A Factorization Law for Entanglement Decay

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We present a simple and general factorization law for quantum systems shared by two parties, which describes the time evolution of entanglement upon passage of either component through an arbitrary noisy channel. The robustness of entanglement-based quantum information processing protocols is thus easily and fully characterized by a single quantity.

Whenever we contemplate the potential technological applications of quantum information theory [1], from secure quantum communication over quantum teleportation [2] to quantum computation [3], we need to worry about the unavoidable and detrimental coupling of any such quantum device to uncontrolled degrees of freedom – typically lumped together under the label "environment". Environment coupling induces decoherence [4–6], i.e., it gradually destroys the phase relationship between quantum states, and thus their ability to interfere. In composite quantum systems, these phase relationships (or "coherences") are at the origin of strong quantum correlations between measurements on distinct system constituents – which then are *entangled*. The promises of quantum information technology rely on exploring precisely these non-classical correlations.

Yet, entanglement is *not* equivalent to many-particle coherences: it is an even stronger property, and hard to quantify – all commonly accepted entanglement measures [7, 8] are *nonlinear* functions of the density matrix which describes the state of the composite quantum system, and in particular the coherences. While an elaborate theory on the time evolution of quantum states under environment coupling is at our hands, virtually no general results on entanglement dynamics have been stated. Hitherto, the time evolution of entanglement always needed to be *deduced* from the time evolution of the state [9–18]. In the present Letter, a *direct* relationship between the initial and final entanglement of an arbitrary bipartite state of two qubits (basic units of quantum information) subject to incoherent dynamics in one system component is derived, which, as illustrated in Fig. 1, renders the solution of the corresponding state evolution equation obsolete. Our result can be directly applied to input/output processes, such as gates used in sequential quantum computing. Moreover, it allows to infer the evolution of entanglement under certain time-continuous influences of the environment, e.g. phase- and amplitude damping.

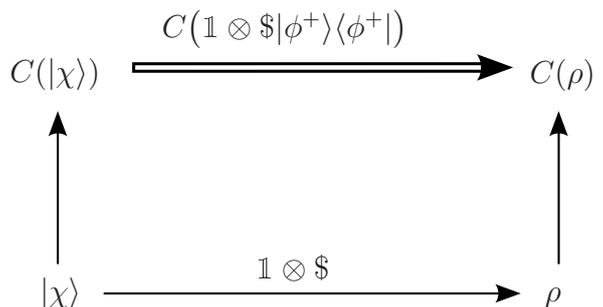Let us consider entangled states of qubit pairs, with

FIG. 1: From state evolution to entanglement dynamics. Hitherto, the time evolution of entanglement C under open system dynamics had to be deduced from the time evolution of the state $|\chi\rangle$. However, it turns out that knowlegde of the entanglement evolution of the maximally entangled state $|\phi^+\rangle$ under the channel $\mathbb{1} \otimes \$$ suffices to establish a direct mapping of $C(|\chi\rangle)$ onto $C(\rho)$, without detour over the state.

one qubit being subject to an arbitrary channel $\$$ – which may represent the influence of an environment, of a measurement, or of both. In order to illustrate the situation, we consider a source which emits a particle to the left and another one to the right. Each particle on its own carries one qubit of quantum information (in general a superposition or mixture of two basis states $|0\rangle$ and $|1\rangle$). We therefore also refer to the particles as "left" and "right" qubit. Let the particles leaving the source be in a pure state $|\chi\rangle$:

$$|\chi\rangle = \sqrt{\omega}|00\rangle + \sqrt{1-\omega}|11\rangle \ , \tag{1}$$

with $0 \leq \omega \leq 1$, i.e., for values of ω between zero and one the particles are in a coherent superposition of both being in state $|0\rangle$ and both being in state $|1\rangle$. Any pure state can be written in this form, modulo local unitary operations. Since our results are not affected by these local unitaries, as we will see below, this choice of the pure initial state does not impose a restriction of generality.
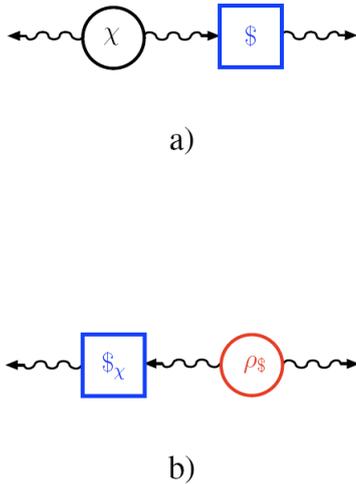
a)

b)

FIG. 2: Entanglement decay scenarios. a) Laboratory scenario: one of the qubits of the initial state $|\chi\rangle$, the "right" one, undergoes the action of a general quantum channel $. b) Dual scenario: the same end result is obtained by interchanging the role of states and channels: the "left" qubit of a mixed state $\rho_\$$ undergoes the action of the quantum channel $\$_\chi$. Circles represent sources of entangled states, squares symbolize channels.

We quantify entanglement by concurrence C [19], which implies for the state $|\chi\rangle$: $C(|\chi\rangle) = 2\sqrt{\omega(1-\omega)}$. For ω in (1) equal to zero or one, the state's entanglement and hence its concurrence vanishes; $\omega = 1/2$ implies $|\chi\rangle = |\phi^+\rangle$, one of the maximally entangled Bell states, with maximal concurrence one.

Now the right qubit traverses an arbitrary quantum channel $, as illustrated in Fig. 2a), and we want to derive the qubits' entanglement hereafter. To do so, note that the qubits' final state must be the same as in a dual picture [20], where the roles of initial state and channel are interchanged, as depicted in Fig. 2b). Thus, the two-qubit state $|\chi\rangle$ is identified with a qubit channel $\$_\chi$, and the qubit channel $ with a two-qubit state $\rho_\$$; symbolically:

$$\frac{(\mathbb{1} \otimes \$)\,|\chi\rangle\langle\chi|}{p'} = \frac{(\$_\chi \otimes \mathbb{1})\,\rho_\$}{p} .\qquad (2)$$

Here, $p' = \mathrm{Tr}[(\mathbb{1} \otimes \$)\,|\chi\rangle\langle\chi|]$ and $p = \mathrm{Tr}[(\$_\chi \otimes \mathbb{1})\,\rho_\$]$ are the probabilities for channels $ and $\$_\chi$ to act on the states $|\chi\rangle\langle\chi|$ and $\rho_\$$, respectively. Thus, we also account for non-trace-preserving channels, where the particle number is not conserved.

We now need to determine $\$_\chi$ and $\rho_\$$ explicitly. For this purpose, we first remember that Quantum Teleportation [2] is a means to transfer the state of one system to another one, in principle with perfect fidelity. Consequently, teleporting the right qubit of the state $|\chi\rangle$ assisted by the maximally entangled state $|\phi^+\rangle$ leaves the
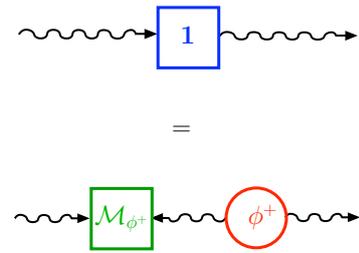

FIG. 3: Quantum Teleportation identity. The teleportation protocol transfers the incoming state from the left to the outgoing state on the right [23]. The procedure consists of a composite Bell measurement on the incoming qubit and on one qubit of an auxiliary, maximally entangled state $|\phi^+\rangle$. We restrict to the case where the measurement results in a projection $\mathcal{M}_{\phi^+}$ on the state $|\phi^+\rangle$.

state $|\chi\rangle$ invariant. This invariance is depicted in Fig. 3. We therefore obtain the same final state as in the situation we considered so far (Fig. 2a)) – if we replace the source preparing state $|\chi\rangle$ by a source which prepares $|\chi\rangle$ followed by a teleportation of the right qubit as shown in Fig. 4. Let us now consider the source of the qubit pair in state $|\phi^+\rangle$, which we inserted with the teleportation. The succession of processes influencing the left qubit and those acting on the right qubit of the pair can be altered without consequences for the final state. For this reason we can replace the source producing the state $|\phi^+\rangle$ together with the channel $ acting on the right qubit by yet another source which immediately prepares the state

$$\rho_\$:= (\mathbb{1} \otimes \$)|\phi^+\rangle\langle\phi^+|/p'' ,\qquad (3)$$

where $p'' = \mathrm{Tr}[(\mathbb{1} \otimes \$)|\phi^+\rangle\langle\phi^+|]$, see Fig. 4. The resulting scheme in Fig. 5 transfers entanglement between the qubit pairs prepared in states $|\chi\rangle$ and $\rho_\$$ to entanglement between the left qubit of the first pair and the right qubit of the second pair. This scheme is called entanglement swapping [2, 21, 22].

Finally, we define $\$_\chi$ to be the channel corresponding to the change of the left qubit of $\rho_\$$ in Fig. 5, which includes a projection $\mathcal{M}_{\phi^+}$ of the left qubit of state $\rho_\$$ and the right qubit of $|\chi\rangle$ on $|\phi^+\rangle$. Channel $\$_\chi$ can be interpreted as imperfect teleportation assisted by state $|\chi\rangle$, leaves the resulting state in general non-normalized, and can be expressed in the particularly simple form:

$$(\$_\chi \otimes \mathbb{1})\,\rho_\$ = (M \otimes \mathbb{1})\,\rho_\$\,(M^\dagger \otimes \mathbb{1}) ,\qquad (4)$$

with $M = \left(\sqrt{\omega}|0\rangle\langle0| + \sqrt{1-\omega}|1\rangle\langle1|\right)/\sqrt{2}$. The normalized final state $(\$_\chi \otimes \mathbb{1})\rho_\$/p$ is the same as $(\mathbb{1} \otimes \$)|\chi\rangle\langle\chi|/p'$, as spelled out by (2) and in Fig. 2, but the entanglement evolution induced by the particular channel $\$_\chi$ can be deduced more easily, as we will now demonstrate. The concurrence C of the final state
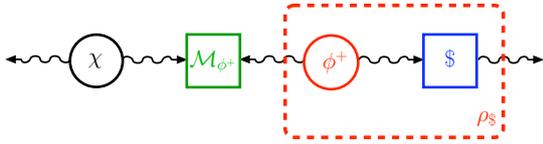
FIG. 4: Building in the teleportation identity. The right qubit of $|\chi\rangle$ undergoes the action of a quantum channel $\$$, after intermediate teleportation. The maximally entangled state $|\phi^+\rangle$ together with the action of the channel $\mathbb{1} \otimes \$$ yield the source of the mixed state $\rho_\$$.
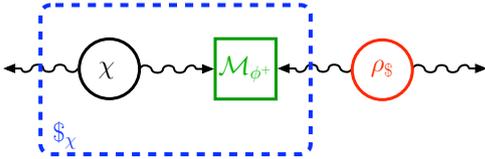

FIG. 5: Entanglement swapping. Transfer of entanglement between the qubit pairs of $|\chi\rangle$ and $\rho_\$$, respectively, to entanglement between the outgoing pair of qubits. The Bell measurement $\mathcal{M}_{\phi^+}$ together with the source of the entangled state $|\chi\rangle$ constitute the quantum channel $\$_\chi$ for the left qubit of state $\rho_\$$.

$\rho' = (\mathbb{1} \otimes \$)|\chi\rangle\langle\chi|/p'$ is given by

$$C(\rho') = \max\left\{0, \sqrt{\xi_1} - \sqrt{\xi_2} - \sqrt{\xi_3} - \sqrt{\xi_4}\right\}, \quad (5)$$

where the ξ_i are the eigenvalues of the matrix $\rho' \cdot \tilde\rho'$, in decreasing order, with $\tilde\rho' = (\sigma_y \otimes \sigma_y) \cdot \rho'^* \cdot (\sigma_y \otimes \sigma_y)$, and ρ'^* the complex conjugate of ρ', in the canonical basis. In order to evaluate this expression, we use relation (2) together with (4). We write explicitly:

$$\rho' \cdot \tilde\rho' = \frac{1}{p^2}(M \otimes \mathbb{1})\rho_\$ \cdot [M\sigma_y M \otimes \sigma_y] \cdot \rho_\$^* \cdot [M\sigma_y \otimes \sigma_y], \quad (6)$$

where we employed that $M = M^\dagger = M^*$. For invertable M [24], it follows that the eigenvalues of $\rho' \cdot \tilde\rho'$ and $\rho_\$ \cdot \tilde\rho_\$$ are proportional, since

$$
\begin{aligned}
&\det\left[\rho' \cdot \tilde\rho' - \xi\mathbb{1}\right] \\
&= \det\left[(M \otimes \mathbb{1})^{-1}\right]\det\left[M \otimes \mathbb{1}\right]\det\left[\rho' \cdot \tilde\rho' - \xi\mathbb{1}\right] \\
&= \det\left[(M \otimes \mathbb{1})^{-1}\rho' \cdot \tilde\rho'(M \otimes \mathbb{1}) - \xi\mathbb{1}\right] \\
&= \frac{1}{4p^2}\omega(1-\omega)\det\left[\rho_\$ \cdot \tilde\rho_\$ - \xi\mathbb{1}\right],
\end{aligned}
\quad (7)
$$

where we used $M\sigma_y M = \sqrt{\omega(1-\omega)}\sigma_y/2$ in order to obtain the last equality. Eq. (5), together with the definitions of $\rho_\$ = (\mathbb{1} \otimes \$)|\phi^+\rangle\langle\phi^+|/p''$, $\rho' = (\mathbb{1} \otimes \$)|\chi\rangle\langle\chi|/p'$, and $C(|\chi\rangle) = 2\sqrt{\omega(1-\omega)}$, thus lead [25] to our central result:

$$C\left[(\mathbb{1} \otimes \$)|\chi\rangle\langle\chi|\right] = C\left[(\mathbb{1} \otimes \$)|\phi^+\rangle\langle\phi^+|\right]C(|\chi\rangle) \quad (8)$$

– the entanglement reduction under a one-sided noisy channel is *independent* of the initial state $|\chi\rangle$ and *completely determined* by the channel's action on the maximally entangled state. Thus, if we know the time evolution of the Bell state's entanglement, we know it for *any* pure initial state [26]. This result can also be interpreted in terms of entanglement swapping between a pure state $|\chi\rangle$ and a mixed state $\rho_\$$, leading to the final state ρ', due to the equivalence of the processes represented in Figs. 2 and 5.

The factorization law (8) can be generalized for mixed initial states ρ_0, by virtue of the convexity of entanglement monotones such as concurrence, and given an optimal pure state decomposition $\rho_0 = \sum_j p_j|\psi_j\rangle\langle\psi_j|$, in the sense that the average concurrence over this pure state decomposition is minimal [27]. It then immediately follows, by convexity, that $C\left[(\mathbb{1} \otimes \$)\rho_0\right] = C\left[\sum_j p_j(\mathbb{1} \otimes \$)|\psi_j\rangle\langle\psi_j|\right] \leq \sum_j p_j C\left[(\mathbb{1} \otimes \$)|\psi_j\rangle\langle\psi_j|\right]$, and application of (8) leaves us with

$$C\left[(\mathbb{1} \otimes \$)\rho_0\right] \leq C\left[(\mathbb{1} \otimes \$)|\phi^+\rangle\langle\phi^+|\right]C(\rho_0). \quad (9)$$

This inequality holds for all one-sided channels $\$$, and has an immediate generalization for local two-sided channels $\$_1 \otimes \$_2 = (\$_1 \otimes \mathbb{1})(\mathbb{1} \otimes \$_2)$:

$$
\begin{aligned}
C\left[(\$_1 \otimes \$_2)\rho_0\right] \leq\ & C\left[(\$_1 \otimes \mathbb{1})|\phi^+\rangle\langle\phi^+|\right] \\
& \times C\left[(\mathbb{1} \otimes \$_2)|\phi^+\rangle\langle\phi^+|\right]C(\rho_0).
\end{aligned}
\quad (10)
$$

The concurrence after passage through a two-sided channel is thus bounded from above, which immediately implies a sufficient criterion for finite-time disentanglement [9, 10, 13] of *arbitrary initial states*, in terms of the evolution of the concurrence of the maximally entangled state under either one of the one-sided channels (e.g., choose $\$_1$ or $\$_2$ induced by infinite temperature or depolarizing environments).

Let us finally identify relevant cases when equality in (9) holds. For that purpose, we consider mixed states that are obtained after the application of a one-sided channel to an arbitrary pure state, $\rho_0 = (\mathbb{1} \otimes \$)|\psi_0\rangle\langle\psi_0|$. This occurs, for instance, if the qubit originally prepared in a pure state suffers amplitude decay, and the resulting mixed state again is subject to decay dynamics. This is tantamount to the concatenation of channels on one side, $(\mathbb{1} \otimes \$_2)(\mathbb{1} \otimes \$_1)|\psi_0\rangle\langle\psi_0|$, what can be lumped together as one channel which combines both actions, $(\mathbb{1} \otimes \$_2)(\mathbb{1} \otimes \$_1) = \mathbb{1} \otimes \$_{2,1}$. In a similar vein as for (10), and using the factorization relation (8) for pure states, we deduce

$$
\begin{aligned}
C\left[(\mathbb{1} \otimes \$_{2,1})|\phi^+\rangle\langle\phi^+|\right] \leq\ & C\left[(\mathbb{1} \otimes \$_2)|\phi^+\rangle\langle\phi^+|\right] \\
& \times C\left[(\mathbb{1} \otimes \$_1)|\phi^+\rangle\langle\phi^+|\right].
\end{aligned}
\quad (11)
$$

The initial state's concurrence rescales both sides of the equation by the same amount, and therefore it is omitted in above equation. It is now sufficient to investigate

the time dependence of the maximally entangled state's concurrence under the concatenated channels (much as for the evaluation of (8)): if all of these are of the form $C(t) = \exp(-\Gamma t)$ (which is the case, e.g., for $\$_1$ an amplitude decay and $\$_2$ a dephasing channel), then equality holds in (9), with $\$ = \$_1$, $\rho_0 = (\mathbb{1} \otimes \$_2)|\psi_0\rangle\langle\psi_0|$, and $C(\rho_t) = \exp(-\Gamma_1 t)C(\rho_0)$ (and, equivalently, for the roles of channels 1 and 2 interchanged).

In conclusion, equations (8,9,10,11) provide us with the first closed expression for the time evolution of a bipartite entangled state under general local, single- and two-sided channels, without recourse to the time evolution of the underlying quantum state itself. This is a general result inherited from the Jamiołkowski isomorphism [20], which is here "lifted" *from state to entanglement evolution* (see Fig. 1), and eases the experimental characterization of entanglement dynamics under unknown channels dramatically: instead of exploring the time-dependent action of the channel on *all* initial states, it suffices to probe the entanglement evolution of the maximally entangled state alone.

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[23] Here only the measurement of $|\phi^+\rangle$ is used. The remaining three possible measurement outcomes would imply local operations on the outgoing qubit, which can be incorporated in the action of the channel $\$$.
[24] This can be assumed without loss of generality, since otherwise, for $\omega = 0$ or $\omega = 1$, and hence $M\sigma_y M = 0$, concurrence vanishes.
[25] In this last equation we used that $C(\alpha\rho) = \alpha C(\rho)$ for positive real α and $p' = 4p\,p''$. The latter can be derived straightforwardly, by careful comparison of the individual normalization factors. In order to compute the entanglement of the normalized final state $C\left[(\mathbb{1} \otimes \$)\,|\chi\rangle\langle\chi|\right]$ has to be divided by p'.
[26] Note that our choice (1) of $|\chi\rangle$ does not restrict the generality of relation (8). An initial state $|\chi'\rangle\langle\chi'| = (\mathcal{U}_a \otimes \mathcal{U}_b)|\chi\rangle\langle\chi|$ leads to the same result (8) since unitary operation $\mathcal{U}_a$ does not affect the concurrence of the final state and unitary operation $\mathcal{U}_b$ can be incorporated into the channel: $C\left[(\mathbb{1} \otimes \$)\,|\chi'\rangle\langle\chi'|\right] = C\left[(\mathbb{1} \otimes \$\mathcal{U}_b)\,|\chi\rangle\langle\chi|\right] = C\left[(\mathbb{1} \otimes \$\mathcal{U}_b)\,|\phi^+\rangle\langle\phi^+|\right]\,C(|\chi\rangle) = C\left[(\mathbb{1} \otimes \$)\,|\phi^+\rangle\langle\phi^+|\right]\,C(|\chi'\rangle)$.
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