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QCD-LIKE PROPERTIES OF ANOMALOUS DIMENSIONS IN THE $\mathcal{N}$=4 SUPERSYMMETRIC YANG–MILLS THEORY

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We show how closed formulas for the anomalous dimensions of two classes of operators in the $\mathcal{N}$=4 supersymmetric Yang–Mills theory can be derived either based on numerical investigation or based on QCD-inspired assumptions. We consider the case of twist-3 "gauge" operators for which we completely prove the reciprocity.

Keywords: integrable quantum field theory, Bethe ansatz

1. Introduction

The supersymmetric Yang–Mills (SYM) theory with the maximal supersymmetry $\mathcal{N} = 4$ is dual to the type-IIB superstring theory on $\mathrm{AdS}_5 \times S^5$ and plays a central role in the AdS/CFT correspondence [1]. Recent developments in the study of the duality between the planar large-N limit of the gauge theory and the free string theory are based on the development of analytic tools that use both the integrability of the string theory [2] and an internal integrability of the superconformal theory [3]. In the latter case, the scale dependence of renormalized composite operators is governed (even at higher loops) by a local, integrable, super-spin-chain Hamiltonian whose interaction range increases with the loop order [4], [5]. This allows using the asymptotic Bethe equations [5] to calculate anomalous dimensions of single-trace operators of the general form

$$\mathcal{O} = \mathrm{Tr}\biggl(\prod_{i=1}^{L} D^{n_i} X_i\biggr) + \text{permutations}, \tag{1}$$

where X_i are elementary fields in certain subsectors of the full $\mathcal{N}$=4 SYM theory and D^n are covariant derivatives. The Bethe equations provide the anomalous dimensions of $\mathcal{O}$ as a perturbative series in the 't Hooft coupling g,

$$\gamma_{\mathcal{O}}(g) = \sum_{n\geq 0} c_n(\mathcal{O}) g^{2n}, \tag{2}$$

and allow using the method for complicated multiloop calculations. But the asymptotic nature of these expansions imposes a serious limitation of possible "wrappings," for which $\gamma_{\mathcal{O}}$ is actually calculable up to terms of the order $\mathcal{O}(g^{2L})$ (where L is the length of the operator). While the wrapping problem disappears in the thermodynamic limit in which L or the Lorentz spin $N = \sum_i \{n_i\}$ tends to infinity, a more serious

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limitation appears when attempting to investigate the dependence on the above parameters at a fixed perturbative order. In general, the Bethe equations give not the expansion coefficients c_n in (2) as functions of L and $\{n_i\}$ but just sequences of numerical (sometimes rational) values for each given operator. To find closed formulas for the anomalous dimensions of interest and thus open the way for a deeper investigation of their properties, we must resort either to conjectures that need further numerical confirmation or, as in the case of $\gamma(N)$, to some general (typically QCD-inspired) assumption.

Operators of the form $\mathcal{O}_L = \operatorname{Tr}\mathcal{F}^L$ built of L components of the stress tensor of the self-dual Yang–Mills field strength were considered in [6].[1] Beyond the one-loop approximation, the method proposed in [9] can be used for a sufficiently effective (although not parametric) computation of the sequence $\{c_n(L)\}$ for any given L. It turns out that $c_n(L)$ is not linear in L for $n \geq 2$. Because nontrivial finite corrections appear starting from the two-loop level, the ratio $\gamma(L)/L$ cannot be expected to have a simple expression at a finite L. Nevertheless, a certain unexpected structure arises. A careful investigation of the exact numerical expression for the sequences $\{c_n(L)\}$ at a fixed n with varying L was crucial for conjecturing and deriving closed expressions up to five loops for $c_n(L)$ in (2). Such a closed formula for the multiloop size dependence has no counterpart in existing calculations for other operators in the various subsectors of the $\mathcal{N}$=4 SYM theory. For example, the two-loop anomalous dimension takes the remarkably simple form

$$\gamma_L(g) = 2L + 3Lg^2 + \left(-\frac{51}{8} + \frac{9}{8}\frac{1}{(-1)^L 2^{L-1} + 1}\right)Lg^4 + \cdots \tag{3}$$

with exponentially suppressed corrections to the trivial linear scaling with L. We note that this expression becomes

$$\frac{\gamma_L(g)}{L} = f_0(g) + g^4 h(gL) e^{-L\log 2} + \mathcal{O}(e^{-2L\log 2}) \tag{4}$$

in the large-L limit, which shows that the size corrections to the thermodynamic limit are characterized by a finite g-independent correlation length $\xi = 1/\log 2$ and by the combination gL as a natural scaling variable for the prefactor. Such $\mathcal{O}(2^{-L})$ corrections are not related to the much smaller wrapping effects of the order $\mathcal{O}(\lambda^L)$, and it would be interesting to understand them from the standpoint of the spin-chain interpretation of the dilatation operator H.[2]

In what follows, we focus on a second particularly interesting class of operators, the so-called "quasi-partonic" twist operators [10]. These are single-trace operators of type (1) constructed with an arbitrary number of derivatives (projected on the light cone in this case) distributed among (collinear) twist-1 fundamental fields X (scalars, gauginos, or gauge fields) such that the operator length L coincides with the operator twist. Quasipartonic twist operators are interesting because they are similar to twist operators in QCD. Indeed, although the $\mathcal{N}$=4 SYM theory and QCD differ in many details, a comparative analysis of their properties has been crucial for a deeper understanding of both. Integrability itself, as a basis for the evolution of composite operators, was first discovered in studying planar QCD [11]. Conformal symmetry, unbroken in QCD at the one-loop level, does not seem a necessary condition for integrability, as discussed in [12]–[15], but it plays an important role by imposing selection rules and multiplet structures. Moreover, a (somewhat hidden) consequence of conformal symmetry can explain the structure of the large-spin expansion of the anomalous dimensions of twist operators. It is likely that QCD would benefit significantly from an ultimate all-loop solution of its superconformal version because this would provide a representation

[1] These operators are exact eigenstates of the one-loop dilatation operator and can be mapped to the ferromagnetic states of an integrable spin-1 chain [7], [8]. At the two-loop level and beyond, they mix with the other $psu(2,2|4)$ fields.

[2] In particular, a natural explanation for the exponential corrections could take length-changing processes into account as suggested in [9], and an explicit two-loop calculation of H would be important for clarifying these issues.

for the "dominant" part of the perturbative gluon dynamics[3] (see, e.g., [16]).

The maximum transcendentality principle is a first interesting example of such an interplay between the two theories. It was proposed in [18] based on the structure of the two-loop anomalous dimension of $\mathcal{N}=4$ twist-2 operators in the $sl(2)$ sector that the three-loop answer could be obtained by including the "most transcendental terms" from the three-loop nonsinglet QCD anomalous dimension derived in [19]. The conjectured three-loop formula was independently confirmed in the framework of the Bethe ansatz equations [20] and also with a space–time approach [21]. The principle according to which $\gamma^{(n)}(N)$ at n loops is a linear combination of Euler–Zagier harmonic sums of transcendentality $\tau = 2n - 1$ was the key for deriving closed multiloop N-dependent expressions for the anomalous dimension of special twist operators [20], [22]–[26]. The functions $\gamma(N)$ beyond the one-loop level can be systematically derived using the Baxter approach.[4] Recent analytic attempts were discussed in [27], [28].

Closed expressions for twist anomalous dimensions are crucial for investigating their physical content, which can be extracted using known facts for the QCD twist-2 operators arising in the analysis of deep inelastic scattering [29], [30]. In that context, the total spin N, in Mellin space, is dual to the Bjorken variable x, and two opposite regimes naturally emerge: $x \to 0$ and $x \to 1$. The first regime is captured by the BFKL equation [31] and can be analyzed by considering the Regge poles of $\gamma(N)$ analytically continued to negative (unphysical) spin values. The BFKL equation was the crucial test device for detecting wrapping effects when using the Bethe equations to calculate anomalous dimensions of short operators [22].[5]

Here, we study the properties of the second (quasielastic) regime, which in the Mellin space is equivalent to the large-N limit. These properties can be obtained from the large-N behavior of known three-loop twist-2 QCD results (and also from general results for higher twists [33]). They can be summarized as follows.

1. The leading large-N behavior of anomalous dimensions for twist operators is logarithmic

$$\gamma(N) = 2\Gamma(\alpha_s)\log N + \mathcal{O}(N^0), \quad N \to \infty, \tag{5}$$

and is governed by the so-called cusp anomaly $\Gamma(\alpha_s)$, which is a universal function of the coupling constant related to soft gluon emission [33]–[35] and appears as a cusp anomalous dimension governing the renormalization of a light-cone Wilson loop. Integrability techniques have significantly deepened this knowledge, providing an integral equation that furnishes the all-order weak-coupling expansion[6] of $\Gamma(\alpha_s)$ [40], [41].

2. It is known that the subleading terms satisfy (three-loop) hidden relations, the Moch–Vermaseren–Vogt (MVV) constraints [19]. In the twist-2 QCD case, such relations are connected with the space–time reciprocity of deep inelastic scattering and its crossed version of e^+e^- annihilation into hadrons. The reciprocity in the twist-2 case holds for the Dokshitzer–Marchesini–Salam (DMS) evolution kernel simultaneously governing the distribution and fragmentation functions [42].[7] In [44], the MVV relations were extended to an infinite set of higher-order relations in the $1/N$ expansion, and it was noted that they originate from the invariance under the $sl(2,\mathbb{R})$ subgroup.[8]

[3]Other notable common issues between the $\mathcal{N}=4$ SYM theory and QCD, such as their infrared structure, were reviewed in [17].

[4]This was reported in a talk by S. Zieme at the Albert-Einstein-Institut, Potsdam, based on work in progress by A. V. Kotikov, A. Rej, and S. Zieme.

[5]In the BFKL picture, an interesting interpretation of the spin-chain magnon was recently given in [32].

[6]The calculation was extended to the strong coupling in the explicit case of the $sl(2)$ sector [36] (also see [37]) and allows further generalizations [38], [39].

[7]The DSM evolution kernel was recently confirmed in [43].

[8]Quasipartonic operators can be classified according to representations of the collinear $sl(2,\mathbb{R})$ subgroup of the $SO(2,4)$ conformal group labeled by the so-called conformal spin $j = (N+\Delta)/2$ [10], where Δ is the scaling dimension of the operator. It can hence be concluded that the anomalous dimension $\gamma = \Delta - N - L$ should be a function of the Lorentz spin N only through its dependence on the conformal spin j. Because $\Delta = N + L + \gamma(N,L)$, this then leads to a relation of type (6), where the function $\mathcal{P}$ depends on the twist L.

More specifically, a suitable generalization of the analysis in [42], [44] to the $\mathcal{N}$=4 SYM case assumes that $\gamma(N)$ satisfies the nonlinear equation

$$\gamma(N) = \mathcal{P}\left(N + \frac{1}{2}\gamma(N)\right) \tag{6}$$

at all orders, where the function $\mathcal{P}$ has a large-N expansion in integer powers of J^2 of the form

$$\mathcal{P}(N) = \sum_n \frac{a_n(\log J)}{J^{2n}} \tag{7}$$

with the Casimir J of the collinear conformal subgroup $SL(2,\mathbb{R}) \subset SO(4,2)$, i.e., $J^2 = (N+Ls-1)(N+Ls)$, where $s = 1/2, 1, 3/2$ distinguishes the scalar, spinor, and vector cases [10]. If expansion (7) holds, then we say that $\mathcal{P}$ is a reciprocity-respecting (RR) kernel.[9] Beyond the one-loop level, testing the reciprocity requires knowing the multiloop anomalous dimensions as closed functions of N. These are currently available in the cases of twists 2 and 3. Three-loop tests of reciprocity for QCD and for the universal twist-2 supermultiplet in the $\mathcal{N}$=4 SYM theory were discussed in [42], [44]. The twist-3 anomalous dimension in the $sl(2)$ sector was tested at the four-loop level in [25]. It was proved in [45] that even the wrapping-affected four-loop result for the twist-2 operators [22] is RR in the sense of (7). This certainly suggests some important structure in the Bethe ansatz that deserves deeper understanding. Indeed, while a relation to the underlying conformal symmetry was suggested for (6) in [44], there is no obvious explanation for property (7).

Below, we illustrate the example of a four-loop anomalous dimension obtained in a closed formula using a (generalized) maximum transcendentality principle, and we analyze its structure to verify the RR relations. The example of twist-3 gluonic operators is interesting for various reasons. First, in contrast to the quasipartonic operators built with scalars and gauginos, which belong to closed sectors and therefore scale autonomously at all loops,[10] the description as a gluonic operator is only correct at the one-loop level [47] with mixing effects at higher orders (see the discussion in [26]). Second, in the twist-3 case, operators built with scalars, gauginos, or gauge fields are not related by supersymmetry, in contrast to the twist-2 case, where all channels are in a single supermultiplet.[11] As a consequence, there are various universality classes of anomalous dimensions and also a generalized form of the maximum transcendentality principle in this richer multiplet structure.

We note that it is natural to use the AdS/CFT correspondence to investigate the presence of MVV-like relations at a strong coupling. Because the planar perturbation theory should converge, such an organized structure of subleading terms in the large-spin expansion should also be seen in the energies of the semiclassical string states corresponding to twist operators. This analysis, initiated in [44] for the folded string at the classical level, was recently extended [48] to configurations (spiky strings) that should correspond to twist operators with a higher dimension and at the one-loop level in the string perturbation theory. Remarkably, the large-spin expansion of the classical string energy happens to have exactly the same structure as that of $\gamma(N)$ in the perturbative gauge theory and respects an MVV-like relation at the one-loop level. This strongly indicates that these relations hold in not only the weak-coupling (gauge theory) but also the strong-coupling (string theory) perturbative expansions and confirms that a solid explanation of their origin is needed.

[9] The name *reciprocity* comes from the formulation of this property for the Mellin transform: $\widetilde{P}(x) = -x\widetilde{P}(1/x)$, where $\mathcal{P}(N) = \int_0^1 dx\, x^{S-1}\widetilde{P}(x)$.

[10] Operators built of scalars belong to the $\mathcal{N}$=4 $sl(2)$ subsector, which is closed at all orders. Operators built of gauginos appear in the closed $sl(2|1)$ subsector, where there is mixing between scalars and fermions but not for the maximally fermionic component [46], which is the one of interest for the class of quasipartonic operators.

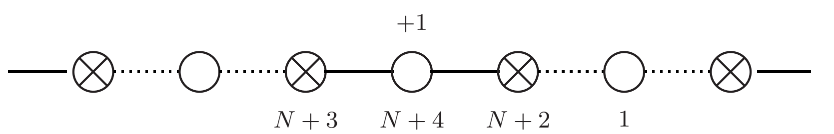
[11] Remarkably, such a twist-2 universality class is inherited in the gaugino sector [24].

2. Analysis and results

2.1. Closed formulas for the anomalous dimension. At the one-loop level, the gluonic sector is described by the $XXX_{-3/2}$ closed spin chain, and the anomalous dimension is known as an exact solution of the Baxter equation. At higher orders, we perturbatively solve the long-range Bethe equations, whose compact form is

$$\left(\frac{u_j + iV_{k_j}/2}{u_j - iV_{k_j}/2}\right)^L = \prod_{\substack{\ell=1,\\ \ell\neq j}}^{K} \frac{u_j - u_\ell + iM_{k_j,k_\ell}/2}{u_j - u_\ell - iM_{k_j,k_\ell}/2}, \tag{8}$$

where $M_{k\ell}$ is the Cartan matrix of the algebra and V_k are the Dynkin labels of the spin representation carried by each site of the chain.[12] The excitation numbers K_i of the Bethe roots u_i can be computed [4] from the quantum numbers of the superconformal state associated with the twist-3 gluonic operator under consideration. To identify the correct superconformal primary describing this sector, we can use the superconformal properties of the (maximally symmetric) tensor product of three singletons [49]. This was done in [26], where the Dynkin diagram associated with the Cartan matrix in (8) for the considered case was found to be



(9)

The number above the diagram indicates the spin representation, and the numbers below are the root excitation numbers of the superconformal primary. Using the one-loop solution as an input,[13] we can expand the Bethe equations in the coupling constant g order by order in the perturbation theory. The equations for the quantum corrections to the one-loop roots are linear and therefore numerically solvable with high precision. The resulting anomalous dimension has rational coefficients in its loop expansions, and these coefficients can be easily and unambiguously identified using the methods discussed in [22], [23]. To find a suitable closed analytic formula for the first loops, we can assume a generalized form of the maximum transcendentality principle for the anomalous dimension. Inspired by the one-loop result [49], where not all terms have a constant degree of transcendentality[14] and by similar QCD calculations [50], we can use the ansatz generalizing the one-loop result

$$\gamma_n = \sum_{\tau=0}^{2n-1} \gamma_n^{(\tau)}, \qquad \gamma_n^{(\tau)} = \sum_{k+\ell=\tau} \frac{\mathcal{H}_{\tau,\ell}(n)}{(n+1)^k}, \qquad n = \frac{N}{2} + 1, \tag{10}$$

where $\mathcal{H}_{\tau,\ell}(n)$ is a combination of harmonic sums with the homogeneous fixed transcendentality ℓ. The terms with $k = 0$ have maximum transcendentality; all the others have subleading transcendentality. The three-loop result was found in [26]. In [45], we computed a long list of values for the four-loop anomalous dimension $\gamma_4(n)$ as exact rational numbers obtained from the perturbative expansion of the long-range Bethe equations. We matched them against general ansatz (10). A very large number of possible terms appear with unknown coefficients. To reduce them, we imposed some structural properties emerging from the analysis of the three-loop result (the complete results can be found in [45]). We here present only the

[12] Together with Eqs. (8), we must consider the additional constraint coming from the cyclicity of the spin chain.
[13] See [26] for an explanation of the necessary (backtraced) dualization of the Bethe roots.
[14] An alternative standpoint is to adopt the maximum transcendentality principle in a related non-canonical basis of harmonic sums.

term with maximal trancendentality[15]

$$\mathcal{H}_{7,7} = \frac{S_7}{2} + 7S_{1,6} + 15S_{2,5} - 5S_{3,4} - 29S_{4,3} - 21S_{5,2} - 5S_{6,1} - 40S_{1,1,5} -$$
$$- 32S_{1,2,4} + 24S_{1,3,3} + 32S_{1,4,2} - 32S_{2,1,4} + 20S_{2,2,3} + 40S_{2,3,2} + 4S_{2,4,1} +$$
$$+ 24S_{3,1,3} + 44S_{3,2,2} + 24S_{3,3,1} + 36S_{4,1,2} + 36S_{4,2,1} + 24S_{5,1,1} + 80S_{1,1,1,4} -$$
$$- 16S_{1,1,3,2} + 32S_{1,1,4,1} - 24S_{1,2,2,2} + 16S_{1,2,3,1} - 24S_{1,3,1,2} - 24S_{1,3,2,1} -$$
$$- 24S_{1,4,1,1} - 24S_{2,1,2,2} + 16S_{2,1,3,1} - 24S_{2,2,1,2} - 24S_{2,2,2,1} - 24S_{2,3,1,1} -$$
$$- 24S_{3,1,1,2} - 24S_{3,1,2,1} - 24S_{3,2,1,1} - 24S_{4,1,1,1} - 64S_{1,1,1,3,1},$$

where $S_a \equiv S_a(n)$, $n = N/2 + 1$, are the nested harmonic sums defined by

$$S_a(N) = \sum_{n=1}^{N} \frac{1}{n^a}, \qquad S_{a,\mathbf{b}}(N) = \sum_{n=1}^{N} \frac{1}{n^a} S_{\mathbf{b}}(n). \tag{11}$$

2.2. Reciprocity-respecting formulas. Proving the reciprocity for the gluonic operators amounts to first deriving the $\mathcal{P}$ function by inverting relation (6), which in terms of the perturbative expansions $\mathcal{P} = \sum_{k=1}^{\infty} \mathcal{P}_k g^{2k}$ and $\gamma = \sum_{k=1}^{\infty} \gamma_k g^{2k}$ eventually becomes

$$\mathcal{P}_1 = \gamma_1, \qquad \mathcal{P}_2 = \gamma_2 - \frac{1}{8}(\gamma_1^2)', \qquad \ldots. \tag{12}$$

We must then verify parity invariance (7) with respect to the quadratic Casimir, which in this case is

$$J^2 = N^2 + 8N + \frac{63}{4} = 4n(n+2) + \frac{15}{4}. \tag{13}$$

The constant in (13) is irrelevant to the proof, and we can define an effective Casimir $J^2_{\text{eff}} = n(n+2)$. Remarkably, a complete proof of reciprocity at the four-loop level can be given in closed form. For this, the following observations are useful (see [45] for their proofs).

Theorem 1. *Let $f(n)$ be RR with respect to $J^2 = n(n+1)$. Then the combination $\tilde{f}(n) = f(n) + f(n+1)$ is RR with respect to J^2_{eff}.*

We then consider the *linear map* defined on linear combinations of simple S sums by

$$\Phi_a(S_{b,\mathbf{c}}) = S_{a,b,\mathbf{c}} - \frac{1}{2} S_{a+b,\mathbf{c}},$$

and set

$$I_a = S_a, \qquad I_{a_1,a_2,\ldots,a_n} = \Phi_{a_1}(\Phi_{a_2}(\cdots \Phi_{a_{n-1}}(S_{a_n}))).$$

Theorem 2. *The combinations $I_{a_1,\ldots,a_n}$ with odd $a_1, \ldots, a_n$ have a large-N RR expansion*

$$I_{a_1,\ldots,a_n} = \sum_{\ell=0}^{\infty} \frac{P_\ell(\log J^2)}{J^{2\ell}}, \tag{14}$$

where $J^2 = N(N+1)$ and P_ℓ is a polynomial.

[15]The next transcendentality-7 term comes from the contribution of the so-called dressing factor in the Bethe equations and consists of a combination of harmonic sums of transcendentality 4 multiplied by the characteristic ζ_3 contribution.

These observations can be used to write the function $\mathcal{P}$ in (12) in a manifestly RR form. For example, at the three-loop level, we have

$$\mathcal{P}_3 = \frac{\tilde{I}_3}{2(n+1)^2} + \frac{3\tilde{I}_5}{2} - 4\tilde{I}_{1,1,3} + \frac{2}{(n+1)^4} - 4\tilde{I}_{1,3} + \frac{\pi^2\tilde{I}_3}{6} - 2\tilde{I}_3 + 4\tilde{I}_{1,1}\zeta_3 - \\ - \frac{\zeta_3}{(n+1)^2} - \frac{4}{(n+1)^2} + 4\zeta_3\tilde{I}_1 + \frac{4\pi^4\tilde{I}_1}{45} + 4\zeta_3 + \frac{8\pi^4}{45} + \frac{4\pi^2}{3} + 32, \tag{15}$$

where $\tilde{I}_{\boldsymbol{a}} \equiv \tilde{I}_{\boldsymbol{a}}(n) = \tilde{I}_{\boldsymbol{a}}(n) + \tilde{I}_{\boldsymbol{a}}(n+1)$. As explained in detail in [45], the reciprocity is obvious because formula (15) is a combination of invariants $\tilde{I}_a$ (such as in Theorem 2)[16] and factors $(n+1)^{-p}$ with even p.[17] In this case, the expression is automatically RR with respect to $n(n+2)$. Analogous manifestly RR expressions for $\mathcal{P}$ up to the four-loop level were obtained in [45].

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