

# Cosmological Quantum Billiards <sup>1</sup>

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The mini-superspace quantization of  $D = 11$  supergravity is equivalent to the quantization of a  $E_{10}/K(E_{10})$  coset space sigma model, when the latter is restricted to the  $E_{10}$  Cartan subalgebra. As a consequence, the wavefunctions solving the relevant mini-superspace Wheeler-DeWitt equation involve automorphic (Maass wave) forms under the modular group  $W^+(E_{10}) \cong PSL_2(0)$ . Using Dirichlet boundary conditions on the billiard domain a general inequality for the Laplace eigenvalues of these automorphic forms is derived, entailing a wave function of the universe that is generically complex and always tends to zero when approaching the initial singularity. The significance of these properties for the nature of singularities in quantum cosmology in comparison with other approaches is discussed. The present approach also offers interesting new perspectives on some long standing issues in canonical quantum gravity.

## 1 Introduction

The present contribution is based on [1], and elaborates on several issues and arguments that were not fully spelled out there. In that work, a first step was taken towards quantization of the one-dimensional ‘geodesic’  $E_{10}/K(E_{10})$  coset

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model which had been proposed in [2] as a model of M-theory. Here,  $E_{10}$  denotes the hyperbolic Kac–Moody group  $E_{10}$  which is an infinite-dimensional extension of the exceptional Lie group  $E_8$ , and plays a similarly distinguished role among the infinite-dimensional Lie algebras as  $E_8$  does among the finite-dimensional ones. The proposal of [2] had its roots both in the appearance of so-called ‘hidden symmetries’ of exceptional type in the dimensional reduction of maximal supergravity to lower dimensions [3], as well as in the celebrated analysis of Belinskii, Khalatnikov and Lifshitz (BKL) [4] of the gravitational field equations in the vicinity of a generic space-like (cosmological) singularity. According to the basic hypothesis underlying this analysis the causal decoupling of spatial points near the spacelike singularity<sup>2</sup> effectively leads to a dimensional reduction whereby the equations of motion become ultralocal in space, and the dynamics should therefore be describable in terms of a (continuous) superposition of one-dimensional systems, one for each spatial point. More specifically, in this approximation the dynamics at each spatial can be described by a sequence of Kasner regimes, such that in the strict limit towards the singularity, the Kasner behavior is interspersed with hard reflections of the logarithms of the spatial scale factors off infinite potential walls [5, 6]. This generic behavior has been termed ‘cosmological billiards’. The geometry of the billiard table and the configuration of the walls (‘cushions’) of the billiard table are determined by the dimension and the matter content of the theory [7, 8]. Likewise the occurrence or non-occurrence of chaotic oscillations near the singularity depends on this configuration. In particular, for  $D = 11$  supergravity it was shown in [9] that the billiard domain is the fundamental Weyl chamber  $\mathcal{C}$  of the ‘maximally extended’ hyperbolic Kac–Moody group  $E_{10}$ . The volume of this fundamental Weyl chamber is finite, implying chaotic behavior [9, 10]. The emergence of the hyperbolic Kac–Moody algebra  $E_{10}$  in this context is also in line with its conjectured appearance in the dimensional reduction of  $D = 11$  supergravity to one time dimension [11].

Ref. [2] goes beyond the standard BKL analysis, as well as the original conjecture [11], in that it establishes a correspondence at the classical level between a truncated gradient expansion of the  $D = 11$  supergravity equations of motion near the spacelike singularity and an expansion in heights of roots of a one-dimensional constrained ‘geodesic’  $E_{10}/K(E_{10})$  coset space model. The cosmological billiards approximation then corresponds to the restriction of this coset model to the Cartan subalgebra of  $E_{10}$ . Going beyond this billiard approximation involves bringing in spatial dependence, in such a way that a ‘small tension’ expansion in spatial gradients *à la* BKL gets converted into a level expansion of the  $E_{10}$  Lie algebra. However, the correspondence between the field equations on the one hand, and the  $E_{10}/K(E_{10})$  model on the other hand, codified in a ‘dictionary’, has so far only been shown to work up to first order spatial gradients. The proper inclusion of higher order spatial gradients, and thus the emergence of a space-time field theory from a ‘pre-geometrical’ scheme, remains

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<sup>2</sup>This is the same decoupling that later came to be associated with the so-called ‘horizon problem’ of inflationary cosmology.

an outstanding problem, in spite of the fact that the  $E_{10}$  Lie algebra contains all the ‘gradient representations’ that would be needed for a Taylor expansion around a given spatial point [2].

Quantizing M-theory in the  $E_{10}$  framework thus amounts to setting up and solving a Wheeler-DeWitt equation for the full  $E_{10}/K(E_{10})$  model, and imposing the subsidiary constraints corresponding to the canonical (diffeomorphism, Gauss,...) constraints of the usual canonical approach. As a very first step, ref. [1] solves the quantum constraints for  $D = 11$  supergravity for the ten spatial scale factors, and for their fermionic ‘superpartners’, in compliance with the supersymmetry constraint. The resulting *quantum cosmological billiard* is a variant of the ‘minisuperspace’ quantization of gravity pioneered in [12, 13, 14] and further developed in many works, see in particular [15, 16, 17, 18, 19, 20] and references therein.<sup>3</sup> The essential new ingredient in the present construction is the *arithmetic structure* provided by  $E_{10}$  and its Weyl group, whose relevance in the context of Einstein gravity was pointed out and explored in [21]; see also [22] for an ansatz based on superconformal quantum mechanics, which has some similarities with the present approach. We believe that the key issues with the proposal of [2], in particular the conjectured emergence of classical space-time out of a pre-geometrical quantum gravity phase, and the question how the ‘dictionary’ of [2] can be extended to the full  $E_{10}$  algebra, cannot be resolved within the framework of classical field equations, but will require quantization of the  $E_{10}/K(E_{10})$  coset model. Equally important, the ‘resolution’ of the cosmological singularity, a key issue of modern research in canonical quantum gravity, is expected to involve quantum theory in an essential way. In addition, it will almost certainly require new concepts besides the transition to the quantum theory, with the question of what happens to classical space-time near the singularity as the core problem.

The main results reported in [1] and in the present contribution are

- The quantum cosmological billiard problem is well-posed and a Hilbert space (with positive definite metric) can be defined.
- The solutions to the bosonic Wheeler-DeWitt (WDW) equation of  $D = 11$  supergravity can be described as odd Maass wave forms of the ‘modular group’  $W^+(E_{10}) \cong PSL(2, 0)$ .
- With the appropriate Dirichlet boundary conditions all solutions of the WDW equation, a.k.a. ‘wavefunctions of the universe’, can be shown generally to vanish rapidly near a space-like singularity while remaining complex and oscillating.
- The analysis can be extended to include the fermionic degrees of freedom without changing the conclusions.

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<sup>3</sup>We remind readers that the ‘super’ in ‘minisuperspace’ refers to Wheeler’s moduli space of 3-geometries and is a purely bosonic concept. It therefore has no relation to the notion of ‘supersymmetry’ relating bosons and fermions, which is also discussed in this article.

Possible extensions and the eventual significance of these results for quantum cosmology are discussed at the end of this article.

## 2 Minisuperspace quantization

We first consider the bosonic variables. We proceed from the following metric ansatz, as appropriate for a cosmological billiard for a  $(d+1)$ -dimensional space-time

$$ds^2 = -N^2 dt^2 + \sum_{a=1}^d e^{-2\beta^a} \theta^a \otimes \theta^a, \quad (1)$$

where we keep the spatial dimension  $d$  arbitrary (but always  $d \geq 3$ ) for the moment, and will specialize to  $D = 11$  supergravity (and thus  $d = 10$ ) only later.  $\theta^a \equiv \mathcal{N}^a_m dx^m$  is a spatial frame in an Iwasawa decomposition of the metric, as explained in [7]. Substituting the above ansatz into the Einstein action, one arrives at the kinetic term

$$\mathcal{L}_{\text{kin}} = \frac{1}{2} n^{-1} \sum_{a,b=1}^d \dot{\beta}^a G_{ab} \dot{\beta}^b \quad (2)$$

in terms of the new lapse  $n \equiv N/\sqrt{g}$  (the spatial volume is  $\sqrt{g} = \exp[-\sum_a \beta^a]$ ) and the Lorentzian DeWitt metric

$$\dot{\beta}^a G_{ab} \dot{\beta}^b \equiv \sum_{a=1}^d (\dot{\beta}^a)^2 - \left( \sum_{a=1}^d \dot{\beta}^a \right)^2. \quad (3)$$

As is well known, the DeWitt metric, with the factor  $(-1)$  in front of the second term, is distinguished by several uniqueness properties that are discussed in [23]. Here, it will be essential that for  $d = 10$  this metric coincides with the restriction of the Cartan–Killing metric of  $E_{10}$  to its Cartan subalgebra. It can now be shown [7] that the remaining contributions to the Hamiltonian constraint at a given spatial point can be combined into an ‘effective potential’ of the generic form

$$V_{\text{eff}} = \sum_A c_A(Q, P, \partial\beta, \partial Q) \exp(-2w_A(\beta)) \quad (4)$$

where  $(Q, P)$  are the (canonical) variables corresponding to all degrees of freedom other than the scale factors  $\beta^a$ , the  $\partial$  stands for *spatial* gradients, and  $w_A$  in the exponent are linear forms, called *wall forms*,

$$w_A(\beta) \equiv G_{ab} w_A^a \beta^b \quad (5)$$

with the DeWitt metric  $G_{ab}$  introduced above. In the limit towards the singularity  $\beta \rightarrow \infty$ , the exponential walls become ‘sharp’, and the dynamics is dominated by a set of ‘nearest’ walls. These make up the ‘cushions’ of a billiard table,

and can be viewed as the result of ‘integrating out’ the off-diagonal metric and the matter degrees of freedom, as explained in [7]. In the strict limit towards the singularity they are simply given by timelike hyperplanes in the forward lightcone in  $\beta$ -space, which are determined by the linear equations  $w_A(\beta) = 0$ . The spatial ultralocality of the BKL limit thus reduces the gravitational model to a classical mechanics system of a relativistic billiard ball described by the  $\beta^a$  variables moving on straight null lines in the Lorentzian space with metric  $G_{ab}$  until hitting a billiard table wall corresponding to a hyperplane. The straight line segments of the billiard motion are the Kasner regimes, while the reflections are usually referred to as ‘Kasner bounces’. We repeat that there is one such system for each spatial point  $\mathbf{x}$ , and these systems are all decoupled.

The canonical bosonic variables of the billiard system are  $\beta^a$  and their canonically conjugate momenta  $\pi_a$ , *viz.*

$$\pi_a := \frac{\partial \mathcal{L}}{\partial \dot{\beta}^a} = G_{ab} \dot{\beta}^b \quad (6)$$

where we set  $n = 1$  from now on.<sup>4</sup> The Hamiltonian is

$$\mathcal{H}_0 = \frac{1}{2} \pi_a G^{ab} \pi_b \quad (7)$$

with the inverse DeWitt metric  $G^{ab}$ . The effective potential (4) has disappeared as we have taken the BKL limit. Before quantisation, we perform the following change of variables by means of which the billiard motion is projected onto the unit hyperboloid in  $\beta$ -space [7]

$$\beta^a = \rho \omega^a, \quad \omega^a G_{ab} \omega^b = -1, \quad \rho^2 = -\beta^a G_{ab} \beta^b, \quad (8)$$

where  $\rho$  is the ‘radial’ direction in the future light-cone and  $\omega^a = \omega^a(z)$  are expressible as functions of  $d - 1$  coordinates  $z$  on the unit hyperboloid. The limit towards the singularity is  $\rho \rightarrow \infty$  in these variables. The Wheeler-DeWitt operator thus takes the form

$$\mathcal{H}_0 \equiv G^{ab} \partial_a \partial_b = -\rho^{1-d} \frac{\partial}{\partial \rho} \left( \rho^{d-1} \frac{\partial}{\partial \rho} \right) + \rho^{-2} \Delta_{\text{LB}}, \quad (9)$$

where  $\Delta_{\text{LB}}$  is the Laplace–Beltrami operator on the  $(d - 1)$ -dimensional unit hyperboloid. We emphasize that ordering ambiguities are entirely absent in this expression, as is manifest in the  $\beta^a$  variables in terms of which the WDW equation is just the free Klein-Gordon equation. The same holds true for the variables  $(\rho, \omega^a(z))$  because the expression in the new coordinates is unambiguously determined by the coordinate transformation (8) (as it would be in any other coordinate system).

The mini-superspace WDW equation therefore reads

$$\mathcal{H}_0 \Phi(\rho, z) = 0 \quad (10)$$

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<sup>4</sup>With this choice of gauge,  $t$  becomes a ‘Zeno-like’ time coordinate, for which the singularity is at  $t = +\infty$ . This time is related to physical (proper) time  $T$  by  $t \sim -\log T$ .

for the ‘wavefunction of the universe’  $\Phi(\rho, z)$ . As usual (see e.g. [20]) one can adopt  $\rho$  as a time coordinate in the initially ‘timeless’ WDW equation, with the standard (Klein–Gordon-like) invariant inner product

$$(\Phi_1, \Phi_2) = i \int d\Sigma^a \Phi_1^* \overleftrightarrow{\partial}_a \Phi_2 \quad (11)$$

where the integral is to be taken over a spacelike hypersurface inside the forward lightcone in  $\beta$ -space. ‘Invariance’ means that this scalar product does not depend on the shape of the spacelike hypersurface, so we can for instance choose any of the unit hyperboloids  $\rho = \text{const}$ .

In order to construct solutions we separate variables by means of the ansatz  $\Phi(\rho, z) = R(\rho)F(z)$  [14, 16]. For any eigenfunction  $F(z)$  obeying

$$-\Delta_{\text{LB}}F(z) = EF(z) \quad (12)$$

the associated radial equation is solved by

$$R_{\pm}(\rho) = \rho^{-\frac{d-2}{2}} e^{\pm i \sqrt{E - (\frac{d-2}{2})^2} \log \rho}. \quad (13)$$

Positive frequency waves emanating from the singularity correspond to  $R_-(\rho)$  and have positive inner product (11). It is important here that one can consistently restrict to positive norm wave functions: the potential which might scatter an initially positive norm state into a negative norm state, is here effectively replaced by a set of boundary conditions on the wavefunction, and hence there is no ‘Klein paradox’. As we will see the same feature continues to hold for the full  $E_{10}$  WDW operator, in marked contrast to the standard WDW operator.

To study the eigenvalues of the Laplace–Beltrami operator on the unit hyperboloid we use a generalized upper half plane model  $z = (\vec{u}, v)$  for the unit hyperboloid with coordinates  $\vec{u} \in \mathbb{R}^{d-2}$  and  $v \in \mathbb{R}_{>0}$ . The relevant coordinate transformation is obtained by first diagonalizing the DeWitt metric (3) in terms of Minkowskian coordinates  $\tilde{\beta}^a$ , such that

$$G_{ab} \beta^a \beta^b \equiv -\tilde{\beta}^+ \tilde{\beta}^- + \sum_{j=1}^{d-2} \tilde{\beta}^j \tilde{\beta}^j, \quad (14)$$

where we have used the last two directions for forming light-cone coordinates. Then the forward unit hyperboloid (with  $\tilde{\beta}^{\pm} > 0$ ) is coordinatized by

$$\tilde{\beta}^+ = \frac{1}{v}, \quad \tilde{\beta}^- = v + \frac{\vec{u}^2}{v}, \quad \tilde{\beta}^j = \frac{u^j}{v} \quad (v > 0) \quad (15)$$

The metric induced on the unit hyperboloid is easily calculated to be the Poincaré metric on the generalized upper half plane

$$ds^2 = \frac{dv^2 + d\vec{u}^2}{v^2} \quad \Rightarrow \quad d\text{vol}(z) \equiv v^{1-d} dv d^{d-2}u \quad (16)$$

such that the Laplace-Beltrami operator becomes

$$\Delta_{\text{LB}} = v^{d-1} \partial_v (v^{3-d} \partial_v) + v^2 \partial_u^2. \quad (17)$$

For the spectral problem we must specify boundary conditions. For the cosmological billiard, these are provided by infinite ('sharp') potential walls which encapsulate the effect of spatial inhomogeneities and matter fields near the spacelike singularity, as explained above (see [7, 8] for details). Following the original suggestion of [14], we are thus led to impose the vanishing of the wavefunction on the boundary of the fundamental domain specified by these walls. Accordingly, let  $F(z)$  be any function on the hyperboloid satisfying (12) with *Dirichlet conditions* at the boundaries of this domain (in contrast to [16, 21], where Neumann boundary conditions are assumed). A direct generalization of the arguments on page 28 of Ref. [24] gives

$$-(\Delta_{\text{LB}} F, F) \geq \int dv d^{d-2} u v^{3-d} (\partial_v F)^2 \quad (18)$$

with (12) and (17). Considering also

$$\begin{aligned} (F, F) &\equiv \int d\text{vol}(z) F^2(z) = \int dv d^{d-2} u v^{1-d} F^2 \\ &= \frac{2}{d-2} \int dv d^{d-2} u v^{2-d} F \partial_v F, \end{aligned} \quad (19)$$

the use of the Cauchy-Schwarz inequality entails

$$E \geq \left( \frac{d-2}{2} \right)^2. \quad (20)$$

From the explicit solution (13) we thus conclude that  $R_{\pm}(\rho) \rightarrow 0$  when  $\rho \rightarrow \infty$ , and therefore *the full wavefunction and all its  $\rho$  derivatives tend to zero near the singularity*. Evidently, this result hinges on the peculiar form of the differential operator in the  $(\rho, z)$  variables in (9), which itself is uniquely determined by the form of the operator in  $\beta$ -coordinates. It would not be valid if we were allowed to move around the  $\rho$  factors in the differential operator of (9).

While the wave function would also vanish for Neumann boundary conditions (for which  $E \geq 0$ ) with the given ordering, the inequality (20) furthermore ensures that the full wavefunction is generically complex and oscillating. Let us point out here that this result may be of relevance to a long standing issue in canonical gravity, namely the question why and how the *real* WDW equation should give rise to *complex* wave functions [25, 26, 27]. As explained there, the complexity of the wave function is intimately linked to the emergence of a *directed* time in canonical gravity. More specifically, admitting only positive norm wave functions corresponds to choosing an 'arrow of time' (real wave functions have vanishing norm with the product (11), and would thus not select a time direction). Let us repeat that restricting to positive norm states would

be inconsistent for the standard WDW equation with a potential even in mini-superspace quantisation. Here, the potential has effectively disappeared in the BKL limit, leaving its trace only via the boundary conditions, so the restriction is consistent.

### 3 Automorphy and the $E_{10}$ Weyl group

Whereas the discussion above was valid for gravity in any space-time of dimension  $d + 1$  we now focus on maximal supergravity in eleven space-time dimensions. For the bosonic sector of maximal supergravity, the wavefunctions can be further analyzed by exploiting the underlying symmetry encoded in the Weyl group  $W(E_{10})$  and its arithmetic properties, and in particular the new links between hyperbolic Weyl groups and generalized modular groups uncovered in [28]. The Weyl reflections that the classical particle is subjected to when colliding with one of the walls are norm preserving, and therefore the reflections can be projected to any hyperboloid of constant  $\rho$ , inducing a non-linear action on the co-ordinates  $z$  (given in (23) below for the fundamental reflections). For physical amplitudes to be invariant under the Weyl group, the full wavefunction must transform as follows

$$\Phi(\beta) = \pm\Phi(w_I(\beta)) \quad \Leftrightarrow \quad \Phi(\rho, z) = \pm\Phi(\rho, w_I(z)) \quad (21)$$

for the ten generating fundamental reflections  $w_I$  of  $W(E_{10})$ , labeled by  $I = -1, 0, 1, \dots, 8$ . Restricting the wavefunction to the fundamental Weyl chamber, one easily checks that the plus sign in (21) corresponds to Neumann boundary conditions, and the minus sign to Dirichlet conditions (which we adopt here). From (21) it follows that  $\Phi(\rho, z)$  is invariant under *even* Weyl transformations  $s \in W^+(E_{10})$  irrespective of the chosen boundary conditions.

Choosing coordinates as in (15) the relevant variables now live in a nine-dimensional ‘octonionic upper half plane’ with coordinate

$$z = u + iv \quad , \quad u \equiv \bar{u} \in \mathbb{O} \quad (22)$$

where  $\mathbb{O}$  is the non-commutative and non-associative algebra of *octonions*, while  $v$  is still real and positive. Next, we recall [29, 30, 28] that the 240 roots of  $E_8$  can be represented by unit octonions; more precisely, these are the 240 *units* in the non-commutative and non-associative ring of *integral octonions*  $\mathbb{O}$  called ‘*octavians*’, see [30]. Denoting by  $\varepsilon_j$  (for  $j = 1, \dots, 8$ ) the eight simple roots and by  $\theta$  the highest root of  $E_8$ , respectively, expressed as unit octonions, we arrive at the following *modular realization* of the  $E_{10}$  Weyl transformations on the nine-dimensional unit hyperboloid: the ten fundamental reflections of  $W(E_{10})$  act as

$$w_{-1}(z) = \frac{1}{\bar{z}} \quad , \quad w_0(z) = -\theta\bar{z}\theta + \theta \quad , \quad w_j(z) = -\varepsilon_j\bar{z}\varepsilon_j \quad (23)$$

where  $\bar{z} := \bar{u} - iv$ , with  $iu = \bar{u}i$  in accordance with Cayley–Dickson doubling



[30].<sup>5</sup> Observe that, despite the non-associativity of the octonions, there is no need to put parentheses in (23) by virtue of the alternativity of the octonions.

In the present context, the formulas (23) represent the most general (and most sophisticated!) modular realization of a Weyl group, but there are corresponding versions for the other division algebras, with the quaternions  $\mathbb{H}$  for  $d = 6$ , and the complex numbers  $\mathbb{C}$  for  $d = 4$  (with corresponding ‘integers’, see [28] for details). The simplest case is  $\mathbb{A} = \mathbb{R}$  which corresponds to pure gravity in four spacetime dimensions ( $d = 3$ ). In this case  $u \in \mathbb{R}$ , and the formulas (23) reduce to the ones familiar from complex analysis, namely

$$z \mapsto \frac{1}{\bar{z}}, \quad z \mapsto -\bar{z} + 1, \quad z \mapsto -\bar{z}, \quad (24)$$

generating the group  $PGL_2(\mathbb{Z})$ . For *even* Weyl transformations, we re-obtain the standard modular group  $PSL_2(\mathbb{Z})$  generated by

$$S(z) \equiv (w_{-1}w_1)(z) = -1/z, \quad T(z) \equiv (w_0w_1)(z) = z + 1. \quad (25)$$

Consequently, for pure gravity in four space-time dimensions, the relevant eigenfunctions of the mini-superspace WDW operator are automorphic forms with respect to the standard modular group  $PSL_2(\mathbb{Z})$ , as already pointed out in [21].

For the maximally supersymmetric theory, on the other hand, the even Weyl group  $W^+(E_{10})$  is isomorphic to the ‘modular group’  $PSL_2(0)$  over the octavians, where  $PSL_2(0)$  is *defined* by iterating the action of (23) an even number of times [28]. Accordingly, for maximal supergravity the bosonic wavefunctions  $\Phi(\rho, z)$  are *odd Maass wave forms* for  $PSL_2(0)$ , that is, invariant eigenfunctions of the Laplace–Beltrami operator transforming with a minus sign in (21) under the extension  $W(E_{10})$  of  $PSL_2(0)$ . Properly understanding the ‘modular group’  $PSL_2(0)$  and the associated modular functions remains an outstanding mathematical challenge, see [31] for an introduction (and [24] for the  $PSL_2(\mathbb{Z})$  theory). For the groups  $PSL_2(\mathbb{Z})$  and  $PSL_2(\mathbb{Z}[i])$  the (purely discrete) spectra of odd Maass wave forms have been investigated numerically in [32, 33, 34, 35, 36].

One important feature of (23) should be emphasized: supplementing the seven imaginary units of  $\mathbb{O}$  by another imaginary unit  $i$ , it would appear that we have to enlarge the octonions to *sedonions*, a system of hypercomplex numbers with 15 imaginary units, which by Hurwitz’ theorem is no longer a division algebra (that is, has zero divisors). Remarkably, however, the formulas (23) are such that with the iterated action of (23) we never need to introduce any further imaginary units beyond  $i$  and the seven octonionic ones. In other words, the transformations (23) do not move  $z$  out of the 9-dimensional generalized upper

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<sup>5</sup>We recall that the Dickson doubling of a normed algebra  $\mathbb{A}$  with conjugation associates to doubled elements  $a + ib, c + id \in \mathbb{A} + i\mathbb{A}$  the product (see [30])

$$(a + ib)(c + id) = (ac - \bar{d}b) + i(cb + \bar{a}d).$$

The Hurwitz theorem states that, starting from the real numbers  $\mathbb{R}$ , this process generates the division algebras  $\mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}$  of the real, complex, quaternionic and octonionic numbers, respectively. Further application generates algebras with zero divisors.

half plane. In particular, the doubling rule ensures that  $\bar{z}z = v^2 + |u|^2 \in \mathbb{R}_+$  so that the inverse  $1/\bar{z}$  also stays in this plane.

By modular invariance, the wavefunctions can be restricted to the fundamental domain of the action of  $W(E_{10})$  and, conversely, their modular property defines them on the whole hyperboloid. The Klein–Gordon inner product (11) must likewise be restricted to the fundamental chamber

$$(\Phi_1, \Phi_2) = i \int_{\mathcal{F}} d\text{vol}(z) \rho^{d-1} \Phi_1^* \overleftrightarrow{\partial}_\rho \Phi_2. \quad (26)$$

where  $\mathcal{F}$  is the intersection of  $\mathcal{C}$  with the unit hyperboloid; accordingly, the ‘cushions’ of the billiard are obtained by intersecting the hyperplanes  $w_A(\beta) = 0$  with the unit hyperboloid, such that for pure Einstein gravity ( $d = 3$ ) one ends up with the projected billiard table shown in Figure 1. The restriction of the scalar product to the fundamental domain is necessary, as the integral over the whole hyperboloid would be infinite for functions obeying (21), and the product would be ill-defined. This infinity is analogous to the one that arises in the calculation of the one-loop amplitude in string theory, and there as well, the product is rendered finite upon ‘division’ by the modular group  $PSL_2(\mathbb{Z})$ . We thus have at hand an analog of this mechanism in canonical quantum gravity. We note that modular invariance is a distinctive feature of string theory not shared by the quantum field theory of pointlike particles and arguably the ‘real’ reason behind the conjectured finiteness of string theory.

## 4 Classical and quantum chaos

The fundamental region defined by the billiard walls as a subset of the unit hyperboloid (a hyperbolic space of constant negative curvature) has finite volume

$$\text{Vol}(\mathcal{F}) = \int_{\mathcal{F}} d\text{vol}(z) < \infty \quad (27)$$

in spite of the fact that the domain  $\mathcal{F}$  extends to infinity (has a ‘cusp’). The finiteness of the fundamental domain is a consequence of the hyperbolicity of the Kac–Moody algebra  $E_{10}$  (whereas fundamental domains for non-hyperbolic indefinite Kac–Moody algebras have infinite volume). Such finite volume billiards have been known and studied for a long time, as they are known to exhibit classical chaos. They are also the prototypical examples for studying the transition from classical to quantum chaos, and especially the question of how the presence of classical chaos is reflected in the spectra of the corresponding (unitary) Hamiltonian operators, see for example [37, 38, 39, 40]. One of the remarkable results of these investigations is that there is a qualitative difference between the wave functions of classically ergodic and classically periodic orbits: The latter have very drastic (density) fluctuations whereas the former appear more like randomized Gaussians [37] and can be called quantum ergodic. Another notable feature is the appearance of so-called ‘scars’ as remnants of classically periodic

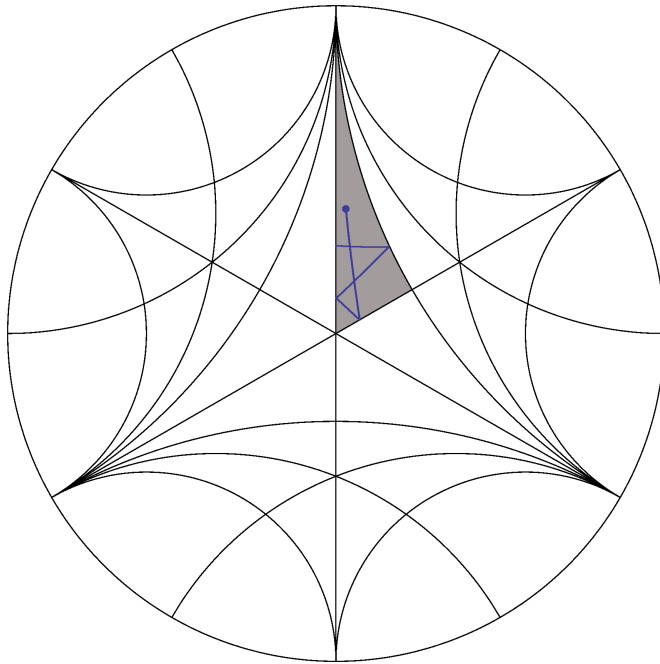


Figure 1: *The projected billiard table of standard general relativity (in grey) on two-dimensional hyperbolic space. There are infinitely many equivalent tables related by Weyl reflections.*

orbits [41, 39].<sup>6</sup> These are regions of (relative) high probability in position space which appear related to the positions of classically periodic orbits.

However, there are two main differences between these studies and the cosmological (quantum) billiards considered here, *viz.*

- The cosmological billiard is *relativistic*, that is, the classical evolution follows a Klein Gordon-like equation, instead of a non-relativistic Schrödinger equation (but see [43] for a discussion of relativistic neutrino billiards).
- In the  $\beta$ -space description, the walls of the billiard move away from one another, and one would thus have to solve the equation with *time-dependent* boundary conditions in the  $\beta$  variables. By contrast, the projection (8) allows to reformulate the problem with static boundary conditions, at the expense of modifying the  $\rho$ -dependent part of the relativistic Hamiltonian  $-\partial_\rho^2$  to the right hand side of (9). For pure Einstein gravity in four dimensions, the resulting fixed walls billiard system is displayed in figure 1.

Within this new setting it would be of much interest to study the fate of a generic wavepacket in our cosmological billiard system. The expectation is that

<sup>6</sup>See [21] for a possible physical interpretation of these ‘scars’ in quantum cosmology.

the quantum theory ‘washes out’ the classical chaos in the sense that any initially localized wavepacket will eventually disperse when approaching the singularity, such that the asymptotic (for  $\rho \rightarrow \infty$ ) wave function will be spread evenly over  $\mathcal{F}$ . For the non-relativistic case some studies of the evolution of wavepackets can be found in [42], where the focus was on non-generic classically periodic configurations.

The particular case of interest in M-theory cosmology possesses a nine-dimensional fundamental domain and has apparently not been considered in the literature so far. The chaotic quantum billiard being merely the quantum theory of a *finite-dimensional* subsystem, corresponding to the Cartan subalgebra of an infinite-dimensional Kac–Moody algebra, such a study would however represent only a first step towards the quantization of the full system, as already mentioned. There will thus arise many new issues, such as for example the link between a formally integrable system in infinitely many variables, and the chaoticity of a finite dimensional system obtained from it by projection to finitely many variables (some comments on this issue can be found in [7]).

## 5 Supersymmetry

The quantum billiard analysis can be extended to maximal *supergravity*, with  $d = 10$ , by supplementing the bosonic degrees of freedom with a vector-spinor, the gravitino. In the  $E_{10}$  approach, the latter corresponds to a spinorial representation of the ‘R-symmetry’ group  $K(E_{10})$  in terms of which the Rarita–Schwinger equation of  $D = 11$  supergravity (with the usual truncations) can be re-written as a  $K(E_{10})$  covariant ‘Dirac equation’ [44, 45, 46]. When restricting to the diagonal metric degrees of freedom, the gravitino  $\psi_\mu$  of  $D = 11$  supergravity performs a separate classical<sup>7</sup> fermionic billiard motion [47]. This is most easily expressed in a supersymmetry gauge  $\psi_t = \Gamma_t \Gamma^a \psi_a$  [44] and in the variables [47] (with  $\Gamma_* = \Gamma^1 \dots \Gamma^{10}$ )

$$\varphi^a = g^{1/4} \Gamma_* \Gamma^a \psi^a \quad (\text{no sum on } a = 1, \dots, 10). \quad (28)$$

(recall that  $g \equiv \exp(-2 \sum \beta^a)$ ). Using (28) in conjunction with Eqn. (6.3) of [46] the Dirac brackets between two gravitino variables become

$$\{\varphi_\alpha^a, \varphi_\beta^b\}_{\text{D.B.}} = -2i G^{ab} \delta_{\alpha\beta} \quad (29)$$

where we have written out the 32 real spinor components using the indices  $\alpha, \beta$ . We stress that it is precisely the inverse DeWitt metric  $G^{ab}$ , see (3), which appears in this equation!

The fermionic and bosonic variables are linked by the supersymmetry constraint

$$\mathcal{S}_\alpha \equiv \sum_{a,b=1}^{10} \dot{\beta}^a G_{ab} \varphi_\alpha^b = \sum_{a=1}^{10} \pi_a \varphi_\alpha^a = 0. \quad (30)$$

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<sup>7</sup>In the sense that the gravitino is treated as a classical variable, not as an operator.

The supersymmetry constraint implies the Hamiltonian constraint  $\mathcal{H}_0 = 0$  by closure of the algebra

$$\frac{1}{4} \{\mathcal{S}_\alpha, \mathcal{S}_\beta\}_{\text{D.B.}} = -i\delta_{\alpha\beta} \mathcal{H}_0. \quad (31)$$

In order to quantize this system we rewrite the 320 real gravitino components  $\varphi_\alpha^a$  in terms of 160 complex ones according to

$$\tilde{\varphi}_A^a := \varphi_A^a + i\varphi_{A+16}^a \quad (32)$$

for  $A, B, .. = 1, \dots, 16$ , and replace the Dirac brackets (29) by canonical anti-commutators

$$\{\tilde{\varphi}_A^a, (\tilde{\varphi}^\dagger)_B^b\} = 2G^{ab}\delta_{AB}, \quad \{\tilde{\varphi}_A^a, \tilde{\varphi}_B^b\} = \{(\tilde{\varphi}^\dagger)_A^a, (\tilde{\varphi}^\dagger)_B^b\} = 0 \quad (33)$$

to obtain a fermionic Fock space of dimension  $2^{160}$  by application of the fermionic creation operators  $(\tilde{\varphi}_A^a)^\dagger$  to the vacuum  $|\Omega\rangle$  defined by

$$\tilde{\varphi}_A^a |\Omega\rangle = 0 \quad (34)$$

For the supersymmetry constraint this amounts to the redefinition  $\tilde{\mathcal{S}}_A = \mathcal{S}_A + i\mathcal{S}_{A+16}$ . Because (34) implies  $\tilde{\mathcal{S}}_A |\Omega\rangle = 0$ , the quantum supersymmetry constraint is then solved by

$$|\Psi\rangle = \prod_{A=1}^{16} \tilde{\mathcal{S}}_A^\dagger \left( \Phi(\rho, z) |\Omega\rangle \right), \quad (35)$$

if and only if the function  $\Phi(\rho, z)$  is a solution of the WDW equation  $\mathcal{H}_0\Phi = 0$ . While this solution is close to the ‘bottom of the Dirac sea’, there is an analogous one ‘close to the top’ with  $\tilde{\mathcal{S}}_A^\dagger$  replaced by  $\tilde{\mathcal{S}}_A$  and  $|\Omega\rangle$  by the completely filled state.

In existing studies of the fermionic sector so far the fermions have been treated ‘quasi-classically’, that is, as Grassmann valued  $c$ -numbers. However, the correspondence between the full supergravity equations of motion and the fermionic extension of the  $E_{10}/K(E_{10})$  model, as far as it has been established, is lacking inasmuch as the relevant spinorial representations of the ‘R-symmetry’ group  $K(E_{10})$  identified to date are all *unfaithful*, and hence cannot capture the full fermionic dynamics of M-theory (see [46] for a detailed discussion of this problem). Again, quantisation may be essential here; in fact, a satisfactory solution and incorporation of all the fermionic degrees of freedom into the  $E_{10}$  model may require some kind of ‘bosonization’ of the fermionic degrees of freedom. This would also be in accord with the fact that fermions are intrinsically quantum objects.

## 6 Outlook

The cosmological billiards description takes into account the dependence on spatial inhomogeneities and matter degrees of freedom only in a very rudimentary way via the infinite potential walls. It would thus be desirable to develop

an approximation scheme for the quantum state in line with the ‘small tension’ expansion proposed in [2], and thereby hopefully resolve the difficulties encountered in extending the ‘dictionary’ of [2] to higher order spatial gradients and heights of roots in a quantum mechanical context. In the strict BKL approximation, the full wavefunction is expected to factorize as

$$|\Psi_{\text{full}}\rangle \sim \prod_{\mathbf{x}} |\Psi_{\mathbf{x}}\rangle, \quad (36)$$

near the singularity into a formal ultralocal product over wavefunctions of type (35), one for each spatial point, with independent bosonic wavefunctions  $\Phi_{\mathbf{x}}(\rho(\mathbf{x}), z(\mathbf{x}))$  and *space-dependent* metric variables  $\beta^a(\mathbf{x}) \equiv (\rho(\mathbf{x}), z(\mathbf{x}))$ . At first sight, it may seem paradoxical that (36) should become a better and better approximation near the singularity, as all the dynamics gets concentrated in a continuous superposition of Cartan subalgebras, whereas one would expect the full tower of  $E_{10}$  Lie algebra states, rather than just the Cartan subalgebra degrees of freedom, to become excited — in analogy with string cosmology, where one would expect the full tower of string states to become relevant near the singularity. However, the apparent paradox may well turn out to be the crux of the matter: the task is to replace the formal expression (36) involving a formal continuous product over all spatial points by a wavefunction which solely depends on the (infinite) tower of  $E_{10}$  degrees of freedom, and where all spatial dependence is discarded. It is this step which would effectively implement the de-emergence of space and time near the cosmological singularity, and their replacement by purely algebraic concepts [44, 48].

For this purpose we will have to generalize the mini-superspace Hamiltonian (9) to the full  $E_{10}$  Lie algebra. In fact, as a consequence of the uniqueness of the quadratic Casimir operator on  $E_{10}$ , there is a *unique*  $E_{10}$  extension of the billiard Hamiltonian (9) given by

$$\mathcal{H}_0 \rightarrow \mathcal{H} = \mathcal{H}_0 + \sum_{\alpha \in \Delta_+(E_{10})} \sum_{s=1}^{\text{mult}(\alpha)} e^{-2\alpha(\beta)} \Pi_{\alpha,s}^2. \quad (37)$$

where the first sum runs over the positive roots  $\alpha$  of  $E_{10}$  and the second one over a basis of the possibly degenerate root space of  $\alpha$ . Due to our lack of a manageable realization of the  $E_{10}$  algebra, this is a highly formal expression, but we can nevertheless note two important features: like the mini-superspace Hamiltonian (9) this operator is free of ordering ambiguities by the uniqueness of the  $E_{10}$  Casimir operator, and it has the form of a *free* Klein-Gordon operator, albeit in infinitely many dimensions. The absence of potential terms is due to the fact that in the approach of [2] (as far as it has been worked out, at least) the spatial gradients, which give rise to the ‘potential’  $\propto R^{(3)}$  in the standard WDW equation, are here replaced by *time derivatives of dual fields*. Accordingly, the contributions to the potential are replaced by momentum-like operators  $\propto \Pi^2$ . It is for this reason that the restriction to positive norm wave functions may still be consistent for the full  $E_{10}$  theory — unlike for the standard WDW equation

(but we note that so far no one has succeeded in generalizing the mini-superspace scalar product (11) to the full theory).

Nevertheless, the Hamiltonian (37) is not the complete story because, as in standard canonical gravity, the extended system requires additional constraints. For the  $E_{10}$  model their complete form is not known, but a first step towards their incorporation was taken in [49] where a correspondence was established at low levels between the classical canonical constraints of  $D = 11$  supergravity (in particular, the diffeomorphism and Gauss constraints) on the one hand, and a set of constraints that can be consistently imposed on the  $E_{10}/K(E_{10})$  coset space dynamics on the other (see [50] for more recent results in this direction). The fact that the latter can be cast in a ‘Sugawara-like’ form quadratic in the  $E_{10}$  Noether charges [49] would make them particularly amenable for the implementation on a quantum wavefunction. In addition, one would expect that  $PSL_2(0)$  must be replaced by a much larger ‘modular group’ whose action extends beyond the Cartan subalgebra degrees of freedom all the way into  $E_{10}$ , perhaps along the lines suggested in [51].

As noted above, the inequality (20) implies that  $\Phi(\rho, z) \rightarrow 0$  for  $\rho \rightarrow \infty$ , and hence the wavefunction  $\Psi$  vanishes at the singularity, in such a way that the norm is preserved. Its oscillatory nature entails that it cannot be analytically extended beyond the singularity, a result whose implications for the question of singularity resolution in quantum cosmology remain to be explored. At least formally, these conclusions remain valid in the full theory: the extra contribution in (37) extending  $\mathcal{H}_0$  to the full Hamiltonian  $\mathcal{H}$  are positive, as follows from the manifest positivity of the  $E_{10}$  Casimir operator on the complement of the Cartan subalgebra of  $E_{10}$ . Hence the inequality (20) is further strengthened and thus constitutes a lower bound also for the full Hamiltonian.

To put our results in perspective we recall that the mechanism usually invoked to resolve singularities in canonical approaches to quantum geometrodynamics would be to replace the classical ‘trajectory’ in the moduli space of 3-geometries (that is, WDW superspace) by a quantum mechanical wave functional which ‘smears’ over the singular 3-geometries. By contrast, the present work suggests a very different picture, namely the ‘resolution’ of the singularity via *the effective disappearance (de-emergence) of space-time* near the singularity (see also [48]). The singularity would thus become effectively ‘unreachable’. This behavior is very different from other possible mechanisms, such as the Hartle–Hawking no boundary proposal [52], or cosmic bounce scenarios of the type considered recently in the context of minisuperspace loop quantum cosmology [53, 54, 55], both of which require continuing the cosmic wavepacket into and beyond the singularity at  $\rho = \infty$ . In contrast to these models, the exponentially growing complexity of the  $E_{10}$  Lie algebra suggests that it may turn out to be impossible to ‘resolve’ the quantum equations as one gets closer and closer to the singularity. In other words, there may appear an element of *non-computability* (in a mathematically precise sense) that may forever screen the big bang from a complete resolution.

A key question for singularity resolution concerns the role of observables, and their behavior near the singularity. While no observables (in the sense

of Dirac) are known for canonical gravity, we here only remark that for the  $E_{10}/K(E_{10})$  coset model the conserved  $E_{10}$  Noether charges do constitute an infinite set of observables, as these charges can be shown to commute with the full  $E_{10}$  Hamiltonian (37). The expectation values of these charges are the only quantities that remain well-defined and can be sensibly computed in the deep quantum regime, where the  $E_{10}/K(E_{10})$  coset model is expected to replace space-time based quantum field theory. In the final analysis these charges would thus replace geometric quantities (such as the curvature scalar) which blow up at the singularity, but which are not canonical observables.

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