Renormalization of a tensorial field theory on the homogeneous space $SU(2)/U(1)$

Vincent Lahoche[1], Daniele Oriti[2]

Abstract

We study the renormalization of a general field theory on the homogenous space $(SU(2)/U(1))^{\times d}$ with tensorial interaction and gauge invariance under the diagonal action of $SU(2)$. We derive the power counting for arbitrary d. For the case $d = 4$, we prove perturbative renormalizability to all orders via multi-scale analysis, study both the renormalised and effective perturbation series, and establish the asymptotic freedom of the model. We also outline a general power counting for the homogeneous space $(SO(D)/SO(D-1))^{\times d}$, of direct interest for quantum gravity models in general dimensions, and point out the obstructions to the direct generalisation of our results to these cases.

1 Introduction

Group field theories [1] (GFT) are a candidate formalism for the fundamental degrees of freedom of quantum spacetime (the 'atoms of space'), and an approach to quantum gravity which merges insights and mathematical structures from loop quantum gravity and spin foam models [2], simplicial quantum gravity and tensor models [3], which have all achieved remarkable progress in recent years.

They share with loop quantum gravity the general structure of quantum states, associated to graphs labeled by group-theoretic data, and indeed can be seen as a 2nd quantized, Fock space-based reformulation of both its kinematics and its operator dynamics [4]. And they encode and complete the covariant definition of the same quantum dynamics, formulated in terms of spin foam models, which in fact appear generically as GFT Feynman amplitudes [5]. The same amplitudes can be recast in the form of simplicial gravity path integrals [6, 7], clarifying their discrete geometric content, and substantiating further the analysis of the quantum geometry of loop quantum gravity states and spin foam amplitudes [8]. At the same time, they are based on the same combinatorial structures (in their action, Feynman graphs and transition amplitudes) of tensor models, which they enrich by adding group-theoretic data. The hope is that this nice interplay between combinatorics and algebra, in a quantum field theory setting, will prove powerful enough to explain from first principles the emergence of spacetime and geometry from more fundamental entities [9].

Indeed, not only they merge the key elements of these related approaches (and thus most results obtained in them), but group field theories offer a promising mathematical context for tackling some of their outstanding open issues, thanks to QFT methods, most notably renormalization. In particular, they allow to identify stringent criteria for constraining spin foam model building, for controlling quantisation ambiguities in both spin foam and canonical formulations of loop quantum gravity, and for ensuring consistency of the resulting quantum dynamics. These issues, in fact, translate into the problem of proving perturbative renormalizability of their GFT reformulation, since the GFT action encodes the choice of operator spin network dynamics and the GFT Feynman amplitudes coincide with spin foam models. The issue of controlling the sum over spin foam complexes, which completes the definition of spin foam models, and of defining the full quantum spin network dynamics, encoded in a projection operator onto physical states or in their partition function, on the other hand, translates into the problem of making sense of the corresponding non-perturbative GFT dynamics and of unravelling the macroscopic phase diagram (and interesting phase transitions) of the theory. This is the problem of the continuum limit

[1]vincent.lahoche@th.u-psud.fr; LPT-UMR 8627, Université Paris 11, 91405 Orsay Cedex, France, EU.
[2]daniele.oriti@aei.mpg.de, Max Planck Institute for Gravitational Physics, Albert Einstein Institute, Am Mühlenberg 1, 14476, Potsdam, Germany

of the theory, which is in many ways *the* outstanding issue of the whole approach (alongside the physical issue of extracting the effective dynamics of the theory in the same continuum limit). Again, QFT tools become available thanks to the GFT reformulation, be it in the form of constructive renormalisation or of functional renormalisation group techniques.

It is here that the input from tensor models has proven most relevant, in particular, the large body of recent results on coloured tensor models [10], where the use of colour labels on combinatorial structures ensures a greater control over their topology, and an analytic understanding of their scaling limits. In turn, this led to important results about the universality classes of tensor models, and to a precise suggestion for the class of allowed tensor interactions: those satisfying a 'tensor invariance'criterion, which can be seen as the tensor analogue of the notion of locality in standard quantum field theory on flat spacetime.

This becomes particularly relevant for group field theories and their renormalisation analysis. In fact, by treating GFT fields as quantum geometrically-enriched tensors, one has then a prescription for the relevant theory space that the renormalisation group flow should explore. This defines the class of GFTs known as *tensorial group field theories* (TGFTs), where most work on renormalisation has been carried out (after the very first step in this research direction [12]).

Beside constructive analyses [13] and the first FRG studies [14], most developments up to now concerned perturbative renormalizability of TGFT models. Such development can be seen as progressively approaching TGFT models for 4d quantum gravity, as developed in the spin foam context. The first models [15, 16] that have been shown to be renormalizable to all orders in perturbation theory were Abelian ones, with the TGFT field defined on several copies of a $U(1)$ group manifold (the number of copies matching the dimensionality of the cellular complex arising in their Feynman expansion). Next [17, 18] came Abelian models incorporating a gauge invariance condition in their amplitudes, which turns them into lattice gauge theories and proper spin foam models, and gives the states of the theory the structure of spin networks. Then came the first proof of perturbative renormalizability at all orders of a non-abelian model, based on $SU(2)$, with the same gauge invariance [19]. For many of these models, the renormalizability analysis was completed by the computation of the beta functions, with very interesting results on their asymptotic freedom (or safety) [20].

A bulk of solid work and understanding has therefore already accumulated. The stage is now set for tackling full-blown 4d quantum gravity models, as developed in the spin foam context. Some results on radiative corrections in the simplicial setting (where more is known also in the 3d case [21]) are available [22], but we lack any systematic analysis, like the ones mentioned in the TGFT setting. Beside a better geometric understanding of the 'tensor invariance'condition, this requires a generalisation to higher-dimensional non-abelian groups, i.e. $SO(4)$ or the even more interesting non-compact Lorentz group $SO(3,1)$, and, most important, the imposition of additional constraints on the amplitudes, the so-called 'simplicity constraints'(see [2, 7] and references therein). Depending on the exact model considered (i.e. the chosen way of imposing the simplicity constraints and the value of the so-called Immirzi parameter), these have the effect of reducing the initial domain of the GFT fields from the Lorentz group (or its euclidean counterpart $SO(4)$) to its homogeneous space $SO(3,1)/SO(3)$ (or $SO(4)/SO(3) \simeq \mathcal{S}_3$), or to another sub- manifold of the same group. A renormalizability analysis of TGFT models of 4d quantum gravity requires therefore an extension of the known results and techniques from simple group manifolds to these more complicated domains.

In the present paper, we perform one more step towards establishing the renormalizability of 4d quantum gravity TGFT models, by studying the renormalization of a TGFT model on the homogeneous space $(SU(2)/U(1))^d$, endowed with the additional gauge invariance conditions characterising spin foam models. The imposition of the constraints reducing the field variables to the homogeneous space is obtained in a covariant manner, using the formalism developed in [7]. By rigorous multi-scale analysis, we prove renormalizability to all orders in perturbation theory of the model for $d = 4$ (in $d = 3$ our results imply super-renormalizability). For the same model, we also compute both the renormalised and effective perturbative series, analyse the 2-point and 4-point correlation functions, compute the beta function and establish asymptotic freedom at one-loop order. Moreover, we generalise several of our results to arbitrary homogeneous spaces of the type $SO(D)/SO(D-1) \simeq \mathcal{S}_{D-1}$; in particular we establish a general Abelian power counting and classify such models in terms of their potential renormalizability, as seen from the Abelian power counting, for various choices of D and d. However, we also discuss why this can be a misleading classification, since the exact power counting of other non-abelian models may deviate from the Abelian one, and what aspects of the analysis need to be carried out in detail for these cases in order to really prove (or disprove) their perturbative renormalizability.

The model is defined in detail in Section 2. In Section 3 we provide an equivalent definition of the same model in terms of projections onto the homogeneous space, which is more elegant and lends itself immediately to the higher-dimensional generalisation. We then set-up the multi-scale analysis of the model, in Section 4, and obtain the Abelian power counting. The analysis of perturbative renormalizability of the model is performed in Section 5, while in Section 6 we go beyond this to study the full renormalisation flow of the model, computing also the renormalised and effective series. In Section 7, we report the study of the beta function at one-loop, and the proof of asymptotic freedom to the same order.

2 The model

2.1 Tensorial quantum field theory on $\mathcal{S}_2^{\times d}$

We consider a tensorial quantum field theory on d copies of the homogeneous space $SU(2)/U(1)$, which is isomorphic to the two dimensional sphere $\mathcal{S}_2$. The phase space of the theory is the cotangent bundle $(\mathcal{T}^*\mathcal{S}_2)^{\times d} \cong (\mathcal{S}_2 \times \mathbb{R}^2)^{\times d}$. The (complex) field $\psi \in L_2(\mathcal{S}_2^{\times d})$, assumed to be square-integrable, is defined as

$$\psi : [SU(2)/U(1)]^d \to \mathbb{C}$$
$$(x_1...x_D) \in [SU(2)/U(1)]^d \to \psi(x_1...x_D)$$
$$\int_{\mathcal{S}_2^{\times d}} \prod_{i=1}^{d} d^2x_i \bar{\psi}(x_1, ..., x_d)\psi(x_1, ..., x_d) < \infty$$

while its quantum dynamics is defined by the partition function

$$\mathcal{Z} = \int d\mu_C(\psi, \bar{\psi}) e^{-S_{int}(\psi,\bar{\psi})}, \tag{1}$$

where we have defined a Gaussian measure encoding the kinetic part of the classical action, and defining the free 2-point function, to be detailed below. The interaction part of the action S_{int} is constructed with all the "trace invariant" contractions

$$S_{int} = \sum_b \lambda_b \mathrm{Tr}_b(\psi, \bar{\psi}), \tag{2}$$

as graphically illustrated by the bipartite regular (i.e. strictly d-valent) graphs, with links coloured with d colors at each vertex, in the figure below, where the black and white vertices correspond to the fields ψ and $\bar{\psi}$. Indeed, it can be shown that each such trace invariant monomial of fields can be put in unique correspondence with one such bipartite coloured graph (called a "bubble"). Such precise characterisation of the allowed interactions by combinatorial properties is important for a recipe characterisation of the theory space we work with, and for the renormalisation analysis of the corresponding field theories. The parameters λ_b are the coupling constants associated to each of the interaction terms. At this stage, all such interaction bubbles are possible, but as we will see later, the renormalizability criteria drastically limit the allowed interactions, as needed to have a predictive field theory.

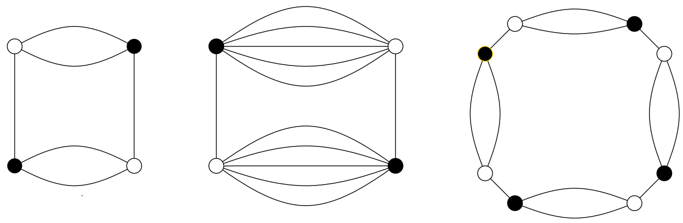

Figure 1: Example of interaction bubbles

In a coordinate system on the 2-dimensional manifold $\mathcal{S}_2$, the Gaussian measure can be defined by the choice of a kinetic action as follows

$$d\mu_C(\psi,\bar{\psi}) := e^{-S_{kin}[\bar{\psi},\psi]} d\psi d\bar{\psi} \tag{3}$$

with, in our case:

$$S_{kin}[\bar{\psi},\psi] := \int_{[\mathcal{S}_2]^D} \prod_{i=1}^{d} d^2x_i \sqrt{|g|}\bar{\psi}(\vec{x})(-\Delta + m^2)\psi(\vec{x}). \tag{4}$$

Δ is the Laplacian operator on the 2-sphere of unit radius. In an Euclidean differential manifold with metric g, the Laplacian is defined by the following generic formula in local coordinates

$$\Delta = \frac{1}{\sqrt{g}}\partial_i \left[\sqrt{g}g^{ij}\partial_j\right] \tag{5}$$

or, explicitly for the unit 2-sphere, in local spherical coordinates (θ,ϕ):

$$\Delta = \frac{\partial^2}{\partial\theta^2} + \frac{1}{\tan\theta}\frac{\partial}{\partial\theta} + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}. \tag{6}$$

As a continuous function on $\mathcal{S}_2^{\times d}$, the field ψ can be expanded in spherical harmonics $Y_{l,m}(\theta,\phi)$, which form a complete basis of L^2-functions on the 2-sphere

$$Y_{l,m}(\theta,\phi) = \sqrt{\frac{2l+1}{4\pi}\frac{(l-m)!}{(l+m)!}} P_l^m(\cos\theta)e^{im\phi}, \tag{7}$$

as

$$\psi(\{\theta_i,\phi_i\}) = \sum_{\{l_i,m_i\}} \prod_{i=1}^{D} t_{\{l_i,m_i\}} Y_{l_i,m_i}(\theta_i,\phi_i). \tag{8}$$

where the components

$$t_{\{l_i,m_i\}} = \iint \prod_{i=1}^{d} d\cos\theta_i d\phi_i Y^*_{l,m}(\theta_i,\phi_i)\psi(\{\theta_i,\phi_i\}), \tag{9}$$

are independent of the m_i. The spherical harmonics are solutions of the Laplace equation

$$-\Delta Y_{l,m}(\theta,\phi) = l(l+1)Y_{l,m}(\theta,\phi), \tag{10}$$

and we can invert the operator $-\Delta + m^2$ in the kinetic term to give the propagator (or covariance)

$$C_0(\{\theta_i,\phi_i,\theta'_i,\phi'_i\}) = \int d\mu_{C_0}\bar{\psi}(\{\theta_i,\phi_i\})\psi(\{\theta'_i,\phi'_i\}) = \sum_{\{l_i,m_i\}} C_{\{l_i,m_i\}} \prod_{i=1}^{d} Y^*_{l_i,m_i}(\theta_i,\phi_i)Y_{l_i,m_i}(\theta'_i,\phi'_i),$$

with

$$C_{0\,\{l_i,m_i\}} := \frac{1}{\sum_i l_i(l_i+1) + m^2}, \tag{11}$$

which does not depend on m_i. To define properly the theory, we introduce a cut-off regularized propagator via Schwinger regularization:

$$C_{0\,\Lambda}(\{\theta_i,\phi_i,\theta'_i,\phi'_i\}) = \int_{1/\Lambda^2}^{+\infty} d\alpha e^{-\alpha m^2} \times \sum_{\{l_i,m_i\}} \prod_{i=1}^{d} e^{-\alpha l_i(l_i+1)} Y^*_{l_i,m_i}(\theta_i,\phi_i) Y_{l_i,m_i}(\theta'_i,\phi'_i), \tag{12}$$

in terms of the heat kernel:

$$K_\alpha(\{\theta_i,\phi_i,\theta'_i,\phi'_i\}) = \sum_{\{l_i,m_i\}} \prod_{i=1}^{d} e^{-\alpha l_i(l_i+1)} Y^*_{l_i,m_i}(\theta_i,\phi_i) Y_{l_i,m_i}(\theta'_i,\phi'_i) \tag{13}$$

verifying the heat equation

$$\frac{\partial}{\partial\alpha} K_\alpha = \Delta K_\alpha \tag{14}$$

with initial conditions:

$$K_{\alpha=0}(\theta,\phi;\theta',\phi') = \delta(\cos\theta - \cos\theta')\delta(\phi-\phi'). \tag{15}$$

From the definition of the heat kernel as a sum over markovian paths on the manifold $\mathcal{S}_2$ one deduces the composition law

$$\int \sin\theta d\theta d\phi K_{\alpha_1}(\{\theta',\phi',\theta,\phi\}) K_{\alpha_2}(\{\theta,\phi,\theta'',\phi''\}) = K_{\alpha_1+\alpha_2}(\{\theta',\phi',\theta'',\phi''\}). \tag{16}$$

This is in turn the key property to obtain the expression for the Feynman amplitudes entering the perturbative expansion of the N-point correlation functions S_N in Feynman graphs (with an example given in the figure (2) below):

$$S_N = \sum_G \frac{1}{s(G)} \Big(\prod_{b\in G} \lambda_b \Big) \mathcal{A}_G. \tag{17}$$

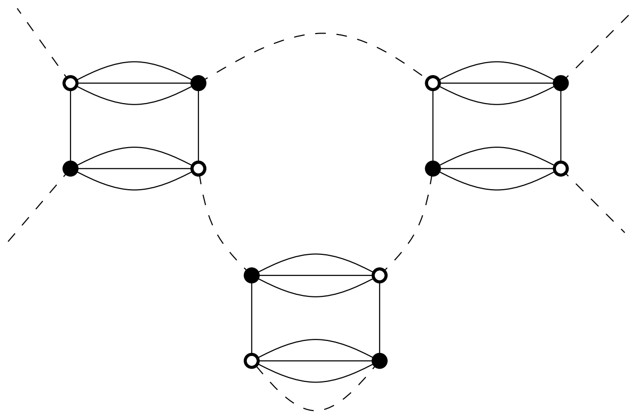

Figure 2: Example of Feynman graph

where $s(G)$, the symmetry factor, is just the dimension of the (discrete) proper symmetry group of the graph G, and the amplitude $\mathcal{A}_G$ is

$$\mathcal{A}_G = \left[\prod_{l\in L(G)} \int_{1/\Lambda^2}^{\infty} d\alpha_l e^{-\alpha_l m^2} \right] \times \prod_{f\in F(G)} \sum_{l_f} (2l_f+1) e^{-\alpha(f) l_f(l_f+1)} \tag{18}$$
$$\times \left(\prod_{f\in F_{ext}(G)} \sum_{l_f,m_f} e^{-\alpha(f)l_f(l_f+1)} Y^*_{l_f,m_f}(\theta_{s(f)},\phi_{s(f)}) Y_{l_f,m_f}(\theta_{t(f)},\phi_{t(f)}) \right),$$

where s and t map open faces to their boundary variables, and $\alpha(f) := \sum_{l\in\partial f} \alpha_l$. ϵ_{ef} is the incidence matrix, which contains the information on whether a line belongs to the boundary of a face and their relative orientation: $\epsilon_{ef} = 0$ if $e \notin \partial f$, $+1$ or -1 if $e \in \partial f$.

2.2 Closure constraint

We will impose an additional condition on our field, which we call "closure constraint". It can be understood as a gauge symmetry for the field, which reduces the manifold $\mathcal{S}_2^{\times d}$ as:

$$[SU(2)/U(1)]^d \rightarrow [SU(2)/U(1)]^d/SU(2), \tag{19}$$

identifying the field components up to a global $SU(2)$ group action. More precisely, if we denote the action of the group element $g \in SU(2)$ on the field ψ as $\hat{\mathcal{R}}(g) \triangleright \psi$, where $\hat{\mathcal{R}}$ and $\triangleright$ are defined by the explicit group action of $SU(2)$, the closure constraint identifies, for a given ψ, all the elements $\hat{\mathcal{R}}(g) \triangleright \psi \, \forall g \in SU(2)$. In other words, the projective field space $\mathbb{P}_G \wp$ is the set $\{\hat{\mathcal{R}}(SU(2)) \triangleright \psi\} =: \mathbb{P}_G L_2(\mathcal{S}_2^{\times d})$.

Let us specify further the group action. We observe that the 2-sphere admits a natural embedding in $\mathbb{R}^3$, and, using this, into $SO(3)$:

$$\pi : \mathcal{S}_2 \rightarrow SO(3) \tag{20}$$

$$(\theta, \phi) \rightarrow \pi(\theta, \phi) \in SO(3) \tag{21}$$

with the following explicit expression in local coordinates (θ, ϕ):

$$\pi(\theta, \phi)[\hat{z}] = \vec{n}(\theta, \phi) \tag{22}$$

where:

$$\vec{n} : (\theta, \phi) \rightarrow \vec{n}(\theta, \phi) := (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta) \in \mathbb{R}^3 \tag{23}$$

Hence, $\pi(\theta, \phi)$ is the rotation of $SO(3)$ mapping the $\hat{z}$ axis in the direction $\vec{n}$ [3].

Starting from our field on $\mathcal{S}_2^{\times d}$, this mapping enable us to define a new field on $SO(3)^{\times d}$, $\tilde{\psi} \in L_2(SO(3)^{\times d})$ such as $\pi^* \tilde{\psi} := \psi$.

For the field $\tilde{\psi} \in L_2(SO(3)^{\times d})$, there are a natural right action of the group $SO(3)$. Hence, we can define the gauge symmetry as the identification of all the fields up to a global right action of $SO(3)$. More concretely, we introduce the symmetric rotation $\hat{R}$ on $L_2(SO(3)^{\times d})$, such that $\hat{R}(g) = \hat{R}(-g) \, \forall g \in SU(2)$ (it is more convenient to work with a compact simply connected group). Hence, the operator $\hat{R}$ can be understood as a function on $SO(3)$. For any $\mathcal{R} \in SO(3)$ we can therefore define the transformation law:

$$\begin{aligned} \hat{R}(g) &: L_2(SO(3)^{\times d}) \rightarrow L_2(SO(3)^{\times d}) \\ [\hat{R}(g)\tilde{\psi}](\{\pi(\theta_i, \phi_i)\}) &:= \tilde{\psi}(\{\pi(\theta_i, \phi_i)\mathcal{R}(g)\}), \end{aligned} \tag{24}$$

Now, we can clarify the definition of the action $\triangleright$ introduced before. More precisely, $\hat{\mathcal{R}}(g)$ acts on ψ as:

$$\begin{aligned} \hat{\mathcal{R}}(g) \triangleright \psi(\{\theta_i, \phi_i\}) &:= \pi^*[\hat{R}(g)\tilde{\psi}](\{\theta_i, \phi_i\}) \\ &= \psi(\pi^{-1}[(\pi(\theta_i, \phi_i))\mathcal{R}(g)]), \end{aligned} \tag{25}$$

implying :

$$\hat{\mathcal{R}}(g) \triangleright \equiv \pi_*^{-1} \hat{I} \circ \hat{R}(\mathcal{R}(g))\pi_*, \tag{26}$$

$$\hat{\mathcal{R}}(g) \triangleright : \mathbb{P}_G L_2(\mathcal{S}_2^{\times d}) \rightarrow \mathbb{P}_G L_2(\mathcal{S}_2^{\times d}), \tag{27}$$

where $\mathcal{R}(g) \in SO(3)$ is the unique element of $SO(3)$ associated to $g \in SU(2)$ ($\mathcal{R}(g) = \mathcal{R}(-g)$) and $\hat{I}$ is the identity operator on $\mathbb{P}_G L_2(\mathcal{S}_2^{\times d})$.

[3] This is a unique group element, up to an initial rotation around $\hat{z}$ and a final rotation around $\vec{n}$: $\pi(\theta, \phi) \sim \mathcal{R}_{\vec{n}} \pi(\theta, \phi) \mathcal{R}_{\hat{z}}$

The gauge symmetry, or closure constraint, is simply

$$\hat{\mathcal{R}}(g) \triangleright \psi = \psi \qquad \forall g \in SU(2) \tag{28}$$

If the model is defined in terms of fields satisfying this symmetry, we can not define easily an explicit S_{kin}, because the propagator of the theory, taken to be the fundamental entity, is not, strictly speaking, invertible on the space of fields. But the Wick theorem states that the Gaussian measure, and with it, the perturbative expansion of the quantum theory, is well defined as long as the 2-point function at $\lambda_b = 0$ (the propagator $C_\Lambda(\{\theta_i, \phi_i\}, \{\theta'_i, \phi'_i\})$) is properly defined. We choose:

$$\int d\mu_C(\psi, \bar{\psi})\psi(\{\theta_i, \phi_i\})\bar{\psi}(\{\theta'_i, \phi'_i\}) = \int_{SU(2)} dg \int d\mu_{C_0}(\psi, \bar{\psi})\hat{\mathcal{R}}(g) \triangleright \psi(\{\theta_i, \phi_i\})\bar{\psi}(\{\theta'_i, \phi'_i\}), \tag{29}$$

or

$$\int d\mu_C(\psi, \bar{\psi})\psi(\{\theta_i, \phi_i\})\bar{\psi}(\{\theta'_i, \phi'_i\}) = \int_{SU(2)} dg \int_{1/\Lambda^2}^{\infty} d\alpha e^{-\alpha m^2} \prod_{i=1}^{d} K_\alpha(\{\pi^{-1}[(\pi(\theta_i, \phi_i))\mathcal{R}(g)]; \theta'_i, \phi'_i\}). \tag{30}$$

We can obtain an explicit expression for this constrained propagator. Using the expression for the heat kernel (13) and the decomposition,

$$Y_{l,m}((\pi(\theta_i, \phi_i))\mathcal{R}(g)\hat{z}) = \sum_{m'=-l}^{+l} D^{(l)}_{m'm}[\mathcal{R}(g)^{-1}\pi(\theta_i, \phi_i)^{-1}]Y_{l,m'}(\hat{z}),$$

where $D^{(l)}$ is the well known Wigner matrix defined, in the usual Dirac notation for the canonical basis of angular momentum, as:

$$D^{(l)}_{mm'}[\mathcal{R}(g)] := \langle m, l|\hat{\mathcal{R}}(g)|l, m'\rangle \quad l \in \mathbb{N},$$

we obtain, using the fact that $Y^{l\,*}_m = \left[\frac{2l+1}{4\pi}\right]^{1/2} D^{(l)}_{m0}$, and that $Y^l_m(0, \phi) = \left[\frac{2l+1}{4\pi}\right]^{1/2}\delta_{m,0}$:

$$\int d\mu_C(\psi, \bar{\psi})\bar{\psi}(\{\theta_i, \phi_i\})\psi(\{\theta'_i, \phi'_i\}) = \int_{SU(2)} dg \int_{1/\Lambda^2}^{\infty} d\alpha e^{-\alpha m^2} \times \sum_{\{l_i\}} \prod_{i=1}^{d} e^{-\alpha l_i(l_i+1)} \frac{2l+1}{4\pi} D^{(l_i)}_{00}[\mathcal{R}(g)\pi(\theta_i, \phi_i)^{-1}\pi(\theta'_i, \phi'_i)]. \tag{31}$$

Note that the integral over the group of a product of such representation matrices defines a resolution of the identity in the space of intertwiners (invariant tensors) of the group $SO(3)$:

$$\int dg \prod_i D^{(l_i)}_{m_i m'_i}[\mathcal{R}(g)] \in \mathrm{inv}(SO(3)) \quad .$$

2.3 Regularized parametric representation of correlation functions

We wish to obtain now the expression of the N-point correlation functions in perturbative expansion. The argument involving (16) is still valid, and not affected by the closure constraint. Using the "addition formula":

$$\sum_{m=-l}^{+l} Y^*_{l,m}(\theta, \phi)Y_{l,m}(\theta', \phi') = \frac{2l+1}{4\pi} P_l(\vec{u} \cdot \vec{u}'), \tag{32}$$

where $\vec{u}$ (resp $\vec{u}'$) is the unit vector pointing on the 2-sphere of radius unity in the direction (θ, ϕ) (resp (θ', ϕ')), we deduce, using the explicit expression (31), the expression for the Feynman amplitudes of the constrained theory:

$$\mathcal{A}_G = \left[\prod_{l \in L(G)} \int_{1/\Lambda^2}^{\infty} d\alpha_l e^{-\alpha_l m^2} \int_{[SU(2)]^{|L(G)|}} \prod_{l \in L(G)} dh_l \right] \tag{33}$$
$$\times \left(\prod_{f \in F_{int}(G)} \sum_{l_f} (2l_f + 1) D_{00}^{(l_f)} \left[\mathcal{R}\left(\Pi_{l \in \partial f} h^{\epsilon_{ef}} \right) \right] e^{-\alpha(f) l_f (l_f+1)} \right)$$
$$\times \left(\prod_{f \in F_{ext}(G)} \sum_{l_f} e^{-\alpha(f) l_f (l_f+1)} \frac{2l_f + 1}{4\pi} D_{00}^{(l_f)} \left(\mathcal{R}\left(\Pi_{l \in \partial f} h^{\epsilon_{ef}} \right) \pi(\theta_{s(f)}, \phi_{s(f)})^{-1} \pi(\theta_{t(f)}, \phi_{t(f)}) \right) \right).$$

From this amplitude, we deduce the following proposition:

Proposition 1 *: The amplitude $\mathcal{A}_G$ for a connected graph G has a $SO(3)^{\times |V(G)|}$ gauge symmetry, which allows to fix variables along a spanning tree $\mathcal{T} \subset G$, such as $h_e = 1 \, \forall e \in L(\mathcal{T})$.*

Proof: Simply note that the expression **??** is invariant under the transformation:

$$h_e \to g_{t(e)} h_e g_{s(e)}^{-1}, \tag{34}$$

where $t(e)$ and $s(e)$ are the target and source vertex of an oriented edge e (with the additional rule that one of the two group elements is the identity for open lines). Because of this invariance, $|V(G)|$ gauge variables can be freely redefined, one more than the $|V(G)| - 1$ lines of a spanning tree of the graph.

□

We also deduce immediately the following lemma,

Lemma 1 *: Let a spanning tree $\mathcal{T} \subset G$ of a connected graph G, and $R(G)$ the rank of the incidence matrix ϵ_{ef}. Then, under the graph contraction $G \to G/\mathcal{T}$ the rank is unchanged.*

We will recall the precise definition of contraction in a colored graph in section 5.

3 Equivalent formulation via projections

The problem of the above formulation, defined directly on the homogeneous space, is that we lose the explicit group structure of the group field theory. This leads to some practical difficulties in dealing with the theory, in particular in studying the divergence structure of its Feynman amplitudes and its renormalisability, following what has been done in previous works. These difficulties are mainly due to the fact that elements of the homogeneous space do not compose via multiplication to other elements of the homogeneous space. The way to proceed is to recast the field theory as a field theory on (several copies of) SU(2), but with the fields subject to constraints effectively projecting them to the homogeneous space. This way one can perform all calculations using the standard SU(2) formalism. This is indeed well-known and already used in the GFT formulation of constrained spin foam models for 4d quantum gravity, in particular the BC model[23, 24, 25, 26]. An ensuing subtle point is that special care should be payed to the compatibility between the constraints projecting the field onto the homogeneous space and the gauge invariance condition to be satisfied by the same fields. more precisely, the constraints have to be imposed *covariantly* with respect to the diagonal group action. This was also realised in the context of GFTs and spin foam models for 4d quantum gravity [27, 23, 7], and a properly covariant construction was identified, which we now describe in some detail.

3.1 Constrained representation

We choose an element of the Lie algebra $su(2)$, σ_z for instance, and note that the set of group elements $g \in SU(2)$, such as $g\sigma_z g^{-1} = \sigma_z$, the stabilizer group, is isomorphic to the group $U(1)$:

$$\mathcal{H}_{\sigma_z} := \{g = e^{i\theta\sigma_z} \; \forall\theta \in [0, 2\pi[\,\} \sim U(1). \tag{35}$$

Now we can simply define a field theory for a new field $\Psi : SU(2)^{\times d} \to \mathbb{C}$ with the constraint :

$$\Psi(g_1, ...g_i, ..., g_d) = \Psi(g_1 h_1, ..., g_i h_i, ...g_d h_d) \; \forall (h_1...h_d) \in \mathcal{H}_{\sigma_z}^{\times d}. \tag{36}$$

For this new field we define the partition function:

$$\mathcal{Z} = \int d\mu_{\tilde{C}_\Lambda}(\Psi, \bar{\Psi}) e^{-S_{int}(\Psi,\bar{\Psi})}, \tag{37}$$

where, as in the previous construction, S_{int} is a sum of tensorial invariants. The only difference between the two formulations is that in the first one, the fields have $2d$ variables, while the new field has $3d$ variables, with a constraint which reduces the number of degrees of freedom from $3d$ to $3d - d = 2d$, so that we left in the end with the same degrees of freedom.

The covariance $\tilde{C}_\Lambda$ for this model is defined as:

$$\int d\mu_{\tilde{C}_\Lambda} \Psi(\{g_i\}) \bar{\Psi}(\{g_i'\}) := \int_{\mathcal{H}_{\sigma_z}^{\times d}} \prod_{i=1}^{d} dh_i \int_{1/\Lambda^2}^{+\infty} d\alpha e^{-\alpha m^2} \prod_{i=1}^{d} K_\alpha(g_i h_i g_i'^{-1}). \tag{38}$$

from which we deduce the Feynman expansion of a N-point function S_N, indexed by graphs G:

$$\mathcal{A}_G = \left[\prod_{e\in\mathcal{L}(G)} \int_{1/\Lambda^2}^{\infty} d\alpha_e e^{-\alpha_e m^2} \prod_{i=1}^{d} dh_{i_e}\right] \tag{39}$$

$$\times \left(\prod_{f\in F(G)} K_{\alpha(f)}\left(\vec{\prod}_{e\in\partial f} h_{i(f)_e}^{\epsilon_{ef}}\right)\right) \times \left(\prod_{f\in F_{ext}(G)} K_{\alpha(f)}\left(g_{s(f)} \vec{\prod}_{e\in\partial f} h_{i(f)_e}^{\epsilon_{ef}} g_{t(f)}^{-1}\right)\right),$$

where $i(f)$ is the color of the face f and K_α is the solution of the heat equation on $SU(2)$ (given by the same formula (14) with Δ replaced by the Laplace operator on $SU(2)$). This equation replaces the formula (39).

We now confirm briefly the equivalence of the two constructions at the dynamical level. This can be seen immediately noting that the spherical harmonics are just the Wigner representation matrices for $SU(2)$ integrated over a one-dimensional subgroup isomorphic to $U(1)$. Indeed, the heat kernel is a class function on $SU(2)$ and, by virtue of the Peter-Weyl theorem, it can be expanded on the (class invariant) basis of characters as:

$$K_\alpha(g_1 g_2^{-1}) := \sum_{j\in\mathbb{N}/2} (2j+1) e^{-4\alpha j(j+1)} \chi^j(g_1 g_2^{-1}), \tag{40}$$

where the characters $\chi^j := \mathrm{Tr}_j D^{(j)}$ of the irreducible representation j, verify :

$$\Delta_{SU(2)} \chi^j(g) = -4j(j+1)\chi^j(g). \tag{41}$$

Now, in the Euler angles parametrization

$$\chi^j(g e^{i\sigma_z\theta}) = \sum_m D_{mm}^{(j)}(g e^{i\sigma_z\theta}) = \sum_m \langle m, j| e^{i\gamma J_z} e^{i\beta J_y} e^{i(\alpha+\theta)J_z} |j, m\rangle,$$

and:

$$\int_0^{2\pi} \frac{d\theta}{2\pi} \chi^j(g e^{i\sigma_z\theta}) = \langle 0, j | e^{i\beta J_y} | j, 0\rangle = D^{(j)}_{00}(g).$$

Note that because $m = 0$, j is necessarily an integer. When applying the previous result to $\int d\theta \chi(g_1 e^{i\theta\sigma_z} g_2^{-1})$, we find:

$$\int_0^{2\pi} \frac{d\theta}{2\pi} \chi^j(g_1 e^{i\sigma_z\theta} g_2^{-1}) = D^{(j)}_{00}(g_2^{-1}g_1) = \sum_m D^{(j)}_{0m}(g_2^{-1}) D^{(j)}_{m0}(g_1) = \sum_m D^{(j)*}_{m0}(g_2) D^{(j)}_{m0}(g_1).$$

Hence, because of the relation : $Y^{l*}_m = \left[\frac{2l+1}{4\pi}\right]^{1/2} D^{(l)}_{m0}$, the equivalence between the two representations (up to a change of normalization of α and m : $\alpha \to \alpha/4$, $m \to 2m$) follows easily.

We now turn to the imposition of the gauge invariance (closure) constraint in this formulation. The aim is to combine the constraint (36) with a global constraint of the form $\psi(g_1, ...g_d) = \psi(g_1 l, ..., g_d l)\, \forall l \in SU(2)$. We first define the two transformations

$$\hat{T}_l : \Psi(g_1, ..., g_d) \to \Psi(g_1 l, ..., g_d l) \tag{42}$$

$$\hat{t}^{(i)}_{h_i} : \Psi(g_1, ..., g_d) \to \Psi(g_1, ..., g_i h_i, ..., g_d), \tag{43}$$

satisfying :

$$\hat{T}_l \circ \hat{t}^{(i)}_{h_i} = \hat{t}^{(i)}_{l^{-1}h_i l} \circ \hat{T}_l \tag{44}$$

Hence, by defining

$$\Psi_{\sigma_z}(g_1, ..., g_d) := \int_{\mathcal{H}_{\sigma_z}^{\times d}} \prod_{i=1}^d dh_i \hat{t}^{(i)}_{h_i}[\Psi](g_1, ..., g_d), \tag{45}$$

we have:

$$\hat{T}_l[\Psi_{\sigma_z}](g_1, ..., g_d) = \Psi_{l^{-1}\sigma_z l}(g_1, ..., g_d), \tag{46}$$

with, for any $k \in su(2)$:

$$\Psi_k(g_1, ..., g_d) := \int_{\mathcal{H}_k^{\times d}} \prod_{i=1}^d dh_i \hat{t}^{(i)}_{h_i}[\Psi](g_1, ...g_d). \tag{47}$$

Then, to include the closure constraint, we recast the theory in terms of the field: $\Psi_k(\{g_i\})$ from $[SU(2)]^d \times \mathfrak{su}(2)$ to $\mathbb{C}$, and impose the constraint:

$$\Psi_k(g_1, g_2, ..., g_d) = \Psi_{h^{-1}kh}(g_1 h, ..., g_d h). \tag{48}$$

We define the partition function:

$$\mathcal{Z} = \int d\mu_C(\Psi, \bar{\Psi}) e^{-S_{int}(\Psi, \bar{\Psi})}, \tag{49}$$

with an interaction of the form:

$$S_{int}(\Psi, \bar{\Psi}) = \sum_b \lambda_b \mathrm{Tr}_b \left(\int_{su(2)} dk \Psi_k, \int_{su(2)} dk' \bar{\Psi}_{k'} \right), \tag{50}$$

and the propagator:

$$\int d\mu_C(\Psi, \bar{\Psi}) \Psi_k(\{g_i\}) \bar{\Psi}_{k'}(\{g_i'\}) = \delta_{k,k'} \int_{SU(2)} dl \int d\mu_{\tilde{C}}(\Psi, \bar{\Psi}) \Psi_k(\{g_i l\}) \bar{\Psi}_k(\{g_i'\}), \tag{51}$$

which gives, explicitly, using (38),

$$\int d\mu_C(\Psi,\bar{\Psi})\Psi_k(\{g_i\})\bar{\Psi}_{k'}(\{g'_i\}) := \delta_{k,k'}\int_{SU(2)} dl \int_{\mathcal{H}^{\times d}_{l^{-1}kl}} \prod_{i=1}^{d} dh_i dh'_i \int_{1/\Lambda^2}^{+\infty} d\alpha e^{-\alpha m^2}\prod_{i=1}^{d} K_\alpha(g_i h_i l h_i'^{-1} g_i'^{-1})$$

$$:= \delta_{k,l^{-1}k'l}\int_{SU(2)} dl \int_{\mathcal{H}^{\times d}_{k}} \prod_{i=1}^{d} dh_i \int_{1/\Lambda^2}^{+\infty} d\alpha e^{-\alpha m^2}\prod_{i=1}^{d} K_\alpha(g_i l h_i g_i'^{-1}) \quad . \tag{52}$$

Because of the form of the action (50), we can define a new "effective" field at the graph level (we sum over all unit vectors k),

$$\psi = \int_{\mathcal{S}_2} dk \Psi_k : [SU(2)]^{\times d} \to \mathbb{C}, \tag{53}$$

satisfying the closure constraint, as it can be easily proven, using the invariance of the Haar measure. Let $h \in SU(2)$,

$$\begin{aligned}\hat{T}_h[\psi](g_1,...,g_d) &= \int_{su(2)} dk \Psi_k(g_1 h,...,g_d h) \\ &= \int_{su(2)} dh^{-1}kh \Psi_{h^{-1}kh}(g_1 h,...,g_d h) \\ &= \int_{su(2)} dhkh^{-1} \Psi_k(g_1,...,g_d) \\ &= \int_{su(2)} dk \Psi_k(g_1,...,g_d) = \psi(g_1,...,g_d),\end{aligned} \tag{54}$$

and has the covariance:

$$\int d\mu_C(\Psi,\bar{\Psi})\psi(\{g_i\})\bar{\psi}(\{g'_i\}) = \int dk \int_{SU(2)} dl \int_{\mathcal{H}^{\times d}_{k}} \prod_{i=1}^{d} dh_i \int_{1/\Lambda^2}^{\infty} d\alpha e^{-\alpha m^2}\prod_{i=1}^{d} K_\alpha(g_i l h_i (g'_i)^{-1}). \tag{55}$$

Using the same strategy as in the previous section, we find the Feynman amplitude $\mathcal{A}_G$ entering the expansion of the N-point function:

$$\begin{aligned}\mathcal{A}_G =& \Big[\prod_{e\in L(G)} \int d\alpha_e e^{-\alpha_e m^2}\int dl_e \int dk_e \prod_{i=1}^{d} D^{k_e} h_{i,e}\Big] \\ &\times \Big(\prod_{f\in F(G)} K_{\alpha(f)}\big(\prod_{e\in\partial f}(l_e h_{i(f)})^{\epsilon_{ef}}\big)\Big) \\ &\times \Big(\prod_{f\in F_{ext}(G)} K_{\alpha(f)}\big(\prod_{e\in\partial f} g_{s(f)}(l_e h_{i(f)})^{\epsilon_{ef}} g^{-1}_{t(f)}\big)\Big)\end{aligned} \tag{56}$$

where s and t map open faces to their boundary variables, ϵ is the adjacency matrix, $i(f)$ is the "color" of the face f, and

$$D^k l_i := dl_i \delta(k - l_i k (l_i)^{-1}) \tag{57}$$

which reduces the integration over $SU(2)^{\times d}$ to $U_k(1)^{\times d} \equiv \mathcal{H}_k^{\times d}$. Because of the integration over k_e, we deduce that the graph amplitude has the same gauge invariance as the amplitude in the first formulation. Hence, the proposition 1 and its corollary 1 hold true.

This formulation is more convenient for the study of the renormalizability of the model, and it also lends itself more easily to generalisation to other homogeneous spaces $SO(D)/SO(D-1) \simeq \mathcal{S}_{D-1}$, making clearer the role of the group manifold dimension parametrized by D.

3.2 Geometrical interpretation

Before we move to the renormalisation of this model, let us give some more information on its geometric interpretation, which also motivates its interest from a quantum gravity perspective. The closure constraint admits a ‘geometrical interpretation which can be easily understood with the mathematical tool of the non-commutative (group) Fourier transform originated in the quantum group literature [28, ?], introduced in the GFT context in [6], after being first used in the spin foam context in [29], and developed, in particular for the case of $SU(2)$, from the more mathematical perspective in [28, 30, 31]. This non-commutative Fourier Transform is a functional mapping from a (usually but not necessarily) compact group G into its Lie algebra $\mathfrak{g}$, sending any square-integrable function on G to a non-commutative function on $\mathfrak{g}$. For the group $SU(2)$, the mapping is between $SU(2)$ and the $\mathbb{R}^3$ space, dual to its Lie algebra $\mathfrak{su}(2)$. Consider a function ϕ on $SU(2)$, its Fourier transform is defined as:

$$\hat{\phi} := \int_{SU(2)} dg\phi(g)e^{\mathrm{Tr}(|g|x)} \qquad x \in \mathfrak{su}(2), \tag{58}$$

where Tr is the trace in the fundamental representation and $|g| = \mathrm{Sign}\big[\mathrm{Tr}(g)\big]g$, ensuring that the basis functions e_g are trivially on $SO(3)$, because : $e_g = e_{-g}$. Note that this condition also concerns the function ϕ: $\phi(g) = \phi(-g)$, which can be understood as a field on $SO(3)$.

The inverse Fourier Transform is formally given by:

$$\phi(g) = \frac{1}{\pi}\int_{\mathbb{R}^3} d^3x\big[\hat{\phi} \star e_{g^{-1}}\big](x), \tag{59}$$

where the $\star$-product is dual to the convolution product on $SU(2)$:

$$\hat{\phi} \star \hat{\psi}(x) = \int_{SU(2)} dg\, e_g(x)\big(\phi \circ \psi\big)(g)$$

and is compatible with the group structure, in the sense that:

$$e_{g_1} \star e_{g_2}(x) = e_{g_1 g_2}(x), \quad \forall g_1, g_2 \in SU(2). \tag{60}$$

The Fourier Transform can easily be extended to any function on $[SU(2)]^d$ as:

$$\hat{\phi}(x_1, ..., x_d) := \int_{[SU(2)]^d} [dg]^d\, \phi(g_1, ..., g_d) \prod_{i=1}^{d} e_{g_i}(x_i). \tag{61}$$

It can thus be immediately applied to the GFT field Ψ_k introduced above, and verifying the constraints (42) and (43). Note that, because of the symmetry $\hat{T}_l[\Psi_k] = \Psi_k$, the field Ψ_k verifies : $\Psi_k(\vec{g}) = \Psi_k(-\vec{g})$, and is effectively a field on $[SO(3)]^d$.

We now have a look at the constraints (42) and (43) successively. Starting with (42), we find:

$$\begin{aligned} &\int_{[SU(2)^d]} [dg]^d \int_{SU(2)} dl\, \hat{T}_l[\Psi_k](g_1, ..., g_d) \prod_{i=1}^{d} e_{g_i}(x_i) \\ &= \int_{[SU(2)^d]} [dg]^d \Psi_k(g_1, ..., g_d) \prod_{i=1}^{d} e_{g_i}(x_i)\, \vec{\star}\, \delta_0\left(\sum_{i=1}^{d} x_i\right) \\ &= \hat{\Psi}_k(x_1, ..., x_d)\, \vec{\star}\, \delta_0\left(\sum_{i=1}^{d} x_i\right). \end{aligned} \tag{62}$$

where the $\vec{\star}$-product distributes the $\star$-product between all the e_g basis functions, and:

$$\delta_0\left(\sum_{i=1}^{d} x_i\right) := \int_{SU(2)} dl \prod_{i=1}^{d} e_l(x_i), \tag{63}$$

verifying, for any function of one variable $\hat{\phi}$,

$$\int d^3y(\delta_0 \star \hat{\phi})(y) = \int d^3y(\hat{\phi} \star \delta_0)(y) = \hat{\phi}(0). \tag{64}$$

Hence, the representation of the right projector $\hat{P} = \int dl\hat{T}_l$ is a simple non-commutative multiplication:

$$\hat{P}[\hat{\Psi}_k](x_1, ..., x_d) := \hat{\Psi}_k(x_1, ..., x_d) \,\vec{\star}\, \delta_0\left(\sum_{i=1}^{d} x_i\right) \quad . \tag{65}$$

We now move on to the constraint (43). Consider the operators $\hat{t}^{(i)}_{h_i}$, acting on the i-th variable of Ψ_k. We wish to compute the Fourier Transform of $\prod_i \int_{h_i \in U_k(1)} dh_i \hat{t}^{(i)}_{h_i}[\Psi_k]$, and we find that it is proportional to:

$$\int_{[SU(2)^d]} [dg]^d \Psi_k(g_1, ..., g_d) \prod_{i=1}^{d} e_{g_i}(x_i) \,\vec{\star} \prod_{i=1}^{d} \delta_0\left(\frac{1}{2}\mathrm{Tr}(kx_i)k\right) = \hat{\Psi}_k(x_1, ..., x_d) \,\vec{\star} \prod_{i=1}^{d} \delta_0\left(\frac{1}{2}\mathrm{Tr}(kx_i)k\right), \tag{66}$$

which follows from the definition of the basis functions e_g, and in particular :

$$\int_{U_k(1)} dh\mathrm{Tr}(hx) = \int_{U_k(1)} dh\mathrm{Tr}\Big[h\Big(\frac{1}{2}\mathrm{Tr}(kx)k\Big)\Big] = \int_{SU(2)} dh\mathrm{Tr}\Big[h\Big(\frac{1}{2}\mathrm{Tr}(kx)k\Big)\Big],$$

and the projector $\hat{S}_k := \prod_i \int_{h_i \in U_k(1)} dh_i \hat{t}^{(i)}_{h_i}$ acts on Ψ_k as:

$$\hat{S}_k[\Psi_k] := \hat{\Psi}_k(x_1, ..., x_d) \,\vec{\star} \prod_{i=1}^{d} \delta_0\left(\frac{1}{2}\mathrm{Tr}(kx_i)k\right) \quad . \tag{67}$$

From the definition (1), it follows that $\Psi_k = \hat{S}_k[\Psi]$, and the non-commutativity (44) implies:

$$\hat{S}_k \circ \hat{P} = \int dl\hat{T}_l \circ \hat{S}_{lkl^{-1}}. \tag{68}$$

The first result (62) explains why the constraint (42) is named "the closure constraint". The second result (66) means that the Fourier variables are forced to be orthogonal to the Lie algebra index k. In the case $d = 3$, the field (quanta) can be interpreted as describing a triangle in $\mathbb{R}^3$, with its Lie algebra variables being its edge vectors. The closure constraint forces in fact these edge vectors to sum to zero, thus the corresponding edges to "close". The constraint (43) implies that this triangle is orthogonal to the unit 3-vector $\vec{k}$, associated with the index $k \in \mathfrak{su}(2)$ of the field. Hence, for example, an interaction represented by the diagram on the left hand side of figure 1 is dual to a topological 2-sphere, triangulated by four triangles, each orthogonal to a vector k, defining the radial direction.

3.3 Extension to $[SO(D)/SO(D-1)]^{\times d}$

In this paper we focus on the field theory on $\mathcal{S}_2^d$, and on its renormalisation. However, most of our construction as well as part of the renormalizability analysis, can easily be extended to the homogeneous space $[SO(D)/SO(D-1)]^{\times d}$, using the projector formulation introduced above. In this section, we reframe the essential results obtained in the previous section for the homogeneous space $[SO(D)/SO(D-1)]^{\times d}$. The extension is straightforward, therefore we give only the essential steps, without too many details.

Let $\{(L_{\mu\nu})_{\rho\sigma}\}$, a basis of anti-symmetric $D\times D$ matrices of the Lie Algebra $\mathfrak{so}(D)$, and $k=\{k_\mu\}$ an unit vector of $\mathbb{R}^D$ (the Greek indices run over $1,...,D$ and label the Euclidean coordinates on $\mathbb{R}^D$). Any element $g\in SO(D)$ can be written as (we use the Einstein convention for sums over Greeks indices):

$$g = e^{\Omega_{\mu\nu}L_{\mu\nu}}, \tag{69}$$

and any element h of the stabilizer group of k, isomorphic to $SO(D-1)$ (denoted $SO_k(D-1)$), can be written as:

$$h = e^{\Omega_{\mu\nu}\mathcal{P}^k_{\mu\mu'}\mathcal{P}^k_{\nu\nu'}L^k_{\mu'\nu'}} \in SO_k(D-1) \tag{70}$$

where $\mathcal{P}^k = \mathbb{I} - k\otimes k$, $(\mathcal{P}^k)^2 = \mathcal{P}^k$ is the projector onto the subspace orthogonal to k. As in section **??**, we define a field theory on $SO(D)^{\times d}$ as a map $\Psi : SO(D)^{\times d} \to \mathbb{C}$, and we reduce the manifold to the homogeneous space $[SO(D)/SO(D-1)]^{\times d}$ imposing the constraint

$$\Psi(g_1,...,g_i,...,g_d) = \Psi(g_1h_1,...,g_ih_i,...,g_dh_d) \quad \forall (h_1,...,h_d)\in [SO_k(D-1)]^{\times d} \quad . \tag{71}$$

At the quantum level, the theory is defined by the choice of a partition function, or in other worlds, by the choice of an action S_{kin} and of a (UV regularized) Gaussian measure $d\mu_{\tilde{C}_\Lambda}$, in the notation of section **??**. As before, the action is a sum of tensorial invariants, built again as in correspondence with coloured bipartite graphs (bubbles). We choose the propagator such as it enforces the constraint (71) at the level of the amplitudes. Similarly to (38), we choose:

$$\int d\mu_{\tilde{C}_\Lambda}\Psi(\{g_i\})\bar{\Psi}(\{g'_i\}) := \int_{SO_k(D-1)^{\times d}} \prod_{i=1}^d dh_i \int_{1/\Lambda^2}^{+\infty} d\alpha e^{-\alpha m^2} \prod_{i=1}^d K_\alpha(g_ih_ig_i'^{-1}) \quad . \tag{72}$$

The closure constraint is then implemented as in section **??**. We first define the operators $\hat{T}_l$ and $\hat{t}^{(i)}_{h_i}$

$$\hat{T}_l : \Psi(g_1,...,g_d) \to \Psi(g_1l,...,g_dl) \tag{73}$$

$$\hat{t}^{(i)}_{h_i} : \Psi(g_1,...,g_d) \to \Psi(g_1,...,g_ih_i,...,g_d) \quad , \tag{74}$$

satisfying again : $\hat{T}_l \circ \hat{t}^{(i)}_{h_i} = \hat{t}^{(i)}_{l^{-1}h_il} \circ \hat{T}_l$. Then, defining

$$\Psi_k(g_1,...,g_d) = \int_{SO_k(D-1)^d} \prod_{i=1}^d dh_i \hat{t}^{(i)}_{h_i}[\Psi](g_1,...,g_d) \quad , \tag{75}$$

we have again

$$\hat{T}_l[\Psi_k](g_1,...,g_d) = \Psi_{\mathcal{R}_l^{-1}[k]}(g_1,...,g_d) \quad , \tag{76}$$

where $\mathcal{R}_l^{-1}[k]$ is the vector k rotated by the $SO(D)$ element l. All the equations after (49) can be applied to this more general case without any change. We define the partition function as

$$\mathcal{Z} = \int d\mu_C(\Psi,\bar{\Psi}) e^{-S_{int}(\Psi,\bar{\Psi})} \quad , \tag{77}$$

with the action

$$S_{int}(\Psi,\bar{\Psi}) = \sum_b \lambda_b \mathrm{Tr}_b\left(\int_{su(2)} dk\Psi_k, \int_{su(2)} dk'\bar{\Psi}_{k'}\right), \tag{78}$$

and the gauge invariant propagator

$$\int d\mu_C(\Psi,\bar{\Psi})\Psi_k(\{g_i\})\bar{\Psi}_{k'}(\{g'_i\}) = \delta_{k,k'}\int_{SO(D)} dl \int d\mu_{\tilde{C}}(\Psi,\bar{\Psi})\Psi_k(\{g_il\})\bar{\Psi}_k(\{g'_i\}) \quad , \tag{79}$$

imposing the closure constraint in each Feynman amplitude. And again we can introduce the effective field ψ:

$$\psi := \int dk \Psi_k : [SO(D)]^d \to \mathbb{C}, \tag{80}$$

which satisfies the closure constraint $\hat{T}_h[\psi](g_1, ..., g_d) = \psi(g_1, ..., g_d) \, \forall h \in SO(D)$, and whose effective propagator can be written as:

$$\int d\mu_C(\Psi, \bar{\Psi}) \psi(\{g_i\}) \bar{\psi}(\{g'_i\}) = \int dk \int_{SU(2)} dl \int_{\mathcal{H}_k^{\times d}} \prod_{i=1}^{d} dh_i \int_{1/\Lambda^2}^{\infty} d\alpha e^{-\alpha m^2} \prod_{i=1}^{d} K_\alpha(g_i l h_i (g'_i)^{-1}) \quad . \tag{81}$$

The Feynman amplitudes for the corresponding TGFT model take then the form (56).

4 Multiscale analysis

In this section we explore the power counting for the divergences of the theory, in order to find renormalizability criteria that would allow to identify the renormalizable interactions (including the super-renormalizable ones). We focus on the $SU(2)/U(1)$ model, but we also try to extend our results to the general $SO(D)/SO(D-1)$ case, whenever possible. We begin by studying the divergences in the Abelian approximation, expected to be optimal from the results obtained recently in [19]. We will give some additional arguments in favour of this intuition in section 3.3, and we will see that the Abelian power counting becomes exact, for the $SU(2)/U(1)$ model, in the next section. We also point out why the same arguments do not generalise trivially to arbitrary dimension D, and what needs to be understood in order to achieve such generalisation.

4.1 Abelian power counting for the $[SU(2)/U(1)]^d$ model

The multiscale analysis is based on the following slice decomposition of the propagator,

$$C_\Lambda = \sum_{i<\rho} C_i \tag{82}$$

where the cut-off Λ is chosen of the form $\Lambda = M^\rho$, $M > 1$, and the propagator "in the slice i" C_i is

$$C_i = \int dk \int_{SU(2)} dh \int_{[SU(2)]^d} \prod_{j=1}^{d} dl_j \delta(k - l_j k(l_j)^{-1}) \int_{M^{-i}}^{M^{-(i-1)}} d\alpha e^{-\alpha m^2} \prod_{i=1}^{d} K_\alpha(g_i h l_i (g'_i)^{-1}). \tag{83}$$

We will prove the following key theorem, given the power counting of the theory and a divergence criterion for a graph amplitude, which is the first step of the perturbative renormalizability analysis at all orders.

Theorem 1 *Consider the previous model on $[SU(2)/U(1)]^{\times d}$ with diagonal $SU(2)$-gauge invariance. The amplitude $\mathcal{A}_{G,\mu}$ for a graph G and scale assignment $\mu = \{i_{l_1}, ..., i_{l_{|L(G)|}}\} \, l_i \in L(G)$, admits the following bound:*

$$|\mathcal{A}_{G,\mu}| \leq K^{|L(G)|} \prod_i \prod_{k=1}^{k(i)} M^{\omega(G_i^k)}, \tag{84}$$

where G_i^k is the k-th connected component of the sub-graph $G_i \subset G$, which contains only the lines of the graph G with a slice $i_l \geq i$, and where the divergence degree $\omega(G_i^k)$ is given by:

$$\omega(G_i^k) = -2|L(G_i^k)| + 2(|F(G_i^k)| - R(G_i^k)). \tag{85}$$

Proof:
The first step is to bound the heat kernel. The heat kernel $K_\alpha(g)$ on $SU(2)$ has a complicated expression (see for example [32]). However, it can be approximated in the "UV" regime, i.e. for large representation labels, by the following bound:

$$C_i(\{g_j\},\{g'_j\}) \leq KM^{(3d-2)i} \int dk \int_{SU(2)} dh \int_{[U_k(1)]^d} \prod_{j=1}^{d} dl_j e^{-\delta M^i \sum_{j=1}^{d} |g_j h l_j g_j'^{-1}|}. \qquad (86)$$

where here $|g_1 g_2^{-1}|$ indicates the geodesic distance (using the standard metric on $SU(2) \simeq \mathcal{S}_3$) between the two group elements g_1 and g_2, and δ, K are two positive constants which can be precisely computed (the values of these constants do not affect the proof).

This result allows us to bound the (multi-)scale decomposition $\mathcal{A}_{G,\mu}$ of the amplitude. The first step is to rewrite in a suitable manner the term $\prod_{l\in L(G)} M^{(3d-2)i_l}$. To this end, note that, trivially: $M^i = \prod_i M$. This allows to rewrite the product over the lines of the graph as: $\prod_{l\in L(G)} M^{(3d-2)i_l} = \prod_{l\in L(G)} \prod_{i=1}^{i_l} M^{(3d-2)}$. Now, we wish to invert the order of the double product. Selecting a scale-assignment i, and a subset of lines in G so that, for each of these lines, the scale assignment is higher than or equal to i, we define the subgraph G_i of G. It follows that

$$\prod_{l\in L(G)} M^{(3d-2)i_l} = \prod_{l\in L(G)} \prod_{i=1}^{i_l} M^{(3d-2)} = \prod_i \prod_{l\in L(G_i)} M^{(3d-2)} \quad .$$

Because the graph G_i is not necessarily connected, we introduce the notation G_i^k for its connected components,so that $G_i = \cup_{k=1}^{k(i)} G_i^k$. It follows that the previous decomposition becomes

$$\prod_{l\in L(G)} M^{(3d-2)i_l} = \prod_i \prod_{l\in L(\cup_{k=1}^{k(i)} G_i^k)} M^{(3d-2)} = \prod_i \prod_{k=1}^{k(i)} \prod_{l\in L(G_i^k)} M^{(3d-2)} = \prod_i \prod_{k=1}^{k(i)} M^{(3d-2)L(G_i^k)} \quad .$$

The second step is to integrate over the group variables along each face of the graph. Using the same decomposition and the compactness of the group $U(1)$, we obtain the following contribution for the internal faces:

$$\prod_i \prod_{k=1}^{k(i)} M^{-3d|L(G_i^k)|+3|F(G_i^k)|} \quad . \qquad (87)$$

Note that, to obtain this formula, we have chosen an optimal tree in each face on which we perform the integrations over the angle variables. This result, combined with the first one provided by the factors $M^{(3d-2)i}$ gives

$$\prod_i \prod_{k=1}^{k(i)} M^{-2|L(G_i^k)|+3|F(G_i^k)|} \quad . \qquad (88)$$

The third and last contribution comes from the remaining integrals

$$\int \prod_e dh_e dk_e \prod_f dl_{k_e}^{c(f)} e^{-\delta M^{i(f)} |\prod_{e\in\partial f} (h_e l_{k_e}^{c(f)})^{\epsilon_{ef}}|} \quad , \qquad (89)$$

and it is at this point that the Abelian approximation intervenes. As showed in [19] for an $SU(2)$ TGFT, the exact power counting is uniformly bounded by its Abelian version, which corresponds to the linearized version of the exact one around identity for all the group elements (i.e. the non-commutativity of the group variables improves the convergence of a graph amplitude compared to the Abelian version). The proof of this fact for our model is the subject of the next section. As we will see below, the Abelian power counting coincides also in our case with the non-Abelian one (for the $SU(2)$-based model) for the leading order (or divergent) graphs, which are the graphs of interest for renormalization.

The Abelian version 89 is

$$\int \prod_{e\in L(G)} d\vec{\lambda}_e dk_e \int \prod_f d\theta_{k_e}^{c(f)} e^{-\delta M^{i(f)} |\sum_e \epsilon_{ef}(\vec{\lambda}_e + \theta_{k_e}^{c(f)} \vec{e}_{k_e})|}. \tag{90}$$

Where $i(f) := \inf\{i_l,\, l \in \partial f\}$ and $c(f)$ is the color of the face f, and where $\vec{e}_k$ is the unit $3d$ vector associated to the unit Lie algebra element k and $|\vec{q}| := \sqrt{\sum_{j=1}^d q_j^2}$ is the $\mathbb{R}^3$ norm (we use here explicitly the trivial isomorphism between the elements of the Lie algebra $su(2)$, and the vectors of $\mathbb{R}^3$).
Integrating over a selected tree $\mathcal{T}_2$ of faces, such that the number of faces in this set equals the rank of the incidence matrix, and in an optimal way, in the sense that the faces of this set proceed recursively from the leaves to the root of the Gallavotti-Nicoló tree, the integral (90) over the $\vec{\lambda}$ and θ variables gives the power counting contribution

$$\prod_i \prod_{k=1}^{k(i)} M^{-3R(G_i^k)}, \tag{91}$$

and the remaining integration

$$\int \prod_{e\in L(G)} dk_e \int \prod_{f\in F/\mathcal{T}_2} d\theta_{k_e}^{c(f)} e^{-\delta M^{i(f)} |\sum_{e\in\partial f} \epsilon_{ef} \theta_{k_e}^{c(f)}|} = \int \prod_{f\in F/\mathcal{T}_2} \int \prod_{e\in\partial f} d\theta_{k_e}^{c(f)} dk_e e^{-\delta M^{i(f)} |\sum_{e\in\partial f} \epsilon_{ef} \theta_{k_e}^{c(f)}|}, \tag{92}$$

gives, up to a positive constant,

$$\prod_{f\in F/\mathcal{T}_2} M^{-i(f)} = \prod_i \prod_k \prod_{f\in F/\mathcal{T}_2(G_i^k)} M^{-1} = \prod_{i,k} M^{-|F(G_i^k)| + R(G_i^k)} \quad , \tag{93}$$

from which we deduce the bound on the amplitude $A_{G,\mu}$:

$$|\mathcal{A}_\mu(G)| \leq K^{|L(G)|} \prod_i \prod_{k=1}^{k(i)} M^{-2|L(G_i^k)| + 2|F(G_i^k)| - 2R(G_i^k)} = K^{|L(G)|} \prod_i \prod_{k=1}^{k(i)} M^{\omega(G_i^k)}. \tag{94}$$

□

4.2 Abelian power counting for models on $[SO(D)/SO(D-1)]^{\times d}$

It is not hard to extend the previous analysis to the homogeneous space $[SO(D)/SO(D-1)]^d \simeq (S)_{D-1}$, allowing to obtain a preliminary classification of potentially just-renormalizable models, for various choices of D and d.

Such classification is of cours valid only to the extent in which the Abelian power counting captures in fact the exact power counting of these non-abelian models. This is however not straightforward, and we have actually reasons not to believe it, as we are going to discuss in the following.

We obtain the following result:

Theorem 2 *The Abelian superficial divergence degree of any Feynman graph G associated to a field theory on $[SO(D)/SO(D-1)]^{\times d}$ with closure constraint is given by*

$$\omega(G) = -2|L(G)| + (D-1)\big[|F(G)| - R(G)\big] \quad . \tag{95}$$

The proof is the exact generalization of the previous one, and we will only give the main steps. Let $\{(L_{\mu\nu})_{\rho\sigma}\}$ be a basis of anti-symmetric $D\times D$ matrices of the Lie Algebra $\mathfrak{so}(D)$, and $k=\{k_\mu\}$ a unit vector of $\mathbb{R}^D$ (the Greek indices run over $1,...,D$ and label the Euclidean coordinates on $\mathbb{R}^D$). Any element $g\in SO(D)$ can be written as (we use the Einstein convention for sums over Greek indices)

$$g=e^{\Omega_{\mu\nu}L_{\mu\nu}} \quad , \tag{96}$$

and any element l of the stabilizer group of k isomorphic to $SO_k(D-1)$ as

$$l=e^{\Omega_{\mu\nu}\mathcal{P}_{\mu\mu'}\mathcal{P}^k_{\nu\nu'}L^k_{\mu'\nu'}} \quad , \tag{97}$$

where $\mathcal{P}^k=\mathbb{I}-k\otimes k$, $(\mathcal{P}^k)^2=\mathcal{P}^k$ is the projector onto the subspace orthogonal to k.

The previous bound (86) for the propagator becomes, for $SO(D)$,

$$C_i(\{g_j\},\{g'_j\})\leq KM^{(dD-2)i}\int dk\int_{SU(2)}dh\int_{[SO_k(D-1)]^d}\prod_{j=1}^{d}dl_j e^{-\delta M^i\sum_{j=1}^{d}|g_jhl_jg_j'^{-1}|} \quad . \tag{98}$$

After integration over group variables g_i, the product (88) becomes

$$\prod_i\prod_{k=1}^{k(i)}M^{-2|L(G_i^k)|+\frac{D(D-1)}{2}|F(G_i^k)|} \tag{99}$$

and the remaining integration (90), in the Abelian approximation, which corresponds to the linearized version of (89), becomes

$$\int\prod_{e\in L(G)}d\lambda_e dk_e\times\int\prod_f d\theta_{k_e}^{c(f)}e^{-\delta M^{i(f)}\left|\sum_e\epsilon_{ef}\left(\lambda_{e,\mu\nu}+\theta^{c(f)}_{k_e,\mu'\nu'}\mathcal{P}^{k_e}_{\mu'\mu}\mathcal{P}^{k_e}_{\nu'\nu}\right)L_{\mu\nu}\right|} \quad . \tag{100}$$

The integration over the λ_e variables replaces (91) by:

$$\prod_i\prod_{k=1}^{k(i)}M^{-\frac{D(D-1)}{2}R(G_i^k)} \tag{101}$$

and the remaining integration over the $\theta_{k_e}^{c(f)}$ gives, instead of (93),

$$\prod_{i,k}M^{-\frac{(D-1)(D-2)}{2}\left[|F(G_i^k)|-R(G_i^k)\right]} \quad . \tag{102}$$

Combining the results (99), (100) and (102), we obtain the divergence degree (95).

4.3 Optimal bound and Abelian power counting

In this section we want to examine further the validity of the Abelian power counting for our non-Abelian model. To this end, we will study the behaviour of the integral (89)[4]. In order to simplify the reasoning, we choose the orientations of faces and lines such that $\epsilon_{ef}\leq 0$. A moment of reflection shows that it is always possible to do so: one just has to exploit the bipartite structure of the Feynman graphs, and choose

[4]Our analysis is close to the one of [33] for the $SU(2)$ case.

the orientation of lines from black to white vertices, for instance. Hence, we will studying the behaviour in Λ of the simpler integral:

$$\mathcal{I}_\Lambda = \int \prod_e dh_e dk_e \prod_f dl_{k_e}^{c(f)} e^{-\Lambda^2 \left| \prod_{e\in\partial f} h_e l_{k_e}^{c(f)} \right|^2}, \tag{103}$$

in the large Λ limit. Because of the normalization of each integration measure, $\mathcal{I}$ goes to zero when $\Lambda \to \infty$, and we expect a behavior of the type $\Lambda^{-\Omega(G)}$. The aim is therefore to find Ω, or, at least, an optimal bound for it. In addition, note that the integral is absolutely convergent, and trivially bounded by 1.

The large Λ limit enforces the relations:

$$\prod_{e\in\partial f} h_e l_{k_e}^{c(f)} = \mathbb{I}, \tag{104}$$

and the strategy is to expand the exponent in the vicinity of these solutions, and integrating around them, by the Laplace method. Let $x = \{\bar{h}_e, \bar{l}_e^{c(f)}\}$ a point in the space of solutions of (104), expected to be a manifold with *a priori* many connected parts, eventually of null dimension (a single point). We define $\mathcal{A} = \{h_e\}$ the set of group variables attached to each line and H_f the map from $SO(D)^{\times|L|}$ to $SO(D)^{\times|F|}$ defined by :

$$H_f(\mathcal{A}) = \prod_{e\in\partial f} h_e l_{k_e}^{c(f)} \tag{105}$$

whose differential around x is:

$$dH_f(x) = \sum_{e\in\partial f} \mathrm{Ad}_{\left\{\prod_{e'\in\partial f | e'<e} \bar{h}_{e'} \bar{l}_{e'}^{c(f)}\right\}} \left[\hat{\delta}_e + \bar{h}_e \hat{\delta}_e^f \bar{h}_e^{-1}\right] =: \sum_{e\in\partial f} L_{fe} \left[\hat{\delta}_e + \bar{h}_e \hat{\delta}_e^f \bar{h}_e^{-1}\right], \tag{106}$$

where $\hat{\delta}_e$ and $\hat{\delta}_e^f$, living in the Lie Algebra $\mathfrak{su}(2)$, are the right variations of h_e and $l_e^{c(f)}$ respectively. Defining $\hat{\delta}_e^{f\,\sharp} = \bar{h}_e \hat{\delta}_e^f \bar{h}_e^{-1}$, we obtain, around x:

$$\mathcal{I}_\Lambda(x) = \int \prod_{e,f} d\hat{\delta}_e d\hat{\delta}_e^f \prod_f e^{-\Lambda^2 |\sum_{e\in\partial f} L_{fe}(\hat{\delta}_e + \hat{\delta}_e^{f\,\sharp})|^2}. \tag{107}$$

In order to integrate it, we introduce the quantities $\hat{\delta}_f$ and $\hat{\delta}_f^\sharp$ as:

$$\hat{\delta}_f = \sum_{e\in\partial f} L_{fe}\hat{\delta}_e \qquad\qquad \hat{\delta}_f^\sharp = \sum_{e\in\partial f} L_{fe}\delta_e^{f\,\sharp} \quad , \tag{108}$$

and the notations $\hat{\delta}_{f\,\parallel}$ and $\hat{\delta}_{f\,\perp}$, designating respectively the components parallel and orthogonal to $\hat{\delta}_f^\sharp$. Inserting this in (107), we find

$$\mathcal{I}_\Lambda(x) = \int \prod_{e,f} d\hat{\delta}_e e^{-\Lambda^2 |\hat{\delta}_{f\,\perp}|^2} \int d\hat{\delta}_e^f \prod_f e^{-\Lambda^2 |\hat{\delta}_{f\,\parallel} + \hat{\delta}_f^\sharp|^2} \quad , \tag{109}$$

which behaves as

$$\begin{aligned} \mathcal{I}_\Lambda(x) &\sim \left|\det[L]_{\ker(L)_\perp}\right|^{-1} \Lambda^{-\left\{\dim\left[SO(D)\right] - \dim\left[SO(D-1)\right]\right\}\mathrm{rk}[L]} \Lambda^{-\dim\left[SO(D-1)\right]|F|} \\ &= \left|\det[L]_{\ker(L)_\perp}\right|^{-1} \Lambda^{-(D-1)\mathrm{rk}[L]} \Lambda^{-\frac{(D-1)(D-2)}{2}|F|} \quad , \end{aligned} \tag{110}$$

where $\mathrm{rk}[L]$ is the rank of L, and the notation $\det[L]_{\ker(L)_\perp}$ indicates the determinant over the complementary space $\ker(L)_\perp$ of $\ker(L)$. Because the rank $\mathrm{rk}[L]$ is at least equal to the rank of ϵ_{ef}, the previous bound in always bounded by its Abelian version. The sum over x, however can eventually spoil this result. As explained before, the support of this sum splits into continuous and discrete components, and the integral over the continuous component can be ill-defined. However, these singularities occur when the determinant vanishes, and because the integral is absolutely convergent, it is a snag of the Laplace method. Moreover, for these points, the co-dimension of the kernel of L becomes bigger than the co-dimension of the kernel on the other points. Hence, presumably these singularities do not affect the conclusion.

This result is important for the rest of this paper, because it allows to find some just-renormalizable models only from the Abelian divergent degree. However, it imply that the Abelian power counting is pessimistic, and as a result, that the list of just-renormalizable models obtained using the Abelian divergent degree is certainly incomplete. In addition, the flatness condition 104 is different of the one obtained in standard TGFT, which is $\prod_{e\in\partial f} h_e = \mathbb{I}$. We will return on this subtlety in a future section.

5 Renormalisability

This section is devoted to a detailed analysis of the divergence degree given by (95). The aim is to determine for which values of d and D, and for which value of the maximal degree v_{max} of interactions, the theory is just-renormalizable (obviously, a stronger degree of convergence would indicate super-renormalizability). Recently, an analysis of this type has been make in [19] for TGFT with gauge invariance (but no other constraints) on group manifolds, for which a classification table has been obtained. We make here the same work for our models on the homogeneous space $[SO(D)/SO(D-1)]^{\times d}$. This work can also be taken as preliminary step towards a similar analysis for TGFT models for quantum gravity, obtained by constraining models of quantum BF theory, the additional constraints there having a similar effect as the projection to a homogeneous space (and in the Barrett-Crane-type models being exactly such projections on $SO(4)/SO(3)$). However, we emphasize in advance that the difficult issue in applying this classification to $SO(D)/SO(D-1)$ models lies in showing that the exact power counting is well captured by the Abelian one. We will return to this point in the following.

5.1 Basics on colored graphs

This section give some definitions and properties of colored graphs. Most of these properties are well-known in tensor model literature, so we simply adopt them and refer to, say, [10] for their proof. We begin with the following lemma:

Lemma 2 *Consider a connected graph G, with F and R respectively its number of faces and the rank of the incidence matrix ϵ_{ef}. Under contraction of a spanning tree $\mathcal{T}$, F and R do not change.*

Proof: Because $\mathcal{T}$ is a spanning tree, its lines bound faces with a number of boundary lines bigger or equal to two, so their number does not change under contraction. For the rank, its invariance is assured by the corollary 1.

□

From [19], we adopt the following definitions.

Definition 1 *(contraction operation). Let G be a Feynman graph and $L_0 = \{l_i\} \subset L(G)$ an ordered subset of dotted (i.e. propagation) lines in G (including tadpole lines). The graph G/L_0 is obtain from G by the following steps:*

considering the dotted line $l_i \in L_0$:

• *deleting the line l_i and its two (black and white) end vertices and all the colored lines joining these two vertice;*
• *identifying the colored line linked to the deleted black vertex with the corresponding line linked to the white vertex;*
• *repeating the same steps for l_{i+1}, and so on.*

Definition 2 *For a connected graph G with $|L|$ lines, $|V|$ vertices and a spanning tree $\mathcal{T} \subset G$, we call tensorial rosette or simply rosette, the contracted graph $G/\mathcal{T}$ with one vertex and $|L| - |V| + 1$ lines.*

The figure 3 below illustrates the definition 1 in a simple example.

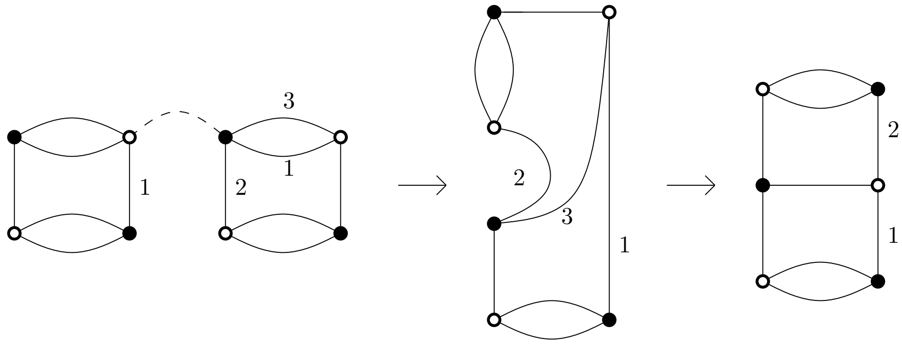

Figure 3: Contraction of a dotted line between two vertices

Definition 3 *Consider a Feynman graph G. The colored extension G_c of this graph is the bipartite regular graph for which:*

• *the vertices are partitioned in the form $\mathcal{V}(G_c) = V \cup \bar{V}$, where V (respectively $\bar{V}$) is the set of black (respectively white) vertices;*
• *the set of lines $\mathcal{E}(G_c)$ is formed by all the lines (colored plus dotted) joining any pair $\{v, \bar{v}\} \in V \times \bar{V}$; by definition, the dotted lines have color 0;*
• *the set of faces is of the form $\mathcal{F}(G_c) = F \cup F_c^{\neq 0}$, where F is the set of faces in G, i.e. the set of faces of the form f_{0i} with boundary lines of color 0 and i $(i \neq 0)$, and $F_c^{\neq 0}$ is the set of faces of the form f_{ij} with boundary lines of color i and $j\,(i \neq j; i, j \neq 0)$;*

Definition 4 *Consider a colored extension G_c. A k-dipole d_k is a set of k colored lines necessarily including the color 0 and linking two vertices v and $\bar{v}$. An example is depicted on the figure 4 below.*

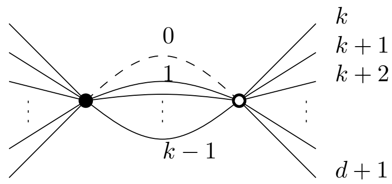

Figure 4: Example of k-dipole

In addition, we recall the following three definitions about colored graphs:

Definition 5 ***(jacket)*** *Consider a colored extension G_c in dimension d. A jacket $\mathcal{J}$ is a 2-subcomplex of G_c, labeled by a $(d+1)$-cycle τ, such that $\mathcal{J}$ has the same number of lines and vertices as G_c, but only a subset of its faces: $\mathcal{F}_{\mathcal{J}} = \{f \in \mathcal{F}_{G_c} | f = (\tau^q(0), \tau^{q+1}(0)), q \in \mathbb{Z}_{D+1}\}$.*

A jacket is a ribbon graph, corresponding to a sub-manifold of dimension 2 and of Euler-Poincaré characteristic given by $\chi(\mathcal{J}) = |F_{\mathcal{J}}| - |\mathcal{E}_{\mathcal{J}}| + |\mathcal{V}_{\mathcal{J}}| = 2 - 2g_{\mathcal{J}}$, where $g_{\mathcal{J}}$ is the genus of the surface.

Definition 6 ***(Degree)*** *The degree $\varpi(G_c)$ of a colored extension G_c is the sum over all the degrees of its jackets:*

$$\varpi(G_c) = \sum_{\mathcal{J}} g_{\mathcal{J}} \quad \Rightarrow \quad \varpi(G_c) \geq 0$$

Definition 7 *The graphs whose degree is equal to zero are called melonic graphs.*

In addition to these definitions, we have the three following lemmas:

Lemma 3 *The melonic graphs are dual to a d-dimensional sphere.*

Lemma 4 *In dimension d, the degree $\varpi(G_c)$ is related to the number of bi-colored faces and to the number of black (or white) vertices p, by the following two relations:*

$$|\mathcal{F}| = \frac{d(d-1)}{2}p + d - \frac{2}{(d-1)!}\varpi(G_c)$$

$$\varpi(G_c) = \frac{(d-1)!}{2}(p + d - \mathcal{B}^{[d]}) + \sum_{i;\rho} \varpi(\mathcal{B}^{\hat{i}}_{(\rho)}).$$

In addition, we can show that $p + d - \mathcal{B}^{[d]} \geq 0$.

Note that, in this lemma, the sum over i in the second relation includes the color 0. In addition, $\mathcal{B}^{\hat{i}}_{(\rho)}$ is the connected component ρ of the sub-graph obtained from G_c by deleting all the lines with color i (including the color 0). $\mathcal{B}^{[d]}$ is the number of these connected sub-graphs. These sub-graphs are the so-called "d-bubbles". From this lemma, we easily deduce the following proposition:

Proposition 2 *Under any* 1*-dipole contraction, the degree of a graph is unchanged.*

With this material at hand, we now move on to the renormalizability analysis, which is the object of the next section.

5.2 Renormalizability

We have seen that, for these models, the divergence degree of a graph grows with the number of faces. The first question is: which are the graphs that have a maximum number of faces? To answer this question, consider a vacuum graph G and its colored extension G_c. We can choose a tree $\mathcal{T}$ in G and build the rosette $G/\mathcal{T} = \hat{G}$, $\hat{G}_c$ being its colored extension. Then, from the lemma 4, we have:

$$|F(\hat{G})| = (d-1)|L(\hat{G})| + 1 - \Delta(\hat{G}) \quad , \tag{111}$$

where we have used the fact that $L(\hat{G}) = p$ in the lemma 4, and where

$$\Delta(\hat{G}) := \frac{2}{(d-2)!}\Big[\frac{1}{d-1}\varpi(\hat{G}) - \varpi(\hat{G}^0)\Big] \quad ,$$

where $\hat{G}^0$ is the d-bubble of color 0 obtained from $\hat{G}$ by deleting all the lines of color 0. Note that because the rosette $\hat{G}$ have only one vertex, $\hat{G}^0$ have only one connected component. Because one can prove that $\varpi(\hat{G}) \geq d\varpi(\hat{G}^0)$, we easily deduce that $|F|$ is bounded by:

$$(d-1)p + 1 - \frac{2}{(d-1)!}\varpi(\hat{G}) \leq |F| \leq (d-1)p + 1 - \frac{2}{(d-1)!}\varpi(\hat{G}^0).$$

The number of faces is then maximal when $\varpi(\hat{G}) = 0$, implying $\varpi(\hat{G}^0) = 0$ from the lemma 4. Hence we deduce that the number of faces is maximal for the melonic (colored extension) graphs. This result is actually a key one for all the TGFT models that have been studied to date. In addition to the formal definition given by 7, the melonic graphs have an iterative definition. From the simplest melon with $p = 1$ (the so-called "supermelon") given in figure 5 below

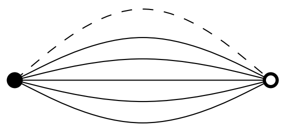

Figure 5: The supermelon graph

we obtain the refined melons of order p by replacing an edge by a d-dipole as in the figure 6 below.

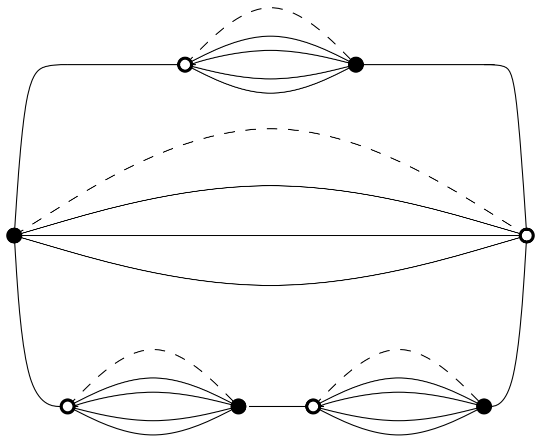

Figure 6: Melonic recursion: d-dipole insertion.

Note that for a rosette, this recursion procedure excludes the line of color 0 because the d-bubble $\hat{G}^0$ has just one connected component.
Now we can turn to the analysis of the rank, the other main contribution to the divergence degree. It is obvious from the previous recursion that for a melonic rosette of order p, the rank is just equal to p, the number of lines $L(\hat{G})$ in $\hat{G}$. Hence, the rank is maximal for the same melonic graphs, and each insertion of a d-dipole increases $|F| - R$ by $d-2$. It follows that, for a melonic rosette graph:

$$|F(\hat{G})| - R(\hat{G}) = (d-2)|L(\hat{G})| + 1. \tag{112}$$

It is tempting to think that this is also the optimal bound for $|F| - R$ for arbitrary graphs. Indeed, we can prove that this is the case by recursion. Starting from the order $p = 1$, the unique connected vacuum graph $\mathcal{M}_1$ is the supermelon in figure 5. For the next order $p = 2$, we wish to add one black vertex and one white vertex, or, in other words, a new dotted line. Each line carries at least d faces $f^{0i}\, i \neq 0$ of length one, and can increase the rank at least of $+1$. Because of the connectivity constraint, it seems that one colored line must be sacrificed, and bound a common face for the two dotted lines. Hence, the maximal number of faces is $2d-1 = 2(d-1)+1$, in accordance with the formula (112). Concerning the rank, if we wish to minimize this variation in the step $p = 1 \to p = 2$, the only possibility is to exclude the creation of a k-dipole for $k > 1$, and so to create a new face f^{0i}. Hence, we loose $d-1$ faces, and, if $d > 2$, this possibility does not correspond to the leading order. Privileging the graphs with the maximum number of faces is then more advantageous, and the connectivity constraint implies that the only possibility is a melonic graph $\mathcal{M}_2$ as depicted in the figure 7.

The same argument survives at order p. Starting with a melonic graph of order p, $\mathcal{M}_p$, we move from the order p to the order $p+1$ by adding a dotted line. This dotted line can carry at least d faces, but one is necessarily common with another dotted line, ensuring the graph connectivity. Hence, adding a new line increases at least by $d-1$ the number of faces, and the optimal graph corresponds to the melon $\mathcal{M}_{p+1}$. As to the rank, it can at least increase by 1. It is clear that the optimal graphs for the rank and the faces are incompatible, because to minimize the rank variation, so $\Delta R = 0$, it is necessary that no k-dipole (for $k > 1$) and no face is created by the new line. Then, we loose $d-1$ lines compared to the melonic graphs, and this solution does not correspond to a leading order graph.

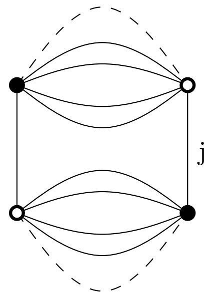

Figure 7: The most divergent graph at the order $p = 2$.

Hence, the melonic rosettes correspond to the most divergent graphs, i.e. the leading order of the perturbative expansion. Because $L(\hat{G}) = |L(G)| - |V(G)| + 1$, we deduce the following result:

Proposition 3 *Let G be a vacuum Feynman graph. Its divergence degree is bounded by*

$$\omega(G) \leq \omega_{melo}(G) \tag{113}$$

with

$$\omega_{melo}(G) := \big[(D-1)(d-2)-2\big]|L(G)| - (D-1)(d-2)(|V(G)|-1)+1 \qquad , \tag{114}$$

which correspond to the divergence degree of a melonic rosette graph.

We are tempted to deduce from this theorem that the leading order graphs are also melonic. In fact, this property can be seen as an elementary property of the melonic graphs. It follows from another elementary result concerning the degree $\varpi(G_c)$, i.e. that it is invariant under 1-dipole contraction [10, 19]. Yet, it is obvious that the tree contraction given by the rosette in the previous proposition is a succession of 1-dipole contractions, then:

Corollary 1 *The leading order graphs are melonic: their degree ϖ vanish.*

From this follows trivislly that

Corollary 2 *Only the melonic interactions contribute to the leading order graphs.*

It remains to consider the non-vacuum graphs. It is tempting to think that the leading order graphs are melonic, as in the vacuum case. This conclusion is right, but must be made more precise, and the result is the following proposition:

Proposition 4 *Let G_N be a non-vacuum graph with N external lines. Its divergence degree is bounded as:*

$$\omega(G_N) \leq \big[(D-1)(d-2)-2\big]|L(G)| - (D-1)(d-2)(|V(G)|-1) \tag{115}$$

with equality for melonic graphs. In addition, an external mono-color face connects all the external black and white vertices.

Proof: Let $\hat{G}$ be a vacuum graph containing G_N in the following sense. $G \subset \hat{G}$ means that $\hat{G}$ contains all the vertices of G_N and has the same connectivity, and that a face of $\hat{G}$ is the sum of the internal and external faces of G_N. In addition, this inclusion supposes that G_N can be obtained from $\hat{G}$ by cutting some internal lines. Because of proposition 3, it follows that $\hat{G}$ is a leading order graph if and only if it is melonic. Starting from this graph, we wish to build the most divergent graph with the same number of external lines as G_N. Suppose that G_N has 2 external lines. From $\hat{G}$, we begin by selecting a spanning tree $\mathcal{T}$ and we construct the rosette $\bar{\hat{G}}$. Now, the simplest way to obtain a non-vacuum graph from the rosette is to cut an internal dotted line. An internal line is necessarily a dipole line, and carries d faces. In addition, the rank has to be decreased by 1, and $|F|-R$ decreases by $d-1=(d-2)+1$. Another (more complicated) way to build a 2-points graph is to cut n internal dotted lines, to select 2 half dotted lines, and to reconnect the $2(n-1)$ remaining half dotted lines in an optimal way. But observe that cutting these n lines decreases $|F|-R$ at most by $(d-2)n+1$. Hence, we must at least reconstruct $(d-1)(n-1)$ faces and increase the rank by $(n-1)$, obtaining in a more involved way exactly the same result. It is then obvious, from the recursive definition of the melonic graphs and their inherited connectivity, that any reconnecting procedure, which does not correspond to the cutting of a singular dipole, increases the length of the d externals faces, and necessarily reduces the number of internal faces. Hence, cutting a single dipole is the more face-economic way to obtaine a 2-point graph which respects the melonicity condition, and thus the maximization of the divergence degree. Therefore, from the rosette $\bar{\hat{G}}$, we obtain the leading order 2-points graphs $\bar{G}_2^{melo}$ by cutting one dipole, and the divergence degree of G_2 is bounded by:

$$\omega_{melo}(G_2)\leq -2|L(G_2)|+(D-1)(d-2)\big[|L(\bar{\hat{G}})|-1\big]=-2|L(G_2)|+(D-1)(d-2)|L(\bar{G}_2)|$$

which bounds also the divergence degree of any 2-point graph.

Now, from the 2-point graph, we would like to build the 4-point graph. As previously, we start from a rosette graph $\bar{G}_2^{melo}$. The same argument as before shows that the leading order graphs are obtained by cutting a new dipole. But a new subtlety appears. Indeed, because of the special connectivity of the melonic graphs, two given lines in $\bar{G}_2^{melo}$ can be the boundary of one and only one internal face. If two lines do not have common faces, they are called "face disconnected", and if we cut two "face-disconnected" lines, we loose $2d$ faces against $2(d-1)+1$ if they are "face-connected". Hence, it follows that the leading order graphs with 4 external lines are melonic with an external face of length upper than 2 in the colored extension graph. The same argument can be applied with N external lines. Then, from the complete graph $\bar{\hat{G}}$, $|F|-R$ decreases by $(N/2)(d-1)+1$ when we cut face-connected lines. Hence,

$$\begin{aligned}\omega_{melo}(G_N)\leq&-2|L(G_N)|+(D-1)(d-2)\big[|L(\bar{\hat{G}})|-N/2\big]\\ =&-2|L(G_N)|+(D-1)(d-2)|L(\bar{G}_N)|,\end{aligned}$$

and an external mono-color face connects all the external black and white vertices.

□

From proposition 4, we can easily deduce a criterion for just-renormalizability. Remember that a field theory is said to be "just-renormalizable" if its divergence degree does not increase with the number of vertices. Because of the following "topological" relationship:

$$|L(G)|=\sum_{k=1}^{k_{max}}kn_k(G)-N_{ext}/2\qquad |V(G)|=\sum_{k=1}^{k_{max}}n_k(G),\tag{116}$$

where n_k is the number of vertices of degree k in G (with k black (or white) vertices in their corresponding bubble interaction vertex), we deduce from the previous theorem 3:

$$\begin{aligned}\omega_{melo}(G):=&(D-1)(d-2)-\big[(D-1)(d-2)-2\big]\frac{N_{ext}}{2}\\ &+\sum_{k=1}^{k_{max}}\big(\big[(D-1)(d-2)-2\big]k-(D-1)(d-2)\big)n_k(G).\end{aligned}$$

Hence, renormalizability is ensured if, and only if, the maximal value k_{max} for the degree of the interactions does not exceed

$$k_R = \frac{(D-1)(d-2)}{(D-1)(d-2)-2}. \tag{117}$$

This result allows to classify the just- and super-renormalizable TGFT models, on the basis of the Abelian power counting. The super-renormalizable models are those for which $k_{max} < k_R$, such that the divergence degree decreases with the number of vertices, implying that only a finite number of graphs needs to be renormalized. It is only when the divergence degree does not depend on the order of the pertubative expansion, i.e. when the higher degree k_{max} equals k_R, that the theory is said to be "just-renormalizable", and that the divergences can be taken care of by a renormalization procedure, implying the definition of a finite number of counter-terms. The table 5.2 below lists some potentially just-renormalizable TGFT models, in the class we have been considering and on the basis of the Abelian power counting only.

Type	d	D	k_R	ω_{melo}
A	3	4	3	$3-N/2-2n_1-n_2$
B	4	3	2	$4-N-2n_1$
C	5	2	3	$3-N/2-2n_1-n_2$
D	6	2	2	$4-N-2n_1$

Table 1: Table of potentially just-renormalizable theories

Note that some promising models for quantum gravity are absent of this table. This is the case, for example, of models on $SO(4)/SO(3)$ in dimension 4, the TGFT counterpart of the simplicial ones studied in [24, 25, 23] and which have been a source of inspiration for this paper. Their absence is due to the rather 'pessimistic'Abelian divergence degree, which, as discussed in section 4.3, is always higher than the exact divergence degree. Hence, if all the models in the table are certainly just-renormalizable, this classification certainly does not exhaust the class of just-renormalizable models. To emphasize this point, we adopt for the following definition:

Definition 8 *Any model which is just-renormalizable on the basis of the Abelian power counting only is said to be Abelian just-renormalizable. Obviously, any Abelian just-renormalizable model is also just-renormalizable according to the general definition.*

In the rest of this paper, however, we study only the Abelian just-renormalizable quartic melonic model on $[SU(2)/U(1)]^{\times 4}$, clarifying further why in this case the Abelian power counting is actually exact.

5.3 Exact power counting and melons

As explained in [19] in the case of a TGFT on $SU(2)$, the Abelian power counting bounds the exact one, and becomes equal to it for the melonic graphs. This fact follows from a special property of melons, which are said to be "contractible", up to a spanning tree contraction, and from the properties of $SU(2)$ holonomies. More precisely:

Definition 9 *A subgraph H of a graph G is said to be "contractible" if it verifies*

$$\forall f \in F(H) \quad \prod_{e \in \partial f} h_e^{\epsilon_{ef}} = \mathbb{I} \Longrightarrow h_e = \mathbb{I}.$$

Hence, in the language of lattice gauge theory, for contractible graphs, the flatness of the holonomies imply triviality of connections, and it is obvious to see that any melon graph contracted with a spanning tree (a melonic rosette, called "melopole" in the literature) is contractible. And because, for this class of TGFT models (based on simple group manifolds, with Laplacian kinetic term and tensor invariant interaction with trivial interaction kernels), the divergences are dominated by flat holonomies, the power counting becomes exact for these graphs.

The issue now is whether this conclusion is exact for our models on $SU(2)/U(1)$. A careful investigation shows that this is not so obviously true. As seen in section **??**, the flatness condition for holonomies is replaced by equation (104), involving additional group variables attached to each face in addition of the connections attached to each lines, imposing the projection onto the homogensous space. Because of this face dependence, the contractibility is not enough to guarantee the equivalence with the exact power counting in the case of melopoles. However, because intuitively melonic graphs have a lot of short faces (i.e. those with few boundary lines), it seems that they are at least those for which the exact power counting is the closest to the Abelian one. More precisely, if the faces are essentially of length one, we expect that the exact power counting goes to its Abelian approximation, as we will see in section 7 for the simplest melopole. Another situation, in which the exact power counting goes to its Abelian version, occurs when the faces share the same lines, and an example is given in section 7. More precisely, if a set of lines l carries $d-1$ faces, it follows from gauge invariance that:

$$\prod_{e\in\partial f} h_e l_{k_e}^{c(f)} = \prod_{e\in\partial f} h_e \prod_{e\in\partial f} l_{k_e}^{c(f)} \qquad \forall f \supset l \tag{118}$$

and this case can therefore be reduced to the case of a melopole with one line. Hence, only a subset of melons have a divergence degree given by the Abelian power counting:

Proposition 5 *Consider a potentially divergent melonic graph with an internal face of color i (with two or four external lines); if it contains only faces of length one, or lines carrying $d-1$ faces, then it divergence degree corresponds to the Abelian one. We called this type of melonic graph an "Abelian melon".*

An example is given in 8 below for the Abelian just-renormalizable melonic ϕ^4 model on $[SU(2)/U(1)]^{\times 4}$.

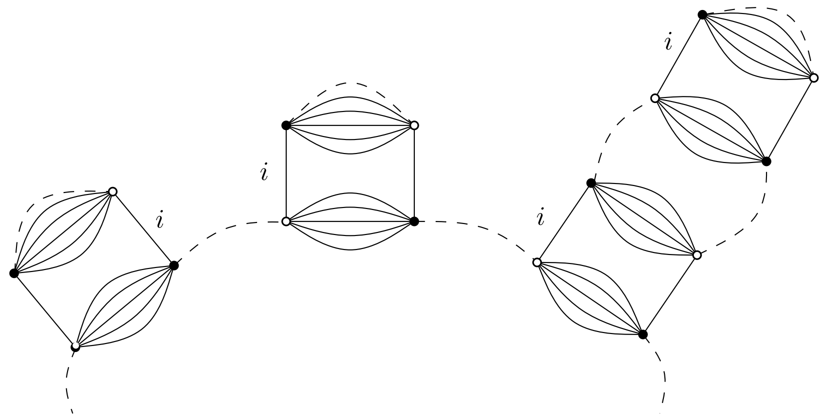

Figure 8: Example of Abelian melon for the renormalizable ϕ^4 model on $[SU(2)/U(1)]^{\times 4}$

5.4 Essentials about the just-renormalizable T^4 model on $[SU(2)/U(1)]^{\times 4}$

The previous analysis shows that the special model in dimension 4 with group $SU(2)$ and quartic melonic interaction is just-renormalizable. The interactions of this model are of the form depicted in the figure 9 below. There are exactly four interactions of this type, one for each choice of the color of the intermediate lines between the two 3-dipoles. In the following, each interaction bubble b_i will be labeled by the color of these intermediate lines.

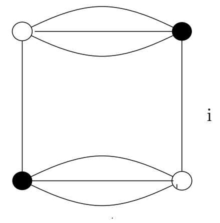

Figure 9: The melonic ϕ^4 interaction b_i

Normally, each bubble b_i can appear in the interaction part of the action with its own coupling but in the following we will limit our attention to the simplest case where all the interaction bubbles have the same coupling. Hence :

$$S_{int} = \lambda \sum_{i=1}^{4} \mathrm{Tr}_{b_i}(\bar{\psi}, \psi). \tag{119}$$

As for the Gaussian measure, it, is given by the formula (38) (or (31) in the first formulation), and allows to write the regularized generating functional as:

$$\mathcal{Z}_\Lambda[J, \bar{J}] := \int d\mu_{C_\Lambda} e^{-S_{int}(\bar{\psi},\psi)+\langle \bar{J},\psi\rangle+\langle\bar{\psi},J\rangle}, \tag{120}$$

where:

$$\langle \bar{J}, \psi\rangle := \int [dg]^4 \bar{J}(g_1, g_2, g_3, g_4)\psi(g_1, g_2, g_3, g_4) \quad . \tag{121}$$

In its minimal prescription, the aim of the renormalization procedure is to give a sense to the limit $\Lambda \to \infty$ perturbatively. This issue will be considered in the following two sections.

From the previous section, a question remains: if it is now clear that the most divergent graphs are the so-called melon graphs, it is not obvious that all the divergent graphs are melonics. In other words, we have not proved that the melonic graphs contain all the divergences occurring in the graph expansion of correlation functions, and we dedicate the end of this section to the answer to this question.

The form of the interaction bubbles in figure 9 allows to use a very useful representation of the theory, the “intermediate field representation”, from which the problem can be translated into a simple recursion relation. The building rules of the intermediate field representation are the following. From the basic properties of the Gaussian integration, the generating functional (120) can be formally rewritten as:

$$\mathcal{Z}_\Lambda[J, \bar{J}] = \int d\mu_{C_\Lambda} \prod_{i=1}^{4} d\mu_1(\sigma_i) e^{i\sqrt{2\lambda}\langle\bar{\psi},\Sigma\psi\rangle+\langle \bar{J},\psi\rangle+\langle\bar{\psi},J\rangle} \quad , \tag{122}$$

where $d\mu_1(\sigma_i) := e^{-\mathrm{tr}(\sigma^2)}d\sigma$ is the Gaussian measure for Hermitian matrices σ_i, $\Sigma = \sum_i \Sigma_i$, and:

$$\Sigma_i = \mathbb{I} \otimes \cdots \otimes \sigma_i \otimes \cdots \otimes \mathbb{I} \quad . \tag{123}$$

Now, the Gaussian integration over ψ and $\bar{\psi}$ can be performed, leading to the following effective multi-matrix model:

$$\mathcal{Z}_\Lambda[J, \bar{J}] = \prod_{i=1}^{4} d\mu_1(\sigma_i) e^{-\mathrm{Tr}\ln(1-i\sqrt{2\lambda}C\Sigma)+\langle \bar{J},RJ\rangle}, \tag{124}$$

where : $R := (1 - i\sqrt{2\lambda}C\Sigma)^{-1}C$. The Feynman rules are the following: expanding the logarithm, we generate interactions with one, two,... n-external points, and a typical Feynman graph is composed of several of these vertices, connected to each other by matrix lines. These matrix lines, of color 1 to 4, are depicted by a wavy line, and the vertices, to which they are hooked, by a grey disk, as in figure 10. In this figure, one of the grey disks has a dotted arrow line (a cilium). This grey disc does not correspond to an interaction generated by the logarithm expansion, but comes from the expansion of the R operator defined before (124), and a ciliated disk with n external wavy lines correspond to the term of degree n in the expansion in powers of Σ.

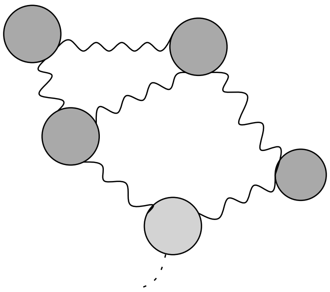

Figure 10: A Feynman graph in the intermediate field representation

This representation has been studied in detail in several recent papers, e.g. [**?**]. An important result about this representation is that the leading order graphs, the melons of the original representation, appear as trees in the intermediate field representation. More precisely:

Proposition 6 *In the intermediate field representation, the melon graphs are trees with one or two external wavy lines, all of the same color, and connected to the same external face.*

This can be easily proven by recursion. Now, we will use this property to see if all the divergent graphs are melons. Starting from a leading order graph (a tree) with l wavy lines (to each wavy line corresponds a vertex of the original representation, and the number of these wavy lines is equivalent to the power of λ associated to the graph), we will investigate the different ways to build a graph with $l+1$ wavy lines, and the possibility that one of these gives a non-melonic divergent graph. Two of these ways are depicted in figure 11. They correspond to the addition of a tadpole graph over a grey disk, or of a "field correction" on a wavy line. But neither of them affects the tree structure of the starting graph, which remains a melon.

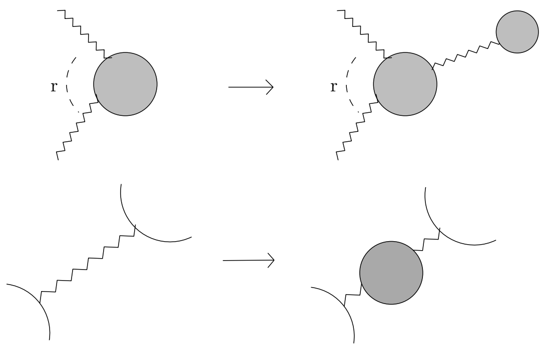

Figure 11: Addition of a tadpole over a grey disk and of a "field correction" over a wavy line

The two operations depicted in figure 12 are more promising. They are both a deviation from the melonicity, because both affect the tree structure of the starting graph (because of the connectivity of the starting graph, the second operation necessarily builds a loop). As a result, only these two operations can give us a non-melonic divergent graph, and we will examine these two possibilities.

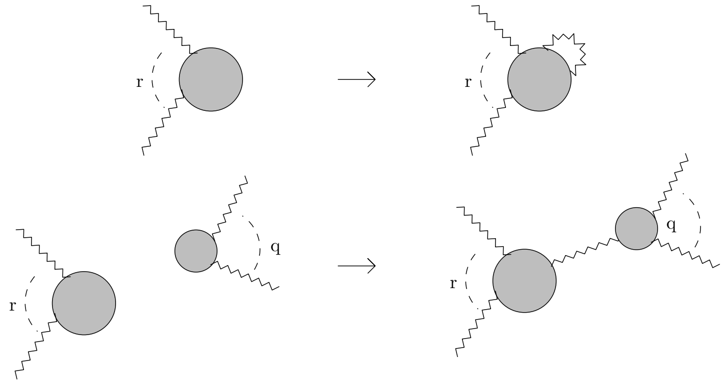

Figure 12: Addition of a self-loop and of a wavy line between two disks.

In the two cases, we increase the number of field lines by two (the field lines are the dotted lines of the original formulation, which are hidden inside the disks in the intermediate field representation). The number of faces increases only by one, and the rank does not change. Hence, the total variation of the divergence degree is:

$$\delta\omega \leq \delta\omega_{Abelian} = -2\delta L + 2\delta(F-R) \leq -2 \quad , \tag{125}$$

where $\omega_{Abelian}$ is the Abelian divergence degree computed previously. Because of the Abelian divergence degree is $4-N$ for a graph with N external lines, any graph with $N \leq 2$ becomes superficially convergent, and we have proven the following result:

Proposition 7 *All the divergent graphs with four external lines of the melonic ϕ^4 model on $[SU(2)/U(1)]^{\times 4}$ are melonic.*

The divergent 2-points graphs, however, can include submelonic contributions, and we expect that this is indeed the case. This result does not affect the analysis of the rest of the paper, because, in all cases, the melonic contributions are the most divergent graphs, and if the corresponding divergences are subtracted by counter-terms, all the sub-divergences are automatically subtracted, and affect only the mass parameter.

As a remark, note that, conversely, for models of type A in table 5.2, all divergent graphs are melonic, and the melonic sector alone contains all the divergences of the model. This can be seen as follows. From equation (111) and proposition 4, it follows that $|F|-R$ can be written as

$$|F(G)| - R(G) = (d-2)(|L(G)| - |V(G)| + 1) + \rho \quad , \tag{126}$$

with $\rho \leq 0$ for non-vacuum graphs, and $\rho < 0$ for non-melonic graphs. By the same methods deployed above to obtain the Abelian classification, we find that the divergence degree can be written as:

$$\omega(G) = 3 - N/2 - 2n_1 - n_2 + 3\rho, \tag{127}$$

and is bounded by $-N/2$ for non-melonic graphs. Our conclusion about the Abelian just-renormalizable model ϕ^4 in $d=4$ is fundamental for the following reason. The recent literature on TGFT renormalization has shown that melonic graphs are interesting for two (closely related) essential reasons. The first one is that the Abelian power counting become exact for these graphs, and the second one is that they are said to be “tracial”, in the sens that any connected graph remains connected under contraction of a melonic subgraph - a property which is essential for renormalization. In our case, however, we have seen that divergent sub-melonic contributions can occur, but only for 2-points graphs. And fortunately, for such graphs, contraction does not change the connectivity, as we will see explicitly in the next section 6.1, and the renormalization procedure can be defined in a worst-case-scenario context, in which all the Abelian divergent graphs are regarded as “dangerous”.

6 Renormalization of the model

6.1 Divergent graphs and renormalized amplitude

The table 5.2 shows that in the present case, the divergence degree of a Feynman graph G with N external lines is bounded by $4-N$ (with equality for melonic graphs). Hence, *a priori*, only the 2 and 4-points functions are potentially dangerous. Then, we adopt the following definitions:

Definition 10 *Consider a Feynman graph G and let $h \subseteq G$ be a subgraph of G with n_h external lines. The subgraph h is said to be:*

- *superficially convergent if $\omega_{Abelian}(h) < 0$.*
- *superficially divergent or dangerous if $\omega_{Abelian}(h) \geq 0 \Rightarrow n_h \leq 4$.*

Definition 11 *Consider a Feynman graph G. The set of divergent subgraphs $D(G) = \{h \subseteq G | \omega(h) \geq 0\}$, eventually including G itself, is the so-called divergent or Zimmerman forest. In addition, the Zimmerman forest is an inclusion forest, in the sense that, taking two elements $h_1, h_2 \in D(G)$, they are either included one into the other, or vertices and lines disjoint.*

The definition 10 is motivated by the following theorem, which states that the Feynman amplitude is finite if it does not contain any subdivergent graph in the sense of the definition 10 :

Theorem 3 ("Weinberg" uniform) *Consider a completely convergent graph G, i.e. a graph with no subdivergences. Its corresponding Feynman amplitude $\mathcal{A}_G$ has the following bounds:*

$$|\mathcal{A}_G| \leq K^{|V(G)|}, \ K \in \mathbb{R}^+. \tag{128}$$

Proof. The proof is standard in renormalization theory, and we will only give the main steps. The first step is to note that, when $N > 4$, one has $4 - N \leq -N/3$ (the graph with five external lines does not exist). Hence, for a given scale attribution μ, the graph amplitude $\mathcal{A}_G$ verifies the following trivial bounds:

$$|A_{G\mu}| \leq \prod_{i,k} M^{-N(G_i^{(k)})/3} \quad .$$

Now, we define:

Definition 12

$$i_b(\mu) = \sup_{l \in L_b(G)} i_l(\mu) \qquad e_b(\mu) = \inf_{l \in L_b(G)} i_l(\mu) \quad ,$$

where b stands for a vertex bubble $b \in G$, and $L_b(G)$ is the set of its external lines. Note that b touches a connected subgraph $G_i^{(k)}$ if and only if $i \leq i_b(\mu)$, and is an external vertex if $e_b < i \leq i_b$. Therefore, because each vertex touches at most 4 subgraphs:

$$\prod_{i,k} M^{-N(G_i^{(k)})/3} \leq \prod_{i,k} \prod_{\substack{b \in G_i^{(k)} \\ e_b < i \leq i_b}} M^{1/12} \quad ,$$

and

$$|A_{G\mu}| \leq K^{l(G)} \prod_b M^{-\frac{|i_b(\mu) - e_b(\mu)|}{12}} \quad .$$

Using the fact that there are at most 4 half-lines, and thus $6 = 4 \times 3/2$ pairs of half-lines hooked to a given vertex, and that, for two lines l and l' of a bubble b, $|e_b - i_b| \geq |i_l - i_{l'}|$, we obtain:

$$|A_{G\mu}| \leq K^{l(G)} \prod_b \prod_{(l,l') \in L_b \times L_b} M^{-\frac{|i_l - i_{l'}|}{72}} \quad .$$

This expression implies directly the finiteness of A_G. To understand why, observe that we can choose a total ordering of the lines $L(G) = \{l_1, ..., l_{|L(G)|}\}$ such that l_1 is hooked to an external vertex b_0 and that each subset $\{l_1, ..., l_m\}$, $m \leq |L(G)|$ is connected. Hence, for any line l_j, we can choose $l_{p(j)}$ (with $p(j) < j$) a line sharing a vertex with l_j, from which we deduce:

$$\prod_b \prod_{(l,l')\in L_b\times L_b} M^{-\frac{|i_l(\mu)-i_{l'}(\mu)|}{78}} \leq \prod_{j=1}^{|L(G)|} M^{-\frac{|i_{l_j}-i_{l_{p(j)}}|}{72}} \quad .$$

Because

$$\sum_{i_{l_j}} M^{-\frac{|i_{l_j}-i_{l_{p(j)}}|}{72}} \leq \sum_{i_{l_j}\geq i_{l_{p(j)}}} M^{-\frac{|i_{l_j}-i_{l_{p(j)}}|}{72}} \leq \frac{1}{1-M^{-1/72}} \quad ,$$

we have

$$|A_{G\,\mu}| \leq K^{l(G)} \sum_{\mu=\{i_1,...,i_{l(G)}\}} \prod_{j=1}^{|L(G)|} M^{-\frac{|i_{l_j}-i_{l_{p(j)}}|}{72}} \leq K'^{l(G)} \quad .$$

□

When the graphs contain some dangerous subgraphs, the proof given above breaks down, and the finiteness of the sum over scale attribution is not guaranteed. Therefore, the case of the presence of these subgraphs must be considered in details.

- **N=2** . We start with the case of a subdivergent graph with two external lines. The situation is depicted in figure 13 below. The structure of the amplitude $\mathcal{A}_{G\mu}$ for the scale attribution μ is

$$\mathcal{A}_{\mu G} = \int \prod_l d\bar{g}_{1l} d\bar{g}_{2l} dg_{1l} dg_{2l} \bar{\mathcal{A}}_{\mu,G}(\{\bar{g}_{1l}\},\{\bar{g}_{2l}\}) C_{i_1}(\{g_{1l},\}\{\bar{g}_{1l}\}) C_{i_2}(\{g_{2l},\}\{\bar{g}_{2l}\}) \mathcal{M}_j(g_{11}, g_{21}) \quad , \tag{129}$$

where $\mathcal{M}_j$ is the 2-points subgraph depicted in figure 13, j its scale, i.e. the scale of its highest line, and $\bar{\mathcal{A}}_{\mu,G}$ is a completely convergent amplitude. The scale attribution is chosen such as $j > i_1, i_2$, and the subgraph $\mathcal{M}_j$ is said to be "high". This is typically the region in which this graph is potentially divergent. Obviously from the propagator structure one has $\mathcal{M}_j(g, g') = \mathcal{M}_j(gg'^{-1})$.

Now, we define the real parameter t as follows: $g_{21}(t) = g_{11} \exp\left(tX_{g_{11}^{-1}g_{21}}\right)$ $t \in [0, 1]$, where $X_g \in su(2)$ is the Lie algebra element such that $g = e^{X_g} \in SU(2)$. Therefore, we can define

$$\mathcal{A}_{\mu G}(t) = \int \prod_l d\bar{g}_{1l} d\bar{g}_{2l} dg_{1l} dg_{2l} \bar{\mathcal{A}}_{\mu,G}(\{\bar{g}_{1l}\},\{\bar{g}_{2l}\}) C_{i_1}(\{g_{1l}\},\{\bar{g}_{1l}\}) C_{i_2}(g_{21}(t),\{g_{2l\,l\neq 1}\},\{\bar{g}_{2l}\}) \mathcal{M}_j(g_{11}, g_{21}) \quad , \tag{130}$$

and $\mathcal{A}_{\mu G} = \mathcal{A}_{\mu G}(t = 1)$. We introduce the $*$ application $\tau_{\mathcal{M}}$ such as:

$$\tau^*_{\mathcal{M}} \mathcal{A}_{\mu G}(t) := \sum_{n=0}^{\omega(\mathcal{M})} \frac{1}{n!} \frac{d^n \mathcal{A}_{\mu G}}{dt^n}(t = 0) \quad , \tag{131}$$

and the aim, motivated by the usual quantum field theory, is to prove that

$$\mathcal{A}^R_{\mu,G} = (1 - \tau^*_{\mathcal{M}})\mathcal{A}_{\mu G}|_{t=1} = \int_0^1 dt \frac{(1-t)^{\omega(\mathcal{M}_j)}}{\omega(\mathcal{M}_j)!} \frac{d^{\omega(\mathcal{M}_j)+1} \mathcal{A}_{\mu G}(t)}{dt^{\omega(\mathcal{M}_j)+1}} \tag{132}$$

is finite. The first key result is the following obvious bound of the Lie derivative of the propagator in the slice i:

$$\left[\mathcal{L}_{X_g,g_1}\right]^k C_i(\{g_i\},\{g_i'\}) \leq |X_g|^k K M^{(3d-2+k)i} \quad , \tag{133}$$

where $||$ is the Killing norm of the $su(2)$ elements, and $\mathcal{L}_{X_g,g_1}$ is the Lie derivative of the variable g_1 in the direction X_g, defined as : $\mathcal{L}_{X_g,g_1} f(g) := f'(ge^{tX_g})|_{t=0}$. Because of the bounds (133), the derivative appearing in (132) increases the bound of the propagator by a factor $M^{(\omega(\mathcal{M}_j)+1)i_2}$. Furthermore, because the norm of $X_{g_1^{-1}g_2}$ scales as M^{-j} with M, the power $|X_{g_1^{-1}g_2}|^{\omega(\mathcal{M}_j)+1}$ is bounded by $KM^{-(\omega(\mathcal{M}_j)+1)j}$. Hence, the total exponential decay is of the type

$$\omega(\mathcal{M}_j) + (\omega(\mathcal{M}_j)+1)(i_2 - j) < -1 \quad , \tag{134}$$

and the subgraph is made superficially convergent in the sens of the definition 10.

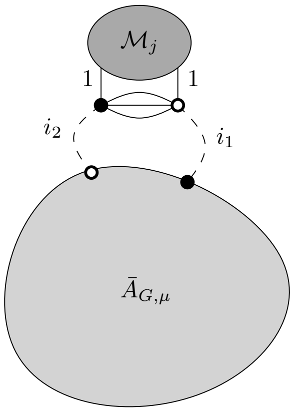

Figure 13: Sub-divergent 2-points graph

• **N=4** We now turn to the melonic subgraphs with four external lines. A typical sub-divergence of this type is depicted in figure 14 below, in which the graphs $\bar{\mathcal{A}}^{(i)}_{G,\mu}$ are free of sub-divergences.

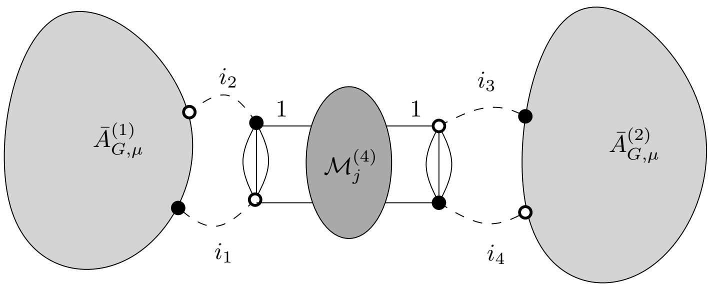

Figure 14: Sub-divergent 4-points graph

As in the previous case, we begin by writing the amplitude in terms of the three blocs defined in figure 14. We define, with the same notations as in the case $N = 2$:

$$\begin{aligned} \mathcal{A}_{G,\mu}(t) = \int \prod_{l=1}^{4}\prod_{k=1}^{4} dg_{lk} d\bar{g}_{lk} \mathcal{A}^{(1)}_{G,\mu}(\{\bar{g}_{1l}\},\{\bar{g}_{2l}\}) \\ \times \prod_{k=1}^{2} C_{i_k}(\{\bar{g}_{kl}\}; g_{k1}(t), \{g_{kl\, l\neq 0}\}) \prod_{k=3}^{4} C_{i_k}(g_{k1}(t), \{g_{kl\, l\neq 0}\}; \{\bar{g}_{kl}\}) \\ \times \mathcal{A}^{(2)}_{G,\mu}(\{\bar{g}_{3l}\},\{\bar{g}_{4l}\}) \mathcal{M}^{(4)}_j(g_{11}, g_{12}; g_{13}, g_{14}) \qquad . \end{aligned} \tag{135}$$

As in the case of the 2-point function, we introduce the $*$ operator $\tau_{\mathcal{M}^{(4)}}$ whose action is defined by the equation (132). The same argument as in the previous section can be applied to this case (there are two terms instead of one after differentiation with respect to t), and the conclusion is unchanged: in the domain $j > i_k\ \forall k$, the subgraph $\mathcal{M}^{(4)}$ is made superficially convergent in the sense of the definition 10.

The previous analysis motivates the following definition:

Definition 13 *The renormalized amplitude $\mathcal{A}_G^R$ associated with the graph G is deduced from the bare amplitude $\mathcal{A}_G$ though the Zimmermann formula (or forest formula):*

$$\mathcal{A}_G^R := \sum_{\mathbb{F}\subset D(G)} \prod_{\gamma\in\mathbb{F}} (-\tau_\gamma^*)\mathcal{A}_G, \tag{136}$$

where $D(G)$ is the Zimmerman forest.

The explicit form of the counter-term $\tau^*_{\mathcal{M}}\mathcal{A}_{G,\mu}$ is of interest for the next section. As in the previous paragraph, we start with the case $N = 2$. From the definition (132), we have three terms, corresponding to the zeroth, first and second derivative with respect to t.

• Because $\mathcal{M}_j$ is a class function, in the sense that $\mathcal{M}_j(g_1, g_2) \equiv F(g_1 g_2^{-1})$ and $F(g) = F(g^{-1}) = F(hgh^{-1})\ \forall h \in SU(2)$, the zero derivative can be written as:

$$\tau^{1*}_{\mathcal{M}}\mathcal{A}_{G,\mu} = \left\{\int_{SU(2)} dg\mathcal{M}_j(g)\right\}\mathcal{A}_{G/\mathcal{M},\mu} \quad , \tag{137}$$

where $G/\mathcal{M}, \mu$ is the graph obtained from G by cutting the two lines of color 1 linked to the melon $\mathcal{M}$, and joining to one another the two half lines of color 1 linked to $\bar{\mathcal{A}}_{G,\mu}$. In the usual terminology, the term in square brackets corresponds to the mass renormalization. Note that this term includes presumably some sub-melonic contributions, as mentioned at the end of section 5.4.

• The case of the first derivative with respect to t is easier, because the property $F(g) = F(g^{-1})$ trivially implies $\mathcal{A}'_{G,\mu}(0) = 0$.

• The last case, involving two derivative terms with respect to t, also exploits the class properties of $\mathcal{M}$. Noting that $\sum_\nu(\mathcal{L}_{i\sigma_\nu})^2 \equiv \Delta_{SU(2)}$, where the $\sigma_\nu\ \nu = 1, 2, 3$ are the usual Pauli matrices, we can easily show that

$$\begin{aligned}\tau^{2*}_{\mathcal{M}}\mathcal{A}_{G,\mu} = &\left\{\frac{1}{3}\int_{SU(2)} dg\mathcal{M}_j(g)|X_g|^2\right\} \\ &\times \int\prod_l d\bar{g}_{1l}d\bar{g}_{2l}dg_{1l}dg_{2l}\bar{\mathcal{A}}_{\mu,G}(\{\bar{g}_{1l}\},\{\bar{g}_{2l}\}) \\ &\qquad\times C_{i_1}(\{g_{1l}\},\{\bar{g}_{1l}\})\Delta_{g_{11}}C_{i_2}(g_{11},\{g_{2l\,l\neq 1}\},\{\bar{g}_{2l}\}) \quad ,\end{aligned} \tag{138}$$

where the term in square brackets corresponds to the so-called wave function renormalization term, and gives the "first deviation from locality", in the sense that the combination with the Laplacian operator does not correspond exactly to an "invariant trace".

The case of the 4-point function follows the same pattern, but is simpler because only one term appears in the Taylor expansion: the zeroth derivative term. It follows that the divergent term can be written as

$$\tau^{1*}_{\mathcal{M}}\mathcal{A}_{G,\mu} = \left\{\int dg dg' \mathcal{M}_j^{(4)}(g, g')\right\} \times \mathcal{A}_{G/\mathcal{M}^{(4)},\mu} \quad . \tag{139}$$

where $G/\mathcal{M}^{(4)}$ is the (connected) contracted graph obtained from G in the procedure detailed previously, and depicted in figure 6.1 below. This counter-term gives the coupling constant renormalization.

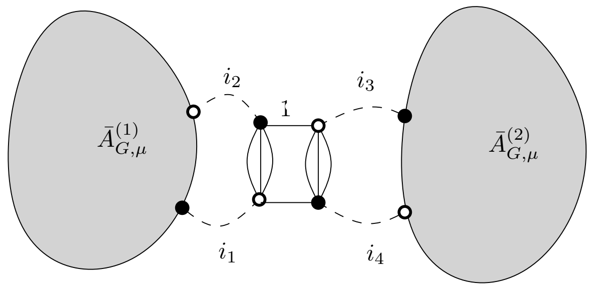

Figure 15: Contraction of a 4-points subgraph

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6.2 Counter-terms and renormalized series

The previous calculations show that the divergent parts of the dangerous graphs are all of the form of the interaction and of the kinetic terms in the action, without gauge symmetry. Its a familiar situation in quantum field theory, justifying the definition of the *renormalized theory*, which can be operationally identified as follows. We begin by defining the renormalized quantities, coupling constant, mass and field, labeled by an index "r", as

$$\psi = Z^{1/2}(\Lambda)\psi_r \quad ,$$
$$\lambda = Z^{-2}(\Lambda)Z_\lambda^{1/2}(\Lambda)\lambda_r = \mathcal{Z}_\lambda^{1/2}\lambda_r \quad ,$$
$$m = Z^{-1/2}(\Lambda)Z_m^{1/2}(\Lambda)m_r = \mathcal{Z}_m^{1/2}m_r \quad ,$$

in terms of which the UV-regularized partition function can be rewritten as:

$$\int d\mu_{C(Z^{-1/2}Z_m^{1/2}m_r)}(Z^{1/2}\psi_r, Z^{1/2}\bar{\psi}_r)e^{Z_\lambda^{1/2}\lambda_r\sum_{i=1}^4 Tr_{b_i}(\bar{\psi}_r,\psi_r)}$$
$$= \int d\mu_{C(Z^{-1/2}Z_m^{1/2}m_r)/Z}(\psi_r, \bar{\psi}_r)e^{Z_\lambda^{1/2}\lambda_r\sum_{i=1}^4 Tr_{b_i}(\bar{\psi}_r,\psi_r)} \quad .$$

The covariance can be expressed as follows:

$$\int d\mu_{C(Z^{-1/2}Z_m^{1/2}m_r)/Z}(\psi_r, \bar{\psi}_r)\psi_r(\{\theta_i,\phi_i\})\bar{\psi}_r(\{\theta'_i,\phi'_i\})$$
$$= 4\pi\int dg \sum_{\{l_i,m_i,m'_i\}} \frac{\prod_{i=1}^4 \hat{R}(g)\triangleright Y_{l_i,m'_i}(\theta_i,\phi_i)Y^*_{l_i,m_i}(\theta'_i,\phi'_i)}{Z\big[\sum_i l_i(l_i+1) + Z^{-1}Z_m m_r^2\big]}$$
$$= 4\pi\int dg \sum_{\{l_i,m_i,m'_i\}} \frac{\prod_{i=1}^4 \hat{R}(g)\triangleright Y_{l_i,m'_i}(\theta_i,\phi_i)Y^*_{l_i,m_i}(\theta'_i,\phi'_i)}{\sum_i l_i(l_i+1) + m_r^2}$$
$$\times \frac{1}{1+\dfrac{\delta_Z\sum_i l_i(l_i+1)+\delta_m m_r^2}{\sum_i l_i(l_i+1)+m_r^2}} \quad ,$$

where, in the last expression, $\delta_Z := Z-1$ and $\delta_m := Z_m - 1$, and the factor 4π comes from the choice of normalization for spherical harmonics. By expanding any Feynman graph in δ_Z and δ_m, we generate all the so-called "counter-terms" needed to define a finite amplitude in the $\Lambda\to\infty$ limit.

The previous propagator has the form:

$$C' = \frac{C}{1+\Delta C}, \tag{140}$$

where C is the propagator defined by (31), in which the bare mass parameter is replaced by the renormalized mass parameter m_r. From the basic properties of the Gaussian integration (in particular, the Wick theorem) we can prove the two following results, even in the case in which the covariance C' does not admit an explicit kinetic action term (i.e. an explicit inverse):

Proposition 8 *For non-normalized Gaussian measure $d\mu_C$, the covariance $C' = C - \dfrac{\Delta C}{1+\Delta C}$ satisfies the following equation:*

$$\int d\mu_{C'}(\psi,\bar\psi)\prod_{j=1}^{N}\bar\psi^j(\{g_j\})\psi^j(\{g'_j\}) \tag{141}$$
$$= \det\left[\frac{C}{C'}\right]\int d\mu_C(\psi,\bar\psi)e^{-\int\bar\psi\Delta\psi}\prod_{j=1}^{N}\bar\psi^j(\{g_j\})\psi^j(\{g'_j\}) \quad ,$$

where

$$\int\bar\psi\Delta\psi := \int[dg]^6\bar\psi(\vec g)(-\delta_Z\Delta_{\vec g}+\delta_m m_r^2)\psi(\vec g) \qquad . \tag{142}$$

Corollary 3 *The previous proposition ensures that, even if C does not correspond to an explicit kinetic term, the counter-terms in Δ can be included in the interaction part S_{int}, and, for normalized Gaussian measure:*

$$\int d\mu_{C'}(\psi,\bar\psi)e^{-S_{int}[\bar\psi,\psi]}\prod_{j=1}^{N}\bar\psi^j(\{g_j\})\psi^j(\{g'_j\}) \tag{143}$$
$$= \int d\mu_C(\psi,\bar\psi)e^{-\int\bar\psi\Delta\psi - S_{int}[\bar\psi,\psi]}\prod_{j=1}^{N}\bar\psi^j(\{g_j\})\psi^j(\{g'_j\}),$$

where normalized Gaussian measure means that $\int d\mu_C = 1$.

In the Feynman expansion of a correlation function, the counter-terms in S_{int} allow to define an operational renormalization procedure generating all the subtractions of the divergent parts of each divergent subgraph. Hence, in the Feynman expansion, each amplitude corresponds to a renormalized one defined in definition (13). Then, the Feynman expansion of an arbitrary correlation function S_N can be reorganized in the following way, namely the "renormalized series":

$$S_N = \sum_G \frac{1}{s(G)}\lambda_r^{|V(G)|}\mathcal{A}_G. \tag{144}$$

This result is exactly what we obtain from the renormalized generating functional with the previous prescription for counter-terms.

6.3 Bounds on the renormalized series

The finiteness of the renormalized amplitude can be proved rigorously. In fact, we can prove that, when the graph contains some subdivergences, the renormalized amplitude $\mathcal{A}_G^R$ is finite, but increases dramatically as the factorial of the number of divergent forest. Proving this theorem requires to define precisely the so-called "dangerous" and "safe" divergent forests:

Definition 14 Dangerous and safe forests *Consider a graph G, $\mathcal{A}_{G,\mu}$ the corresponding amplitude for the scale attribution μ, and D_G the set of divergent forests. Consider then $H \subset D_G$. We define i_H and e_H as:*

$$e_H = \sup\{i_l | l \in H/\mathbb{A}_{D_G}(H)\} \quad i_H = \inf\{i_l | l \in L_H \cap \mathbb{B}_{D_G}(H)\}$$

where L_H is the set of external lines of H, $\mathbb{B}_{D_G}$ is the ancestor of H in $D_G \cup H$ and $\mathbb{A}_{D_G}(H)$ the descendant, such as $\mathbb{A}_{D_G}(H) = \cup_{h; g \supset h \in D_G} h$. H is said to be "compatible" with D_G, in the sense that $D_G \cup H$ is still a forest.

The safe forest $\mathbb{F}_\mu$ is then the complementary in D_G of the set $D_\mu(\mathbb{F}_\mu)$ of dangerous or high subgraphs in G with respect to the scale assignment μ, defined as : $D_\mu = \{H \in D_G | e_H > i_H\}$.

This definition allows to rewrite the renormalized amplitude as:

$$\mathcal{A}_G^R = \sum_{f \in D_G} A_{G,f}^R \quad , \tag{145}$$

with:

$$A_{G,f}^R := \sum_{\mu | f \in \mathbb{F}_\mu} \prod_{g \in f} (-\tau_g^*) \prod_{h \in D_\mu(f)} (1 - \tau_h^*) \mathcal{A}_{G,\mu} \quad , \tag{146}$$

or

$$A_{G,f}^R := \sum_{\mu | f \in \mathbb{F}_\mu} \prod_{g \in f} (-\tau_g^*) \prod_{g \in f \cup \{G\}} \prod_{\substack{h \in D_\mu(f) \\ \mathbb{B}_f(h) = g}} (1 - \tau_h^*) \mathcal{A}_{G,\mu} \quad . \tag{147}$$

Beginning with the contractions over the safe forest f, we obtain, after appropriate organization of the successive contractions:

$$\prod_{g \in f} (-\tau_g^*) \mathcal{A}_{G,\mu} = \prod_{g \in f \cup \{G\}} \nu_\mu(g/\mathbb{A}_f(g)) \quad , \tag{148}$$

where $\nu_\mu(g)$ is the discarded part of the amplitude. Note that all these terms are not exactly disconnected, because the contraction of the 2-points graphs reveal a non-local operator, which acts on another contracted component. From the multiscale analysis, it follows that:

$$\Big| \prod_{g \in f \cup \{G\}} \nu_\mu(g/\mathbb{A}_f(g)) \Big| \leq \prod_{g \in f \cup \{G\}} \prod_{i,k} M^{\omega \left[(g/\mathbb{A}_f(g))_i^k \right]} \quad . \tag{149}$$

Now, observe that the contraction over the high divergent graphs only affects the components $g/\mathbb{A}_f(g)$. It follows then, from the analysis of the previous paragraph, that the decay of a renormalized graph g is at most $M^{-|e_g - i_g|}$. Hence, the renormalized amplitude is bounded by:

$$|A_G^R| \leq \sum_{\mu | f \in \mathbb{F}_\mu} \prod_{g \in f \cup \{G\}} \prod_{i,k} M^{\omega' \left[(g/\mathbb{A}_f(g))_i^k \right]} \quad , \tag{150}$$

where :

$$\omega' \left[(g/\mathbb{A}_f(g))_i^k \right] := \inf \left(-1, \omega \left[(g/\mathbb{A}_f(g))_i^k \right] \right) \quad ,$$

except when $(g/\mathbb{A}_f(g))_i^k = g/\mathbb{A}_f(g)$, in which case

$$\omega' \left[(g/\mathbb{A}_f(g))_i^k \right] = 0 \quad .$$

From the decay factor of equation (150), we can extract the factor $M^{-\delta i_{max}(\mu)}$, where $i_{max}(\mu) := \sup(\mu)$. With the rest of the decay, we can sum over each component $g/\mathbb{A}_f(g)$ $g \in f$, as in the proof of the Weinberg theorem. Because of the following bound:

$$\prod_{g \in f \cup \{G\}} K^{|V(g/\mathcal{A}_f(g))|} \leq K'^{|V(G)|} \quad , \tag{151}$$

the sum over internal scale assignments in each $g/\mathbb{A}_f(g)$ is bounded by $K'^{|V(g/\mathcal{A}_f(g))|}$. The remaining sum over i_{max} is bounded by:

$$\sum_{i_{max}} (i_{max})^{|f|} M^{-\delta i_{max}} \leq |f|! K^{|f|} \quad , \tag{152}$$

where $|f|$ is the cardinality of the set f. Because the number of sub-forests in a graph G can be bounded by $2^{|D_G|}$, we finally deduce the following theorem:

Theorem 4 (BPH uniform) *Consider a Feynman graph G of order $|V(G)|$. The renormalized amplitude $\mathcal{A}_G^R$ has the following bound:*

$$|\mathcal{A}_G^R| \leq K^{|V(G)|} |D_G|!, \;\; K \in \mathbb{R}^+ \quad , \tag{153}$$

where $|D(G)|$ is the cardinality of the divergent forest set in G.

As announced, the amplitude is finite but arbitrarily large, increasing dramatically with the size of the divergent forest. This is the known problem of *renormalons*, which implies that the convergence of the renormalized series (144) is not guaranteed in a perturbative approach. To prove its convergence we would need the help of the constructive theory and of Borel summability technology, which is not the focus of this paper. To solve this technical difficulty, we use the effective series, defined in the next section, which is renormalons-free. This result, added to the asymptotic freedom, proved in section 7 at the one loop order, confirms the convergence of the effective series i.e. the perturbative series expressed in terms of the effective amplitudes and effective coupling.

6.4 The effective series

The effective series is a more physical approach of renormalization, closely related to the Wilson approach. It is a way to solve the renormalons problem and to ensure the convergence of the perturbative series in many cases, as we will see below. The basic idea is the following. Consider a graph G and its bare amplitude $A_\mu(G)$ at scale attribution μ, as defined above. As we have seen before, in this graph, there are some divergent graphs, which form the set $D(G)$. But in fact, only a subset of these subgraphs is potentially dangerous, the subset noted $D_\mu(G)$ in the previous section. The argument is that only this subset needs to be renormalized, and the effective amplitude $A_\mu^{eff}(G)$ is defined by

$$A_\mu^{eff}(G) := \prod_{\gamma \in D_\mu} (1 - \tau_\gamma^*) A_\mu(G) \quad , \tag{154}$$

about which we have the following theorem [19, 34]:

Theorem 5 (Existence of the effective expansion)*: Consider the formal (bare) power series defined by:*

$$S_N^\Lambda = \sum_{G,\mu} \frac{1}{s(G)} \left(\prod_{b \in \mathcal{V}(G)} \Big(-\lambda_b^{(\Lambda)} \Big) \right) A_\mu(G) \quad , \tag{155}$$

where $\mathcal{V}(G)$ is the set of vertices in G including all the interactions compatible with the just-renormalizability criterion and $\lambda_b^{(\Lambda)}$ their coupling constants. This series can be rewritten in a more convenient form in terms of the effective amplitudes:

$$S_N^\Lambda = \sum_{G,\mu} \frac{1}{s(G)} \left(\prod_{b \in \mathcal{V}(G)} \big(-\lambda_{b,e_b(G,\mu)}^{(\Lambda)} \big) \right) A_\mu^{eff}(G) \quad , \tag{156}$$

where the $\lambda^{(\Lambda)}_{b,e_b(G,\mu)}$ *are the effective couplings, generated by the local part of the high divergent subgraphs. They obey the following inductive relation*

$$-\lambda^{(\Lambda)}_{b,i} = -\lambda^{(\Lambda)}_{b,i+1} + \sum_{\substack{(\mathcal{H},\mu,\hat{S})\hat{S}\neq\emptyset \\ \phi_i(\mathcal{H},\mu,\hat{S})=(b,\mu,\emptyset)}} \frac{1}{s(\mathcal{H})} \left(\prod_{b'\in\mathcal{V}(\mathcal{H})} \left(-\lambda^{(\Lambda)}_{b',i'_b(\mathcal{H},\mu)} \right) \right) \times \left(\prod_{m\in D^{i+1}_\mu\setminus\hat{S}} (1-\tau^*_m) \right) \prod_{M\in\hat{S}} \tau^*_M A_\mu(\mathcal{H}) \quad , \tag{157}$$

with $e_b = \sup\{\mu_l, l \,\text{hooked to}\, b\}$.

The notation introduced above will be defined precisely in the proof, for which we give only the main steps, referring to [19, 34] for details.

Proof (Sketch)

The basic idea is to introduce an intermediate step between the bare and the effective series as follows. We consider a slice i and define:

$$S^\Lambda_N = \sum_{G,\mu} \frac{1}{s(G)} \left(\prod_{b\in\mathcal{V}(G)} (-\lambda^{(\Lambda)}_{b,sup(i,i_b(G,\mu))} \right) A^{eff,i}_\mu(G) \quad , \tag{158}$$

where

$$A^{eff,i}_\mu(G) := \prod_{\gamma\in D^i_\mu} (1-\tau^*_\gamma) A_\mu(G) \quad , \tag{159}$$

and

$$D^{i+1}_\mu(G) = \{m \in D(G) | i_m > i\} \quad i_m := \inf\{\mu_l, l\, hooked \quad to\, b\} \quad .$$

It is obvious that, if $i = \rho$, where $\Lambda = M^\rho$, the effective series reduces to the bare one. Assuming this is true at scale $i+1$, we can prove it at scale i by induction, by multiplying the effective amplitude at scale $i+1$ by a suitable form of the identity, adding and subtracting the counter-terms in $D^i_\mu(G) \setminus D^{i+1}_\mu(G) = \{m \in D(G) | i_m = i+1\}$, which changes $A^{eff,i+1}_\mu(G)$ into $A^{eff,i}_\mu(G)$,

$$A^{eff,i}_\mu(G) := \prod_{\substack{S\subseteq D^i_\mu\setminus D^{i+1}_\mu \\ S\neq\emptyset}} \prod_{M\in S} (1-\tau^*_M+\tau^*_M) \prod_{\gamma\in D^i_\mu} (1-\tau^*_\gamma) A_\mu(G) \quad .$$

The completely subtracted piece changes $A^{eff,i+1}_\mu(G)$ into $A^{eff,i}_\mu(G)$, and the second one is developed as a sum over S as follows:

$$S^\Lambda_N = \sum_{\substack{(G,\mu,S) \\ S\subseteq D^i_\mu\setminus D^{i+1}_\mu}} \frac{1}{s(G)} \left(\prod_{b\in\mathcal{V}(G)} (-\lambda^{(\Lambda)}_{b,sup(i+1,i_b(G,\mu))} \right) A^{eff,i}_{\mu,S}(G) \quad ,$$

with

$$A^{eff,i}_{\mu,S} := \prod_{M\in S} (-\tau^*_M) \prod_{m\in D^i\setminus S} (1-\tau^*_m) A_\mu(G) \quad ,$$

and in particular $A^{eff,i}_{\mu,\emptyset} = A^{eff,i}_\mu$. A subtlety appears in this case because the 2-point divergent graphs (with degree $\omega = 2$) introduce two counter-terms, one for the mass and one for the wave-function. For this reason we modify the previous definition of S, and introduce the new definition:

$$\hat{S} = \{(M, k_M) | M \in S, k_M \in 0,2, k_M \leq \omega(M)\} \quad .$$

Secondly, we introduce the *collapse* ϕ_i which sends the triplets $(G, \mu, \hat{S})$ to its contracted version $(\mathcal{G}', \mu', \emptyset)$, such that the previous sum can be rewritten as a sum on $\mathcal{G}'$

$$S_N^\Lambda = \sum_{\mathcal{G}',\mu'} \sum_{\substack{\{(G,\mu,S)\}= \\ \phi_i^{-1}(\mathcal{G}',\mu',\emptyset)}} \frac{A^{eff,i}_{\mu,S}(G)}{s(G)} \left(\prod_{b\in\mathcal{V}(G)} (-\lambda^{(\Lambda)}_{b,sup(i+1,i_b(G,\mu))} \right) \quad . \tag{160}$$

Decomposing

$$\prod_{M\in\hat{S}} (-\tau^*_M) = \prod_{b'\in\mathcal{V}(G)} \left(\prod_{M\in\hat{S},\, M\subset\phi_i^{-1}(b')} (-\tau^*_M) \right)$$

in the sum (160), we find that it gives exactly the effective sum at scale i given by (158), if the coupling satisfies the recursive relation of the theorem.

□

The coupling recursion defines a discrete flow, for which the initial data are, as usual in standard quantum field theory, imposed by the 1PI functions at zero momenta.

The main interest of the effective series is that all these amplitudes are bounded in the form [19]

$$|A^{eff}(G)| \le K^{V(G)} \quad , \tag{161}$$

a result which can be directly deduced from the theorem 4 proved in the previous section, in the special case where the set of inoffensive forest is empty. Remarkably, in the previous bound, renormalons do not appear.

Another important fact about the effective series and effective coupling constants is their relationship with the renormalized series. In fact, if we define the renormalized coupling by $\lambda_r := \lambda_{-1}$, and if we reframe the effective series in terms of the renormalized coupling, we find exactly the renormalized series.

7 Running of the coupling constant and asymptotic freedom

In this section we study the behavior of the one particle irreducible part (1PI) of the 2- and 4-point correlation functions at the leading (melonic) order. From these results, we deduce the beta function, describing the behavior of the effective coupling λ_{eff} with respect to the UV regulator Λ. It turns out that this beta function is negative, meaning that the theory is asymptotically free at least at one-loop order.

7.1 Divergences of the 2-points function

At one-loop order, the divergences are due to the melonic tadpole diagrams, an example of which is depicted in figure 16, and corresponds to the following amputated Feynman amplitude:

$$\begin{aligned}
\mathcal{A}^{(4)}_{\mathcal{M}}(g_t^{-1}g_s) &= \int_{1/\Lambda^2}^{+\infty} d\alpha e^{-\alpha m^2} \int dl \int dk \int_{U_k(1)^{\times 4}} [dh]^4 \\
&\quad \times K_\alpha\big(lh_1\big) K_\alpha\big(lh_2\big) K_\alpha\big(lh_3\big) K_\alpha\big(g_t^{-1}g_s lh_4\big) \\
&= \sum_{l_4,m_4} \int_{1/\Lambda^2}^{+\infty} d\alpha e^{-\alpha m^2} \int dl \int dk \int_{U_k(1)^{\times 3}} [dh]^3 (2l_4+1) \\
&\quad \times K_\alpha\big(lh_1\big) K_\alpha\big(lh_2\big) K_\alpha\big(lh_3\big) D^{(l_4)}_{m_4,0}[l] D^{(l_4)}_{0,m_4}[g_t^{-1}g_s] e^{-4\alpha l_4(l_4+1)} \quad ,
\end{aligned} \tag{162}$$

where we have performed the integration over h_4, and used of the cyclic permutation invariance of the heat kernel : $K_\alpha(g_1g_2) = K_\alpha(g_2g_1)$. We define:

$$\mathcal{A}^{(4)}_{\mathcal{M};l_4,m} = \int_{1/\Lambda^2}^{+\infty} d\alpha e^{-\alpha m^2} \int dl \int dk \int_{U_k(1)^{\times 3}} [dh]^3 \\ \times K_\alpha\big(lh_1\big) K_\alpha\big(lh_2\big) K_\alpha\big(lh_3\big) D^{(l_4)}_{m,0}[l] e^{-4\alpha l_4(l_4+1)}, \tag{163}$$

such that $\mathcal{A}^{(4)}_{\mathcal{M}}(g_t^{-1}g_s) = \sum_{l_4,m} \mathcal{A}^{(4)}_{\mathcal{M};l_4,m}(2l_4+1) D^{(l_4)}_{0,m}[g_t^{-1}g_s]$.

The aim is to extract the divergent part of the previous expression.

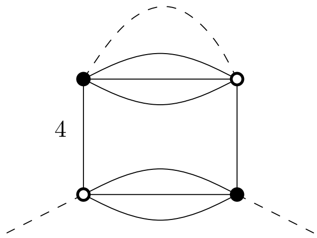

Figure 16: Tadpole contribution to the one-loop 1PI 2-point function

As the divergences occur in the vicinity of $\alpha = 0$, we can make use of the corresponding approximation for the heat kernels appearing in (162)

$$K_\alpha(g = e^{iX}) \underset{\alpha\to 0}{\longrightarrow} \big(4\pi\alpha\big)^{-3/2} e^{-\frac{\langle X,X\rangle}{4\alpha}} \quad , \tag{164}$$

where $\langle .,. \rangle : \mathrm{su}(2) \to \mathbb{R}$ is the (normalized) Killing form on the Lie algebra. The vicinity of $\alpha = 0$ forces us, in the saddle point approximation, to evaluate only the fluctuations around the identity for each group arguments :

$$e^{X_l} e^{Y_{h_i}} \approx \mathbb{I} \quad where \quad l =: e^{X_l} \ , h_i =: e^{Y_{h_i}} \quad . \tag{165}$$

Hence, the integral in (163) behaves as

$$\left(\frac{1}{4\pi\alpha}\right)^{9/2} \int_{\mathbb{R}^2} d^2x \int_{\mathbb{R}} \prod_{i=1}^{3} dy_i e^{-\frac{3x_1^2+3x_2^2+(x_3-y_1)^2+(x_3-y_2)^2+(x_3-y_3)^2}{4\alpha}} \sim \alpha^{-2} \quad ,$$

where $\sum_j ix_j\hat{J}_j := X_l$ and $iy_j\vec{\hat{J}}\cdot\vec{k} = Y_j$. Note that the power of α coincides with the divergence degree computed previously. Hence, we confirm its validity in this simple example.

The divergences come from the negative powers of α. In order to extract these divergent parts, we use the general group manifold analog of the following identity for the Gaussian distribution on $\mathbb{R}^d$:

$$e^{-\vec{x}^2/4\alpha} \underset{\alpha\to 0}{=} (4\pi\alpha)^{d/2}\delta(\vec{x}) + (4\pi\alpha)^{d/2}\alpha\Delta\delta(\vec{x}) + \mathcal{O}(\alpha^{\frac{d+2}{2}}) \quad , \tag{166}$$

where Δ is the Laplacian on $\mathbb{R}^d$. This relation can be extended to the general group case as follows: for any function $f(g)$ on $SU(2)$, and with X_g such that $g = e^{X_g}$, we have:

$$\int dg e^{-\frac{\langle X_g,X_g\rangle}{4\alpha}} f(g) \underset{\alpha\to 0}{=} \tag{167}$$
$$\int dg e^{-\frac{\langle X_g,X_g\rangle}{4\alpha}} \left[f(\mathbb{I}) + \frac{df(e^{tX_g})}{dt}\Big|_{t=0} + \frac{1}{2}\frac{d^2f(e^{tX_g})}{dt^2}\Big|_{t=0} + \cdots \right]$$
$$= \int dg e^{-\frac{\langle X_g,X_g\rangle}{4\alpha}} \left[f(\mathbb{I}) + \mathcal{L}_{X_g} f(\mathbb{I}) + \frac{1}{2}\mathcal{L}_{X_g}\mathcal{L}_{X_g} f(\mathbb{I}) + \cdots \right]$$
$$= (4\pi\alpha)^{3/2} f(\mathbb{I}) + (4\pi\alpha)^{3/2}\alpha\Delta_{SU(2)} f(\mathbb{I}) + \mathcal{O}(\alpha^{5/2}) \quad ,$$

where in the last step we have used the reflection symmetry of the Gaussian, the relation $\mathcal{L}_{X_g} = \sum_\nu X_g^\nu \mathcal{L}_{i\sigma_\nu}$, and the definition of the Laplacian : $\Delta_{SU(2)} = \sum_\nu \left(\mathcal{L}_{i\sigma_\nu}\right)^2$. This result allows to extract the divergent part of (163). From the power counting, it follows that the divergent contributions are all contained in:

$$\mathcal{A}^{(4)\,\infty}_{\mathcal{M};l_4,m} := \int_{1/\Lambda^2}^{+\infty} d\alpha e^{-\alpha m^2} \int dl \int dk \int [dh]^3 \prod_{i=1}^{3} K_\alpha(lh_i)$$
$$\times \left[1 + \frac{1}{2}\sum_{\mu,\nu} X_l^\mu X_l^\nu \mathcal{L}_{i\sigma_\mu}\mathcal{L}_{i\sigma_\nu}\right] D^{(l_4)}_{m0}(\mathbb{I})\big(1 - 4\alpha l_4(l_4+1)\big) \qquad . \tag{168}$$

Because of the gauge symmetry, any term involving $X_l^\mu X_l^\nu \mathcal{L}_{\sigma_\mu}\mathcal{L}_{\sigma_\mu}$ vanishes when $\mu \neq \nu$, and does not depend on ν. Hence, the sum can be replaced by $(X_l^1)^2 \sum_\nu (\mathcal{L}_{i\sigma_\nu})^2 = (X_l^1)^2 \Delta_{SU(2)}$, and because of the relation

$$\Delta_{SU(2)} D^{l_4}_{m0}(\mathbb{I}) = -4l_4(l_4+1) D^{l_4}_{m0}(\mathbb{I}) = -4l_4(l_4+1)\delta_{m0} \quad ,$$

we find:

$$\mathcal{A}^{(4)\,\infty}_{\mathcal{M};l_4,m} = \delta_{m0}\int_{1/\Lambda^2}^{+\infty} d\alpha e^{-\alpha m^2} \frac{1}{3}\Big(\frac{1}{4\pi\alpha}\Big)^2 \left[1 - \frac{16}{3}\alpha l_4(l_4+1))\right] = \frac{\delta_{m0}}{48\pi^2}\left[\mathcal{I}_1 - \frac{16}{3}\mathcal{I}_2\, l_4(l_4+1)\right] \qquad .$$

Note that, as explained above, the terms which do not appear in this expression have a power of α higher than -1, and do not diverge when $\Lambda \to \infty$. From the definition of (163), we finally obtain:

$$\mathcal{A}^{(4)\,\infty}_{\mathcal{M}}(g_t^{-1}g_s) = \sum_{l_4} \frac{1}{48\pi^2}\left[\mathcal{I}_1 + \frac{4}{3}\mathcal{I}_2\,\Delta_4\right](2l_4+1) D^{(l_4)}_{0,0}[g_t^{-1}g_s]$$
$$= \frac{1}{48\pi^2}\left[\mathcal{I}_1 + \frac{4}{3}\mathcal{I}_2\,\Delta_4\right]\int dk_4 dh_4 K_0[g_s h_4 g_t^{-1}] \quad ,$$

where we have used the fact that $\Delta_4 D^{(l_4)}_{0,0}[g_t^{-1}g_s] = -4l_4(l_4+1)D^{(l_4)}_{0,0}[g_t^{-1}g_s]$, and introduced $\mathcal{I}_1$ and $\mathcal{I}_2$, defined as:

$$\mathcal{I}_1 := \int_{1/\Lambda^2}^{+\infty} \frac{d\alpha}{\alpha^2} e^{-\alpha m^2} \tag{169}$$
$$\mathcal{I}_2 := \int_{1/\Lambda^2}^{+\infty} \frac{d\alpha}{\alpha} e^{-\alpha m^2} \qquad . \tag{170}$$

We obtain a similar expression to (169) for each of the four colors of the intermediate lines. Hence, the complete amplitude $\Gamma^{(2)}_\infty(\{g_{t_i}\},\{g_{s_i}\})$ is (a global factor 2 comes from the Wick-theorem):

$$\Gamma^{(2)}_\infty(\{g_{t_i}\},\{g_{s_i}\}) = -\lambda \int \prod_l dg_l dg'_l \sum_{i=1}^{4} \mathcal{A}^{(i)\,\infty}_{\mathcal{M}}(g_i^{-1}g'_i) \tag{171}$$
$$\prod_{l\neq i}\delta(g_l^{-1}g'_l) C_\Lambda(\{g_{t_k}\},\{g'_k\}) C_\Lambda(\{g_k\},\{g_{s_k}\})$$
$$= -\frac{\lambda}{6\pi^2}\left[\mathcal{I}_1 + \frac{1}{3}\mathcal{I}_2\,\Delta_{SU(2)^{\times 4}}\right] C_\Lambda(\{g_{t_k}\},\{g_s\}) \quad ,$$

where the first term is interpreted as a mass renormalization and the second one as a wave function renormalization.

7.2 4-points function

We now move on to the computation of the 4-point function. At one-loop order, the leading (or divergent) graphs are of the type depicted in figure 17 below, with one melonic loop between two vertices.

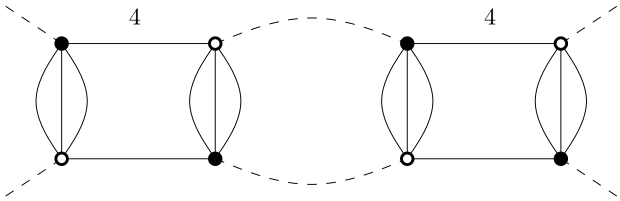

Figure 17: Melonic contribution to the 1PI 4-points function at the one-loop order

The amputated Feynman amplitude can be easily obtained from the Feynman rules and from the previous calculation. We find:

$$\mathcal{A}^{(4)}_{\mathcal{M}_4}(g_s,g_t,g'_s,g'_t) = \int_{1/\Lambda^2}^{+\infty} d\alpha_1 d\alpha_2 e^{-(\alpha_1+\alpha_2)m^2} \int dl_1 dl_2 \int dk_1 dk_2 \int [dh^1]^3[dh^2]^3 \prod_{i=1}^{3} K_{\alpha_1+\alpha_2}\big(l_2 h_i^2 l_1 h_i^1\big)$$
$$\times \sum_{\substack{l_4,m_4\\ l'_4,m'_4}} (2l_4+1)(2l'_4+1) D^{(l_4)}_{m_4 0}(l_1) D^{(l'_4)}_{m'_4 0}(l_2) \Big[D^{(l_4)}_{0 m_4}\big(g_t^{-1}g_s\big) D^{(l'_4)}_{0 m'_4}\big(g_t'^{-1}g'_s\big) + g_s \longleftrightarrow g'_s \Big] \quad .$$

To extract the divergent part of this amplitude, we consider the Hepp sector $\alpha_1 \geq \alpha_2$, and define the new variables α and β as:

$$\alpha_1 = \alpha \tag{172}$$
$$\alpha_2 - \frac{1}{\Lambda^2} = \beta\Big(\alpha_1 - \frac{1}{\Lambda^2}\Big) \tag{173}$$

such that $\alpha \in [1/\Lambda^2, \infty]$ and $\beta \in [0,1]$. In terms of these new variables, the Lebesgue measure becomes:$d\alpha_1 d\alpha_2 = \alpha d\alpha d\beta$. Because of the presence of α, it follows, from the previous computation, that the divergent contribution comes from the approximation:

$$K_\alpha(g) \sim \delta(g), \tag{174}$$

and using the same approximation scheme as before, we deduce that the divergent part of the 4-point amplitude, defined as the contribution involving a negative power of α, is:

$$\mathcal{A}^{(4)\,\infty}_{\mathcal{M}_4}(g_s,g_t,g'_s,g'_t) = \frac{\lambda^2}{48\pi^2}\mathcal{I}_2 \int dk_4 dh_4 \Big[K_0\big(g_t^{-1}g_s h_4\big)\delta\big(g_t'^{-1}g'_s\big) + g_s \longleftrightarrow g'_s \Big] \quad . \tag{175}$$

7.3 Renormalization and asymptotic freedom

In order to extract the dangerous part of the expression obtained above, i.e. the terms involving a positive or null power of Λ, we study the behavior of the integrals $\mathcal{I}_1$ and $\mathcal{I}_2$. Firstly, observe that an integration by part gives:

$$\mathcal{I}_1 = \Lambda^2 e^{-m^2/\Lambda^2} - m^2 \mathcal{I}_2 \quad .$$

Secondly, observe that the divergence of $\mathcal{I}_2$ is at most logarithmic. Hence,

$$\mathcal{I}_2 = A\ln(\Lambda) + \mathcal{O}(1/\Lambda) \quad .$$

By differentiating the two members of this equality, we obtain $A = 2$, and finally:

$$\mathcal{I}_1 \sim \Lambda^2 - 2m^2\ln(\Lambda) \tag{176}$$
$$\mathcal{I}_2 \sim 2\ln(\Lambda) \quad . \tag{177}$$

Hence, for the 1PI 2-point function, the dangerous part Γ^2_{div} is equal to

$$\Gamma^{(2)}_\infty(\{g_{t_i}\},\{g_{s_i}\}) = \frac{-\lambda}{6\pi^2}\left[\Lambda^2 - 2m^2\ln(\Lambda) + \frac{2}{3}\ln(\Lambda)\,\Delta_{SU(2)^{\times 4}}\right] C_\Lambda(\{g_{t_k}\},\{g_s\}) \quad , \tag{178}$$

which fixes the divergent (or essential) parts of the mass and wave function counter-terms as:

$$\delta m^2_{div} := -\frac{\lambda}{6\pi^2}\left[\Lambda^2 - 2m^2\ln(\Lambda)\right] \tag{179}$$

$$\delta Z_{div} := \frac{\lambda}{9\pi^2}\ln(\Lambda) \quad . \tag{180}$$

Similarly, from the expression (175), we deduce that the divergence is exactly compensated by an interaction of the initial form with an intermediate line of color 4, if this interaction is proportional to the counter-term $\delta\lambda$, with:

$$\delta\lambda = \frac{\lambda^2}{12\pi^2}\ln(\Lambda). \tag{181}$$

This result allows to obtain the dependence of the effective coupling on Λ. Indeed, the effective coupling includes the effect of the wave-function renormalization. Hence:

$$\lambda_{eff}(\Lambda) := \frac{\lambda + \delta\lambda}{Z^2} \quad . \tag{182}$$

By differentiating the two terms, and using the relations (178) and (181), we find:

$$\Lambda\frac{d\lambda_{eff}}{d\Lambda} = -\frac{5}{36\pi^2}\lambda^2_{eff} \quad , \tag{183}$$

where the minus sign means that the model is asymptotically free.

8 Conclusion

We have studied the renormalization of a TGFT model on the homogeneous space $(SU(2)/U(1))^d$, endowed with the additional gauge invariance condition, using multi-scale methods. We have proven renormalizability to all orders in perturbation theory for the model with melonic quartic interactions in $d = 4$ (and, implicitly, super-renormalizability for the model in $d = 3$). This is the first example of a renormalization analysis for a TGFT model on a homogeneous space, rather than a group manifold, and a promising step forward towards 4d gravity models, which have similar formulations.

For the same model, we have also computed both the renormalised and effective perturbative series, and established its asymptotic freedom at one-loop order, by the analysis of the 2-point and 4-point correlation functions. This is another interesting result, because it support the view that asymptotic freedom is generic in TGFTs, and even survives stepping out of the simple group-based setup to move to homogeneous spaces. Clearly, however, more work is needed to confirm such general expectation.

Whenever possible, we have also generalised our construction and results to arbitrary homogeneous spaces of the type $SO(D)/SO(D-1) \simeq \mathcal{S}_{D-1}$. This included a general Abelian power counting, and a corresponding classification of potentially just-renormalizable models, for various choices of D and d. However, as we pointed out, the exact power counting of such more general non-abelian models may deviate from the Abelian one, and a more detailed case-by-case analysis needs to be carried out in order to prove (or disprove) their perturbative renormalizability.

To keep moving in the direction of 4d quantum gravity models, as defined in the spin foam context, is our next goal. In particular, the mentioned detailed analysis of divergences and exact power counting should be performed for TGFTs on the homogeneous space $(SO(4)/SO(3))^d$, the case $d = 4$ corresponding to the so-called Barrett-Crane imposition of the simplicity constraints reducing topological BF theory to

gravity (see [23] and references therein), defining interesting 4d quantum gravity models (in absence of the Immirzi parameter). We expect the results of [33] to be a good basis for such generalisation. The Lorentzian counterpart of these models would of course be the next target. After this, one would have the proper understanding and basis to tackle the deformation of such models induced by the Immirzi parameter, which brings out of the homogeneous space setting to more general sub-manifolds of the $SO(4)$ (or $SO(3,1)$) group manifold (see [7].
It is clear that the path towards a renormalizable quantum field theory for the 'atoms of space'is still long, but it should be also clear that we are making steady and important progress along it.

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