

Recipe theorems for polynomial invariants on ribbon graphs with half-edges

Remi C. Avohou, Joseph Ben Geloun, and Mahouton N. Hounkonnou

ABSTRACT. We provide recipe theorems for the Bollobàs and Riordan polynomial \mathcal{R} defined on classes of ribbon graphs with half-edges introduced in arXiv:1310.3708[math.GT]. We also define a generalized transition polynomial Q on this new category of ribbon graphs and establish a relationship between Q and \mathcal{R} .

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1. Introduction

The Bollobàs-Riordan (BR) polynomial [3] is a four-variable polynomial generalizing Tutte polynomial [16] from simple graphs to ribbon graphs. Ribbon graphs are often called neighborhoods of graphs embedded into surfaces. Like Tutte polynomial, the BR polynomial satisfies a contraction/deletion recurrence relation and it is a universal invariant. The universality property of these invariants means that any invariant of graphs satisfying the same relations of contraction and deletion can be calculated from those. The universality can be also discussed in other contexts, for example, in statistical mechanics [14] and even in quantum field theory [6, 5, 15].

In [10], the authors provide a “recipe theorem” for the BR polynomial intimately close to its universality property [4, 8, 9]. This theorem is a very useful tool in order to evaluate any function F on these ribbon graphs in terms of the BR polynomial itself, once one imposes that F satisfies the same contraction/deletion recurrence relation and few more properties of the BR polynomial. Both proofs of the universality property and the recipe theorem of the BR polynomial are based on the property of the contraction/deletion relation and the understanding of other ingredients such as chord diagrams associated with one-vertex ribbon graphs.

In a subsequent work [1], another BR polynomial \mathcal{R} is introduced on a new class of ribbon graphs called half-edged ribbon graphs (HERGs). A half-edge or half-ribbon (HR) is a ribbon incident to a unique vertex without forming a loop. The presence of HRs in a ribbon graph

have notable combinatorial properties which makes the polynomial found in [1] a nontrivial one. Furthermore, the presence HRs also allows one to define the cut operation of a ribbon edge which differs from the usual edge deletion. The new polynomial \mathcal{R} found on HERGs then satisfies a contraction/cut recurrence relation. There exists another interesting operation on ribbon graphs which consists in moving the HRs on the boundary of these graphs [2]. It has been proved that one can quotient the action of these HR moves and get the so-called HR-equivalent classes of ribbon graphs with half-ribbons. There exists a natural extension of \mathcal{R} on these equivalence classes. The main result in [2] is the proof of the universal property of the polynomial \mathcal{R} on HERGs and of its extension to HR-equivalent classes of ribbon graphs. This statement relies on the understanding and generalization of the tools necessary to the proof of the universality of the original BR polynomial.

The polynomial \mathcal{R} is universal on HERGs or on HR-equivalent ribbon graphs (in the following discussion, we will call these HR-classes or simply classes). HR-classes are in a sense more fundamental. Indeed, as far as one is concerned with the evaluation of \mathcal{R} on a HERG \mathcal{G} , we have $\mathcal{R}(\mathcal{G}) = \mathcal{R}([\mathcal{G}])$ where $[\mathcal{G}]$ is the HR-class of \mathcal{G} . Hence, we might not need to encode all the information about the positions of the HRs on a given ribbon graph before evaluating its invariant \mathcal{R} . We can address the next question related to the existence of a universal property of \mathcal{R} : “Can one provide a recipe theorem for \mathcal{R} on these HR-classes of ribbon graphs?”. Answering this question is one of the main purposes of this paper.

In the present work which should be considered as a companion paper of [2], we will provide recipe theorems (Theorems 4 and 6) for computing any function F on HR-classes satisfying the contraction/cut rule from the knowledge of \mathcal{R} . Our proof is then different from the one introduced in [10]. Indeed, the authors of this contribution based their proof on four items among which (item 2 therein) the factorization property of the BR polynomial when evaluated on one-point-joint ribbon graphs. To be clearer, consider G_1 and G_2 two distinct ribbon graphs (without any half-edge consideration) $R(G_1 \cdot G_2) = R(G_1)R(G_2)$, where R is the BR polynomial in the original sense of [3] and $G_1 \cdot G_2$ is the one-point-joint operation of G_1 and G_2 . With this property, the way to evaluate R on particular chord diagrams becomes simple. It turns out that the polynomial \mathcal{R} defined on half-edged ribbon graphs does not satisfy the same property, namely, $\mathcal{R}(G_1 \cdot G_2) \neq \mathcal{R}(G_1)\mathcal{R}(G_2)$, for G_1 and G_2 ribbon graphs with half-edges. This makes the polynomial \mathcal{R} a radically different invariant. We have identified new sets of conditions (replacing in particular the failing item 2) under which a new genuine recipe can be provided. In fact, one of the recipes found here can be considered as truly fundamental in the following sense: our method yielding Theorem 6 can be slightly adjusted for the polynomial R and we will obtain the recipe as determined in [10]. Finally, as a second main result of this work, we introduce a generalized version of the transition polynomial [10, 8] for HERGs and establish another main result (Theorem 8) which provides a relationship between the transition polynomial found and \mathcal{R} . The extension of this invariant to HR-classes is immediate.

This paper is organized as follows. In section 2, we recall some results on the BR polynomial for HERGs and for HR-classes and its universality property obtained in [2]. The reader must be aware of the basics of ribbon graphs as found in [3] or [8] (but for notations closer to the present paper, we refer to [2]). In section 3, we give the first main result which is the proof of recipe theorems for this polynomial. We finally define, in section 4, the generalized transition polynomial Q on HERGs via the medial graph construction (associated with HERGs) and find a relationship between Q and the BR polynomial \mathcal{R} .

2. Polynomial invariants

This section recalls the main results of [2]:

- the definition of ribbon graphs with half-edges (studied originally in [11]) then precise the HR-equivalent classes that we will consider in this work;
- the definition of a polynomial invariant on HR-classes of ribbon graphs and its universality property.

DEFINITION 1 (Half-ribbon edges [1]). A half-ribbon edge (or simply half-ribbon, denoted henceforth HR) is a ribbon incident to a unique vertex by a unique segment and without forming loops. A HR has two segments one touching a vertex and another free or external segment. The end-points of any free segment are called external points of the HR (see Figure 1).

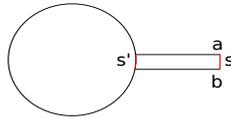


FIGURE 1. A HR with two end segments (in red): s' touching the vertex and s external; the ends a and b of s are the external points.

DEFINITION 2 (Cut of a ribbon edge [11]). Let \mathcal{G} be a ribbon graph and e be an edge in \mathcal{G} . The cut graph $\mathcal{G} \vee e$ is the graph obtained by removing e and let two HRs attached at the end vertices of e . If e is a self-loop, the two HRs are on the same vertex. (See an illustration in Figure 2.)



FIGURE 2. Cutting a ribbon edge.

In a ribbon graph, we can identify three different kinds of edges: the bridge, the loop and the regular edge. Consider an edge e of a ribbon graph \mathcal{G} , e is called a bridge in \mathcal{G} if its removal disconnects a component of \mathcal{G} . If the two ends of e are incident to the same vertex v of \mathcal{G} , e is called a loop in \mathcal{G} . A loop e is trivial if there is no cycle in \mathcal{G} which can be contracted to form a loop f interlaced with e . The edge e is a regular edge of \mathcal{G} if it is neither a bridge nor a loop.

There are twisted and untwisted ribbon edges (see illustration in Figure 3). A loop e attached to a vertex v of a ribbon graph \mathcal{G} is twisted if $v \cup e$ forms a Möbius band as opposed to an annulus (an untwisted loop).

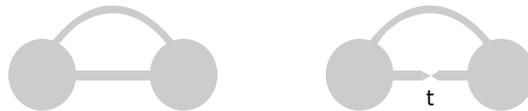


FIGURE 3. Untwisted (left) and twisted (right) edge notations.

The notion of a half-edged ribbon graph (HERG) may be now introduced.

DEFINITION 3 (Ribbon graph with HRs [1]). • A ribbon graph \mathcal{G} with HRs is a ribbon graph $\mathcal{G}(\mathcal{V}, \mathcal{E})$ with a set \mathfrak{f} of HRs defined by the disjoint union of \mathfrak{f}^1 the set of HRs obtained only from the cut of all edges of \mathcal{G} and a set \mathfrak{f}^0 of additional HRs together with a relation which associates with each additional HR a unique vertex. We denote a ribbon graph with set \mathfrak{f}^0 of additional HRs as $\mathcal{G}(\mathcal{V}, \mathcal{E}, \mathfrak{f}^0)$. (See Figure 4.)

• A c -subgraph A of $\mathcal{G}(\mathcal{V}, \mathcal{E}, \mathfrak{f}^0)$ is defined as a ribbon graph with HRs $A(\mathcal{V}_A, \mathcal{E}_A, \mathfrak{f}_A^0)$ the vertex set of which is a subset of \mathcal{V} , the edge set of which is a subset of \mathcal{E} together with their end vertices. Call \mathcal{E}'_A the set of edges incident to the vertices of A and not contained in \mathcal{E}_A . The HR set of A contains a subset of \mathfrak{f}^0 plus additional HRs attached to the vertices of A obtained by cutting all edges in \mathcal{E}'_A . In symbols, $\mathcal{E}_A \subseteq \mathcal{E}$ and $\mathcal{V}_A \subseteq \mathcal{V}$, $\mathfrak{f}_A^0 = \mathfrak{f}_A^{0;0} \cup \mathfrak{f}_A^{0;1}(\mathcal{E}_A)$ with $\mathfrak{f}_A^{0;0} \subseteq \mathfrak{f}^0$ and $\mathfrak{f}_A^{0;1}(\mathcal{E}_A) \subseteq \mathfrak{f}^1$, where $\mathfrak{f}_A^{0;1}(\mathcal{E}_A)$ is the set of HRs obtained by cutting all edges in \mathcal{E}'_A and incident to vertices of A . We write $A \subseteq \mathcal{G}$. (See a c -subgraph A illustrated in Figure 4.)

• A spanning c -subgraph A of $\mathcal{G}(\mathcal{V}, \mathcal{E}, \mathfrak{f}^0)$ is defined as a c -subgraph $A(\mathcal{V}_A, \mathcal{E}_A, \mathfrak{f}_A^0)$ of \mathcal{G} with all vertices and all additional HRs of \mathcal{G} . Hence $\mathcal{E}_A \subseteq \mathcal{E}$ and $\mathcal{V}_A = \mathcal{V}$, $\mathfrak{f}_A^0 = \mathfrak{f}^0 \cup \mathfrak{f}_A^{0;1}(\mathcal{E}_A)$. We write $A \in \mathcal{G}$. (See \tilde{A} in Figure 4.)

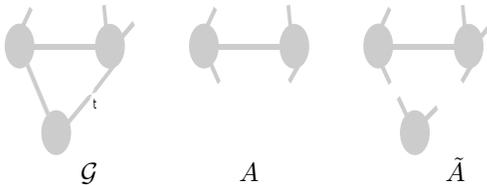


FIGURE 4. A ribbon graph with HRs \mathcal{G} , a c -subgraph A and a spanning c -subgraph \tilde{A} .

A cutting-spanning subgraph can be obtained as follows: cut a subset of edges of a given graph. Then consider the spanning subgraph formed by the resulting graph. The set of HRs of this subgraph is the disjoint union of the set of HRs of the initial graph (\mathfrak{f}^0) plus an additional set induced by the cut of the edges.

Cutting an edge of a HERG brings some modifications on the boundary faces of this graph. We obtain new boundary faces (following the contour of the HRs) which are different from the ones which follow only the boundary of ribbon edges.

DEFINITION 4 (Closed and open faces [7]). Consider $\mathcal{G}(\mathcal{V}, \mathcal{E}, \mathfrak{f}^0)$ a ribbon graph with HRs.

• A closed or internal face is a boundary face component of a ribbon graph (regarded as a geometric ribbon) which never passes through any free segment of additional HRs. The set of closed faces is denoted \mathcal{F}_{int} .

• An open or external face is a boundary face component leaving an external point of some HR rejoining another external point. The set of open faces is denoted \mathcal{F}_{ext} .

• The two boundary lines of a ribbon edge or a HR are called strands. Each strand belongs either to a closed or to open face.

• The set of faces \mathcal{F} of a graph is defined by $\mathcal{F}_{\text{int}} \cup \mathcal{F}_{\text{ext}}$.

• A graph is said to be open if $\mathcal{F}_{\text{ext}} \neq \emptyset$ i.e. $\mathfrak{f}^0 \neq \emptyset$. It is closed otherwise.

Open and closed faces are illustrated in Figure 5.

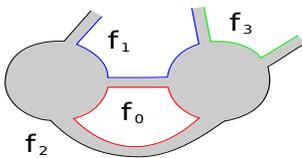


FIGURE 5. A ribbon graph with set of internal faces $\mathcal{F}_{\text{int}} = \{f_0\}$, and set of external faces $\mathcal{F}_{\text{ext}} = \{f_1, f_2, f_3\}$.

DEFINITION 5 (Boundary graph [7]). • The boundary $\partial\mathcal{G}$ of a ribbon graph $\mathcal{G}(\mathcal{V}, \mathcal{E}, \mathfrak{f}^0)$ is a simple graph $\partial\mathcal{G}(\mathcal{V}_\partial, \mathcal{E}_\partial)$ such that \mathcal{V}_∂ is one-to-one with \mathfrak{f}^0 and \mathcal{E}_∂ is one-to-one with \mathcal{F}_{ext} .

• The boundary graph of a closed graph is empty.

We obtain the boundary $\partial\mathcal{G}$ of the graph \mathcal{G} by inserting a vertex of valence two at each HR, the external faces of \mathcal{G} are incident to these vertices (see an illustration in Figure 6). The graph resulting after the insertion of 2-valent vertices at each HR is called pinched ribbon graph [7].

The notion of cut of an edge as an operation on HERGs has been already introduced. In addition, the notions of edge contraction and deletion keep their ordinary meaning as operations on HERGs. (See [2] for further details.)

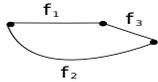


FIGURE 6. The boundary graph associated with the ribbon graph in Figure 5.

So far, the definition of HERGs is purely combinatorial. We can certainly comment on the geometrical meaning of these ribbon structures (this will be used in section 4).

Consider a graph with half-edges (HEG). A HEG \mathcal{G} can be “cellularly embedded” in a surface Σ with punctures (with boundary circles) in the following sense:

- Removing all half-edges from \mathcal{G} we get \mathcal{G}' , a simple graph, which is then cellularly embedded in Σ such that each connected component of $\Sigma \setminus \mathcal{G}'$ is homeomorphic either to a disc or to discs with holes;

- Each of the half-edges of \mathcal{G} is embedded in Σ , respecting of course their incidence relation, and ends on a different puncture (but can be on the same boundary circle).

Then a HERG makes sense as the neighborhood of a HEG cellularly embedded in a punctured surface as defined above.

We now recall the polynomial invariant introduced in [2].

DEFINITION 6 (BR polynomial for HERGs [2]). *Let $\mathcal{G}(\mathcal{V}, \mathcal{E}, \mathfrak{f}^0)$ be a HERG. We define the ribbon graph polynomial of \mathcal{G} to be*

$$\mathcal{R}_{\mathcal{G}}(X, Y, Z, S, W, T) = \sum_{A \in \mathcal{G}} (X-1)^{r(A)-r(\mathcal{G})} (Y-1)^{n(A)} Z^{k(A)-F_{\text{int}}(A)+n(A)} S^{C_{\partial}(A)} W^{t(A)} T^{f(A)}, \quad (1)$$

considered as an element of the quotient of $\mathbb{Z}[X, Y, Z, S, W, T]$ by the ideal generated by $W^2 - W$, and where $r(A)$, $n(A)$, $k(A)$, and $t(A)$ are, respectively, the rank, the nullity, the number of connected components, and the parameter which characterizes the orientability of A as a surface. If A is orientable, then $t(A) = 0$, otherwise, $t(A) = 1$. By definition, $r(A) = |\mathcal{V}| - k(A)$ and $n(A) = |\mathcal{E}(A)| - r(A)$. Furthermore, $C_{\partial}(A) = |\mathcal{C}_{\partial}(A)|$ is the number of connected components of the boundary of A , $F_{\text{int}}(A) = |\mathcal{F}_{\text{int}}(A)|$ and $f(A)$ the number of HRs of A .

THEOREM 1 (Contraction and cut on BR polynomial [2]). *Let $\mathcal{G}(\mathcal{V}, \mathcal{E}, \mathfrak{f}^0)$ be a HERG. Then, for a regular edge e ,*

$$\mathcal{R}_{\mathcal{G}} = \mathcal{R}_{\mathcal{G}_{\vee e}} + \mathcal{R}_{\mathcal{G}/e}, \quad (2)$$

for a bridge e , we have

$$\mathcal{R}_{\mathcal{G}} = (X-1)\mathcal{R}_{\mathcal{G}_{\vee e}} + \mathcal{R}_{\mathcal{G}/e}; \quad (3)$$

for a trivial twisted self-loop e , the following holds

$$\mathcal{R}_{\mathcal{G}} = \mathcal{R}_{\mathcal{G}_{\vee e}} + (Y-1)ZW \mathcal{R}_{\mathcal{G}/e}, \quad (4)$$

whereas for a trivial untwisted self-loop e , we have

$$\mathcal{R}_{\mathcal{G}} = \mathcal{R}_{\mathcal{G}_{\vee e}} + (Y-1)\mathcal{R}_{\mathcal{G}/e}. \quad (5)$$

The change of variable $S \rightarrow Z^{-1}$ leads to another polynomial \mathcal{R}' for HERGs extending the BR polynomial. For a graph \mathcal{G} , we write

$$\mathcal{R}_{\mathcal{G}}(X, Y, Z, Z^{-1}, S, W, T) = \mathcal{R}'_{\mathcal{G}}(X, Y, Z, W, T), \quad (6)$$

where \mathcal{R} is given by (1).

COROLLARY 1 (Contraction and cut on BR polynomial \mathcal{R}' [2]). *Let $\mathcal{G}(\mathcal{V}, \mathcal{E}, \mathfrak{f}^0)$ be a HERG. Then, for a regular edge e ,*

$$\mathcal{R}'_{\mathcal{G}} = \mathcal{R}'_{\mathcal{G}_{\vee e}} + \mathcal{R}'_{\mathcal{G}/e}, \quad \mathcal{R}'_{\mathcal{G}_{\vee e}} = T^2 \mathcal{R}'_{\mathcal{G}-e}; \quad (7)$$

for a bridge e , we have $\mathcal{R}'_{\mathcal{G}/e} = \mathcal{R}'_{\mathcal{G}-e} = T^{-2} \mathcal{R}'_{\mathcal{G}\vee e}$

$$\mathcal{R}'_{\mathcal{G}} = [(X-1)T^2 + 1] \mathcal{R}'_{\mathcal{G}/e}; \quad (8)$$

for a trivial twisted self-loop, $\mathcal{R}'_{\mathcal{G}-e} = T^{-2} \mathcal{R}'_{\mathcal{G}\vee e}$ and

$$\mathcal{R}'_{\mathcal{G}} = [T^2 + (Y-1)ZW] \mathcal{R}'_{\mathcal{G}-e}, \quad (9)$$

whereas for a trivial untwisted self-loop, we have $\mathcal{R}'_{\mathcal{G}-e} = T^{-2} \mathcal{R}'_{\mathcal{G}\vee e}$ and

$$\mathcal{R}'_{\mathcal{G}} = [T^2 + (Y-1)] \mathcal{R}'_{\mathcal{G}-e}. \quad (10)$$

Graph operations such as the disjoint union and the one-point-joint ($\mathcal{G}_1 \sqcup \mathcal{G}_2$ and $\mathcal{G}_1 \cdot_{v_1, v_2} \mathcal{G}_2$, respectively) extend to HERGs [1]. The product $\mathcal{G}_1 \cdot_{v_1, v_2} \mathcal{G}_2$ at the vertex resulting from merging v_1 and v_2 keeps its usual sense and respects the cyclic order of all edges and HRs on the previous vertices v_1 and v_2 . The following proposition holds.

PROPOSITION 1 (Operations on BR polynomials [1]). *Let \mathcal{G}_1 and \mathcal{G}_2 be two disjoint ribbon graphs with HRs, then*

$$\mathcal{R}_{\mathcal{G}_1 \sqcup \mathcal{G}_2} = \mathcal{R}_{\mathcal{G}_1} \mathcal{R}_{\mathcal{G}_2}, \quad \mathcal{R}'_{\mathcal{G}_1 \sqcup \mathcal{G}_2} = \mathcal{R}'_{\mathcal{G}_1} \mathcal{R}'_{\mathcal{G}_2}, \quad (11)$$

$$\mathcal{R}'_{\mathcal{G}_1 \cdot_{v_1, v_2} \mathcal{G}_2} = \mathcal{R}'_{\mathcal{G}_1} \mathcal{R}'_{\mathcal{G}_2}, \quad (12)$$

for any disjoint vertices $v_{1,2}$ in $\mathcal{G}_{1,2}$, respectively.

It turns out that $\mathcal{R}_{\mathcal{G}_1 \cdot_{v_1, v_2} \mathcal{G}_2} \neq \mathcal{R}_{\mathcal{G}_1} \mathcal{R}_{\mathcal{G}_2}$. After a closer inspection (see in Appendix B), we have investigated the breaking terms by giving more structure to the one-point-joint operation. The breaking of this factorization property will have severe consequence on the formulation of a recipe theorem as we will see in the sequel.

We now introduce a new equivalence relation on HERGs.

DEFINITION 7 (HR move operation). *Let $\mathcal{G}(\mathcal{V}, \mathcal{E}, \mathfrak{f}^0)$ be a ribbon graph with HRs. A HR move in \mathcal{G} consists in removing a HR $f \in \mathfrak{f}^0$ from one-vertex v and placing f either on v or on another vertex such that it is called*

- a HR displacement if the boundary connected component where f belongs is not modified (see \mathcal{G}_1 and \mathcal{G}_2 in Figure 7);
- a HR jump if the HR is moved from one boundary connected component to another one, provided the former remains a connected boundary component (see \mathcal{G}_1 and \mathcal{G}_3 or \mathcal{G}_2 and \mathcal{G}_3 in Figure 7).

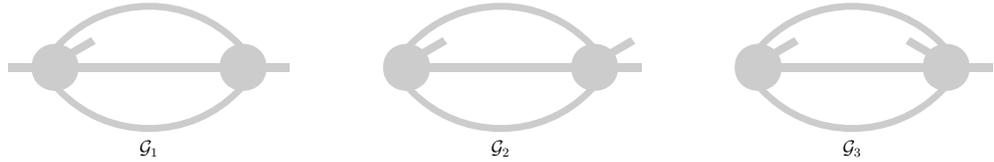


FIGURE 7. Some HR moves.

HR jumps can modify the boundary graph whereas under HR displacements the boundary graph remains unchanged. Nevertheless, the number of connected components of the boundary graph is always preserved under these operations of HR moves.

DEFINITION 8 (HR-equivalence relation). *We say that two ribbon graphs with HRs \mathcal{G} and \mathcal{G}' are HR-equivalent if they are related by a sequence of HR moves.*

We recall the following statements (Lemma 3 and Lemma 4, respectively in [2]):

PROPOSITION 2. *If two ribbon graphs with HRs \mathcal{G} and \mathcal{G}' are HR-equivalent, then for any edge e in \mathcal{G} and \mathcal{G}' , $\mathcal{G} \vee e$ and $\mathcal{G}' \vee e$ are HR-equivalent.*

PROPOSITION 3. For two HR-equivalent ribbon graphs, \mathcal{G} and \mathcal{G}' , $\mathcal{R}(\mathcal{G}) = \mathcal{R}(\mathcal{G}')$ with \mathcal{R} the polynomial defined in (1).

It becomes immediate that the definition of the polynomial \mathcal{R} exports on HR-equivalence classes.

DEFINITION 9 (Polynomial for HR-equivalence classes [2]). Let $\mathcal{G}(\mathcal{V}, \mathcal{E}, \mathfrak{f}^0)$ be a ribbon graph with HRs and $[\mathcal{G}]$ be its HR-equivalence class. We define the polynomial of $[\mathcal{G}]$ to be

$$\mathcal{R}_{[\mathcal{G}]} = \mathcal{R}_{\mathcal{G}}. \quad (13)$$

Thus, for \mathcal{G} a ribbon graph with HRs, $[\mathcal{G}]$ its HR-class and e one of its edges, $\mathcal{R}_{[\mathcal{G} \vee e]} = \mathcal{R}_{\mathcal{G} \vee e}$ and $\mathcal{R}_{[\mathcal{G}/e]} = \mathcal{R}_{\mathcal{G}/e}$. We therefore have the next claim.

THEOREM 2 (Contraction/cut on BR polynomial on classes [2]). Let $\mathcal{G}(\mathcal{V}, \mathcal{E}, \mathfrak{f}^0)$ be a ribbon graph with HRs and $[\mathcal{G}]$ be its HR-equivalence class. Then, for a regular edge e ,

$$\mathcal{R}_{[\mathcal{G}]} = \mathcal{R}_{[\mathcal{G} \vee e]} + \mathcal{R}_{[\mathcal{G}/e]}, \quad (14)$$

for a bridge e , we have

$$\mathcal{R}_{[\mathcal{G}]} = (X - 1)\mathcal{R}_{[\mathcal{G} \vee e]} + \mathcal{R}_{[\mathcal{G}/e]}, \quad (15)$$

for a trivial twisted self-loop e , the following holds

$$\mathcal{R}_{[\mathcal{G}]} = \mathcal{R}_{[\mathcal{G} \vee e]} + (Y - 1)ZW \mathcal{R}_{[\mathcal{G}/e]}, \quad (16)$$

whereas for a trivial untwisted self-loop e , we have

$$\mathcal{R}_{[\mathcal{G}]} = \mathcal{R}_{[\mathcal{G} \vee e]} + (Y - 1)\mathcal{R}_{[\mathcal{G}/e]}. \quad (17)$$

To provide a universality property for \mathcal{R} , let us consider the following expansion of $\mathcal{R}_{\mathcal{G}}$:

$$\mathcal{R}_{\mathcal{G}}(X, Y, Z, S, T, W) = \sum_{i,j,k,l,m} \mathcal{R}_{ijklm}(\mathcal{G})(Y - 1)^i Z^j S^k T^l W^m, \quad (18)$$

$$\mathcal{R}_{ijklm}(\mathcal{G}) := \sum_{A \in \mathcal{G}/n(A)=i, k(A)-F_{\text{int}}(A)+n(A)=j, C_{\partial}(A)=k, f(A)=l, t(A)=m} (X - 1)^{r(\mathcal{G})-r(A)},$$

with \mathcal{R}_{ijklm} a map from the set \mathcal{G}^* of isomorphism classes of connected ribbon graphs with HRs to $\mathbb{Z}[X]$ which extends to \mathfrak{G} the set of HR-equivalence classes of isomorphism classes of connected ribbon graphs with HRs. Then the following statement holds.

THEOREM 3 (Universality of \mathcal{R} on classes [2]). Let \mathfrak{A} be a commutative ring and $x \in \mathfrak{A}$. If a function $\phi : \mathfrak{G} \rightarrow \mathfrak{A}$ satisfies

$$\phi([\mathcal{G}]) = \begin{cases} \phi([\mathcal{G} \vee e]) + \phi([\mathcal{G}/e]) & \text{if } e \text{ is regular,} \\ (x - 1)\phi([\mathcal{G} \vee e]) + \phi([\mathcal{G}/e]) & \text{if } e \text{ is a bridge.} \end{cases} \quad (19)$$

Then there are coefficients $\lambda_{ijklm} \in \mathfrak{A}$, with $i \geq 0$, $0 \leq k \leq i + 1$, $l \geq 0$, $0 \leq m \leq 1$ and $0 \leq j \leq i + 1$ such that

$$\phi([\mathcal{G}]) = \sum_{i,j,k,l,m} \lambda_{ijklm} \mathcal{R}_{ijklm}(x). \quad (20)$$

Chord diagrams. So far, all proofs of universality theorems for topological polynomials on ribbon graph rest on the “projection” of one-vertex ribbon graphs onto the so-called chord-diagrams [3, 10, 2]. These diagrams will play a crucial role in the proof of the following recipe theorems as well. In a way useful to the present context, such a notion of chord diagrams and its main properties must be recalled.

DEFINITION 10 (Chord diagrams [2]). • A *half-chord* on a chord diagram is a segment attached to a unique point on its circle.

• An (open) chord diagram is a chord diagram in sense of [3] with a (nonempty) set of half-chords. In the case where this set is empty, it becomes BR chord diagram.

• A signed (open) chord diagram is an (open) chord diagram with an assignment of a sign “t” or not to each chord.

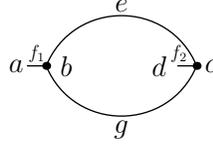


FIGURE 8. Two-vertex ribbon graph with HRs.

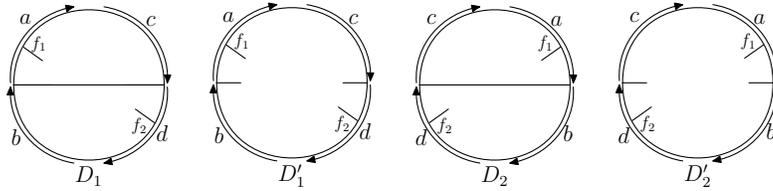


FIGURE 9. Related chords diagrams D_1, D'_1, D_2, D'_2 .

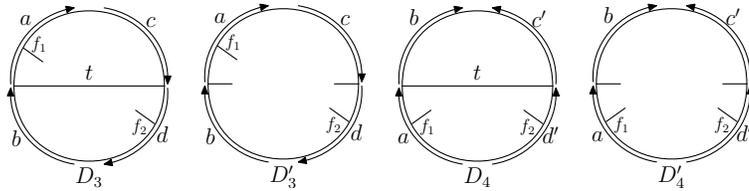


FIGURE 10. Related chords diagrams D_3, D'_3, D_4, D'_4 .

Given a one-vertex HERG, we assign twisted ribbon edges to chords coined by “t” in the corresponding chord diagram.

Consider a two-vertex (embedded) half-edged graph as given in Figure 8. Contracting either e or g yields a one-vertex graph which can be mapped onto a chord diagram. Note that e or g might correspond to twisted edges. The different possibilities are summarized by Figures 9 (D_1 and D_2 if both e and g are not twisted) and 10 (D_3 and D_4 if one of e or g are twisted). Then from each of these configurations, one cuts the chord coming from the edge about which we rotate or twist (this gives the configurations denoted by $D'_{1,2,3,4}$). Detailed explanations can be found in the companion paper [2].

Two signed (open) chord diagrams are said to be related by a rotation about the chord e if they are related as D_1 and D_2 in Figure 9, and related by a twist about e , if they are related as D_3 and D_4 in Figure 10. We now give the definitions of R -equivalent diagrams and the sum of two chord diagrams.

DEFINITION 11 (R -equivalence relation [3]). Two diagrams or signed diagrams D_1 and D_2 are R -equivalent if and only if they are related by a sequence of rotations and twists. We write $D_1 \sim D_2$.

DEFINITION 12 (Sum of diagrams [3]). *The sum of two diagrams or signed diagrams D_1 and D_2 is obtained by choosing a point p_i (not the end-point of a chord or a half-chord) on the boundary of each D_i , joining the boundary circles at these points and then deforming the result until it is again a circle.*

The sum of diagrams can be performed in several ways, if the p_i are chosen differently. However, all of them are R -equivalent. The proof of the following statement can be found in [2] (Lemma 1) which is an adjustment of a similar lemma in [3].

LEMMA 1. *If two diagrams D and D' are both sums of diagrams D_1 and D_2 , then they are R -equivalent.*

The next important notion is that of canonical chord diagrams. Given $i \geq 0$, $0 \leq 2j \leq i$, $0 \leq k \leq i+1$, $l \geq 0$ and $0 \leq m \leq 2$, let $D_{i,j,k,(s;l_1,\dots,l_q),m}$ be the chord diagram consisting of i chords, j pairs of chords intersecting each other, k connected components of the boundary of this diagram, l half-chords ($l = s + \sum_{p=1}^q l_p$) specifically arranged and m negative chords (or twisted chords) intersecting no other chords ($i - 2j - m$ is the number of positive chords intersecting no other chords). This diagram is drawn such that a number $l - s$ of half-chords is partitioned in $(l_p)_{p=1,\dots,q}$ positive chords intersecting no other chords (we call these isolated chords) and s is the rest of the half-chords. All these chords and half-chords are arranged on the circle diagram (see an illustration for $D_{4,1,2,(3;1),1}$ and $D_{5,1,2,(0;1,2),1}$):

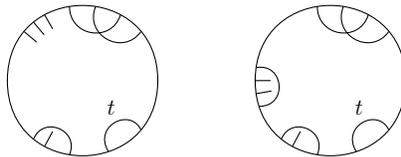


FIGURE 11. Canonical diagrams: $D_{4,1,2,(3;1),1}$ and $D_{5,1,2,(0;1,2),1}$.

It becomes straightforward to perform a quotient on the space of chord diagrams with respect to the HR-equivalence relation. One gets HR-equivalent (canonical) chord diagrams. At this stage, the partition of l half-chords becomes superfluous to emphasize. One can dispose the l flags on the chord diagram as desired and must only respect the number of connected components of the boundary. It can be proved that Lemma 1 still holds in that context.

3. Recipe theorems

This section discusses recipe theorems for the BR polynomial on classes of HR-equivalent ribbon graphs. Then, only in this section, we call HERG or ribbon graph with half-edges (or with HR), a class of HR-equivalent class. The same remark holds for HR-classes of (canonical) chord diagrams which are simply referred to (canonical) chord diagrams. For simplicity, we drop all brackets hence, from now on, $[\mathcal{G}]$ will be denoted simply \mathcal{G} , and D_{ijklm} stands for the HR-equivalent class canonical diagram $[D_{i,j,k,[l],m}]$.

Any (open) diagram is related to a canonical signed chord diagram D_{ijklm} ($0 \leq m \leq 2$) that consists of $i - 2j - k - m$ positive disjoint chords, j pairs of intersecting positive chords, k connected components of the boundary graph, m negative disjoint chords and l half-chords. In this case, we observe that:

$$\begin{aligned} \mathcal{R}(D_{10000}; x, y, z, s, t, w) &= (y - 1) + zst^2, \\ \mathcal{R}(D_{10001}; x, y, z, s, t, w) &= (y - 1)zw + zst^2, \\ \mathcal{R}(D_{21000}; x, y, z, s, t, w) &= (y - 1)^2z^2 + 2(y - 1)(zst)^2 + zst^4, \\ \mathcal{R}(D_{00110}; x, y, z, s, t, w) &= zst^l. \end{aligned} \tag{21}$$

The polynomial \mathcal{R} is not multiplicative with respect to the one-point-join of ribbon graphs with half-edges (see Appendix B for a complete development on this issue). In order to find a

recipe, this factorization property must be replaced by other useful relations. We have identified weaker conditions that \mathcal{R} satisfies which will ensure the existence of a recipe in these cases.

In the following, we denote by $M(A)$ the monomial associated with $A \in \mathcal{G}$ which we recall

$$M(A)(x, y, z, s, t, w) = (x-1)^{r(\mathcal{G})-r(A)}(y-1)^{n(A)}z^{k(A)-F_{\text{int}}(A)+n(A)}s^{C_{\partial}(A)}w^{t(A)}t^{f(A)}. \quad (22)$$

Let γ be the function defined on the set $\{0, 1, 2\}$ by:

$$\begin{cases} \gamma(0) = 0, \\ \gamma(1) = \gamma(2) = 1. \end{cases} \quad (23)$$

LEMMA 2. *Consider a canonical chord diagram with half-chords D_{ijklm} . Then, the following decomposition is valid:*

$$M(D_{ijklm})(x, y, z, s, t, w) = (M(D_{10000}))^{i-2j-m}(M(D_{00110}))^k(M(D_{10001}))^m \times (M(D_{21000}))^j(x, y, z, s, t^{\frac{1}{k}}, w). \quad (24)$$

PROOF. We have $M(D_{ijklm})(x, y, z, s, t, w) = (y-1)^i z^{1-F_{\text{int}}(D_{ijklm})+i} s^k t^l w^{\gamma(m)}$ with γ defined as in (23) and $F_{\text{int}}(D_{ijklm}) = i - 2j - m - k + 1$. Then, $M(D_{ijklm}) = (y-1)^i z^{2j+m+k} s^k t^l w^{\gamma(m)}$. Furthermore, evaluating $M(D_{10000})(x, y, z, s, t^{\frac{1}{k}}, w) = y - 1$, $M(D_{00110})(x, y, z, s, t^{\frac{1}{k}}, w) = z s t^{\frac{1}{k}}$, $M(D_{10001})(x, y, z, s, t^{\frac{1}{k}}, w) = (y-1)z w$ and $M(D_{21000})(x, y, z, s, t^{\frac{1}{k}}, w) = (y-1)^2 z^2$. This ends the proof of (24). \square

From the definition of \mathcal{R} ,

$$\mathcal{R}(D_{ijklm})(x, y, z, s, t, w) = \sum_{D_{i'j'k'l'm'} \in D_{ijklm}} M(D_{i'j'k'l'm'})(x, y, z, s, t, w), \quad (25)$$

because any subgraph of D_{ijklm} is of the form $D_{i'j'k'l'm'}$ with $i' \leq i$. We apply (25) in order to find the following relations:

$$\begin{aligned} \mathcal{R}(D_{10000}) &= M(D_{10000}) + M(D_{00120}), \\ \mathcal{R}(D_{21000}) &= M(D_{21000}) + 2M(D_{10000})(M(D_{00110}))^2 + M(D_{00140}), \\ \mathcal{R}(D_{10001}) &= M(D_{10001}) + M(D_{00120}), \\ \mathcal{R}(D_{00110}) &= M(D_{00110}). \end{aligned} \quad (26)$$

These relations can be inverted to find

$$\begin{aligned} M(D_{10000}) &= \mathcal{R}(D_{10000}) - \mathcal{R}(D_{00120}), \\ M(D_{10001}) &= \mathcal{R}(D_{10001}) - \mathcal{R}(D_{00120}), \\ M(D_{21000}) &= \mathcal{R}(D_{21000}) + 2(\mathcal{R}(D_{10000}) - \mathcal{R}(D_{00120}))(\mathcal{R}(D_{00110}))^2 + \mathcal{R}(D_{00140}), \\ M(D_{00110}) &= \mathcal{R}(D_{00110}). \end{aligned} \quad (27)$$

We immediately discover that, from Lemma 2, the existence of such relations (26) allows us to write in return a generic monomial $M(D_{ijklm})$ in terms of $\mathcal{R}(D_{i'j'k'l'm'})$, for i', j', k', l' and m' well chosen. This means that the system (25) can be inverted. In general $M(D_{ijklm})$ is a non linear function of $\mathcal{R}(D_{ijklm})$. For the simplest chord diagrams (27), we observe that the relation is still linear.

Replacing the relations (26) in (24) and in (25), we obtain $\mathcal{R}(D_{ijklm})$ as function of $\mathcal{R}(D_{10000})$, $\mathcal{R}(D_{10001})$, $\mathcal{R}(D_{21000})$ and $\mathcal{R}(D_{00110})$.

Let \mathcal{M} be a minor closed subset of HERGs containing all HERGs on two vertices (this notion extends [12] to HERGs). Let F be a map from \mathcal{M} to a commutative ring \mathfrak{R} with unity. Suppose that F satisfies the relation:

$$\begin{aligned} F(D_{ijklm}) &= \sum_{D_{i'j'k'l'm'} \in D_{ijklm}} \alpha^{i'-j'+m'+\frac{l'}{k'}} M(D_{i'j'k'l'm'}) \\ &= \sum_{i' \leq i} (M(D_{10000}))^{i'-2j'-m'} (M(D_{00110}))^{k'} (M(D_{10001}))^{m'} (M(D_{21000}))^j(x, y, z, s, t^{\frac{1}{k'}}, w), \end{aligned} \quad (28)$$

with the conditions:

$$\begin{aligned}
 M(D_{10000}) &= \alpha^{-1}[F(D_{10000}) - F(D_{00120})], \\
 M(D_{10001}) &= \alpha^{-1}[F(D_{10001}) - F(D_{00120})], \\
 M(D_{21000}) &= \alpha^{-1}[F(D_{21000}) + (F(D_{10000}) - F(D_{00120}))(F(D_{00110}))^2 + F(D_{00140})], \\
 M(D_{00110}) &= \alpha^{-1}F(D_{00110}).
 \end{aligned} \tag{29}$$

From (29), we are guaranteed that once again the monomials $M(D_{ijklm})$ can be written in terms of $F(D_{i'j'k'l'm'})$, for i', j', k', l' and m' well chosen. Under conditions (28) and (29), the recipe theorem implies that F is strongly related to \mathcal{R} .

THEOREM 4 (First recipe theorem for \mathcal{R}). *Let \mathcal{M} be a minor closed subset of HERGs containing all HERGs on two vertices. Let F be a map from \mathcal{M} to a commutative ring \mathcal{R} with unity. Let $s = F(D_{21000})$, $q = F(D_{10000})$, $r = F(D_{10001})$ and $s_l = F(D_{00110})$ and suppose there are elements $\alpha, x, u, v, w, o \in \mathfrak{R}$ with α a unit such that:*

(1)

$$F(G) = \begin{cases} F(G \vee e) + F(G/e) & \text{if } e \text{ is regular,} \\ (x-1)F(G \vee e) + F(G/e) & \text{if } e \text{ is a bridge.} \end{cases} \tag{30}$$

(2) $F(G \sqcup H) = F(G)F(H)$ where G and H are embedded bouquets with half-edges and F satisfies also (28) and (29);

(3) $F(E) = \alpha^n$ if E is an edgeless graph with n vertices without HRs;

(4) $(q - s_2)^2 u^2 = \alpha[s - 2\alpha^{-2}(q - s_2)s_1^2 - s_4]$, and $(q - s_2)uw = r - s_2$, and $s_l = \alpha(uvo^l)$, and also $w = w^2$. Then

$$F(G) = \alpha^{k(G)} \mathcal{R}(G; x, \alpha^{-1}(q - s_2 + \alpha), u, v, w, o), \tag{31}$$

where $k(G)$ is the number of components of G .

PROOF. The proof of the recipe theorem follows the lines of the universality proof, because as previously noticed, it is a very similar statement. Thus, we will proceed by two inductions, on the number of chords in chord diagrams representing an embedded bouquet graph and then on the number of non-loop edges in a general graph.

From item (1), we obtain

$$F(D_1) - F(D'_1) = F(D_2) - F(D'_2), \text{ and} \tag{32}$$

$$F(D_3) - F(D'_3) = F(D_4) - F(D'_4), \tag{33}$$

where D_1, D_2, D_3 and D_4 are related as in Figures 9 and 10 and $D'_i = D_i \vee e$.

Since the polynomial \mathcal{R} satisfies (32) and (33), $F' = F - \alpha\mathcal{R}$ also satisfies the same equations. Using items (2), (3) and (4) of Theorem 4, the relation (31) holds in the case of chord diagrams with 0 chord.

Assume by induction that $F(D) = \alpha\mathcal{R}(D)$ for any signed chord diagram with fewer than n chords. In this case F' vanishes on signed chord diagrams with fewer than n chords. Using (32) and (33), we have $F'(D) = F'(D_{nijklm})$ where D is related to a canonical diagram D_{nijklm} . Now from (28) and (29) in item (2), we deduce:

$$F(D_{nijklm}) = \alpha\mathcal{R}(D_{nijklm}; x, \alpha^{-1}(q - s_2 + \alpha), u, v, w, o) \tag{34}$$

and

$$F'(D) = F'(D_{nijklm}) = 0. \tag{35}$$

By induction, $F'(D) = 0$ on any signed chord diagram D . Finally the result holds on any rosette ribbon graph. From item (1), the result becomes true on any ribbon graph. \square

The fact that \mathcal{M} contains all ribbon graphs on two vertices is not important in this proof. The important thing is to require that F satisfies (32) and (33) for a chain of rosettes ending in a canonical chord diagram. This gives the so-called ‘‘low fat’’ recipe theorem [10]. Thus, we have

also a low fat recipe for \mathcal{R} . First, we must introduce another minor set. A minor closed set \mathcal{M} of HERGs is said closed under chord operations whenever $D \in \mathcal{M}$ and $D \sim D_{ijklm}$, then there is a finite sequence $D = D_1 \dots D_n = D_{ijklm}$ with $D_i \in \mathcal{M}$ and $D \sim D_{i+1}$ for all i .

THEOREM 5 (The “low fat” recipe theorem for \mathcal{R}). *Theorem 4 holds with “Let \mathcal{M} be a minor closed subset of HERGs containing all HERGs on two vertices and let F map \mathcal{M} to a commutative ring \mathfrak{R} with unity”, replaced by “Let \mathcal{M} be a minor closed subset of HERGs closed under chord operations that contains D_{10000} , D_{10001} , D_{21000} , D_{00110} and let F map \mathcal{M} to a commutative ring \mathfrak{R} with unity, such that F satisfies (32) and (33) whenever the D_i 's are related as in Figures 9 and 10.”*

We also have:

COROLLARY 2. *If F , \mathcal{M} , \mathfrak{R} satisfy conditions of Theorem 5, with both $q - s_2$ and $r - s_2$ being unit of \mathfrak{R} , then $w = 1$, and thus F does not discern orientation by the presence or absence of a single idempotent element.*

PROOF. From the second equation in item (4), we obtain $\frac{r-s_2}{q-s_2} = uw = uw^2 = w \frac{r-s_2}{q-s_2}$. Since $q - s_2$ and $r - s_2$ are units then we infer $w = 1$. Then $F(G) = \alpha^{k(G)} \mathcal{R}(G; x, \alpha^{-1}(q - s_2 + \alpha), u, v, 1, o)$. \square

There exists another way to introduce a recipe that we now detail. Instead of imposing that F satisfies (28) and (29), we can impose another type of condition. This will give rise to another recipe. To be precise the way to evaluate the function is the same, but the full set conditions that must satisfy F is now different. In a very interesting way, the following analysis can be also applied to the BR polynomial R on ribbon graphs (without HRs) and we can recover a recipe theorem without mentioning a factorization property of the function F with respect to the one-point-joint operation. We first need a series of preliminary results and will introduce a specific terminology now.

A “genus loop” of a HR-equivalent ribbon graph \mathcal{G} is defined by exactly two untwisted loops e and e' crossing each other on a vertex and which do not cross any other loops at that vertex (see Figure 12).



FIGURE 12. A genus loop.

LEMMA 3 (Genus loop evaluation). *Let \mathcal{G} be an one-vertex ribbon graph having a genus loop defined by e and e' . Then, the following recurrence relation is obeyed*

$$\mathcal{R}(\mathcal{G}) = \mathcal{R}(\mathcal{G} \vee e) + ((Y - 1)Z)^2 \mathcal{R}(\mathcal{G}/e/e') + (Y - 1) \mathcal{R}((\mathcal{G}/e) \vee e'). \quad (36)$$

PROOF. The proof is quite reminiscent of the ordinary contraction/cut rule with however modifications that we will emphasize. From the ordinary partition of the set of spanning subgraphs of \mathcal{G} between cutting subgraphs containing e and those which do not, it is clear that the sum of monomials of subgraphs which do not contain e are directly mapped to $\mathcal{R}(\mathcal{G} \vee e)$. Now we focus on the rest of subgraphs which contain e and further partition them into those containing e' and those which do not.

A subgraph A , such that $e, e' \in A$, must be mapped onto $A/e/e' \in \mathcal{G}/e/e'$, by contracting successively e and e' . To compare the monomials $M(A)$ and $M(A/e/e')$, we need the following relations (all the rest of the combinatoric numbers are constants)

$$n(A/e/e') = n(A/e) = n(A) - 2. \quad (37)$$

This gives a relation between the two monomials as $M(A) = ((Y - 1)Z)^2 M(A/e/e')$.

On the other hand, to a subgraph A such that $e \in A$ and $e' \notin A$, we assign $(A/e) \vee e'$. The monomials $M(A)$ and $M((A/e) \vee e')$ can be compared as well. We have the basic relations:

$$\begin{aligned} n((A/e) \vee e') &= n(A/e) = n(A) - 1, \\ r((\mathcal{G}/e) \vee e') - r((A/e) \vee e') &= 0 = r(\mathcal{G}/e) - r(A/e) - 1, \\ r(\mathcal{G}/e) - r(A/e) &= r(\mathcal{G}) - r(A) + 1, \\ k((A/e) \vee e') &= k(A/e) = k(A) + 1. \end{aligned} \quad (38)$$

Therefore, $M(A) = (X - 1)^{-1}(Y - 1)M(A/e) = (X - 1)^{-1}(Y - 1)(X - 1)M((A/e) \vee e') = (Y - 1)M((A/e) \vee e')$. Summing over all contributions gives the result. \square

The following statement holds.

THEOREM 6 (Second recipe theorem for \mathcal{R}). *Let \mathcal{M} be a minor closed subset of HERGs containing all HERGs on two vertices. Let F be a map from \mathcal{M} to a commutative ring \mathfrak{R} with unity. Let $s = F(D_{21000})$, $q = F(D_{10000})$, $r = F(D_{10001})$ and $s_1 = F(D_{00110})$ and suppose there are elements $\alpha, x, y, z, u, v, w, o \in \mathfrak{R}$ with α a unit such that:*

(1)

$$F(G) = \begin{cases} F(G \vee e) + F(G/e) & \text{if } e \text{ is regular,} \\ (x - 1)F(G \vee e) + F(G/e) & \text{if } e \text{ is a bridge,} \\ F(G \vee e) + \alpha^{-1}(y - 1)F(G/e) & \text{if } e \text{ is a trivial untwisted loop,} \\ F(G \vee e) + (y - 1)zwF(G/e) & \text{if } e \text{ is a trivial twisted loop,} \\ F(\mathcal{G} \vee e) + ((y - 1)z)^2F(\mathcal{G}/e/e') \\ \quad + \alpha^{-1}(y - 1)F((\mathcal{G}/e) \vee e') & \text{if } (e, e') \text{ defines a genus loop,} \end{cases} \quad (39)$$

(2) $F(G \sqcup H) = F(G)F(H)$ where G and H are embedded bouquets with half-edges.

(3) $F(E) = \alpha^n$ if E is an edgeless graph with n vertices without HRs;

(4) $(q - s_2)^2u^2 = \alpha[s - 2\alpha^{-2}(q - s_2)s_1^2 - s_4]$, and $(q - s_2)uw = r - s_2$, and $s_1 = \alpha(uv^l)$, and also $w = w^2$. Then

$$F(G) = \alpha^{k(G)}\mathcal{R}(G; x, \alpha^{-1}(q - s_2 + \alpha), u, v, w, o), \quad (40)$$

where $k(G)$ is the number of components of G .

Before proving this theorem we shall need a particular relation satisfied by F on canonical chord diagrams. Here we will denote a canonical chord diagram $D_{i,j,k,l,m}$ because we will perform operations on the indices.

LEMMA 4. *For all i, j, k, l and m for an arbitrary chord diagram $D_{i,j,k,l,m}$, and let F a function on HERGs satisfying the condition (1)-(3) and $F(D_{0,0,1,l,0}) = \alpha\mathcal{R}(D_{0,0,1,l,0})$ then we have*

$$F(D_{i,j,k,l,m}) = \alpha\mathcal{R}(D_{i,j,k,l,m}). \quad (41)$$

PROOF. In order to prove our claim, we will use an algorithm which will reduce the number of chords and the complexity of the chord diagrams. From simpler cases, we will be able to conclude.

If $m > 0$, there exists a negative chord e and we have

$$\begin{aligned} F(D_{i,j,k,l,m}) &= F(D_{i,j,k,l,m} \vee e) + (y - 1)zwF(D_{i,j,k,l,m}/e) \\ &= F(D_{i-1,j,k+\varepsilon,l+2,m-1}) + (y - 1)zwF(D_{i-1,j,k,l,m-1}) \end{aligned} \quad (42)$$

where $\varepsilon = 0, 1$ depends on the type of chord diagram. The property (42) reduces the number of negative chords up to 0 (we can use it twice if $m = 2$). We note also that the total number of chords i decreases. After the procedure, one gets a sum of (2 or 4) terms involving new $D_{i',j',k',l',0}$. Note that a similar relation holds exactly for \mathcal{R} . We can therefore concentrate on the evaluation of such canonical diagram $D_{i,j,k,l,0}$.

(A) For chords which do not intercept any other chords and which encloses ℓ HRs, we use the relation in (39) corresponding to trivial untwisted loops. The number of these chords is $i - 2j$. We will contract/cut these chords until none will be left. Considering that the loop e has ℓ (possibly

0) HRs, it can generate $\epsilon = 0$ or 1 connected component of the boundary. For such a chord e enclosing $\ell \geq 0$ chords, we write

$$F(D_{i,j,k,l,0}) = F(D_{i,j,k,l,0} \vee e) + \alpha^{-1}(y-1)F(D_{i,j,k,l,0}/e). \quad (43)$$

Note that the first term $F(D_{i,j,k,l,0} \vee e) = F(D_{i-1,j,k+\varepsilon,l+2,0})$, $\varepsilon = \pm 1, 0$, does not cause any trouble: the number of chords decreases and we will perform again (A). The possible terms generated are of the same form as for the initial $D_{i,j,k,l,0}$ and the following discussion can be applied for $F(D_{i-1,j,k+\varepsilon,l+2,0})$. We then concentrate on the second term. Using in particular (3), and the function $\epsilon(\ell) = 0$, if $\ell = 0$, and $\epsilon(\ell) = 1$, if $\ell > 0$, we write this term as

$$\begin{aligned} \alpha^{-1}(y-1)F(D_{i,j,k,l,0}/e) &= \alpha^{-1}(y-1)F(D_{i-1,j,k-\epsilon(\ell),l-\ell,0})F(D_{0,0,\epsilon(\ell),\ell,0}) \\ &= (y-1)F(D_{i-1,j,k-\epsilon(\ell),l-\ell,0})\mathcal{R}(D_{0,0,\epsilon(\ell),\ell,0}). \end{aligned} \quad (44)$$

The last simplification occurs by definition $F(D_{0,0,1,l,0}) = \alpha\mathcal{R}(D_{0,0,1,l,0})$, $\forall l \geq 0$. Thus the evaluation of F on $D_{i,j,k,l,0}$ with chords enclosing possibly HRs revolves into a sum of terms involving the evaluation of F on $D_{i',j',k',l',0}$ with fewer number of chords times some power of $(y-1)$ and then products of terms involving \mathcal{R} . Evaluating \mathcal{R} itself on the same $D_{i,j,k,l,0}$ will give a similar expression apart from the evaluation of \mathcal{R} on $D_{i',j',k',l',0}$. Then apply again (A).

(B) The last type of chords are genus loops. We adopt the same strategy as above. The argument is lengthier but we can also prove that, using the particular relation adapted to these genus loops in (39) (or Lemma 3), we can recast the evaluation of $F(D_{2j,j,k,l,0})$ a sum of terms of simpler diagrams. We will simply give here the list of arguments which will allow one to achieve the proof.

- Consider (e, e') a genus loop. Use the double recurrence relation and find that the evaluation of F on $D_{2j,j,k,l,0}$ involves the evaluation of F onto

(a) $D_{2j,j,k,l,0} \vee e$ is of the form $D_{2j-1,j-1,k+\varepsilon,l+2,0}$ possesses a untwisted loop. One must reduce this untwisted loop using (A) and find a sum of terms where F evaluates as a sum of simpler terms on a unique canonical diagram $D_{i',j',k',l',0}$.

(b) $D_{2j,j,k,l,0}/e/e'$ which is of the form $D_{2j-2,j-1,k,l,0}$ and we go back to the procedure (B).

(c) $(D_{2j,j,k,l,0}/e) \vee e'$ which is of the form $D_{2j-2,j-1,k+\varepsilon,l+1,0} \sqcup D_{0,0,1,1,0}$, then we use the fact that $\alpha^{-1}F(D_{0,0,1,1,0}) = \mathcal{R}(D_{0,0,1,1,0})$, to recast the evaluation of F on this sector to a single chord diagram $D_{2j-2,j-1,k+\varepsilon,l+1,0}$, and go back to (B).

At the end of the algorithm, $F(D_{i,j,k,l,m})$ is a sum of terms which involves products of the form $\alpha^{-1}f(y-1, z, w)F(D_{0,0,\varepsilon(l'),l',0}) \prod_{\kappa \in K} \mathcal{R}(D_{\kappa})$, where $l' \geq 0$, $f(y-1, z, w)$ is a polynomial function which is a product of different contributions of special edges which have been contracted or cut and K is a family of chord diagrams of the form $D_{\kappa} = D_{0,0,\varepsilon(l),l,0}$. If we apply the same procedure to $\mathcal{R}(D_{i,j,k,l,m})$ each term in this expansion will be of the form $f(y-1, z, w)\mathcal{R}(D_{0,0,\varepsilon(l'),l',0}) \prod_{\kappa \in K} \mathcal{R}(D_{\kappa})$. We conclude that $F(D_{i,j,k,l,m}) = \alpha\mathcal{R}(D_{i,j,k,l,m})$, since one has $F(D_{0,0,\varepsilon(l),l,0}) = \alpha\mathcal{R}(D_{0,0,\varepsilon(l),l,0})$, for all $l \geq 0$. □

The proof of our theorem can be now completed.

PROOF OF THE THEOREM 6. Using Lemma 4, the relation (40) holds in the case of canonical chord diagrams with half-chords.

Using the two first equations in item (1), F satisfies (32) and (33) for any chord diagrams D_1 , D_2 , D_3 and D_4 related as in Figures 9 and 10 and $D'_i = D_i \vee e$. Hence $F' = F - \alpha\mathcal{R}$ also satisfies the same equations since \mathcal{R} satisfies them.

We prove that this theorem holds on any chord diagram using step by step the proof in Theorem 4 and replacing the relations (28) and (29) by (41). Using the two first equations in item (1), the result becomes true on any ribbon graph. □

It is natural to find the restricted polynomial \mathcal{R}' (6) over classes of HR-equivalent pinched ribbon graphs and to show for \mathcal{R}' corresponding recipe theorems (see Appendix A).

4. Topological transition polynomial

We focus now on another interesting extension of the so-called transition polynomial for embedded graphs [10, 8] to HERGs. Note that, in this section, we will introduce this transition polynomial at the level of HERGs and not at the level of their HR-classes. Only at the end, the extension to classes will be quickly discussed.

4.1. Vertex and HEGs states. Consider a 4-regular graph with half-edges and v one of its vertices. A vertex state at a vertex v is a partition (with at most two elements in any subset of this partition) of the half-edges incident to v . Here, the vertex state is an extension of that given in [10] for graphs without half-edges since, in that case, the partition is strictly restricted into pairs of half-edges incident to v . A subset in any such partition containing only one element will correspond to a half-edge which can be linked or not to another half-edge in some other vertex state. Thus here, a vertex state here consists still in a set of disjoint curves. For a given vertex in a HEG, we obtain the usual three (white smoothing, black smoothing and crossing) states obtained when the graph is without half-edges [10] but also more states. These latter are obtained by cutting one or the two arcs given in the first three states considered. There is one particular state which will turn out to be important in the next analysis. We call it the cutting state and it is obtained by cutting the two arcs of a black (or white) smoothing state (see Figure 13). The states that will retain our attention are the black, white smoothing and cutting states. The rest of the states (even the crossing state) will not be used in the following. It might be however interesting to incorporate those states in a more general study.



FIGURE 13. Vertex states of a vertex v (most left) in a HEG: (in that order) the black, white and cutting states of v .

For an abstract graph, the black and white smoothing (together with crossing) states cannot be distinguished. However, for an embedded graph, choosing an appropriate neighborhood of the half-edges in the partitioned vertex leads to different configurations.

Consider a 4-regular graph F with half-edges, and choose a particular configuration of vertex states at each of its vertices. Then one calls this resulting graph configuration a graph state S of F . We denote $c(S)$ the number of components of the graph state S which are closed and $c'(S)$ the number of those which are open. Importantly, in the following, the cutting state will replace the black smoothing state. As a mapping between these two states, we can insert fictitious vertices of degree 2 between particular pairs of half-edges coming from the cutting state, in such a way that these pairs mimic the pairs of half-edges of the black smoothing. Now, $c''(S)$ counts the number of closed curves having at least one fictitious vertex of degree 2. We will call these punctured curves. Note that, removing all the 2-valent vertices from punctured curves, one must end up with the open curves (in other words, open curves issued from the cutting state can be always glued together by 2-valent vertices and must form punctured curves).

Let F be a 4-regular graph with half-edges. A weight system $W(F)$ of F is an assignment of weight valued in a unitary ring \mathfrak{R} to every vertex state of F . The state weight of a graph state S of F with weight system W is $w(S) = \prod_v w(v, S)$ where $w(v, S) \in \mathfrak{R}$ is the vertex state weight of the vertex state at v in the graph state S .

DEFINITION 13. *Let F be a 4-regular graph with half-edges having weight system W with values in the unitary ring \mathfrak{R} . Then the state model formulation of the generalized transition polynomial is given by*

$$q(F; W, a, b, d) = \sum_S w(S) a^{c(S)} b^{c'(S)} d^{c''(S)}, \tag{45}$$

where the sum is over all graph states S of F .

As an easily checked property, one proves that q is multiplicative for the disjoint union operations on 4-regular graphs.

We now define the notion of medial graph with half-edges. If G is a HEG and it is cellularly embedded, its medial graph G_m is constructed by

- placing a vertex of degree 4 on each edge of G and placing a “fictitious” vertex of degree 2 at the end point of each half-edge
- and drawing the edges of the medial graph by following the face boundaries of G .
- then remove all vertices of degree 2.

The insertion of vertices of degree 2 allows to keep track of the open faces and new types of closed components formed by these open faces. By convention, the medial graph of an isolated vertex is an isolated closed face. In the case of n half-edges attached to this isolated vertex, the medial graph is a collection of open faces between with n 2-valent vertices.

A checkerboard coloring of a cellularly embedded HEG is an assignment of the color black or white to each face such that adjacent faces receive different colors. We can color the faces of the medial graph with half-edges containing a vertex of the original graph G black and the remaining white. We keep calling this, the canonical checkerboard coloring of G_m . In Figure 14, we illustrate this construction.

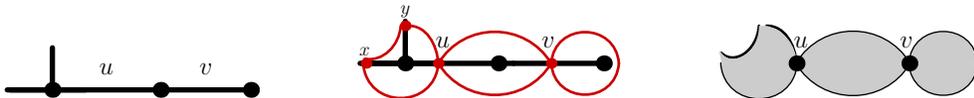


FIGURE 14. A HEG (left), its medial graph construction in red (middle) with insertion of 2-valent vertices x and y which should be removed at the end; its canonical checkerboard coloring (right).

Let G_m be a canonically checkerboard colored medial graph and v one of its vertices. $(G_m)_{wh(v)}$ is the embedded graph with half-edges resulting from taking the white smoothing at the vertex v of G_m , $(G_m)_{bl(v)}$ the embedded graph with half-edges obtained from taking the black smoothing at the vertex v , and $(G_m)_{cut(v)}$ the embedded graphs with half-edges resulting from taking the cutting state at the vertex v . More embedded graphs from other states at v could be introduced but they will be not important for the rest of the present work. Precisely, there are parameters assigned to the other states but, in our next developments, these parameters are set at value 0.

The following proposition is immediate at this stage:

PROPOSITION 4. *Let G be an embedded graph (with half-edges) with embedded, canonically checkerboard colored medial graph G_m , and let e be any edge of G , with v_e the associated vertex in G_m . Then*

- (1) $(G_m)_{bl(v_e)} = (G - e)_m$,
- (2) $(G_m)_{wh(v_e)} = (G/e)_m$,
- (3) $(G_m)_{cut(v_e)} = (G \vee e)_m$.

Consider an embedded graph G with half-edges and its medial graph G_m . A vertex $v \in G_m$ has some state weight given by ordered elements (α, η) , specifying the weight of the white smoothing state (referred as ‘uncut’ vertex state), and cutting state, in that order.

4.2. Topological transition polynomial for graphs with half-edges. We have all prerequisites to introduce a new transition polynomial.

DEFINITION 14. *Let G be an embedded graph with half-edges with canonically checkerboard colored embedded medial graph G_m , and let $W_m(G_m) = (\alpha, \eta)$ be a weight system associated with G_m . Then the topological transition polynomial for G expresses as*

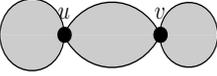
$$Q(G, (\alpha, \eta), a, b, d) := q(G_m; W_m, a, b, d). \quad (46)$$

The following proposition is straightforward.

PROPOSITION 5. *The topological transition polynomial may be computed by repeatedly applying the following linear recursion relation at each vertex $v \in \mathcal{V}(G_m)$, and, when there are no more vertices, evaluating each of the resulting closed curves to an independent variable a , the resulting open curves to an independent variable b , and the resulting punctured curves to an independent variable d :*

$$q(G_m, W_m, a, b, d) = \alpha_v q((G_m)_{wh(v)}, W_m, a, b, d) + \eta_v q((G_m)_{cut(v)}, W_m, a, b, d). \quad (47)$$

Note that, in practice during the calculation of this recurrence rule and in order to be not confused by open curves which might seem not be on the same connected component, it is better to restore the fictitious 2-valent vertices. Applying repeatedly (47) when all the vertices of valence 4 have been decomposed, one removes then from the result all 2-valent vertices and count the number of open and closed curves. We can provide an example as follows: Consider the graph

$\mathcal{G} = \bullet \xrightarrow{u} \bullet \xrightarrow{v} \bullet$, then $\mathcal{G}_m =$ . We apply (47) successively as follows:

$$\begin{aligned} Q(\mathcal{G}, (\boldsymbol{\alpha}, \boldsymbol{\eta}), a, b, d) &= \alpha_u \left(\text{Diagram 1} \right) + \eta_u \left(\text{Diagram 2} \right) \\ &= \alpha_u (\alpha_v \left(\text{Diagram 3} \right) + \eta_v \left(\text{Diagram 4} \right)) + \\ &\quad \eta_u (\alpha_v \left(\text{Diagram 5} \right) + \eta_v \left(\text{Diagram 6} \right)) \\ &= \alpha_u \alpha_v a + \alpha_u \eta_v b^2 d^2 + \eta_u \alpha_v b^2 d^2 + \eta_u \eta_v b^4 d^3. \end{aligned} \quad (48)$$

THEOREM 7. *Let G be an embedded graph with half-edges and $e \in \mathcal{E}(G)$. Then*

$$Q(G; (\boldsymbol{\alpha}, \boldsymbol{\eta}), a, b, d) = \alpha_e Q(G/e; (\boldsymbol{\alpha}, \boldsymbol{\eta}), a, b, d) + \eta_e Q(G \vee e; (\boldsymbol{\alpha}, \boldsymbol{\eta}), a, b, d),$$

where, on the right-hand side, $(\boldsymbol{\alpha}, \boldsymbol{\eta})$ denotes the weight system for G restricted to G/e or $G \vee e$ which is obtained by eliminating the weights for the vertex associated with the edge e .

PROOF. From Definition 14 and Propositions 5 and 4, the result follows. \square

PROPOSITION 6. *In any ribbon graph with half-edges G and G_m its medial graph, we have:*

$$F_{\text{ext}}(G) = f(G) = c'(G_m), \quad C_{\partial}(G) = c''(G_m), \quad (49)$$

where $F_{\text{ext}}(G)$ is the number of external faces, $f(G)$ the number of half-edges and $C_{\partial}(G)$ the number of connected components of the boundary of G ; $c'(G_m)$ is the number of open curves in G_m and $c''(G_m)$ its number of punctured curves.

PROOF. Notice that a connected component of the boundary is obtained by following a certain number of half-edges and external faces which alternate. And to such a connected component we have a unique punctured curve. \square

The following statement holds:

THEOREM 8. *Let \mathcal{G} be a ribbon graph with half-edges with topological medial graph \mathcal{G}_m . Then, fixing all $\alpha_v = \alpha$ and $\eta_v = \eta$, for all v , we have*

$$Q(\mathcal{G}; (\boldsymbol{\alpha}, \boldsymbol{\eta}), a, b, d) = \alpha^{r(\mathcal{G})} \eta^{n(\mathcal{G})} a^{k(\mathcal{G})} \mathcal{R}(\mathcal{G}; \frac{\eta a}{\alpha} + 1, \frac{\alpha a}{\eta} + 1, \frac{1}{a}, d, b, 1). \quad (50)$$

PROOF. Consider \mathcal{G} a ribbon graph possibly with half-edges and \mathcal{G}_m the associated topological medial graph. Let S be a state graph of \mathcal{G}_m , $u(S)$ be its number of uncut vertex states (white-smoothing) and $q(S)$ its number of cut vertex states (cutting state). Introduce a weight system $W_m = (\boldsymbol{\alpha}, \boldsymbol{\eta})$ on \mathcal{G}_m such that $\alpha_v = \alpha$ and $\eta_v = \eta$, for all v , and define $Q(\mathcal{G}; (\boldsymbol{\alpha}, \boldsymbol{\eta}), a, b, d) = q(\mathcal{G}_m; W_m, a, b, d) = \sum_{S \in \mathcal{S}} \alpha^{u(S)} \eta^{q(S)} a^{c(S)} b^{c'(S)} d^{c''(S)}$, with \mathcal{S} the set of graph states. Consider

$A \in \mathcal{G}$, the graph state S associated with A is obtained by taking a cut vertex state for all vertices v_e with $e \notin \mathcal{E}(A)$ and uncut state for the remaining ones, i.e. for v_e such that $e \in \mathcal{E}(A)$. For each uncut (resp. cut) vertex state, we assign an uncut (resp. cut) edge in \mathcal{G} . Hence Q can be written as, given $\mathcal{E}(\mathcal{G}) = E$, $|\mathcal{E}(A)| = |A|$, $|\mathcal{V}(\mathcal{G})| = v(\mathcal{G})$,

$$\begin{aligned} Q(\mathcal{G}; (\boldsymbol{\alpha}, \boldsymbol{\eta}), a, b, d) &= \sum_{A \in \mathcal{G}} \alpha^{|A|} \eta^{|E|-|A|} a^{F_{\text{int}}(A)} b^{F_{\text{ext}}(A)} d^{C_{\partial}(A)} \\ &= \eta^{|E|} \sum_{A \in \mathcal{G}} \left(\frac{\alpha}{\eta}\right)^{|A|} a^{F_{\text{int}}(A)} d^{C_{\partial}(A)} b^{F_{\text{ext}}(A)}. \end{aligned} \quad (51)$$

On the other hand,

$$\begin{aligned} \mathcal{R}(\mathcal{G}; x, y, z, s, t, 1) &= (x-1)^{-k(\mathcal{G})} (y-1)^{-v(\mathcal{G})} z^{-v(\mathcal{G})} \times \\ &\sum_{A \in \mathcal{G}} ((x-1)(y-1)z^2)^{k(A)} ((y-1)z)^{E(A)} z^{-F_{\text{int}}(A)} s^{C_{\partial}(A)} t^{f(A)}. \end{aligned} \quad (52)$$

Setting $z = \frac{1}{a}$, $y = \frac{\alpha a}{\eta} + 1$, $x = \frac{\eta a}{\alpha} + 1$, $s = d$ and $t = b$, one has:

$$\begin{aligned} \mathcal{R}(\mathcal{G}; x, y, z, s, t, 1) &= \left(\frac{\eta}{\alpha}\right)^{v(\mathcal{G})-k(\mathcal{G})} a^{-k(\mathcal{G})} \sum_{A \in \mathcal{G}} \left(\frac{\alpha}{\eta}\right)^{E(A)} a^{F_{\text{int}}(A)} d^{C_{\partial}(A)} b^{f(A)} \\ &= (\eta)^{-n(\mathcal{G})} (\alpha)^{-r(\mathcal{G})} a^{-k(\mathcal{G})} Q(\mathcal{G}; (\boldsymbol{\alpha}, \boldsymbol{\eta}), a, b, d). \end{aligned} \quad (53)$$

□

Let us now discuss how the above properties extend to HR-classes. This can be understood in a direct way: we can set a definition $Q([\mathcal{G}]; -) := Q(\mathcal{G}; -) = \mathcal{R}(\mathcal{G}) = \mathcal{R}([\mathcal{G}])$, and $Q([\mathcal{G}; -]) = q([\mathcal{G}_m], -)$ where we define the class $[\mathcal{G}_m]$ of medial graphs obtained by choosing \mathcal{G}_m the medial graph of \mathcal{G} and all medial graphs obtained from \mathcal{G}_m by displacing the fictitious 2-valent vertices within the same punctured curve or from one punctured curve to the other. We must keep however at least one fictitious 2-valent vertex per punctured curve. A moment of thought leads to the correspondence between these latter moves with the HR-moves. This can be used to achieve the claim $\forall g \in [\mathcal{G}_m]$, $q(g; -) = q(\mathcal{G}_m; -)$. Thus we can define $q([\mathcal{G}_m], -) = q(\mathcal{G}_m; -) = Q(\mathcal{G}; -) = Q([\mathcal{G}]; -)$.

Several other interesting developments can be now undertaken from the polynomial invariants treated in this paper. For instance, we might ask if one can make sense of a duality relation for the generalized transition polynomial defined on HERGs. Finding a dual for a HERG becomes however a nontrivial task. Indeed, two HERGs belonging to the same HR-equivalence class do not always have equivalent dual. These certainly deserves to be elucidated. On another connected domain, significant progresses around matroids [5] and Hopf algebra techniques [6] applied to the Tutte polynomial have been recently highlighted. These studies should find as well an extension for the types of invariants worked out in this paper. Finally, combining some ideas of the present work and Hopf algebra calculations as found in [13], one might be able to prove a universality theorem for polynomial invariants over stranded graphs [1] generalizing ribbon graphs with half-edges.

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Appendix

Appendix A. Recipe theorems for the polynomial on pinched ribbon graphs

We provide here two recipe theorems for the polynomial \mathcal{R}' defined HR-classes of pinched ribbon graphs. The proof of the first recipe theorem is very similar to the proof as found in [10]. This is mainly due to the factorization property of \mathcal{R}' with respect to the one-point-joint operation. We also use particular diagrams called canonical chord-diagrams with half-chords associated with the pinched one-vertex ribbon graphs with half-edges. Notice that half-chords here can be disposed in arbitrary way on the circle. The variable t appears as a deformation parameter during this proof. The second recipe holds for a more general contraction/deletion rule weighted by parameters. It can be considered as the most complete recipe theorem in that matter for \mathcal{R}' . For completeness reasons, we report these theorems for the interested reader.

The change of variable $S \rightarrow Z^{-1}$ in \mathcal{R} leads to the polynomial \mathcal{R}' defined by:

$$\mathcal{R}_{\mathcal{G}}(X, Y, Z, Z^{-1}, S, W, T) = \mathcal{R}'_{\mathcal{G}}(X, Y, Z, W, T). \quad (\text{A.1})$$

A quick look of the expression of \mathcal{R}' , one infers the relation

$$\mathcal{R}'_{\mathcal{G}}(X, Y, Z, W, T) = T^{|\mathfrak{f}^0|} T^{2n(\mathcal{G})} R_{\mathcal{G}}(\tilde{X}, \frac{\tilde{Y}}{T^2}, Z, W), \quad (\text{A.2})$$

where

$$\begin{cases} \tilde{X} = (X - 1)T^2 + 1, \\ \tilde{Y} = Y - 1 + T^2. \end{cases} \quad (\text{A.3})$$

As a polynomial on a different category of graphs, a recipe theorem for \mathcal{R}' on pinched ribbon graphs can be investigated on its own right.

To proceed with, let us define now some canonical chord diagrams with half-chords associated to the HR-classes of pinched one-vertex ribbon graphs. Let D_{ijkl} be the following chord diagram called canonical: it consists of $i - 2j - k$ positive disjoint chords, j pairs of intersecting positive chords, k number of negative disjoint chords and l number of half-chords. Then, we have:

$$\mathcal{R}'(D_{1000}; x, y, z, w, t) = (y - 1) + t^2, \quad (\text{A.4})$$

$$\mathcal{R}'(D_{10010}; x, y, z, w, t) = (y - 1)zw + t^2, \quad (\text{A.5})$$

$$\mathcal{R}'(D_{2100}; x, y, z, w, t) = (y - 1)^2 z^2 + 2(y - 1)t^2 + t^4, \quad (\text{A.6})$$

$$\mathcal{R}'(D_{0001}; x, y, z, w, t) = t. \quad (\text{A.7})$$

Using the fact that the polynomial \mathcal{R}' is multiplicative for the one-point-joint of pinched ribbon graphs, we can write

$$\mathcal{R}'(D_{ijkl}) = [\mathcal{R}'(D_{1000})]^{i-2j-k} [\mathcal{R}'(D_{2100})]^j [\mathcal{R}'(D_{1010})]^k [\mathcal{R}'(D_{0001})]^l. \quad (\text{A.8})$$

THEOREM 9 (Recipe theorem for \mathcal{R}'). *Let \mathcal{M} be a minor closed subset of pinched ribbon graphs containing all pinched ribbon graphs on two vertices. Let F be a map from \mathcal{M} to a commutative ring \mathcal{R} with unity. Let $s = F(D_{2100})$, $q = F(D_{1000})$, $r = F(D_{1010})$ and $s_1 = F(D_{0001})$ and suppose there exist elements $\alpha, x, u, v \in \mathfrak{R}$ with α a unit such that:*

(1)

$$F(G) = \begin{cases} F(G \vee e) + F(G/e) & \text{if } e \text{ is regular,} \\ ((x - 1)(\alpha^{-1} s_1)^2 + 1) F(G/e) & \text{if } e \text{ is a bridge.} \end{cases} \quad (\text{A.9})$$

(2) $F(G \sqcup H) = F(G)F(H)$ and $\alpha F(G \cdot H) = F(G)F(H)$ where G and H are pinched embedded bouquets.

(3) $F(E) = \alpha^n t^m$ if E is a pinched edgeless graph with m half-edges and n vertices;

(4) $(q - s_1^2)^2 u^2 = \alpha[s - 2\alpha^{-2}(q - s_1^2)s_1^2 - s_1^4]$, $(q - s_1^2)uv = r - s_1^2$, and $v = v^2$. Then

$$F(G) = \alpha^{k(G)} \mathcal{R}'(G; x, \alpha^{-1}(q - s_1^2 + \alpha), u, v, \alpha^{-1} s_1), \quad (\text{A.10})$$

where $k(G)$ is the number of components of G .

PROOF. The proof proceeds in two steps: (i) we focus on pinched one-vertex ribbon graphs and then (ii) on general pinched ribbon graphs.

- (i) Using items (2) and (4) of Theorem 9, the relation (A.10) holds in the canonical diagrams with half-chords. Indeed, considering a canonical diagram D_{ijkl} and using the relations in item (4), Theorem 9 is immediately verified for D_{1000} , D_{2100} , D_{1010} and D_{0001} . From item (2), we have:

$$\alpha^{i-j+l-1}F(D_{ijkl}) = [F(D_{1000})]^{i-2j-k}[F(D_{2100})]^j[F(D_{1010})]^k[F(D_{0001})]^l. \quad (\text{A.11})$$

Using the relation (A.11) and the fact that the theorem holds for the diagrams D_{1000} , D_{2100} , D_{1010} and D_{0001} , then it is also true on any canonical diagram D_{ijkl} . From item (1), we obtain the relations:

$$F(D_1) - \mu F(D'_1) = F(D_2) - \mu F(D'_2) \text{ and } , \quad (\text{A.12})$$

$$F(D_3) - \mu F(D'_3) = F(D_4) - \mu F(D'_4), \quad (\text{A.13})$$

where D_1, D_2, D_3 and D_4 are related as shown in Figures 9 and 10¹ with $D'_i = D_i \vee e$, and $\mu = 1$ if there is a chord from $a \cup b$ to $c \cup d$, otherwise $\mu = [(x-1)(\alpha^{-1}s_1)^2 + 1]$. The relation (A.12) holds also for \mathcal{R}' and hence for $F' = F - \alpha\mathcal{R}'$.

By induction, assume that $F(D) = \alpha\mathcal{R}'(D)$ for any signed chord diagram with fewer than n chords. Then F' vanishes on any signed chord diagram with fewer than n chords. From (A.12), we have $F'(D) = F'(D_{nijkl})$ where D is related to a canonical diagram D_{nijkl} . From the following equations:

$$F(D_{nijkl}) = \alpha\mathcal{R}'(D_{nijkl}; x, \alpha^{-1}(q - s_1^2 + \alpha), u, v, \alpha^{-1}s_1) \quad (\text{A.14})$$

obtained using (A.11), and

$$F'(D) = F'(D_{nijkl}) = 0, \quad (\text{A.15})$$

we get, by induction, $F'(D) = 0$ on any signed chord diagram D . Finally, the result holds on any rosette of pinched ribbon graph.

- (ii) Using item (1), the result becomes obvious on any pinched ribbon graph. □

In this proof we remark that, we do not need \mathcal{M} to contain all ribbon graphs on two vertices. In fact, it is sufficient to consider that F satisfies (A.12) and (A.13) for a chain of rosettes terminating in a canonical chord diagram. The next definition will be needed.

We say that a minor closed set \mathcal{M} of pinched ribbon graphs is closed under chord operations whenever $D \in \mathcal{M}$ and $D \sim D_{ijkl}$, then there is a finite sequence $D = D_1 \dots D_n = D_{ijkl}$ with $D_i \in \mathcal{M}$ and $D \sim D_{i+1}$ for all i .

THEOREM 10 (The ‘‘low fat’’ recipe theorem for \mathcal{R}'). *Theorem 9 holds with ‘‘Let \mathcal{M} be a minor closed subset of pinched ribbon graphs containing all ribbon graphs on two vertices and let F map \mathcal{M} to a commutative ring \mathfrak{R} with unity’’, replaced by ‘‘Let \mathcal{M} be a minor closed subset of pinched ribbon graphs closed under chord operations that contains $D_{1000}, D_{1010}, D_{2100}, D_{0001}$ and let F map \mathcal{M} to a commutative ring \mathfrak{R} with unity, such that F satisfies (A.12) and (A.13) whenever the D_i ’s are related as in Figures 9 and 10.’’*

From this point, we also have the following statement:

COROLLARY 3. *If $F, \mathcal{M}, \mathfrak{R}$ satisfy the conditions of Theorem 10, with both $q - s_1^2$ and $r - s_1^2$ being unit of \mathfrak{R} , then $v = 1$, and thus F does not discern orientation by the presence or absence of a single idempotent element.*

PROOF. From the second equation in the above item (4), we infer $\frac{r-s_1^2}{q-s_1^2} = uv = uw^2 = v\frac{r-s_1^2}{q-s_1^2}$. Furthermore, $v = 1$ since $q - s_1^2$ and $r - s_1^2$ are units. Therefore, $F(G) = \alpha^{k(G)}\mathcal{R}'(G; x, \alpha^{-1}(q - s_1^2 + \alpha), u, 1, \alpha^{-1}s_1)$.

¹Notice here that, the position of the half-chords does not matter. We can also use diagrams without half-chords for illustration.

□

There exists another statement about the recipe theorem when the contraction/deletion rule is weighted by parameters. We now provide, in that more general case, a second more generic recipe to construct a function satisfying the weighted recurrence relation.

THEOREM 11 (Second recipe theorem for \mathcal{R}'). *Let \mathcal{M} be a minor closed subset of pinched ribbon graphs containing all pinched ribbon graphs on two vertices. Let F be a map from \mathcal{M} to a commutative ring \mathfrak{R} with unity. Let $s = F(D_{2100})$, $q = F(D_{1000})$ and $r = F(D_{1010})$; and suppose there exist elements $\alpha, x, u, v \in \mathfrak{R}$ with α a unit such that:*

(1)

$$F(G) = \begin{cases} \sigma F(G - e) + \tau F(G/e) & \text{if } e \text{ is regular,} \\ x F(G/e) & \text{if } e \text{ is a bridge.} \end{cases} \quad (\text{A.16})$$

(2) $F(G \sqcup H) = F(G)F(H)$ and $\alpha F(G \cdot H) = F(G)F(H)$ where G and H are pinched embedded bouquets.

(3) $F(E) = \alpha^n T^m$ if E is an edgeless graph with m half-edges and n vertices.

(4) $(q - \alpha\tau)^2 u^2 = \alpha(s - 2\tau q + \alpha\tau^2)$, $(q - \alpha\tau)uv = r - \alpha\tau$, and $v = v^2$. Then

$$\begin{aligned} \sigma^{-r(G)} \tau^{-n(G)} F(G) &= \alpha^{k(G)} R(G; \sigma^{-1}x, \alpha^{-1}\tau^{-1}q, u, v) \\ &= T^{-(|f^0|+2n(G))} \alpha^{k(G)} \mathcal{R}'(\sigma^{-1}[(x - \sigma)T^{-2} + \sigma], T^2(\alpha^{-1}\tau^{-1}q) - T^2 + 1, u, v, T), \end{aligned} \quad (\text{A.17})$$

where $k(G)$, $r(G)$ and $n(G)$ are respectively the number of connected components, rank and nullity of G .

PROOF. This theorem holds on any canonical diagram D_{ijkl} or D_{ijk} (BR diagrams). Using item (1), we have the relation

$$\tau F(D_1) - \mu\sigma F(D'_1) = \tau F(D_2) - \mu\sigma F(D'_2) \quad \text{and} \quad (\text{A.18})$$

$$\tau F(D_3) - \mu\sigma F(D'_3) = \tau F(D_4) - \mu\sigma F(D'_4), \quad (\text{A.19})$$

where D_1, D_2, D_3 and D_4 are related as shown in Figures 9 and 10 with $D'_i = D_i \vee e$, and $\mu = 1$ if there is a chord from $a \cup b$ to $c \cup d$, and otherwise $\mu = x$. The polynomials \mathcal{R}' (respectively R), satisfies (A.18) and (A.19) and hence $F'(G) = \sigma^{-r(G)} \tau^{-n(G)} F(G) - T^{-(|f^0|+2n(G))} \alpha^{k(G)} \mathcal{R}'(G)$ (respectively $F'(G) = \sigma^{-r(G)} \tau^{-n(G)} F(G) - \alpha R(G)$) satisfies the same relation. Assume that $\sigma^{-r(D)} \tau^{-n(D)} F(D) = T^{-(|f^0|+2n(D))} \alpha^{k(D)} \mathcal{R}'(D)$ (respectively $\sigma^{-r(D)} \tau^{-n(D)} F(D) = \alpha R(D)$) for any signed chord diagram with fewer than n chords. Therefore F' vanishes on any signed chord diagram with fewer than n chords. Using (A.18) and (A.19) we have $\tau F'(D) = \tau F'(D_{nijkl})$ (respectively $\tau F'(D) = \tau F'(D_{njk})$) where D is related to a canonical diagram D_{nijkl} (respectively D_{njk}). From the following equations:

$$\tau^{-n} F(D_{nijkl}) = T^{-l-2n} \alpha \mathcal{R}'(D_{nijkl}) = \alpha R(D_{njk}), \quad (\text{A.20})$$

and

$$F'(D) = F'(D_{njk}) = 0, \quad (\text{A.21})$$

we have by induction, $F'(D) = 0$ on any signed chord diagram D . Finally the result holds on any rosette of pinched ribbon graph. Using item (1), the result becomes true on any pinched ribbon graph. □

We can introduce the weaker form of the recipe theorem.

THEOREM 12 (The “low fat” recipe theorem for \mathcal{R}'). *Theorem 11 holds with “Let \mathcal{M} be a minor closed subset of pinched ribbon graphs containing all ribbon graphs on two vertices and let F map \mathcal{M} to a commutative ring \mathfrak{R} with unity”, replaced by “Let \mathcal{M} be a minor closed subset of pinched ribbon graphs closed under chord operations that contains $D_{1000}, D_{1010}, D_{2100}$ and let F map \mathcal{M} to a commutative ring \mathfrak{R} with unity, such that F satisfies (A.18) and (A.19) whenever the D'_i s are related as in Figures 9 and 10.”*

COROLLARY 4. *If F , \mathcal{M} , \mathfrak{R} satisfy the conditions of Theorem 12, with both $q - \alpha\tau$ and $r - \alpha\tau$ being unit of \mathfrak{R} , then $v = 1$, and thus F does not discern orientation by the presence or absence of a single idempotent element.*

PROOF. From the second equation in the above item (4), we infer $\frac{r-\alpha\tau}{q-\alpha\tau} = uv = uv^2 = v\frac{r-\alpha\tau}{q-\alpha\tau}$. Furthermore, $v = 1$ since $q - \alpha\tau$ and $r - \alpha\tau$ are units. Therefore, $\sigma^{-r(G)}\tau^{-n(G)}F(G) = \alpha^{k(G)}\mathcal{R}'(G; \sigma^{-1}x, \alpha^{-1}\tau^{-1}q, u, 1)$. \square

Appendix B. One-point-joint and semi-factorization property of \mathcal{R}

In this appendix, we investigate the precise breaking of the factorization property of \mathcal{R} for the one-point-joint operation of two HERGs or of two HR-classes. First, a precision must be given on the notation $\mathcal{G}_1 \cdot_{v_1, v_2} \mathcal{G}_2$ and we will adopt instead the following notation:

$$\mathcal{G}_1 \cdot_{(v_i, c_i)} \mathcal{G}_2, \quad (\text{B.22})$$

where the pairs (v_i, c_i) , $i = 1, 2$, incorporate the vertex v_i of \mathcal{G}_i where the joining operation is performed and an arc c_i on v_i (which without edges) where v_1 and v_2 merge.

Let us denote $\mathcal{P}_{\mathcal{G}}$ the set of cutting spanning subgraphs of \mathcal{G} . We denote $\mathfrak{h}_{A_i}(c_i)$ the face of $A_i \in \mathcal{G}_i$ which contains the arc c_i in v_i seen as a vertex of A_i . And let us call $\mathcal{P}_{\mathcal{G}_i}(v_i, c_i, \epsilon)$, $\epsilon \in \{0, 1\}$, the set of cutting spanning subgraphs of \mathcal{G}_i such that

$$\begin{aligned} A_i \in \mathcal{P}_{\mathcal{G}_i}(v_i, c_i, 0) \subset \mathcal{P}_{\mathcal{G}_i} &\Leftrightarrow \mathfrak{h}_{A_i}(c_i) \in \mathcal{F}_{\text{int}}(A_i), \\ A_i \in \mathcal{P}_{\mathcal{G}_i}(v_i, c_i, 1) \subset \mathcal{P}_{\mathcal{G}_i} &\Leftrightarrow \mathfrak{h}_{A_i}(c_i) \in \mathcal{F}_{\text{ext}}(A_i). \end{aligned} \quad (\text{B.23})$$

Consider a cutting spanning subgraph A_i of \mathcal{G}_i , when we write $A_1 \cdot_{(v_i, c_i)} A_2$, we implicitly refer to the fact that c_i must be seen as an arc of v_i in A_i (and not anymore as an arc of v_i in \mathcal{G}_i , except when, of course, $A_i = \mathcal{G}_i$).

One must see that the set of cutting spanning subgraphs of $\mathcal{G}_1 \cdot_{(v_i, c_i)} \mathcal{G}_2$ is one-to-one with the set of cutting spanning subgraphs of $\mathcal{G}_1 \sqcup \mathcal{G}_2$, i.e. to each $A_1 \cdot_{(v_i, c_i)} A_2$ corresponds a unique $A_1 \sqcup A_2$, for $A_i \in \mathcal{G}_i$. It follows the decomposition lemma:

LEMMA 5 (Cutting spanning subgraph decomposition). *Let \mathcal{G}_i two disjoint graphs, then the set $\mathcal{P}_{\mathcal{G}_1 \cdot_{(v_i, c_i)} \mathcal{G}_2}$ is the union of two disjoint sets $\mathcal{P}_{\mathcal{G}_1 \cdot_{(v_i, c_i)} \mathcal{G}_2}^0$ and $\mathcal{P}_{\mathcal{G}_1 \cdot_{(v_i, c_i)} \mathcal{G}_2}^1$ such that*

$$\begin{aligned} \mathcal{P}_{\mathcal{G}_1 \cdot_{(v_i, c_i)} \mathcal{G}_2}^0 &\equiv (\mathcal{P}_{\mathcal{G}_1}(v_1, c_1, 0) \cup \mathcal{P}_{\mathcal{G}_2}(v_2, c_2, 0)) \cup (\mathcal{P}_{\mathcal{G}_1}(v_1, c_1, 0) \cup \mathcal{P}_{\mathcal{G}_2}(v_2, c_2, 1)) \\ &\quad \cup (\mathcal{P}_{\mathcal{G}_1}(v_1, c_1, 1) \cup \mathcal{P}_{\mathcal{G}_2}(v_2, c_2, 0)), \\ \mathcal{P}_{\mathcal{G}_1 \cdot_{(v_i, c_i)} \mathcal{G}_2}^1 &\equiv \mathcal{P}_{\mathcal{G}_1}(v_1, c_1, 1) \cup \mathcal{P}_{\mathcal{G}_2}(v_2, c_2, 1). \end{aligned} \quad (\text{B.24})$$

where \equiv means "one-to-one with".

LEMMA 6. *Let $A_i \in \mathcal{G}_i$, then*

$$\begin{aligned} r(A_1 \cdot_{(v_i, c_i)} A_2) &= r(A_1) + r(A_2), \\ n(A_1 \cdot_{(v_i, c_i)} A_2) &= n(A_1) + n(A_2), \\ t(A_1 \cdot_{(v_i, c_i)} A_2) &= t(A_1) + t(A_2), \\ k(A_1 \cdot_{(v_i, c_i)} A_2) &= k(A_1) + k(A_2) - 1, \end{aligned} \quad (\text{B.25})$$

$$\begin{aligned} F_{\text{int}}(A_1 \cdot_{(v_i, c_i)} A_2) &= \begin{cases} F_{\text{int}}(A_1) + F_{\text{int}}(A_2) - 1, & \text{if } A_1 \cdot_{(v_i, c_i)} A_2 \in \mathcal{P}_{\mathcal{G}_1 \cdot_{(v_i, c_i)} \mathcal{G}_2}^0 \\ F_{\text{int}}(A_1) + F_{\text{int}}(A_2), & \text{if } A_1 \cdot_{(v_i, c_i)} A_2 \in \mathcal{P}_{\mathcal{G}_1 \cdot_{(v_i, c_i)} \mathcal{G}_2}^1 \end{cases} \\ C_{\partial}(A_1 \cdot_{(v_i, c_i)} A_2) &= \begin{cases} C_{\partial}(A_1) + C_{\partial}(A_2), & \text{if } A_1 \cdot_{(v_i, c_i)} A_2 \in \mathcal{P}_{\mathcal{G}_1 \cdot_{(v_i, c_i)} \mathcal{G}_2}^0 \\ C_{\partial}(A_1) + C_{\partial}(A_2) - 1, & \text{if } A_1 \cdot_{(v_i, c_i)} A_2 \in \mathcal{P}_{\mathcal{G}_1 \cdot_{(v_i, c_i)} \mathcal{G}_2}^1 \end{cases} \end{aligned} \quad (\text{B.26})$$

PROOF. The relations involving the rank, the nullity, the number of half-edges and number of connected components in (B.25) are directly obtained from the usual additivity of these quantities during the one-point-joint.

The relation involving the number of internal lines and connected components of the boundary can be inferred as follows: in the sector $\mathcal{P}_{\mathcal{G}_1 \cdot (v_i, c_i) \mathcal{G}_2}^1$, after the one-point-joint, one always loses a connected component of the boundary. The contact between A_1 and A_2 happens on two external faces belonging necessarily to two initial and different connected components of the boundary of A_1 and A_2 . Note that closed faces are all preserved. For the sector $\mathcal{P}_{\mathcal{G}_1 \cdot (v_i, c_i) \mathcal{G}_2}^0$, the reasoning is similar: one loses an internal face this time for each subsector $\mathcal{P}_{\mathcal{G}_1}(v_1, c_1, 0) \cup \mathcal{P}_{\mathcal{G}_2}(v_2, c_2, 0)$, $\mathcal{P}_{\mathcal{G}_1}(v_1, c_1, 0) \cup \mathcal{P}_{\mathcal{G}_2}(v_2, c_2, 1)$ and $\mathcal{P}_{\mathcal{G}_1}(v_1, c_1, 1) \cup \mathcal{P}_{\mathcal{G}_2}(v_2, c_2, 0)$. In the end, there is always an internal face which is lost but the number of connected components of the boundary is always preserved for each sector. \square

We are in position to understand how the BR polynomial for HERGs behaves under one-point-joint. Let us adopt a slightly different notation for the same object \mathcal{R} making explicit its dependence on the set of its cutting subgraphs

$$\mathcal{R}(\mathcal{G}; \mathcal{P}_{\mathcal{G}}; x, y, z, s, w, t) = \sum_{A \in \mathcal{P}_{\mathcal{G}}} M_{\mathcal{R}}(A; x, y, z, s, w, t) \quad (\text{B.27})$$

where $M_{\mathcal{R}}$ is the monomial ordinarily associated with each subgraph and defined by \mathcal{R} . Now, let us compute using Lemmas 5 and 6

$$\begin{aligned} & \mathcal{R}(\mathcal{G}_1 \cdot (v_i, c_i) \mathcal{G}_2; \mathcal{P}_{\mathcal{G}_1 \cdot (v_i, c_i) \mathcal{G}_2}) = \sum_{A_1 \cdot (v_i, c_i) A_2 \in \mathcal{P}_{\mathcal{G}_1 \cdot (v_i, c_i) \mathcal{G}_2}^0} M_{\mathcal{R}}(A_1 \cdot (v_i, c_i) A_2; x, y, z, s, w, t) \\ & + \sum_{A_1 \cdot (v_i, c_i) A_2 \in \mathcal{P}_{\mathcal{G}_1 \cdot (v_i, c_i) \mathcal{G}_2}^1} M_{\mathcal{R}}(A_1 \cdot (v_i, c_i) A_2; x, y, z, s, w, t) \\ & = \sum_{A_1 \cdot (v_i, c_i) A_2 \in \mathcal{P}_{\mathcal{G}_1 \cdot (v_i, c_i) \mathcal{G}_2}^0} M_{\mathcal{R}}(A_1; x, y, z, s, w, t) M_{\mathcal{R}}(A_2; x, y, z, s, w, t) \\ & + (zs)^{-1} \sum_{A_1 \cdot (v_i, c_i) A_2 \in \mathcal{P}_{\mathcal{G}_1 \cdot (v_i, c_i) \mathcal{G}_2}^1} M_{\mathcal{R}}(A_1; x, y, z, s, w, t) M_{\mathcal{R}}(A_2; x, y, z, s, w, t) \\ & = \left\{ \sum_{A_1 \in \mathcal{P}_{\mathcal{G}_1}(v_1, c_1; 0)} M_{\mathcal{R}}(A_1; x, y, z, s, w, t) \right\} \left\{ \sum_{A_2 \in \mathcal{P}_{\mathcal{G}_2}(v_2, c_2; 0)} M_{\mathcal{R}}(A_2; x, y, z, s, w, t) \right\} \\ & + \left\{ \sum_{A_1 \in \mathcal{P}_{\mathcal{G}_1}(v_1, c_1; 0)} M_{\mathcal{R}}(A_1; x, y, z, s, w, t) \right\} \left\{ \sum_{A_2 \in \mathcal{P}_{\mathcal{G}_2}(v_2, c_2; 1)} M_{\mathcal{R}}(A_2; x, y, z, s, w, t) \right\} \\ & + \left\{ \sum_{A_1 \in \mathcal{P}_{\mathcal{G}_1}(v_1, c_1; 1)} M_{\mathcal{R}}(A_1; x, y, z, s, w, t) \right\} \left\{ \sum_{A_2 \in \mathcal{P}_{\mathcal{G}_2}(v_2, c_2; 0)} M_{\mathcal{R}}(A_2; x, y, z, s, w, t) \right\} \\ & + (zs)^{-1} \left\{ \sum_{A_1 \in \mathcal{P}_{\mathcal{G}_1}(v_1, c_1; 1)} M_{\mathcal{R}}(A_1; x, y, z, s, w, t) \right\} \left\{ \sum_{A_2 \in \mathcal{P}_{\mathcal{G}_2}(v_2, c_2; 1)} M_{\mathcal{R}}(A_2; x, y, z, s, w, t) \right\}. \end{aligned} \quad (\text{B.28})$$

This computes to give, in suggestive notation where $\mathcal{R}(\mathcal{G}, \mathcal{P})$ means that we compute the sum of monomials only for subgraphs in \mathcal{P} ,

$$\begin{aligned} & \mathcal{R}(\mathcal{G}_1 \cdot (v_i, c_i) \mathcal{G}_2; \mathcal{P}_{\mathcal{G}_1 \cdot (v_i, c_i) \mathcal{G}_2}) = \\ & \mathcal{R}(\mathcal{G}_1; \mathcal{P}_{\mathcal{G}_1}(v_1, c_1; 0)) \mathcal{R}(\mathcal{G}_2; \mathcal{P}_{\mathcal{G}_2}(v_2, c_2; 0)) + \mathcal{R}(\mathcal{G}_1; \mathcal{P}_{\mathcal{G}_1}(v_1, c_1; 1)) \mathcal{R}(\mathcal{G}_2; \mathcal{P}_{\mathcal{G}_2}(v_2, c_2; 0)) \\ & + \mathcal{R}(\mathcal{G}_1; \mathcal{P}_{\mathcal{G}_1}(v_1, c_1; 0)) \mathcal{R}(\mathcal{G}_2; \mathcal{P}_{\mathcal{G}_2}(v_2, c_2; 1)) + (zs)^{-1} \mathcal{R}(\mathcal{G}_1; \mathcal{P}_{\mathcal{G}_1}(v_1, c_1; 1)) \mathcal{R}(\mathcal{G}_2; \mathcal{P}_{\mathcal{G}_2}(v_2, c_2; 1)). \end{aligned} \quad (\text{B.29})$$

Thus, we obtain a broken factorization property of the polynomial \mathcal{R} of the form:

$$\mathcal{R}(\mathcal{G}_1 \cdot (v_i, c_i) \mathcal{G}_2) = \mathcal{R}(\mathcal{G}_1) \mathcal{R}(\mathcal{G}_2; \mathcal{P}_{\mathcal{G}_2}(v_2, c_2; 0))$$

$$\begin{aligned}
& + \left(\mathcal{R}(\mathcal{G}_1; \mathcal{P}_{\mathcal{G}_1}(v_1, c_1; 0)) + (zs)^{-1} \mathcal{R}(\mathcal{G}_1; \mathcal{P}_{\mathcal{G}_1}(v_1, c_1; 1)) \right) \mathcal{R}(\mathcal{G}_2; \mathcal{P}_{\mathcal{G}_2}(v_2, c_2; 1)) \\
& = \mathcal{R}(\mathcal{G}_1) \mathcal{R}(\mathcal{G}_2; \mathcal{P}_{\mathcal{G}_2}(v_2, c_2; 0)) + \tilde{\mathcal{R}}_{sz}(\mathcal{G}_1; v_1, c_1) \mathcal{R}(\mathcal{G}_2; \mathcal{P}_{\mathcal{G}_2}(v_2, c_2; 1)), \tag{B.30}
\end{aligned}$$

where $\tilde{\mathcal{R}}_{sz}(\mathcal{G}; v, c) := \mathcal{R}(\mathcal{G}; \mathcal{P}_{\mathcal{G}}(v, c; 0)) + (zs)^{-1} \mathcal{R}(\mathcal{G}; \mathcal{P}_{\mathcal{G}}(v, c; 1))$ is a deformed version of the $\mathcal{R}(\mathcal{G})$ polynomial depending on a given vertex v and an arc c on it. There exists of course a symmetric relation where the role of \mathcal{G}_1 is played by \mathcal{G}_2 . Finally, we do not see how the polynomial $\mathcal{R}_{\mathcal{G}_1, \mathcal{G}_2}$ factorizes unless $s = z^{-1}$ which maps \mathcal{R} to \mathcal{R}' . Finally, all the above properties extends to HR-classes without ambiguity.

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(R.C.A.) INTERNATIONAL CHAIR IN MATHEMATICAL PHYSICS AND APPLICATIONS, ICMIPA-UNESCO CHAIR, 072BP50, COTONOU, REP. OF BENIN

E-mail address: remi.avohou@cipma.uac.bj

(J.B.G.) MAX-PLANCK INSTITUTE FOR GRAVITATIONAL PHYSICS, ALBERT EINSTEIN INSTITUTE, AM MUHLBERG 1, 14476, POTSDAM, GERMANY, AND INTERNATIONAL CHAIR IN MATHEMATICAL PHYSICS AND APPLICATIONS, ICMIPA-UNESCO CHAIR, 072BP50, COTONOU, REP. OF BENIN

E-mail address: jbengeloun@aei.mpg.de

(M.N.H.) INTERNATIONAL CHAIR IN MATHEMATICAL PHYSICS AND APPLICATIONS, ICMIPA-UNESCO CHAIR, 072BP50, COTONOU, REP. OF BENIN

E-mail address: norbert.hounkonnou@cipma.uac .bj