett. **107**, 031102 (2011), [arXiv:1104.3702]
 - [2] J. Jałmużna, A. Rostworowski, P. Bizoń, Phys. Rev. D **84**, 085021 (2011), [arXiv:1108.4539]
 - [3] F.V. Dimitrakopoulos, B. Freivogel, M. Lippert, I.-S. Yang, [arXiv:1410.1880]
 - [4] V. Balasubramanian, A. Buchel, S.R. Green, L. Lehner, S.L. Liebling, Phys. Rev. Lett. **113**, 071601 (2014), [arXiv:1403.6471]
 - [5] B. Craps, O. Evnin, J. Vanhoof, JHEP **10**, 48 (2014), [arXiv:1407.6273]
 - [6] B. Craps, O. Evnin, J. Vanhoof, JHEP **01**, 108 (2015), [arXiv:1412.3249]
 - [7] P. Basu, C. Krishnan, P.N. Bala Subramanian, [arXiv:1501.07499]
 - [8] http://www.ctc.cam.ac.uk/activities/adsgrav2014/Slides/Slides_Bizon.pdf
 - [9] M. Maliborski, A. Rostworowski, Phys. Rev. Lett. **111**, 051102 (2013), [arXiv:1403.5434]
 - [10] P. Bizoń, A. Rostworowski, [arXiv:1410.2631]
 - [11] M.S. Krol, Math. Methods Appl. Sciences **11**, 649 (1989)
 - [12] C. Sulem, P.-L. Sulem, H. Frisch, J. Comput. Phys. **50**, 138 (1983)
 - [13] P. Bizoń, J. Jałmużna, Phys. Rev. Lett. **111**, 041102 (2013), [arXiv:1306.0317]
 - [14] M.W. Choptuik, Phys. Rev. Lett. **70**, 9 (1993)