ANCIENT SOLUTIONS TO THE RICCI FLOW WITH PINCHED CURVATURE

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1. Introduction

In this note, we study ancient solutions to the Ricci flow on compact manifolds. Recall that a one-parameter family of metrics $g(t)$ on a compact manifold M evolves by the Ricci flow if

$$\frac{\partial}{\partial t} g = -2\,\mathrm{Ric}_{g(t)}.$$

A solution to the Ricci flow is called ancient if it is defined on a time interval $(-\infty, T)$. Ancient solutions typically arise in the study of singularities to the Ricci flow (see e.g. [12], [13], [15], [16]).

P. Daskalopoulos, R. Hamilton, and N. Sešum [8] have recently obtained a complete classification of all ancient solutions to the Ricci flow in dimension 2. (See also [7], where the analogous question for the curve shortening flow is studied.) V. Fateev [9] has constructed an interesting example of an ancient solution in dimension 3. L. Ni [14] showed that any ancient solution to the Ricci flow which is of Type I, κ-noncollapsed, and has nonnegative curvature operator has constant sectional curvature.

In this note, we show that any ancient solution to the Ricci flow in dimension $n \geq 3$ which satisfies a suitable curvature pinching condition must have constant sectional curvature. In dimension 3, we require a uniform lower bound for the Ricci tensor:

Theorem 1. *Let M be a compact three-manifold, and let $g(t)$, $t \in (-\infty, 0)$, be an ancient solution to the Ricci flow on M. Moreover, suppose that there exists a uniform constant $\rho > 0$ such that*

$$\mathrm{Ric}_{g(t)} \geq \rho\, \mathrm{scal}_{g(t)}\, g(t) \geq 0$$

for all $t \in (-\infty, 0)$. Then the manifold $(M, g(t))$ has constant sectional curvature for each $t \in (-\infty, 0)$.

Fateev's example shows that the pinching condition for the Ricci tensor cannot be removed. The proof of Theorem 1 relies on a new interior estimate for the Ricci flow in dimension 3. This estimate is proved using the maximum principle, and will be presented in Section 2.

In dimension $n \geq 4$, we prove the following result:

Theorem 2. *Let M be a compact manifold of dimension $n \geq 4$, and let $g(t)$, $t \in (-\infty, 0)$, be an ancient solution to the Ricci flow on M. Moreover, suppose that there exists a uniform constant $\rho > 0$ with the following property: for each $t \in (-\infty, 0)$, the curvature tensor of $(M, g(t))$ satisfies*

$$\begin{aligned} &R_{g(t)}(e_1, e_3, e_1, e_3) + \lambda^2 \, R_{g(t)}(e_1, e_4, e_1, e_4) \\ &+ R_{g(t)}(e_2, e_3, e_2, e_3) + \lambda^2 \, R_{g(t)}(e_2, e_4, e_2, e_4) \\ &- 2\lambda \, R_{g(t)}(e_1, e_2, e_3, e_4) \geq \rho \, \mathrm{scal}_{g(t)} \geq 0 \end{aligned}$$

for all orthonormal four-frames $\{e_1, e_2, e_3, e_4\}$ and all $\lambda \in [0, 1]$. Then the manifold $(M, g(t))$ has constant sectional curvature for each $t \in (-\infty, 0)$.

Theorem 2 again follows from pointwise curvature estimates which are established using the maximum principle. In dimension $n \geq 4$, the evolution equation for the curvature tensor is much more complicated, and our estimates are not as explicit as in the three-dimensional case. In order to handle the higher dimensional case, we use the invariant curvature conditions introduced in [3] and [5]. These ideas also play a key role in the proof of the Differentiable Sphere Theorem (cf. [5]).

2. Proof of Theorem 1

Proposition 3. *Let M be a compact three-manifold, and let $g(t)$, $t \in [0, T)$, be a solution to the Ricci flow on M. Moreover, suppose that there exists a uniform constant $\rho \in (0, 1)$ such that*

$$\mathrm{Ric}_{g(t)} \geq \rho \, \mathrm{scal}_{g(t)} \, g(t) \geq 0$$

for each $t \in [0, T)$. Then, for each $t \in (0, T)$, the curvature tensor of $(M, g(t))$ satisfies the pointwise estimate

$$|\overset{\circ}{\mathrm{Ric}}_{g(t)}|^2 \leq \Big(\frac{3}{2t}\Big)^{\sigma} \, \mathrm{scal}_{g(t)}^{2-\sigma},$$

where $\sigma = \rho^2$.

Proof. The assertion is trivial if $(M, g(0))$ is Ricci flat. Hence, it suffices to consider the case that $(M, g(0))$ is not Ricci flat. By the maximum principle, the manifold $(M, g(t))$ has strictly positive scalar curvature for all $t \in (0, T)$.

We next define a function $f : M \times (0, T) \to \mathbb{R}$ by

$$f = \mathrm{scal}^{\sigma-2} \, |\overset{\circ}{\mathrm{Ric}}|^2,$$

where $\sigma = \rho^2$. It is easy to see that $f \leq \mathrm{scal}^{\sigma}$. Moreover, it follows from Lemma 10.5 in [10] that

$$\frac{\partial}{\partial t} f \leq \Delta f + \frac{2(1-\sigma)}{\mathrm{scal}} \, \partial_k \mathrm{scal} \, \partial^k f + 2 \, \mathrm{scal}^{\sigma-3} \Big[\sigma \, |\mathrm{Ric}|^2 \, |\overset{\circ}{\mathrm{Ric}}|^2 - 2P \Big],$$

where P is a polynomial expression in the eigenvalues of the Ricci tensor. By assumption, we have $\text{Ric} \geq \rho \, \text{scal} \, g$. Hence, it follows from Lemma 10.7 in [10] that

$$P \geq \sigma \, |\text{Ric}|^2 \, |\overset{\circ}{\text{Ric}}|^2.$$

This implies

$$\begin{aligned} 2P - \sigma \, |\text{Ric}|^2 \, |\overset{\circ}{\text{Ric}}|^2 &\geq \sigma \, |\text{Ric}|^2 \, |\overset{\circ}{\text{Ric}}|^2 \\ &\geq \frac{1}{3} \, \sigma \, \text{scal}^2 \, |\overset{\circ}{\text{Ric}}|^2 \\ &= \frac{1}{3} \, \sigma \, \text{scal}^{4-\sigma} \, f \\ &\geq \frac{1}{3} \, \sigma \, \text{scal}^{3-\sigma} \, f^{1+\frac{1}{\sigma}}. \end{aligned}$$

Putting these facts together, we conclude that

$$\frac{\partial}{\partial t} f \leq \Delta f + \frac{2(1-\sigma)}{\text{scal}} \, \partial_k \text{scal} \, \partial^k f - \frac{2}{3} \, \sigma \, f^{1+\frac{1}{\sigma}}.$$

Using the maximum principle, we obtain

$$f \leq \Big(\frac{3}{2t}\Big)^{\sigma}.$$

This completes the proof.

Corollary 4. *Let M be a compact three-manifold, and let $g(t)$, $t \in (-\infty, 0)$, be an ancient solution to the Ricci flow on M. Moreover, suppose that there exists a uniform constant $\rho \in (0,1)$ such that*

$$\text{Ric}_{g(t)} \geq \rho \, \text{scal}_{g(t)} \, g(t) \geq 0$$

for each $t \in (-\infty, 0)$. Then the manifold $(M, g(t))$ has constant sectional curvature for each $t \in (-\infty, 0)$.

Proof. It follows from Proposition 3 that $|\overset{\circ}{\text{Ric}}_{g(t)}|^2 = 0$ for each $t \in (-\infty, 0)$. Therefore, the manifold $(M, g(t))$ has constant sectional curvature for each $t \in (-\infty, 0)$.

3. The higher dimensional case

In this section, we develop some general tools that will be used in proof of Theorem 2. To that end, we fix an integer $n \geq 4$. Moreover, we denote by $\mathscr{C}_B(\mathbb{R}^n)$ the space of algebraic curvature tensors on $\mathbb{R}^n$. Given any algebraic curvature tensor $R \in \mathscr{C}_B(\mathbb{R}^n)$, we define an algebraic curvature tensor $Q(R) \in \mathscr{C}_B(\mathbb{R}^n)$ by

$$Q(R)_{ijkl} = \sum_{p,q=1}^n R_{ijpq} \, R_{klpq} + 2 \sum_{p,q=1}^n (R_{ipkq} \, R_{jplq} - R_{iplq} \, R_{jpkq}).$$

The expression $Q(R)$ arises naturally in the evolution equation for the curvature tensor under Ricci flow (cf. [11]; see also [4], Section 2.3). The ordinary differential equation $\frac{d}{dt}R = Q(R)$ on the space $\mathscr{C}_B(\mathbb{R}^n)$ will be referred to as the Hamilton ODE.

We next consider a cone $C \subset \mathscr{C}_B(\mathbb{R}^n)$. We say that the cone C has property $(*)$ if the following conditions are met:

(i) C is closed, convex, and $O(n)$-invariant.
(ii) C is transversally invariant under the Hamilton ODE $\frac{d}{dt}R = Q(R)$.
(iii) Every algebraic curvature tensor $R \in C \setminus \{0\}$ has positive scalar curvature.
(iv) The curvature tensor $I_{ijkl} = \delta_{ik}\,\delta_{jl} - \delta_{il}\,\delta_{jk}$ lies in the interior of C.

In the remainder of this section, we assume that $C \subset \mathscr{C}_B(\mathbb{R}^n)$ is a cone satisfying $(*)$. Then $Q(R)$ lies in the interior of the tangent cone T_RC for all $R \in C \setminus \{0\}$. By continuity, we can find a real number $\alpha_0 > 0$ such that

$$Q(R + \alpha\,\mathrm{scal}(R)\,I) - \alpha_0^2\,\mathrm{scal}(R)^2\,I \in T_RC$$

for all $R \in C \setminus \{0\}$ and all $\alpha \in [0, \alpha_0]$. Moreover, there exists a real number $\Lambda > 0$ such that $|\mathrm{Ric}(R)| \leq \Lambda\,\mathrm{scal}(R)$ for all $R \in C$. Let

$$\delta = \min\Big\{\frac{1}{2n(n-1)}, \frac{\alpha_0}{2}, \frac{\alpha_0^2}{4(1+2\Lambda^2)}\Big\} > 0.$$

For each $t \in [0, \delta]$, we define a subset $F(t) \subset \mathscr{C}_B(\mathbb{R}^n)$ by

$$F(t) = \{R \in C : R + (1 - t\,\mathrm{scal}(R))\,I \in C\}.$$

Clearly, $F(t)$ is closed, convex, and $O(n)$-invariant. Moreover, $F(0) = C$.

Lemma 5. *Suppose that R is an algebraic curvature tensor on $\mathbb{R}^n$ such that $R \in C$ and $R + (1 - t\,\mathrm{scal}(R))\,I \in C$ for some $t \in [0, \delta]$. Then*

$$Q(R) - \mathrm{scal}(R)\,I - 2t\,|\mathrm{Ric}(R)|^2\,I$$

lies in the interior of the tangent cone to C at the point $R + (1 - t\,\mathrm{scal}(R))\,I$.

Proof. If $t\,\mathrm{scal}(R) < 1$, then the sum $R + (1 - t\,\mathrm{scal}(R))\,I$ lies in the interior of C. In this case, the assertion is trivial.

Hence, it suffices to consider the case $t\,\mathrm{scal}(R) \geq 1$. For abbreviation, let

$$S = R + (1 - t\,\mathrm{scal}(R))\,I \in C.$$

Since $t \in [0, \delta]$, we have

$$\mathrm{scal}(S) > (1 - n(n-1)t)\,\mathrm{scal}(R) \geq \frac{1}{2}\,\mathrm{scal}(R).$$

Hence, if we put

$$\alpha = \frac{t\,\mathrm{scal}(R) - 1}{\mathrm{scal}(S)},$$

then we have $0 \leq \alpha < 2t \leq \alpha_0$. Since $S \in C \setminus \{0\}$, it follows that

$$Q(S + \alpha\,\mathrm{scal}(S)\,I) - \alpha_0^2\,\mathrm{scal}(S)^2\,I \in T_SC$$

by definition of α_0. We next observe that

$$S + \alpha \, \mathrm{scal}(S) \, I = R$$

and

$$\alpha_0^2 \, \mathrm{scal}(S)^2 > \frac{\alpha_0^2}{4} \, \mathrm{scal}(R)^2 \geq (1 + 2\Lambda^2) \, t \, \mathrm{scal}(R)^2 \geq \mathrm{scal}(R) + 2t \, |\mathrm{Ric}(R)|^2.$$

Putting these facts together, we conclude that

$$Q(R) - \mathrm{scal}(R) \, I - 2t \, |\mathrm{Ric}(R)|^2 \, I$$

lies in the interior of the tangent cone $T_S C$. This completes the proof.

Proposition 6. *Suppose that $R(t)$ is a solution of the Hamilton ODE $\frac{d}{dt} R(t) = Q(R(t))$ which is defined on some time interval $[t_0, t_1] \subset [0, \delta]$. If $R(t_0) \in F(t_0)$, then $R(t) \in F(t)$ for all $t \in [t_0, t_1]$.*

Proof. By assumption, we have $R(t_0) \in C$. Since C is invariant under the Hamilton ODE, we conclude that $R(t) \in C$ for all $t \in [t_0, t_1]$. Hence, it suffices to show that $R(t) + (1 - t \, \mathrm{scal}(R(t))) \, I \in C$ for all $t \in [t_0, t_1]$.

For abbreviation, let

$$S(t) = R(t) + (1 - t \, \mathrm{scal}(R(t))) \, I$$

for all $t \in [t_0, t_1]$. Since $R(t)$ is a solution of the Hamilton ODE, we have

$$\frac{d}{dt} S(t) = Q(R(t)) - \mathrm{scal}(R(t)) \, I - 2t \, |\mathrm{Ric}(R(t))|^2 \, I$$

for all $t \in [t_0, t_1]$. We claim that $S(t) \in C$ for all $t \in [t_0, t_1]$. Suppose this false. We define a real number τ by

$$\tau = \inf\{t \in [t_0, t_1] : S(t) \notin C\}.$$

By definition of τ, we have $\tau \in [0, \delta]$ and $S(\tau) \in C$. Furthermore, we have $R(\tau) \in C$. Hence, Lemma 5 implies that the derivative $\frac{d}{dt} S(t)\big|_{t=\tau}$ lies in the interior of the tangent cone $T_{S(\tau)} C$. By Proposition 5.4 in [4], there exists a real number $\varepsilon > 0$ such that $S(t) \in C$ for all $t \in [\tau, \tau + \varepsilon)$. This contradicts the definition of τ.

Corollary 7. *Let δ be defined as above. Moreover, let $g(t)$, $t \in [0, \delta]$, be a solution to the Ricci flow on a compact n-dimensional manifold M. Finally, we assume that the curvature tensor of $(M, g(0))$ lies in the cone C for all points $p \in M$. Then*

$$R_{g(t)} + (1 - t \, \mathrm{scal}_{g(t)}) \, I \in C$$

for all points $(p, t) \in M \times [0, \delta]$.

Proof. By assumption, the curvature tensor of $(M, g(0))$ lies in the set $F(0)$ for all points $p \in M$. Using Proposition 6 and the maximum principle (cf. [6], Theorem 3), we conclude that the curvature tensor of $(M, g(t))$ lies in the set $F(t)$ for all points $(p, t) \in M \times [0, \delta]$. This proves the assertion.

Corollary 8. *Let $g(t)$, $t \in (-\infty, 0)$, be an ancient solution to the Ricci flow on a compact n-dimensional manifold M. Moreover, suppose that the curvature tensor of $(M, g(t))$ lies in the cone C for all $t \in (-\infty, 0)$. Then*

$$R_{g(t)} - \delta \operatorname{scal}_{g(t)} I \in C$$

for all points $(p, t) \in M \times (-\infty, 0)$.

Proof. Fix a time $\tau \in (-\infty, 0)$ and a real number $\sigma > 0$. We define a one-parameter family of metrics $\tilde{g}(t)$, $t \in [0, \delta]$, by

$$\tilde{g}(t) = \sigma \, g\Big(\frac{t - \delta}{\sigma} + \tau\Big).$$

Clearly, the metrics $\tilde{g}(t)$, $t \in [0, \delta]$, form a solution to the Ricci flow. By assumption, the curvature tensor of $(M, \tilde{g}(0))$ lies in the cone C for all points $p \in M$. Hence, it follows from Corollary 7 that

$$R_{\tilde{g}(\delta)} + (1 - \delta \operatorname{scal}_{\tilde{g}(\delta)}) \, I \in C$$

for all points $p \in M$. This implies

$$R_{g(\tau)} + (\sigma - \delta \operatorname{scal}_{g(\tau)}) \, I \in C$$

for all points $p \in M$. Taking the limit as $\sigma \to 0$, we conclude that

$$R_{g(\tau)} - \delta \operatorname{scal}_{g(\tau)} I \in C$$

for all points $p \in M$. Since $\tau \in (-\infty, 0)$ is arbitrary, the assertion follows.

Theorem 9. *Let $C(s)$, $s \in [0, 1]$, be a family of cones in $\mathscr{C}_B(\mathbb{R}^n)$ satisfying property $(*)$. Moreover, suppose that the cones $C(s)$ vary continuously in s. Finally, let $g(t)$, $t \in (-\infty, 0)$, be an ancient solution to the Ricci flow on a compact n-dimensional manifold M such that $R_{g(t)} \in C(0)$ for all points $(p, t) \in M \times (-\infty, 0)$. Then $R_{g(t)} \in C(1)$ for all $(p, t) \in M \times (-\infty, 0)$.*

Proof. Let $\mathscr{S}$ denote the set of all real numbers $s \in [0, 1]$ with the property that $R_{g(t)} \in C(s)$ for all points $(p, t) \in M \times (-\infty, 0)$. We claim that $\mathscr{S} = [0, 1]$.

Clearly, $\mathscr{S}$ is closed and non-empty. We next show that $\mathscr{S}$ is an open subset of $[0, 1]$. To that end, we fix a real number $s_0 \in \mathscr{S}$. Then $R_{g(t)} \in C(s_0)$ for all points $(p, t) \in M \times (-\infty, 0)$. By Corollary 8, there exists a real number $\delta > 0$ such that

$$R_{g(t)} - \delta \operatorname{scal}_{g(t)} I \in C(s_0)$$

for all points $(p, t) \in M \times (-\infty, 0)$. Since the cones $C(s)$ vary continuously in s, there exists a real number $\varepsilon > 0$ such that $R_{g(t)} \in C(s)$ for all points

$(p,t) \in M \times (-\infty,0)$ and all $s \in [s_0-\varepsilon, s_0+\varepsilon] \cap [0,1]$. Consequently, we have $[s_0-\varepsilon, s_0+\varepsilon] \cap [0,1] \subset \mathscr{S}$. This shows that $\mathscr{S}$ is an open subset of $[0,1]$. Thus, we conclude that $\mathscr{S} = [0,1]$, as claimed.

4. Proof of Theorem 2

We now describe the proof of Theorem 2. As in the previous section, we fix an integer $n \geq 4$. We denote by $\tilde{C}$ and $\hat{C}$ the cones introduced in [3] and [5]. The cone $\tilde{C}$ consists of all algebraic curvature tensors $R \in \mathscr{C}_B(\mathbb{R}^n)$ satisfying

$$\begin{aligned} &R(e_1,e_3,e_1,e_3) + \lambda^2 \, R(e_1,e_4,e_1,e_4) \\ &+ R(e_2,e_3,e_2,e_3) + \lambda^2 \, R(e_2,e_4,e_2,e_4) \\ &- 2\lambda \, R(e_1,e_2,e_3,e_4) \geq 0 \end{aligned}$$

for all orthonormal four-frames $\{e_1,e_2,e_3,e_4\} \subset \mathbb{R}^n$ and all $\lambda \in [0,1]$. Similarly, the cone $\hat{C}$ consists of all algebraic curvature tensors $R \in \mathscr{C}_B(\mathbb{R}^n)$ satisfying

$$\begin{aligned} &R(e_1,e_3,e_1,e_3) + \lambda^2 \, R(e_1,e_4,e_1,e_4) \\ &+ \mu^2 \, R(e_2,e_3,e_2,e_3) + \lambda^2 \mu^2 \, R(e_2,e_4,e_2,e_4) \\ &- 2\lambda\mu \, R(e_1,e_2,e_3,e_4) \geq 0 \end{aligned}$$

for all orthonormal four-frames $\{e_1,e_2,e_3,e_4\} \subset \mathbb{R}^n$ and all $\lambda, \mu \in [0,1]$. The cones $\tilde{C}$ and $\hat{C}$ are both invariant under the Hamilton ODE $\frac{d}{dt} R = Q(R)$. A detailed discussion of these cones can be found in [4], Chapter 7.

We next describe a family of invariant curvature cones interpolating between the cone $\tilde{C}$ and the cone $\hat{C}$. For each $s \in (0,\infty)$, we denote by $\tilde{C}(s)$ the set of all algebraic curvature tensors $R \in \mathscr{C}_B(\mathbb{R}^n)$ such that

$$\begin{aligned} &R(e_1,e_3,e_1,e_3) + \lambda^2 \, R(e_1,e_4,e_1,e_4) \\ &+ \mu^2 \, R(e_2,e_3,e_2,e_3) + \lambda^2 \mu^2 \, R(e_2,e_4,e_2,e_4) \\ &- 2\lambda\mu \, R(e_1,e_2,e_3,e_4) + \frac{1}{s} \, (1-\lambda^2) \, (1-\mu^2) \, \mathrm{scal}(R) \geq 0 \end{aligned}$$

for all orthonormal four-frames $\{e_1,e_2,e_3,e_4\} \subset \mathbb{R}^n$ and all $\lambda, \mu \in [0,1]$. Clearly, $\tilde{C}(s)$ is a closed, convex cone, which is invariant under the natural action of $O(n)$. Moreover, we have $\hat{C} \subset \tilde{C}(s) \subset \tilde{C}$ for each $s \in (0,\infty)$. The following result is an immediate consequence of Proposition 10 in [3]:

Proposition 10. *For each $s \in (0,\infty)$, the cone $\tilde{C}(s)$ is invariant under the Hamilton ODE $\frac{d}{dt} R = Q(R)$.*

Proof. Let us fix a real number $s \in (0,\infty)$. Moreover, let $R(t)$, $t \in [0,T)$, be a solution of the Hamilton ODE such that $R(0) \in \tilde{C}(s)$. We claim that $R(t) \in \tilde{C}(s)$ for all $t \in [0,T)$. Without loss of generality, we may assume that $\text{scal}(R(0)) = s$. This implies

$$\begin{aligned} &R(0)(e_1, e_3, e_1, e_3) + \lambda^2 \, R(0)(e_1, e_4, e_1, e_4) \\ &+ \mu^2 \, R(0)(e_2, e_3, e_2, e_3) + \lambda^2 \mu^2 \, R(0)(e_2, e_4, e_2, e_4) \\ &- 2\lambda\mu \, R(0)(e_1, e_2, e_3, e_4) + (1 - \lambda^2) \, (1 - \mu^2) \geq 0 \end{aligned}$$

for all orthonormal four-frames $\{e_1, e_2, e_3, e_4\} \subset \mathbb{R}^n$ and all $\lambda, \mu \in [0, 1]$. Hence, Proposition 10 in [3] implies that

$$\begin{aligned} &R(t)(e_1, e_3, e_1, e_3) + \lambda^2 \, R(t)(e_1, e_4, e_1, e_4) \\ &+ \mu^2 \, R(t)(e_2, e_3, e_2, e_3) + \lambda^2 \mu^2 \, R(t)(e_2, e_4, e_2, e_4) \\ &- 2\lambda\mu \, R(t)(e_1, e_2, e_3, e_4) + (1 - \lambda^2) \, (1 - \mu^2) \geq 0 \end{aligned}$$

for all orthonormal four-frames $\{e_1, e_2, e_3, e_4\} \subset \mathbb{R}^n$, all $\lambda, \mu \in [0, 1]$, and all $t \in [0, T)$. Since $\text{scal}(R(t)) \geq \text{scal}(R(0)) = s$, we conclude that $R(t) \in \tilde{C}(s)$ for all $t \in [0, T)$.

After these preparations, we now present the proof of Theorem 2.

Theorem 11. *Assume that $g(t)$, $t \in (-\infty, 0)$, is an ancient solution to the Ricci flow on a compact n-dimensional manifold M. Moreover, we assume that there exists a uniform constant $\rho > 0$ such that*

$$R_{g(t)} - \rho \, \text{scal}_{g(t)} \, I \in \hat{C}$$

for all points $(p, t) \in M \times (-\infty, 0)$. Then the manifold $(M, g(t))$ has constant sectional curvature for each $t \in (-\infty, 0)$.

Proof. Consider the one-parameter family of cones $\hat{C}(s)$, $s \in (0, \infty)$, defined in [5]. It is shown in [5] that the cone $\hat{C}(s)$ has property $(*)$ for each $s \in (0, \infty)$. Furthermore, the cones $\hat{C}(s)$ vary continuously in s.

By assumption, there exists a real number $s_0 \in (0, \infty)$ such that $R_{g(t)} \in \hat{C}(s_0)$ for all points $(p, t) \in M \times (-\infty, 0)$. Using Theorem 9, we conclude that $R_{g(t)} \in \hat{C}(s)$ for all points $(p, t) \in M \times (-\infty, 0)$ and all $s \in (0, \infty)$. Consequently, the manifold $(M, g(t))$ has constant sectional curvature for each $t \in (-\infty, 0)$. This completes the proof of Theorem 11.

Theorem 12. *Assume that $g(t)$, $t \in (-\infty, 0)$, is an ancient solution to the Ricci flow on a compact n-dimensional manifold M. Moreover, we assume that there exists a uniform constant $\rho > 0$ such that*

$$R_{g(t)} - \rho \, \text{scal}_{g(t)} \, I \in \tilde{C}$$

for all points $(p, t) \in M \times (-\infty, 0)$. Then the manifold $(M, g(t))$ has constant sectional curvature for each $t \in (-\infty, 0)$.

Proof. By assumption, we have

$$R_{g(t)} - \rho \, \text{scal}_{g(t)} \, I \in \tilde{C}$$

for all points $(p,t) \in M \times (-\infty,0)$. Hence, we can find a real number $s_0 \in (0,\infty)$ such that

$$R_{g(t)} - \frac{1}{2}\,\rho\,\mathrm{scal}_{g(t)}\, I \in \tilde{C}(s_0)$$

for all points $(p,t) \in M \times (-\infty,0)$.

We next consider a pair of real numbers a,b such that $2a = 2b + (n-2)b^2$ and $b \in \big(0, \frac{\sqrt{2n(n-2)+4}-2}{n(n-2)}\big]$. Following [2], we define a linear transformation $\ell_{a,b} : \mathscr{C}_B(\mathbb{R}^n) \to \mathscr{C}_B(\mathbb{R}^n)$ by

$$\ell_{a,b}(R) = R + b\,\mathrm{Ric}(R) \owedge \mathrm{id} + \frac{1}{n}\,(a-b)\,\mathrm{scal}(R)\,\mathrm{id} \owedge \mathrm{id},$$

where $\owedge$ denotes the Kulkarni-Nomizu product; see e.g. [1], Definition 1.110. If we choose $b \in \big(0, \frac{\sqrt{2n(n-2)+4}-2}{n(n-2)}\big]$ sufficiently small, then

$$R_{g(t)} \in \ell_{a,b}(\tilde{C}(s_0))$$

for all points $(p,t) \in M \times (-\infty,0)$.

By Proposition 10, the cone $\tilde{C}(s)$ is invariant under the Hamilton ODE for each $s \in (0,\infty)$. Consequently, the cone $\ell_{a,b}(\tilde{C}(s))$ is transversally invariant under the Hamilton ODE for each $s \in (0,\infty)$ (cf. [2], Proposition 3.2). Therefore, the cone $\ell_{a,b}(\tilde{C}(s))$ has property $(*)$ for each $s \in (0,\infty)$. Using Theorem 9, we conclude that $R_{g(t)} \in \ell_{a,b}(\tilde{C}(s))$ for all points $(p,t) \in M \times (-\infty,0)$ and all $s \in (0,\infty)$. Taking the limit as $s \to \infty$, we obtain $R_{g(t)} \in \ell_{a,b}(\hat{C})$ for all points $(p,t) \in M \times (-\infty,0)$. Hence, it follows from Theorem 11 that $(M,g(t))$ has constant sectional curvature for each $t \in (-\infty,0)$.

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