ANCIENT SOLUTIONS TO THE RICCI FLOW WITH PINCHED CURVATURE

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Abstract

We show that any ancient solution to the Ricci flow which satisfies a suitable curvature pinching condition must have constant sectional curvature.

1. Introduction

In this article, we study ancient solutions to the Ricci flow on compact manifolds. Recall that a one-parameter family of metrics $g(t)$ on a compact manifold M evolves by the Ricci flow if

$$\frac{\partial}{\partial t} g(t) = -2 \operatorname{Ric}_{g(t)}.$$

A solution to the Ricci flow is called *ancient* if it is defined on a time interval $(-\infty, T)$. Ancient solutions typically arise in the study of singularities to the Ricci flow (see, e.g., [15], [16], [19], [20]).

P. Daskalopoulos, R. Hamilton, and N. Šešum [11] have recently obtained a complete classification of all ancient solutions to the Ricci flow in dimension 2 (see also [10], where the analogous question for the curve shortening flow is studied). V. Fateev [12] has constructed an interesting example of an ancient solution in dimension 3. L. Ni [18] showed that any ancient solution to the Ricci flow which is of type I, is κ-noncollapsed, and has positive curvature operator has constant sectional curvature.

In this article, we show that any ancient solution to the Ricci flow in dimension $n \geq 3$ which satisfies a suitable curvature pinching condition must have constant sectional curvature. In dimension 3, we impose a condition on the Ricci tensor of $(M, g(t))$.

THEOREM 1

Let M be a compact three-manifold, and let $g(t)$, $t \in (-\infty, 0)$, be an ancient solution to the Ricci flow on M. Moreover, suppose that there exists a uniform constant $\rho > 0$

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such that

$$\mathrm{Ric}_{g(t)} \geq \rho\, \mathrm{scal}_{g(t)}\, g(t) \geq 0$$

for all $t \in (-\infty, 0)$. *Then the manifold* $(M, g(t))$ *has constant sectional curvature for each* $t \in (-\infty, 0)$.

Fateev's example shows that the pinching condition for the Ricci tensor cannot be removed. The proof of Theorem 1 relies on a new interior estimate for the Ricci flow in dimension 3 (see Proposition 3 below). The proof of this estimate relies on the maximum principle and will be presented in Section 2.

In dimension $n \geq 4$, we prove the following result.

THEOREM 2
Let M *be a compact manifold of dimension* $n \geq 4$, *and let* $g(t)$, $t \in (-\infty, 0)$, *be an ancient solution to the Ricci flow on* M. *Moreover, suppose that there exists a uniform constant* $\rho > 0$ *with the following property: for each* $t \in (-\infty, 0)$, *the curvature tensor of* $(M, g(t))$ *satisfies*

$$\begin{aligned} &R_{g(t)}(e_1, e_3, e_1, e_3) + \lambda^2\, R_{g(t)}(e_1, e_4, e_1, e_4) \\ &\quad + R_{g(t)}(e_2, e_3, e_2, e_3) + \lambda^2\, R_{g(t)}(e_2, e_4, e_2, e_4) \\ &\quad - 2\lambda\, R_{g(t)}(e_1, e_2, e_3, e_4) \geq \rho\, \mathrm{scal}_{g(t)} \geq 0 \end{aligned}$$

for all orthonormal four-frames $\{e_1, e_2, e_3, e_4\}$ *and all* $\lambda \in [0, 1]$. *Then the manifold* $(M, g(t))$ *has constant sectional curvature for each* $t \in (-\infty, 0)$.

Theorem 2 again follows from pointwise curvature estimates which are established using the maximum principle (see Corollary 7 below). In dimension $n \geq 4$, the evolution equation for the curvature tensor is much more complicated, and our estimates are not as explicit as in the three-dimensional case. In order to handle the higher-dimensional case, we use the invariant curvature conditions introduced in [3] and [7]. These ideas also play a key role in the proof of the differentiable sphere theorem (see [7], [8]).

2. Proof of Theorem 1

PROPOSITION 3
Let M *be a compact three-manifold, and let* $g(t)$, $t \in [0, T)$, *be a solution to the Ricci flow on* M. *Moreover, suppose that there exists a uniform constant* $\rho \in (0, 1)$ *such*

that

$$\mathrm{Ric}_{g(t)} \geq \rho \, \mathrm{scal}_{g(t)} \, g(t) \geq 0$$

for each $t \in [0, T)$. Then, for each $t \in (0, T)$, the curvature tensor of $(M, g(t))$ satisfies the pointwise estimate

$$|\overset{\circ}{\mathrm{Ric}}_{g(t)}|^2 \leq \left(\frac{3}{2t}\right)^{\sigma} \mathrm{scal}_{g(t)}^{2-\sigma},$$

where $\sigma = \rho^2$.

Proof

The assertion is trivial if $(M, g(0))$ is Ricci flat. Hence, it suffices to consider the case that $(M, g(0))$ is not Ricci flat. By the maximum principle, the manifold $(M, g(t))$ has strictly positive scalar curvature for all $t \in (0, T)$.

We next define a function $f : M \times (0, T) \to \mathbb{R}$ by

$$f = \mathrm{scal}^{\sigma-2} \, |\overset{\circ}{\mathrm{Ric}}|^2,$$

where $\sigma = \rho^2$. It is easy to see that $f \leq \mathrm{scal}^{\sigma}$. Moreover, it follows from [13, Lemma 10.5] that

$$\frac{\partial}{\partial t} f \leq \Delta f + \frac{2(1-\sigma)}{\mathrm{scal}} \, \partial_k \mathrm{scal} \, \partial^k f + 2 \, \mathrm{scal}^{\sigma-3} \, [\sigma \, |\mathrm{Ric}|^2 \, |\overset{\circ}{\mathrm{Ric}}|^2 - 2P],$$

where P is a polynomial expression in the eigenvalues of the Ricci tensor. By assumption, we have $\mathrm{Ric} \geq \rho \, \mathrm{scal} \, g$. Hence, it follows from [13, Lemma 10.7] that

$$P \geq \rho^2 \, |\mathrm{Ric}|^2 \, |\overset{\circ}{\mathrm{Ric}}|^2.$$

This implies that

$$\begin{aligned} 2P - \sigma \, |\mathrm{Ric}|^2 \, |\overset{\circ}{\mathrm{Ric}}|^2 &\geq \sigma \, |\mathrm{Ric}|^2 \, |\overset{\circ}{\mathrm{Ric}}|^2 \\ &\geq \frac{1}{3} \, \sigma \, \mathrm{scal}^2 \, |\overset{\circ}{\mathrm{Ric}}|^2 \\ &= \frac{1}{3} \, \sigma \, \mathrm{scal}^{4-\sigma} \, f \\ &\geq \frac{1}{3} \, \sigma \, \mathrm{scal}^{3-\sigma} \, f^{1+1/\sigma}. \end{aligned}$$

Putting these facts together, we conclude that

$$\frac{\partial}{\partial t} f \leq \Delta f + \frac{2(1-\sigma)}{\mathrm{scal}} \, \partial_k \mathrm{scal} \, \partial^k f - \frac{2}{3} \, \sigma \, f^{1+1/\sigma}.$$

Using the maximum principle, we obtain

$$f \leq \Big(\frac{3}{2t}\Big)^{\sigma}.$$

This completes the proof. □

COROLLARY 4
Let M be a compact three-manifold, and let $g(t)$, $t \in (-\infty, 0)$, be an ancient solution to the Ricci flow on M. Moreover, suppose that there exists a uniform constant $\rho \in (0, 1)$ such that

$$\mathrm{Ric}_{g(t)} \geq \rho \, \mathrm{scal}_{g(t)} \, g(t) \geq 0$$

for each $t \in (-\infty, 0)$. Then the manifold $(M, g(t))$ has constant sectional curvature for each $t \in (-\infty, 0)$.

Proof
It follows from Proposition 3 that $|\overset{\circ}{\mathrm{Ric}}_{g(t)}|^2 = 0$ for each $t \in (-\infty, 0)$. Therefore, the manifold $(M, g(t))$ has constant sectional curvature for each $t \in (-\infty, 0)$. □

3. The higher-dimensional case
In this section, we develop some general tools that will be used in the proof of Theorem 2. To that end, we fix an integer $n \geq 4$. Moreover, we denote by $\mathcal{C}_B(\mathbb{R}^n)$ the space of algebraic curvature tensors on $\mathbb{R}^n$. Given any algebraic curvature tensor $R \in \mathcal{C}_B(\mathbb{R}^n)$, we define an algebraic curvature tensor $Q(R) \in \mathcal{C}_B(\mathbb{R}^n)$ by

$$Q(R)_{ijkl} = \sum_{p,q=1}^{n} R_{ijpq} \, R_{klpq} + 2 \sum_{p,q=1}^{n} (R_{ipkq} \, R_{jplq} - R_{iplq} \, R_{jpkq}).$$

The expression $Q(R)$ arises naturally in the evolution equation for the curvature tensor under Ricci flow (see [14]; see also [5, Section 2.3]). The ordinary differential equation (ODE) $\frac{d}{dt} R = Q(R)$ on the space $\mathcal{C}_B(\mathbb{R}^n)$ will be referred to as the *Hamilton ODE*.

We next consider a cone $C \subset \mathcal{C}_B(\mathbb{R}^n)$. We say that the cone C has property $(*)$ if the following conditions are met:
(i) C is closed, convex, and $O(n)$-invariant.
(ii) C is transversally invariant under the Hamilton ODE $\frac{d}{dt} R = Q(R)$.
(iii) Every algebraic curvature tensor $R \in C \setminus \{0\}$ has positive scalar curvature.
(iv) The curvature tensor $I_{ijkl} = \delta_{ik} \, \delta_{jl} - \delta_{il} \, \delta_{jk}$ lies in the interior of C.

In the remainder of this section, we assume that $C \subset \mathcal{C}_B(\mathbb{R}^n)$ is a cone satisfying $(*)$. Then $Q(R)$ lies in the interior of the tangent cone $T_R C$ for all $R \in C \setminus \{0\}$. By continuity, we can find a real number $\alpha_0 > 0$ such that

$$Q\big(R + \alpha \operatorname{scal}(R)\, I\big) - \alpha_0^2 \operatorname{scal}(R)^2\, I \in T_R C$$

for all $R \in C \setminus \{0\}$ and all $\alpha \in [0, \alpha_0]$. Moreover, there exists a real number $\Lambda > 0$ such that $|\operatorname{Ric}(R)| \leq \Lambda \operatorname{scal}(R)$ for all $R \in C$. Let

$$\delta = \min\Big\{\frac{1}{2n(n-1)}, \frac{\alpha_0}{2}, \frac{\alpha_0^2}{4(1+2\Lambda^2)}\Big\} > 0.$$

For each $t \in [0, \delta]$, we define a subset $F(t) \subset \mathscr{C}_B(\mathbb{R}^n)$ by

$$F(t) = \big\{R \in C : R + \big(1 - t \operatorname{scal}(R)\big)\, I \in C\big\}.$$

Clearly, $F(t)$ is closed, convex, and $O(n)$-invariant. Moreover, $F(0) = C$.

LEMMA 5
Suppose that R is an algebraic curvature tensor on $\mathbb{R}^n$ with the property that $R \in C$ and $R + (1 - t \operatorname{scal}(R))\, I \in C$ for some $t \in [0, \delta]$. Then

$$Q(R) - \operatorname{scal}(R)\, I - 2t\, |\operatorname{Ric}(R)|^2\, I$$

lies in the interior of the tangent cone to C at the point $R + (1 - t \operatorname{scal}(R))\, I$.

Proof
If $t \operatorname{scal}(R) < 1$, then the sum $R + (1 - t \operatorname{scal}(R))\, I$ lies in the interior of C. In this case, the assertion is trivial.

Hence, it suffices to consider the case $t \operatorname{scal}(R) \geq 1$. For abbreviation, let

$$S = R + \big(1 - t \operatorname{scal}(R)\big)\, I \in C.$$

Since $t \in [0, \delta]$, we have

$$\operatorname{scal}(S) > \big(1 - n(n-1)t\big) \operatorname{scal}(R) \geq \frac{1}{2} \operatorname{scal}(R).$$

Hence, if we put

$$\alpha = \frac{t \operatorname{scal}(R) - 1}{\operatorname{scal}(S)},$$

then we have $0 \leq \alpha < 2t \leq \alpha_0$. Since $S \in C \setminus \{0\}$, it follows that

$$Q\big(S + \alpha \operatorname{scal}(S)\, I\big) - \alpha_0^2 \operatorname{scal}(S)^2\, I \in T_S C$$

by the definition of α_0. We next observe that

$$S + \alpha \operatorname{scal}(S)\, I = R$$

and

$$\alpha_0^2 \operatorname{scal}(S)^2 > \frac{\alpha_0^2}{4} \operatorname{scal}(R)^2 \geq (1 + 2\Lambda^2)\, t \operatorname{scal}(R)^2 \geq \operatorname{scal}(R) + 2t\, |\operatorname{Ric}(R)|^2.$$

Putting these facts together, we conclude that

$$Q(R) - \operatorname{scal}(R)\, I - 2t\, |\operatorname{Ric}(R)|^2\, I$$

lies in the interior of the tangent cone $T_S C$. This completes the proof. □

PROPOSITION 6

Suppose that $R(t)$ is a solution of the Hamilton ODE $\frac{d}{dt} R(t) = Q(R(t))$ which is defined on some time interval $[t_0, t_1] \subset [0, \delta]$. If $R(t_0) \in F(t_0)$, then $R(t) \in F(t)$ for all $t \in [t_0, t_1]$.

Proof

By assumption, we have $R(t_0) \in C$. Since C is invariant under the Hamilton ODE, we conclude that $R(t) \in C$ for all $t \in [t_0, t_1]$. Hence, it suffices to show that $R(t) + \big(1 - t \operatorname{scal}(R(t))\big)\, I \in C$ for all $t \in [t_0, t_1]$.

For abbreviation, let

$$S(t) = R(t) + \big(1 - t \operatorname{scal}(R(t))\big)\, I$$

for all $t \in [t_0, t_1]$. Since $R(t)$ is a solution of the Hamilton ODE, we have

$$\frac{d}{dt} S(t) = Q\big(R(t)\big) - \operatorname{scal}\big(R(t)\big)\, I - 2t\, \big|\operatorname{Ric}\big(R(t)\big)\big|^2\, I$$

for all $t \in [t_0, t_1]$. We claim that $S(t) \in C$ for all $t \in [t_0, t_1]$. Suppose this false. We define a real number τ by

$$\tau = \inf\big\{t \in [t_0, t_1] : S(t) \notin C\big\}.$$

By the definition of τ, we have $\tau \in [0, \delta]$ and $S(\tau) \in C$. Furthermore, we have $R(\tau) \in C$. Hence, Lemma 5 implies that the derivative $\frac{d}{dt} S(t)|_{t=\tau}$ lies in the interior of the tangent cone $T_{S(\tau)} C$. By [5, Proposition 5.4], there exists a real number $\varepsilon > 0$ such that $S(t) \in C$ for all $t \in [\tau, \tau + \varepsilon)$. This contradicts the definition of τ. □

COROLLARY 7

Let δ be defined as above. Moreover, let $g(t)$, $t \in [0, \delta]$, be a solution to the Ricci flow on a compact n-dimensional manifold M. Finally, we assume that the curvature tensor of $(M, g(0))$ lies in the cone C for all points $p \in M$. Then

$$R_{g(t)} + (1 - t\, \mathrm{scal}_{g(t)})\, I \in C$$

for all points $(p, t) \in M \times [0, \delta]$.

Proof

By assumption, the curvature tensor of $(M, g(0))$ lies in the set $F(0)$ for all points $p \in M$. Using Proposition 6 and the maximum principle (see [9, Theorem 3]), we conclude that the curvature tensor of $(M, g(t))$ lies in the set $F(t)$ for all points $(p, t) \in M \times [0, \delta]$. This proves the assertion. □

COROLLARY 8

Let $g(t)$, $t \in (-\infty, 0)$, be an ancient solution to the Ricci flow on a compact n-dimensional manifold M. Moreover, suppose that the curvature tensor of $(M, g(t))$ lies in the cone C for all $t \in (-\infty, 0)$. Then

$$R_{g(t)} - \delta\, \mathrm{scal}_{g(t)}\, I \in C$$

for all points $(p, t) \in M \times (-\infty, 0)$.

Proof

Fix a time $\tau \in (-\infty, 0)$, and fix a real number $\sigma > 0$. We define a one-parameter family of metrics $\tilde{g}(t)$, $t \in [0, \delta]$, by

$$\tilde{g}(t) = \sigma\, g\Big(\frac{t - \delta}{\sigma} + \tau\Big).$$

Clearly, the metrics $\tilde{g}(t)$, $t \in [0, \delta]$, form a solution to the Ricci flow. By assumption, the curvature tensor of $(M, \tilde{g}(0))$ lies in the cone C for all points $p \in M$. Hence, it follows from Corollary 7 that

$$R_{\tilde{g}(\delta)} + (1 - \delta\, \mathrm{scal}_{\tilde{g}(\delta)})\, I \in C$$

for all points $p \in M$. This implies that

$$R_{g(\tau)} + (\sigma - \delta\, \mathrm{scal}_{g(\tau)})\, I \in C$$

for all points $p \in M$. Taking the limit as $\sigma \to 0$, we conclude that

$$R_{g(\tau)} - \delta\, \mathrm{scal}_{g(\tau)}\, I \in C$$

for all points $p \in M$. Since $\tau \in (-\infty, 0)$ is arbitrary, the assertion follows. □

THEOREM 9
Let $C(s)$, $s \in [0, 1]$, be a family of cones in $\mathscr{C}_B(\mathbb{R}^n)$ satisfying property $(*)$. Moreover, suppose that the cones $C(s)$ vary continuously in s. Finally, let $g(t)$, $t \in (-\infty, 0)$, be an ancient solution to the Ricci flow on a compact n-dimensional manifold M such that $R_{g(t)} \in C(0)$ for all points $(p, t) \in M \times (-\infty, 0)$. Then $R_{g(t)} \in C(1)$ for all $(p, t) \in M \times (-\infty, 0)$.

Proof
Let $\mathcal{S}$ denote the set of all real numbers $s \in [0, 1]$ with the property that $R_{g(t)} \in C(s)$ for all points $(p, t) \in M \times (-\infty, 0)$. We claim that $\mathcal{S} = [0, 1]$.

Clearly, $\mathcal{S}$ is closed and nonempty. We next show that $\mathcal{S}$ is an open subset of $[0, 1]$. To that end, we fix a real number $s_0 \in \mathcal{S}$. Then $R_{g(t)} \in C(s_0)$ for all points $(p, t) \in M \times (-\infty, 0)$. By Corollary 8, there exists a real number $\delta > 0$ such that

$$R_{g(t)} - \delta \operatorname{scal}_{g(t)} I \in C(s_0)$$

for all points $(p, t) \in M \times (-\infty, 0)$. Since the cones $C(s)$ vary continuously in s, there exists a real number $\varepsilon > 0$ such that $R_{g(t)} \in C(s)$ for all points $(p, t) \in M \times (-\infty, 0)$ and all $s \in [s_0 - \varepsilon, s_0 + \varepsilon] \cap [0, 1]$. Consequently, we have $[s_0 - \varepsilon, s_0 + \varepsilon] \cap [0, 1] \subset \mathcal{S}$. This shows that $\mathcal{S}$ is an open subset of $[0, 1]$. Thus, we conclude that $\mathcal{S} = [0, 1]$, as claimed. □

4. Proof of Theorem 2

We now describe the proof of Theorem 2. As in the previous section, we fix an integer $n \geq 4$. We denote by $\tilde{C}$ and $\hat{C}$ the cones introduced in [3] and [7]. The cone $\tilde{C}$ consists of all algebraic curvature tensors $R \in \mathscr{C}_B(\mathbb{R}^n)$ satisfying

$$\begin{aligned} &R(e_1, e_3, e_1, e_3) + \lambda^2 R(e_1, e_4, e_1, e_4) \\ &\quad + R(e_2, e_3, e_2, e_3) + \lambda^2 R(e_2, e_4, e_2, e_4) \\ &\quad - 2\lambda R(e_1, e_2, e_3, e_4) \geq 0 \end{aligned}$$

for all orthonormal four-frames $\{e_1, e_2, e_3, e_4\} \subset \mathbb{R}^n$ and all $\lambda \in [0, 1]$. Similarly, the cone $\hat{C}$ consists of all algebraic curvature tensors $R \in \mathscr{C}_B(\mathbb{R}^n)$ satisfying

$$\begin{aligned} &R(e_1, e_3, e_1, e_3) + \lambda^2 R(e_1, e_4, e_1, e_4) \\ &\quad + \mu^2 R(e_2, e_3, e_2, e_3) + \lambda^2\mu^2 R(e_2, e_4, e_2, e_4) \\ &\quad - 2\lambda\mu R(e_1, e_2, e_3, e_4) \geq 0 \end{aligned}$$

for all orthonormal four-frames $\{e_1, e_2, e_3, e_4\} \subset \mathbb{R}^n$ and all $\lambda, \mu \in [0, 1]$. The cones $\tilde{C}$ and $\hat{C}$ are both invariant under the Hamilton ODE $\frac{d}{dt} R = Q(R)$. A detailed discussion of these cones can be found in [5, Chapter 7].

We next describe a family of invariant curvature cones interpolating between the cone $\tilde{C}$ and the cone $\hat{C}$. For each $s \in (0, \infty)$, we denote by $\tilde{C}(s)$ the set of all algebraic curvature tensors $R \in \mathcal{C}_B(\mathbb{R}^n)$ such that

$$\begin{aligned} &R(e_1, e_3, e_1, e_3) + \lambda^2 R(e_1, e_4, e_1, e_4) \\ &\quad + \mu^2 R(e_2, e_3, e_2, e_3) + \lambda^2 \mu^2 R(e_2, e_4, e_2, e_4) \\ &\quad - 2\lambda\mu R(e_1, e_2, e_3, e_4) + \frac{1}{s} (1 - \lambda^2)(1 - \mu^2) \operatorname{scal}(R) \geq 0 \end{aligned}$$

for all orthonormal four-frames $\{e_1, e_2, e_3, e_4\} \subset \mathbb{R}^n$ and all $\lambda, \mu \in [0, 1]$. Clearly, $\tilde{C}(s)$ is a closed, convex cone, which is invariant under the natural action of $O(n)$. Moreover, we have $\hat{C} \subset \tilde{C}(s) \subset \tilde{C}$ for each $s \in (0, \infty)$. The following result is an immediate consequence of [3, Proposition 10].

PROPOSITION 10
For each $s \in (0, \infty)$, the cone $\tilde{C}(s)$ is invariant under the Hamilton ODE $\frac{d}{dt} R = Q(R)$.

Proof
Let us fix a real number $s \in (0, \infty)$. Moreover, let $R(t)$, $t \in [0, T)$, be a solution of the Hamilton ODE such that $R(0) \in \tilde{C}(s)$. We claim that $R(t) \in \tilde{C}(s)$ for all $t \in [0, T)$. Without loss of generality, we may assume that $\operatorname{scal}(R(0)) = s$. This implies that

$$\begin{aligned} &R(0)(e_1, e_3, e_1, e_3) + \lambda^2 R(0)(e_1, e_4, e_1, e_4) \\ &\quad + \mu^2 R(0)(e_2, e_3, e_2, e_3) + \lambda^2 \mu^2 R(0)(e_2, e_4, e_2, e_4) \\ &\quad - 2\lambda\mu R(0)(e_1, e_2, e_3, e_4) + (1 - \lambda^2)(1 - \mu^2) \geq 0 \end{aligned}$$

for all orthonormal four-frames $\{e_1, e_2, e_3, e_4\} \subset \mathbb{R}^n$ and all $\lambda, \mu \in [0, 1]$. Hence, [3, Proposition 10] implies that

$$\begin{aligned} &R(t)(e_1, e_3, e_1, e_3) + \lambda^2 R(t)(e_1, e_4, e_1, e_4) \\ &\quad + \mu^2 R(t)(e_2, e_3, e_2, e_3) + \lambda^2 \mu^2 R(t)(e_2, e_4, e_2, e_4) \\ &\quad - 2\lambda\mu R(t)(e_1, e_2, e_3, e_4) + (1 - \lambda^2)(1 - \mu^2) \geq 0 \end{aligned}$$

for all orthonormal four-frames $\{e_1, e_2, e_3, e_4\} \subset \mathbb{R}^n$, all $\lambda, \mu \in [0, 1]$, and all $t \in [0, T)$. Since $\mathrm{scal}(R(t)) \geq \mathrm{scal}(R(0)) = s$, we conclude that $R(t) \in \tilde{C}(s)$ for all $t \in [0, T)$. □

After these preparations, we now present the proof of Theorem 2.

THEOREM 11
Assume that $g(t)$, $t \in (-\infty, 0)$, is an ancient solution to the Ricci flow on a compact n-dimensional manifold M. Moreover, we assume that there exists a uniform constant $\rho > 0$ such that

$$R_{g(t)} - \rho\, \mathrm{scal}_{g(t)}\, I \in \hat{C}$$

for all points $(p, t) \in M \times (-\infty, 0)$. Then the manifold $(M, g(t))$ has constant sectional curvature for each $t \in (-\infty, 0)$.

Proof
Consider the one-parameter family of cones $\hat{C}(s)$, $s \in (0, \infty)$, defined in [7]. It is shown in [7] that the cone $\hat{C}(s)$ has property $(*)$ for each $s \in (0, \infty)$. Furthermore, the cones $\hat{C}(s)$ vary continuously in s.

By assumption, there exists a real number $s_0 \in (0, \infty)$ such that $R_{g(t)} \in \hat{C}(s_0)$ for all points $(p, t) \in M \times (-\infty, 0)$. Using Theorem 9, we conclude that $R_{g(t)} \in \hat{C}(s)$ for all points $(p, t) \in M \times (-\infty, 0)$ and all $s \in (0, \infty)$. Consequently, the manifold $(M, g(t))$ has constant sectional curvature for each $t \in (-\infty, 0)$. This completes the proof of Theorem 11. □

THEOREM 12
Assume that $g(t)$, $t \in (-\infty, 0)$, is an ancient solution to the Ricci flow on a compact n-dimensional manifold M. Moreover, we assume that there exists a uniform constant $\rho > 0$ such that

$$R_{g(t)} - \rho\, \mathrm{scal}_{g(t)}\, I \in \tilde{C}$$

for all points $(p, t) \in M \times (-\infty, 0)$. Then the manifold $(M, g(t))$ has constant sectional curvature for each $t \in (-\infty, 0)$.

Proof
By assumption, we have

$$R_{g(t)} - \rho\, \mathrm{scal}_{g(t)}\, I \in \tilde{C}$$

for all points $(p, t) \in M \times (-\infty, 0)$. Hence, we can find a real number $s_0 \in (0, \infty)$ such that

$$R_{g(t)} - \frac{1}{2}\,\rho\,\mathrm{scal}_{g(t)}\, I \in \tilde{C}(s_0)$$

for all points $(p, t) \in M \times (-\infty, 0)$.

We next consider a pair of real numbers a, b such that $2a = 2b + (n-2)b^2$ and $b \in \big(0, \frac{\sqrt{2n(n-2)+4}-2}{n(n-2)}\big]$. Following [2], we define a linear transformation $\ell_{a,b} : \mathscr{C}_B(\mathbb{R}^n) \to \mathscr{C}_B(\mathbb{R}^n)$ by

$$\ell_{a,b}(R) = R + b\,\mathrm{Ric}(R) \owedge \mathrm{id} + \frac{1}{n}\,(a-b)\,\mathrm{scal}(R)\,\mathrm{id} \owedge \mathrm{id},$$

where $\owedge$ denotes the Kulkarni-Nomizu product (see, e.g., [1, Definition 1.110]). If we choose $b \in \big(0, \frac{\sqrt{2n(n-2)+4}-2}{n(n-2)}\big]$ sufficiently small, then

$$R_{g(t)} \in \ell_{a,b}\big(\tilde{C}(s_0)\big)$$

for all points $(p, t) \in M \times (-\infty, 0)$.

By Proposition 10, the cone $\tilde{C}(s)$ is invariant under the Hamilton ODE for each $s \in (0, \infty)$. Consequently, the cone $\ell_{a,b}(\tilde{C}(s))$ is transversally invariant under the Hamilton ODE for each $s \in (0, \infty)$ (see [2, Proposition 3.2]). Therefore, the cone $\ell_{a,b}(\tilde{C}(s))$ has property $(*)$ for each $s \in (0, \infty)$. Moreover, if we fix a and b, then the cones $\ell_{a,b}(\tilde{C}(s))$ vary continuously in s. Using Theorem 9, we conclude that $R_{g(t)} \in \ell_{a,b}(\tilde{C}(s))$ for all points $(p, t) \in M \times (-\infty, 0)$ and all $s \in (0, \infty)$. Taking the limit as $s \to \infty$, we obtain $R_{g(t)} \in \ell_{a,b}(\hat{C})$ for all points $(p, t) \in M \times (-\infty, 0)$. Hence, it follows from Theorem 11 that $(M, g(t))$ has constant sectional curvature for each $t \in (-\infty, 0)$. □

5. Ancient solutions satisfying a diameter bound

In this final section, we study ancient solutions to the Ricci flow satisfying a suitable diameter bound. Throughout this section, we assume that M is a compact manifold of dimension n and that $g(t)$, $t \in (-\infty, 0)$, is a solution to the Ricci flow on M. The following proposition is a consequence of the differential Harnack inequality established in [4].

LEMMA 13
Suppose that the curvature tensor of $(M, g(t))$ lies in the cone $\hat{C}$ for each $t \in (-\infty, 0)$. Then

$$\inf_M \mathrm{scal}_{g(\tau/2)} \geq \exp\Big(-\frac{\mathrm{diam}(M, g(\tau))^2}{|\tau|}\Big)\,\sup_M \mathrm{scal}_{g(\tau)} \tag{1}$$

for all $\tau \in (-\infty, 0)$.

Proof
Fix an arbitrary pair of points $p, q \in M$. We can find a smooth path $\gamma : [\tau, \tau/2] \to M$ such that $\gamma(\tau) = p$, $\gamma(\tau/2) = q$, and

$$|\gamma'(t)|_{g(\tau)} = \frac{2\, d_{g(\tau)}(p, q)}{|\tau|}.$$

This implies that

$$|\gamma'(t)|_{g(t)} \leq \frac{2\, d_{g(\tau)}(p, q)}{|\tau|}$$

for all $t \in [\tau, \tau/2]$. Using the trace Harnack inequality (see [4, Proposition 13]), we obtain that

$$\frac{\partial}{\partial t}\mathrm{scal} + 2\, \partial_i \mathrm{scal}\, v^i \geq -2\, \mathrm{Ric}(v, v)$$

for every tangent vector v. Putting $v = \gamma'(t)/2$ gives

$$\begin{aligned}\frac{d}{dt}\mathrm{scal}_{g(t)}\big(\gamma(t)\big) &\geq -\frac{1}{2}\, \mathrm{Ric}_{g(t)}\big(\gamma'(t), \gamma'(t)\big) \\ &\geq -\frac{1}{2}\, \mathrm{scal}_{g(t)}\big(\gamma(t)\big)\, |\gamma'(t)|^2_{g(t)} \\ &\geq -\frac{2\, d_{g(\tau)}(p, q)^2}{|\tau|^2}\, \mathrm{scal}_{g(t)}\big(\gamma(t)\big)\end{aligned}$$

for all $t \in [\tau, \tau/2]$. Thus, we conclude that

$$\mathrm{scal}_{g(\tau/2)}(q) \geq \exp\Big(- \frac{d_{g(\tau)}(p, q)^2}{|\tau|}\Big)\, \mathrm{scal}_{g(\tau)}(p).$$

Since $p, q \in M$ are arbitrary, the assertion follows. □

PROPOSITION 14
Suppose that the curvature tensor of $(M, g(t))$ lies in the cone $\hat{C}$ for each $t \in (-\infty, 0)$. Moreover, suppose that

$$\limsup_{\tau \to -\infty} \frac{1}{\sqrt{|\tau|}}\, \mathrm{diam}\big(M, g(\tau)\big) < \infty.$$

Then

$$\limsup_{\tau \to -\infty} \Big[|\tau|\, \sup_M \mathrm{scal}_{g(\tau)}\Big] < \infty.$$

Proof
Since the solution $g(t)$ is defined until time 0, we have

$$\inf_M \mathrm{scal}_{g(\tau/2)} \leq \frac{n}{|\tau|}$$

for each $\tau \in (-\infty, 0)$ (see, e.g., [5, Proposition 2.19]). Using Lemma 13, we deduce that

$$\sup_M \mathrm{scal}_{g(\tau)} \leq \frac{n}{|\tau|} \exp\Big(\frac{\mathrm{diam}(M, g(\tau))^2}{|\tau|}\Big).$$

From this, the assertion follows. □

Finally, we recall the following result due to B. Kostant (see [17, Corollary 2.2]).

PROPOSITION 15
Let (N, h) be a compact, simply connected Riemannian manifold of dimension $n \neq 5$ which is, topologically, a rational homology sphere. Then the holonomy representation of (N, h) is complete; that is, (N, h) has holonomy group $\mathrm{SO}(n)$.

We now state the main result of this section.

THEOREM 16
Let $g(t)$, $t \in (-\infty, 0)$, be an ancient solution to the Ricci flow on a compact, even-dimensional manifold M. Suppose that the curvature tensor of $(M, g(t))$ lies in the interior of the cone $\hat{C}$ for each $t \in (-\infty, 0)$. Moreover, suppose that

$$\limsup_{\tau \to -\infty} \frac{1}{\sqrt{|\tau|}} \mathrm{diam}\big(M, g(\tau)\big) < \infty.$$

Then $(M, g(t))$ has constant sectional curvature for each $t \in (-\infty, 0)$.

Proof
Suppose the assertion is false. By Theorem 11, we can find a sequence of points $(p_k, \tau_k) \in M \times (-\infty, 0)$ such that $\lim_{k\to\infty} \tau_k = -\infty$ and

$$R_{g(\tau_k)} - \frac{1}{k} \mathrm{scal}_{g(\tau_k)} I \notin \hat{C} \tag{2}$$

at p_k. For each k, we consider the rescaled metrics

$$\tilde{g}_k(t) = \frac{1}{|\tau_k|} g(|\tau_k| t), \quad t \in \Big(-2, -\frac{1}{2}\Big).$$

For each k, the metrics $\tilde{g}_k(t)$, $t \in (-2, -1/2)$, form a solution to the Ricci flow on M. By assumption, the diameter of $(M, \tilde{g}_k(t))$ has uniformly bounded diameter; moreover, it has uniformly bounded curvature by Proposition 14. Since M is even-dimensional, we conclude that the injectivity radius of $(M, \tilde{g}_k(t))$ is uniformly bounded from below.

Hence, after passing to a subsequence if necessary, the sequence $(M, \tilde{g}_k(t))$ converges in the Cheeger-Gromov sense to some limiting solution $(M, \bar{g}(t))$ to the Ricci flow. This limiting solution is defined for all $t \in (-2, -1/2)$. Clearly, the curvature tensor of $(M, \bar{g}(t))$ lies in the cone $\hat{C}$ for each $t \in (-2, -1/2)$. Moreover, it follows from (2) that the curvature tensor of $(M, \bar{g}(-1))$ lies on the boundary of the cone $\hat{C}$ for some point $q \in M$. By [6, Proposition 9], the manifold $(M, \bar{g}(-1))$ has nongeneric holonomy group; that is, $\mathrm{Hol}^0(M, \bar{g}(-1)) \neq \mathrm{SO}(n)$. On the other hand, it follows from the differentiable sphere theorem that the universal cover of $(M, \bar{g}(-1))$ is diffeomorphic to S^n (see [7, Theorem 3]). By Proposition 15, the universal cover of $(M, \bar{g}(-1))$ has holonomy group $\mathrm{SO}(n)$. This is a contradiction. □

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