On static solutions of the Einstein - Scalar Field equations

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In this note we study the Einstein-ScalarField static equations in arbitrary dimensions. We discuss the existence of geodesically complete solutions depending on the form of the scalar field potential $V(\phi)$, and provide full global geometric estimates when the solutions exist. As a special case it is shown that when $V(\phi)$ is the Klein-Gordon potential, i.e. $V(\phi)=m^2|\phi|^2$, geodesically complete solutions are necessarily Ricci-flat, have constant lapse and are vacuum, (that is $\phi=\phi_0$ with $\phi_0=0$ if $m\neq 0$). Hence, if the spatial dimension is three, the only such solutions are either Minkowski or a quotient thereof. For $V(\phi)=m^2|\phi|^2+2\Lambda$, that is, including a vacuum energy or a cosmological constant, it is proved that no geodesically complete solution exists when $\Lambda>0$, whereas when $\Lambda<0$ it is proved that no non-vacuum geodesically complete solution exists unless $m^2<-2\Lambda/(n-1)$, (n is the spatial dimension) and the manifold is non-compact.

I. INTRODUCTION

Matter models that use scalar fields are among the most studied in theoretical physics, and there are good reasons for this to be. The Klein-Gordon equation, for instance, which is given by

$$\nabla_\mu\nabla^\mu\phi=m^2\phi \tag{1}$$

is perhaps the simplest equation for a free-relativistic massive particle that one can think of. More generally than this, one can consider the equations

$$\nabla_\mu\nabla^\mu\phi=m^2\phi+W(\phi) \tag{2}$$

which, depending on the form of $W(\phi)$, can incorporate essentially any type of self interaction while keeping the relativistic structure transparent. Simple form and high diversity is what has characterised models based on scalar fields and has made them very appealing.

Early studies of equations like (2), or (2) coupled to Einstein gravity, paid much attention on the existence of *solitons*. These are static solutions that, because they are localised and everywhere regular, were thought as good candidates for a notion of 'particle'. While this motivation is by now abandoned, the search of solitons in various field theories has found renewed theoretical interests. Einstein-YangMills-Higgs, Einstein-Inflaton and Einstein-ScalarField are examples of systems that, either for motivations in particle physics, cosmology, or black-hole theory, deserve still considerable attention. Reviews on many of these topics can be found in [3] and more recently in [9]. For solitons of the Einstein-ScalarField system, usually called *scalerons*, the reader is referred to [7].

In general terms, this article studies the existence of *geodesically complete* static solutions of the Einstein-ScalarField equations, (that one may naturally call *generalised solitons*), depending on the form of the potential $V(\phi)$ (see the next section for the setup). We do not make any dimensional, global, or asymptotic assumption (like asymptotic flatness) and in this sense several of the conclusions of this paper are the most general they can be.

In basic terms, studying the existence of geodesically complete solutions[1] is important for the following simple reasons. First, (if they are proven to exist), geodesically complete solutions are everywhere regular and inextensible and in this sense they should be considered, (at least in principle), as physically acceptable. Second, and perhaps most importantly, in many contexts nontrivial solutions of this type are indeed proven not to exist ('no-go' theorems). If this is the case, then the *rough* but crucial conclusion that one reaches is that *any* nontrivial static solution, being necessarily geodesically incomplete, must contain either a horizon or a singularity[2].

The results of this article are based on *techniques in comparison-geometry* á la Backry-Emery [10] that to my knowledge were put forward, in a closely related context, first by Case [5], [3]. We opted here to introduce only a main technical Lemma (Lemma III.2 adapted from [5]) from which all the results are deduced. As a main application we discuss thoroughly the special case of the Klein-Gordon field, in the presence, or not, of a cosmological constant. We will also give other less elaborated applications to illustrate the usefulness of the technique. Concerning the main application, what we show is the following. When $V(\phi)$ is the Klein-Gordon potential, i.e. $V(\phi)=m^2|\phi|^2$, geodesically complete solutions are necessarily Ricci-flat, have constant lapse N, and are vacuum, that is $\phi=\phi_0$ with $\phi_0=0$ if $m\neq 0$, (§ Theorem IV.1). Therefore, if the spatial dimension is three, the only such solutions are either Minkowski or a quotient

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[1] We consider only here the geodesic completeness of the spatial-slice where the data set lives.

[2] For instance, the horizon of a Schwarzschild solution of positive mass, and the naked singularity of a Schwarzschild solution of negative mass can be understood as necessary in this way.

[3] It is noteworthy to recall that this type of technique has played also a crucial role in the study of the Ricci flow.

thereof (for vacuum static solutions this was proved by Anderson in [1] by other techniques; see also [5]). When $V(\phi) = m^2|\phi|^2 + 2\Lambda$, that is, including a vacuum energy or a cosmological constant, we prove that no geodesically complete solution exists when $\Lambda > 0$, whereas when $\Lambda < 0$ it is proved that no non-vacuum geodesically complete solution exists unless $m^2 < -2\Lambda/(n-1)$ and unless the manifold is non-compact, (§ Theorem IV.2 and Theorem IV.3). Moreover, in this case, we provide the pointwise estimate $|\nabla\phi|^2 + m^2|\phi|^2 \leq -68\Lambda$ for the energy density (§ Theorem IV.3), the pointwise estimate $|\nabla N|/N \leq 64\sqrt{-\Lambda}$ for the gradient of the lapse (§ Theorem IV.4), and, when the spatial dimension is three, we prove a general pointwise bound on the curvature in terms of $|\Lambda|$ (§ Theorem IV.5). When the spatial dimension is three, vacuum solutions (i.e. $\phi = 0$) other than AdS, were shown to exist by Anderson in [2]. Even in the vacuum case, the pointwise estimates that we obtain seem to be new. The estimates could be useful in theories that study spaces asymptotic to AdS, with or without a scalar field.

One could easily guess that, by following either the line of argument used here for the Klein-Gordon field or by taking other original paths, many other new applications of Lemma III.2 could be found. A bit of this we hoped to convey when we incorporated the Sections IV C and IV D to the text. In Section IV C we enumerate briefly a series of conclusions (mostly 'no-go' theorems) that one can easily reach for the Einstein-*Real*ScalarField system, for several different types of potentials $V(\phi)$ (including the (real)-Sine-Gordon and the (real)-Higgs potentials). In Section IV D instead, we make a simple application to the *Klein-Gordon geons* (as coined by Kaupt [6]). These are, roughly speaking, eigenstates of the Einstein-KleinGordon system where the spacetime metric is static but the scalar field oscillates periodically in time. Spherically symmetric solutions of this type have been studied exhaustively by many authors and are called Mini Boson stars (see [4] and ref. therein). What we prove in Section IV D is that the frequency of oscillation $|\omega|$ of geodesically complete Klein-Gordon geons, must be strictly greater than the boson mass m. For Mini Boson stars, this property was first observed (and proved) by Bizoń and Wasserman ([4], § Lemma 2.5).

The organisation of the article is as follows. In Section II we recall the main static equations of the Einstein-ScalarField system, together with the notation and the terminology. Subsection II A explains the type of manifolds used during the paper, and that have to be read with care to avoid confusion. The main technical estimates to be used in applications are obtained in Section III. All the applications in Section IV are deduced from a main but simple observation that is perhaps worth to mention here it yields light to the nature of the technical Section III. The key point is that, using the static equations and the Böchner type of equation (27), one can obtain expressions for the f-Laplacian ($f = -\ln N$, see next section) of $\psi = |\nabla\phi|^2$ and of $\psi = |\nabla \ln N|^2$ of the form

$$\Delta_f \psi \geq b\psi + c\psi^2, \tag{3}$$

with $b \leq 0$ and $c > 0$. What we do in Section III is to prove that fundamental pointwise estimates can be obtained for such a ψ if the f-Ricci tensor, or Backry-Emery tensor, Ric^1_f (see next section) is bounded below. We obtain thus fundamental 'gradient' estimates for ϕ and $\ln N$ from which all the physical conclusions will follow. The applications are discussed in Section IV which is divided in four subsections: Subs. IV A discusses the Klein-Gordon case (main application), Subs. IV B discusses the Klein-Gordon case in the presence of a cosmological constant, and Subs. IV C and IV D discuss, as we mentioned earlier, applications to real scalar fields and to Klein-Gordon geons respectively.

II. THE STATIC EQUATIONS

We give below the static equations of the Einstein-ComplexScalarField system in spacetime-dimension $n+1$, ($n \geq 2$). We use the following notation: (i) ϕ is the complex scalar field and $\bar{\phi}$ the complex conjugate (ii) ϕ_R is the real part of ϕ and ϕ_I the imaginary part (iii) $|\phi|$ is the norm of ϕ and $|\nabla\phi|$ is the norm of $\nabla\phi$ (i.e. $|\nabla\phi|^2 = \langle\nabla\phi, \nabla\bar{\phi}\rangle$). The potentials that we will consider are of the form $V(\phi_R, \phi_I)$. We will use the shorthand $V(\phi)$. The spacetime metric is assumed to split as $\mathbf{g} = -N^2dt^2 + g$, and the metric g, as well as the lapse $N > 0$, live in a n-dimensional manifold Σ. The relevant data is thus $(\Sigma; N, g; \phi)$.

The static Einstein-(Complex)ScalarField equations are,

$$Ric + \nabla\nabla f - \nabla f \nabla f = \nabla\phi \circ \nabla\bar{\phi} + \frac{V(\phi)}{n-1} g, \tag{4}$$

$$\Delta f - \langle \nabla f, \nabla f \rangle = \frac{V(\phi)}{n-1}, \tag{5}$$

$$\Delta\phi - \langle \nabla f, \nabla\phi \rangle = \frac{\partial V(\phi)}{2}, \tag{6}$$

where $f = -\ln N$, $\nabla\phi \circ \nabla\bar{\phi} = (\nabla\phi\nabla\bar{\phi} + \nabla\bar{\phi}\nabla\phi)/2 = \nabla\phi_R\nabla\phi_R + \nabla\phi_I\nabla\phi_I$, and ∂V is

$$\partial V = \frac{\partial V}{\partial\phi_R} + i\frac{\partial V}{\partial\phi_I} \tag{7}$$

These equations imply, in turn, the following expression for the scalar curvature R (or energy density),

$$R = |\nabla\phi|^2 + V(\phi). \tag{8}$$

The system (4)-(5) arises as the *static* Euler-Langrange equations of the $n+1$-dimensional (spacetime) action[4]

$$\mathcal{S}(\mathbf{g},\phi)=\int\left[R_{\mathbf{g}}-\nabla_\mu\phi\nabla^\mu\bar{\phi}-V(\phi)\right]d\mathrm{v}_{\mathbf{g}} \tag{9}$$

or, also, as the Euler-Lagrange equations of the n-dimensional (spatial) action

$$\mathcal{S}(f,g,\phi)=\int\left[R-|\nabla\phi|^2-V(\phi)\right]e^{-f}d\mathrm{v} \tag{10}$$

A few times below we will use, following [10], the notation

$$Ric^1_f := Ric+\nabla\nabla f-\nabla f\nabla f \tag{11}$$

and

$$\Delta_f\psi=\Delta\psi-\langle\nabla f,\nabla\psi\rangle=0 \tag{12}$$

A. Manifolds

Without any explicit specification, a '*manifold* Σ' is allowed to have boundary or to be boundaryless, and to be compact or non-compact. Whatever the case, (Σ,g) is assumed metrically complete with respect to the standard metric

$$\mathrm{dist}(p,q)=\inf\{\mathrm{length}(\gamma_{pq}):\gamma_{pq}\in\mathcal{C}_{pq}\} \tag{13}$$

where $\mathcal{C}_{pq}$ is the set of smooth curves joining p to q. Hence, if Σ is boundaryless then (Σ,g) is *geodesically complete* by Hopf-Rinow. On the other hand if Σ has boundary then (Σ,g) is *geodesically incomplete* as geodesics can terminate at the boundary. Henceforth, when we say '(Σ,g) *is geodesically complete*', we are saying implicitly that Σ is boundaryless, no matter if Σ is compact or not.

These conventions have to be kept in mind to prevent confusion. For example, the de-Sitter metric

$$g=\left(\frac{1}{1-\Lambda r^2/3}\right)dr^2+r^2d\Omega^2_{n-2},\ N=\sqrt{1-\frac{\Lambda r^2}{3}} \tag{14}$$

is a solution of the static Einstein equations with a positive cosmological constant, although we will show later that there is no such solution which is geodesically complete. The point here is that the de-Sitter solution is defined on a manifold with boundary (the cosmological horizon), hence geodesically incomplete.

It is worth stressing once more that a non-existence result of geodesically complete solutions is important because it says that any inextensible solution (necessarily geodesically incomplete) has always, roughly speaking, either a horizon or a singularity.

[4] Inserting constants in front of R, or $\nabla_\mu\phi\nabla^\mu\phi$ does not change the analysis of this article.

III. THE TECHNICAL LEMMAS

In this section we state and prove Lemma III.2 which is the main technical lemma to be used in applications. We start recalling Theorem A.1 from [10] (in Theorem A.1 use m^{n+1}_H from eq. (3.8) in [10]). In this Theorem and below, d_p is equal to either

$$d_p=\mathrm{dist}(p,\partial\Sigma), \tag{15}$$

if $\partial\Sigma\neq 0$ or

$$d_p=\sup\{\mathrm{dist}(p,x):x\in\Sigma\} \tag{16}$$

if $\partial\Sigma\neq 0$. In particular if Σ is non-compact and boundaryless then $d_p=\infty$, [5]

Theorem III.1. ([10]) *Let (Σ,g) be an n-dimensional Riemannian manifold. Suppose that*

$$Ric+\nabla\nabla f-\nabla f\nabla f\geq(nH)g \tag{17}$$

for some function f and real number H. Let p be a point in $\Sigma\setminus\partial\Sigma$ and let r be the distance function to p, i.e. $r(x)=\mathrm{dist}(x,p)$. Then, at any x such that $r(x)<d_p$ we have

$$\Delta_f r\leq\begin{cases}\dfrac{n\sqrt{H}}{\tan(\sqrt{H}r)} & \text{if } H>0,\\ \frac{n}{r} & \text{if } H=0,\\ \dfrac{n\sqrt{|H|}}{\tanh(\sqrt{|H|}r)} & \text{if } H<0.\end{cases} \tag{18}$$

in the barrier sense[6].

Of course we could have $\partial\Sigma=\emptyset$ in which case $\Sigma\setminus\partial\Sigma=\Sigma$. As seen in [10], this Theorem implies the following generalised Myers's estimate: if $H>0$, then for any point p we have $d_p\leq\pi/\sqrt{H}$. In particular if Σ is non-compact then $\partial\Sigma\neq\emptyset$. We will use this property later.

The following is the main technical Lemma to be used and that is adapted from an estimate due to Case [5].

Lemma III.2. *Let $(\Sigma;g)$, f, H and p be as in Lemma III.1. Let ψ be a real non-negative function such that*

$$\Delta_f\psi\geq b\psi+c\psi^2, \tag{19}$$

[5] A technical remark is here necessary. For $p\in\Sigma\setminus\partial\Sigma$ and $r_p<d_p$ the metric ball $B(p,r_p):=\{q\in\Sigma:\mathrm{dist}(p,q)<r\}$ has the following property: for every q in $B(p,r_p)$, there is at least a length minimising segment joining p to q and entirely inside $B(p,r_p)$. Thus, inside $B(p,r_p)$, the distance function $r(q)=\mathrm{dist}(p,q)$ can be used as any geodesic distance function. These properties may not hold if $r_p>d_p$ and this explain why we need the condition $r_p<d_p$ in Theorem III.1.

[6] This is an important property as it allows us to make analysis as if r were a smooth function. The reader can consult this notion in [8].

with $b \le 0$ and $c > 0$. Then,

$$\psi(p) \le \begin{cases} \frac{1}{c}\left[\frac{4n+24}{d_p^2} - b\right] & \text{if } H \ge 0, \\ \frac{1}{c}\left[\frac{4n\sqrt{|H|}}{d_p \tanh(\sqrt{|H|}d_p)} + \frac{24}{d_p^2} - b\right] & \text{if } H < 0 \end{cases} \tag{20}$$

Proof. For any function χ the following general formula holds

$$\Delta_f(\chi\psi) = \psi\Delta_f\chi + 2\langle\nabla\chi, \nabla\psi\rangle + \chi\Delta_f\psi \tag{21}$$

Thus, if $\chi \ge 0$ and if $\chi\psi$ has a local maximum at q, then we have

$$0 \ge \left[\Delta_f(\chi\psi)\right]\Big|_q \tag{22}$$

$$\ge \left[\psi\Delta_f\chi - 2\frac{|\nabla\chi|^2}{\chi}\psi + b\chi\psi + c\chi\psi^2\right]\Big|_q \tag{23}$$

where to obtain the second inequality we used (19). To simplify notation let $r = r(x) = \text{dist}(x, p)$. Let r_p be a positive number less than d_p. On the ball $B(p, r_p)$ let the function $\chi(x)$ be $\chi(x) = (r_p^2 - r^2(x))^2$. Let q be a point in the closure of $B(p, r_p)$ where the maximum of $\chi\psi$ is achieved. As $(\chi\psi)(q) \ge (\chi\psi)(p) = r_p^4\psi(p)$ we deduce that if $\psi(q) = 0$ then $\psi(p) = 0$. In this case (20) follows. So let us assume that $\psi(q) > 0$ and hence that $q \in B(p, r_p)$. By (22) we have

$$cr_p^4\psi(p) \le c(\chi\psi)(q) \le \left[2\frac{|\nabla\chi|^2}{\chi} - \Delta_f\chi - b\chi\right]\Big|_q \tag{24}$$

$$\le \left[4(r_p^2 - r^2)r\Delta_f r + 4r_p^2 + 20r^2 - br_p^4\right]\Big|_q \tag{25}$$

But if $Ric_f^\alpha \ge nHg$ then $\Delta_f r$ can be estimated from (18). Use this estimation in (25), divide by $c\psi^4(p)$, and take the limit $r_p \to d_p$ to obtain (20) by simple bounds. □

Corollary III.3. *Assume the hypothesis of Lemma III.2 and that $(\Sigma; g)$ is non-compact and geodesically complete. Then,*

$$\psi(p) \le -\frac{b}{c} \tag{26}$$

at any $p \in \Sigma$, regardless of the sign of H.

Proof. If $(\Sigma; g)$ is non-compact and geodesically complete then $d_p = \infty$ and the result follows from (20). □

IV. APPLICATIONS

A. Klein-Gordon

In this section we study the Klein-Gordon potential $V(|\phi|) = m^2|\phi|^2$. The mass is allowed to be zero in which case $V = 0$. The theorem that follows is perhaps the simplest and most elegant application of the estimates of the previous section.

Theorem IV.1. *Let $(\Sigma; N, g, \phi)$ be a geodesically complete solution of the n-dimensional static Einstein-KleinGordon equations. Then, $Ric = 0$, $N = N_0$ and $\phi = \phi_0$, with $\phi_0 = 0$ if $m \ne 0$. In particular if $n = 3$ then $(\Sigma; g)$ is covered by the Euclidean three-space.*

The main Böchner type of formula that we are going to use is

$$\frac{1}{2}\Delta_f|\nabla\chi|^2 = |\nabla\nabla\chi|^2 + \langle\nabla\chi, \nabla(\Delta_f\chi)\rangle \tag{27}$$

$$+ Ric_f^1(\nabla\chi, \nabla\chi) + |\langle\nabla\chi, \nabla f\rangle|^2 \tag{28}$$

which is valid for any real function χ, [10].

Proof. During the proof we make $f = -\ln N$. To start note that if Σ is compact and $m \ne 0$ then $\phi = 0$ by integrating (5) against $N = e^{-f}$. But if $\phi = 0$ then f is constant by integrating (5) against $N^2 = e^{-2f}$. Thus $Ric = 0$ by (5) as claimed. Identical conclusion is reached if $m = 0$ by integrating (5) agains $N^2 = e^{-2f}$ and (6) against ϕ.

Assume then from now on that Σ is non-compact. Recall that we use the notation $\phi = \phi_R + i\phi_I$. From (6) we obtain

$$\Delta_f\chi = m^2\chi \tag{29}$$

for χ equal to either ϕ_R or ϕ_I. Use then these two equations to evaluate (27) with $\chi = \phi_R$ and with $\chi = \phi_I$. Add up the results and get (after discarding a few positive terms)

$$\frac{1}{2}\Delta_f(|\nabla\phi_R|^2 + |\nabla\phi_I|^2) \ge |\nabla\phi_R|^4 + |\nabla\phi_I|^4 \tag{30}$$

Use now $|\nabla\phi|^2 = |\nabla\phi_R|^2 + |\nabla\phi_I|^2$ and the inequality $(x^4 + y^4) \ge (x^2 + y^2)^2/2$ to arrive at

$$\Delta_f|\nabla\phi|^2 \ge |\nabla\phi|^4 \tag{31}$$

It follows then from Corollary III.3 that $\nabla\phi = 0$. Hence $\phi = \phi_0$, and $\phi_0 = 0$ if $m \ne 0$ from (6).

We prove now that the lapse must be constant. From what was proved before we have $Ric_f^1 = 0$ and $\Delta_f f = 0$. Use then (27) with $\chi = f$ to get

$$\frac{1}{2}\Delta_f|\nabla f|^2 \ge |\nabla f|^4 \tag{32}$$

Thus, $\nabla f = 0$ from Corollary III.3 and hence $N = N_0$.

If $f = f_0$ then $Ric = 0$ from $Ric_f^1 = 0$. □

B. Λ-Klein-Gordon

In this section we investigate geodesically complete solutions of the static Einstein-ScalarField equations with potentials of the form $V(\phi)=m^2|\phi|^2+2\Lambda$.

The case $\Lambda=0$ was the one considered in the previous section, therefore we consider below only the cases $\Lambda>0$ and $\Lambda<0$.

$\Lambda>0$: In this case it is easy to see that there are no geodesically complete solutions at all. Indeed, if Σ is compact a contradiction is obtained by integrating (5) against $N=e^{-f}$. On the other hand if Σ is non-compact, then Σ must have boundary because $Ric^1_f\geq(2\Lambda/(n-1))g$, as we already commented after the statement of Theorem III.1. This thus contradicts the assumption that (Σ,g) is geodesically complete.

$\Lambda<0$: As explained in the introduction, there are geodesically complete solutions in this case, therefore the best one can do is to understand the local and global geometry. Our first results shows that geodesically complete solutions with Σ compact do not exist. Our second result uses this information to provide complete estimates on the scalar field ϕ.

Theorem IV.2. *Let $(\Sigma;N,g,\phi)$ be a geodesically complete solution of the static Einstein-ScalarField equations with potential $V(\phi)=m^2|\phi|^2+2\Lambda$, where $\Lambda<0$. Then Σ is non-compact.*

Proof. During the proof we use $f=-\ln N$. Assume that Σ is compact. Then observe that as (6) is equivalent to

$$\nabla(N\nabla\phi)=m^2N\phi \tag{33}$$

we can multiply this equation by $\bar{\phi}$ an integrate over Σ to obtain

$$0=\int_\Sigma N(|\nabla\phi|^2+m^2|\phi|^2)\,dv \tag{34}$$

This implies $\phi=\phi_0$ with $\phi_0=0$ if $m\neq0$. Using this information then note that (5) is equivalent to $\Delta N=(2\Lambda/(n-1))N$. Integrating this over Σ we deduce $\Lambda=0$, and thus a contradiction. □

Theorem IV.3. *Let $(\Sigma;g,N,\phi)$ be a geodesically complete solution of the static Einstein - Scalar Field equations with potential $V(\phi)=m^2|\phi|^2+2\Lambda$, where $\Lambda<0$. Then the following holds:*

(i) if $m^2\geq-2\Lambda/(n-1)$ then ϕ is identically zero, and,

(ii) if $m^2<-2\Lambda/(n-1)$ then,

$$|\nabla\phi|^2\leq\frac{-4\Lambda}{(n-1)},\qquad m^2|\phi|^2\leq-64\Lambda. \tag{35}$$

In particular $R=|\nabla\phi|^2+m^2|\phi|^2+2\Lambda\leq-66\Lambda$, by a coarse estimation.

Proof. During the proof we use $f=-\ln N$. Use (27) with $\chi=\phi_R$ and with $\chi=\phi_I$ and add up the results to obtain (after discarding a few positive terms)

$$\Delta|\nabla\phi|^2\geq2(m^2+\frac{2\Lambda}{n-1})|\nabla\phi|^2+|\nabla\phi|^4 \tag{36}$$

Hence, if $m^2\geq-2\Lambda/(n-1)$ then ϕ is constant by Corollary III.3. But if ϕ is constant and $m^2>0$ then ϕ must be indeed zero by equation (6).

Let us assume then that $m^2<-2\Lambda/(n-1)$. By Corollary III.3 we have

$$|\nabla\phi|^2\leq-2(\frac{2\Lambda}{(n-1)}+m^2)\leq\frac{-4\Lambda}{(n-1)} \tag{37}$$

which shows the first estimate of (35). Using this estimate together with $m^2<-2\Lambda/(n-1)$ we deduce

$$m|\nabla\phi|\leq\frac{\sqrt{8}\sqrt{-\Lambda}}{(n-1)} \tag{38}$$

The convenience of this estimate is the following. If two points p_0 and p are separated by a distance L then

$$m|\phi(p_0)|-m|\phi(p)|\leq|m\phi(p_0)-m\phi(p)| \tag{39}$$
$$=|\int_\gamma m\nabla_{\gamma'}\phi\,ds| \tag{40}$$
$$\leq\frac{\sqrt{8}\sqrt{-\Lambda}}{(n-1)}L \tag{41}$$

where $\gamma(s)$ is a length minimising geodesic segment joining p_0 to p. Hence, if at a point p_0 we have

$$m|\phi(p_0)|\geq8\sqrt{-\Lambda} \tag{42}$$

then

$$m|\phi(p)|\geq5\sqrt{-\Lambda} \tag{43}$$

at every point p of the ball $B(p_0,(n-1)/\sqrt{-\Lambda})$ because using (39) we would have $m|\phi(p)|\geq8\sqrt{-\Lambda}-2\sqrt{2}\sqrt{-\Lambda}\geq5\sqrt{-\Lambda}$. Assume then that (43) holds on $B(p_0,(n-1)/\sqrt{-\Lambda})$. Then by (4) we would have

$$Ric^1_f\geq\left(\frac{-23\Lambda}{n-1}\right)g=(nH)g \tag{44}$$

where the r.h.s is the definition of H. But then the radius of the ball, $(n-1)/\sqrt{-\Lambda}$, should be less or equal than $\pi/\sqrt{H}$, in other words we should have

$$\frac{n-1}{\sqrt{-\Lambda}}\leq\frac{\pi\sqrt{n(n-1)}}{\sqrt{23}\sqrt{-\Lambda}} \tag{45}$$

But his equation doesn't hold for any $n\geq2$. Thus, (42), (hence (43)), cannot hold and we have

$$m^2|\phi|^2\leq-64\Lambda \tag{46}$$

which is the second estimate of (35). □

So far, Theorem IV.3 provides complete estimates for the scalar field ϕ. We occupy now ourselves with the Lorentzian geometry, namely with N and g. As we show below, gradient estimates for $\ln N$ can be provided in any dimension but pointwise curvature estimate only in spatial dimension three. We start proving estimates for N.

Theorem IV.4. *Let $(\Sigma; g, N, \phi)$ be a geodesically complete solution of the static Einstein-ScalarField equations with potential $V(\phi) = m^2|\phi|^2 + 2\Lambda$, where $\Lambda < 0$. Then, the following holds,*

(i) if $m^2 \geq -2\Lambda/(n-1)$, then

$$\frac{|\nabla N|}{N} \leq \sqrt{\frac{-2\Lambda}{n-1}} \tag{47}$$

and,

(ii) if $m^2 < -2\Lambda/(n-1)$ then

$$\frac{|\nabla N|}{N} \leq 64\sqrt{-\Lambda} \tag{48}$$

Proof. During the proof we use $f = -\ln N$. Use (27) with $\chi = f$, discard a pair of terms and obtain

$$\frac{1}{2}\Delta_f|\nabla f|^2 \geq \langle \nabla f, \frac{m^2\nabla(|\phi|^2)}{n-1}\rangle + \frac{2\Lambda}{n-1}|\nabla f|^2 + |\nabla f|^4 \tag{49}$$

If $m^2 \geq -2\Lambda/(n-1)$ then $\nabla\phi = 0$ and the first term in the r.h.s of the previous equation is zero. We can use Corollary III.3 to obtain $|\nabla f|^2 \leq -2\Lambda/(n-1)$, which is (47).

Assume now that $m^2 < -2\Lambda/(n-1)$. We need to bound the first term in the r.h.s of the previous equations. We do this as follows. First write

$$|\langle \nabla f, \frac{m^2\nabla(|\phi|^2)}{n-1}\rangle| = \tag{50}$$

$$= |2m^2(\phi_R\langle \nabla f, \nabla\phi_R\rangle + \phi_I\langle\nabla f, \nabla\phi_I\rangle| \tag{51}$$

$$\leq 2m(m|\phi_R||\nabla\phi_R| + m|\phi_I||\nabla\phi_I|)|\nabla f| \tag{52}$$

Use now Theorem (IV.3) to bound (52) as

$$2m(m|\phi_R||\nabla\phi_R| + m|\phi_I||\nabla\phi_I|)|\nabla f| \leq \frac{128}{n-1}(-\Lambda)^{3/2}|\nabla f| \tag{53}$$

Thus,

$$\langle \nabla f, \frac{m^2\nabla(|\phi|^2)}{n-1}\rangle \geq -\frac{128}{n-1}(-\Lambda)^{3/2}|\nabla f| \tag{54}$$

Hence

$$\frac{1}{2}\Delta_f|\nabla f|^2 \geq -\frac{128}{n-1}(-\Lambda)^{3/2}|\nabla f| + \frac{2\Lambda}{n-1}|\nabla f|^2 + |\nabla f|^4 \tag{55}$$

Making $\psi = |\nabla f|^2$ we can write

$$\Delta_f\psi \geq a\sqrt{\psi} + b\psi + c\psi^2 \tag{56}$$

where $a = -256(-\Lambda)^{3/2}/(n-1)$, $b = 4\Lambda/(n-1)$ and $c = 2$. This equation is not the same as (19) and Corollary III.3 cannot be directly used. However a simple modification of the arguments of Lemma III.2 shows that, if (56) holds, then

$$\psi(p) \leq \max\left\{\left(\frac{a}{b}\right)^2, -\frac{2b}{c}\right\} \tag{57}$$

Using this with the values of a, b and c given before we obtain (48). □

The following theorem proves that, when the spatial dimension is three, the Ricci curvature is bounded by an expression depending only on Λ. The proof uses some advanced elements of Riemannian geometry.

Theorem IV.5. *Let $(\Sigma; g, N, \phi)$ be a geodesically complete solution of the static Einstein-ScalarField equations with potential $V(\phi) = m^2|\phi|^2 + 2\Lambda$, where $\Lambda < 0$ and in spacetime dimension four (i.e. $n = 3$). Then,*

$$|Ric| \leq \mathcal{R}(|\Lambda|) \tag{58}$$

for some non-negative function $\mathcal{R}$.

Proof. In Theorems IV.3 and IV.4 we deduced pointwise bounds for $|\nabla f|$, $|\nabla\phi|$ and for $m^2|\phi|^2$ depending only on Λ. Therefore, recalling (4), the estimate (58) would follow granted we can prove a pointwise estimate of $|\nabla\nabla f|$ depending only on Λ. We prove now that this is possible when $n = 3$.

Let p be an arbitrary point in Σ. Assume that $N(p) = 1$. (If $N(p) \neq 1$ then work with the scaled lapse $N/N(p)$. Observe that the system (4)-(6) is invariant under scalings of the lapse. Below we use therefore $f = -\ln N$ and we assume $N(p) = 1$).

To start note that the estimates of Theorem IV.4 imply[7]

$$|f|(q) \leq K_0(\Lambda) \tag{59}$$

for every q in $B_g(p, 1)$ and where $K_0(\Lambda) = 64\sqrt{-\Lambda}$. Hence we can write

$$K_1(\Lambda)^{-1} \leq N(q) \leq K_1(\Lambda), \tag{60}$$

for every q in $B_g(p, 1)$ and where $K_1(\Lambda) = e^{K_0(\Lambda)}$. As we mentioned earlier, the Theorems IV.3 and IV.4 give us suitable bounds for $|\nabla f|$, $|\nabla\phi|$ and for $m|\phi|$. From such bounds one can write down the coarse estimate

$$|\nabla f| + |\nabla\phi| + m|\phi| \leq K_2(\Lambda) \tag{61}$$

[7] Just integrate $\nabla \ln N$ along radial geodesics and used then the bound $|\nabla \ln N| \leq 64\sqrt{-\Lambda}$.

for some $K_2(\Lambda)$. This is all what we will need later. We will refer to it a couple of times.

From now on we will use the metric

$$\check{g} := N^2 g \tag{62}$$

In terms of the variables $(\check{g}, N, \phi)$, the static equations (4)-(6) are,

$$\check{Ric} = 2\nabla f \nabla f + \nabla\phi \circ \nabla\bar{\phi} + \frac{V(\phi)}{2} e^{2f}\check{g}, \tag{63}$$

$$\check{\Delta} f = \frac{1}{2} V(\phi) e^f, \tag{64}$$

$$\check{\Delta}\phi = \frac{1}{2}\partial V(\phi) e^f, \tag{65}$$

Now, use the bounds (59) and (61) in the formula (63) to deduce that $|\check{Ric}|_{\check{g}}$ is pointwise bounded in $B_g(p,1)$, where the bound depends only on Λ. Thus we have

$$|\check{Ric}|_{\check{g}} \leq K_3(\Lambda) \tag{66}$$

As we are working in dimension three, where the Riemann tensor is made out of the Ricci tensor, the bound (66) implies a bound also for the Riemann tensor $\check{Rm}$ on $B_g(p,1)$ and thus we have,

$$|\check{Rm}|_{\check{g}} \leq K_4(\Lambda) \tag{67}$$

Now, it is direct to see from (60) that one can find $\check{r}_1(\Lambda)$ such that

$$B_{\check{g}}(p, \check{r}_1) \subset B_g(p, 1/2). \tag{68}$$

Moreover, it is a standard fact in Riemannian geometry that a bound on the Riemann tensor as (67) implies that, for some $\check{r}_2(\Lambda) < \check{r}_1(\Lambda)$, the exponential map

$$exp : U(p, \check{r}_2) \to B_{\check{g}}(p, \check{r}_2) \tag{69}$$

is a smooth cover, where in this formula $U(p,\check{r}_2)$ is the ball of radius $\check{r}_2$ in $T_p\Sigma$, (endowed with the metric $\check{g}(p)$, namely $U(p,r) := \{v \in T_p\Sigma : |v|_{\check{g}(p)} \leq r\}$). Provide now $U(p,\check{r}_2)$ with the pull-back metric $\check{g}^* = exp^*\check{g}$. The injectivity radius at p of the space $(U(p,\check{r}_2), \check{g}^*)$ is of course equal to $\check{r}_2$ and the Riemann tensor of $\check{g}^*$ is subject to the same bound (67) as $\check{g}$. Therefore, the *harmonic radius* of the space $(U(p,\check{r}_2), \check{g}^*)$ at p is controlled from below only by Λ, (see [8], § Chp. 10.5.2). To us, the only important consequence of this is that one can make standard elliptic analysis on $(U(p,\check{r}_3), \check{g}^*)$ for a suitable $\check{r}_3(\Lambda) \leq \check{r}_2(\Lambda)$. Hence, we can use the bounds (59)-(61) to obtain Schauder interior elliptic estimates from the elliptic system (64)-(65), (see [8], § Chp. 10.2). Doing so we get

$$|\check{\nabla}\check{\nabla} f|_{\check{g}}(p) \leq K_5(\Lambda) \tag{70}$$

Use now the expression,

$$\check{\nabla}_i\check{\nabla}_j f = \nabla_i\nabla_j f + 2\nabla_j f \nabla_i f - |\nabla f|^2_g g_{ij} \tag{71}$$

and the bounds (70), (59) and (61), to deduce directly the bound

$$|\nabla\nabla f|_g(p) \leq K_6(\Lambda) \tag{72}$$

as wished. □

C. Real scalar fields

General interesting results can be obtained when ϕ is real. The following theorem, gives a simple condition for $V(\phi)$ that forces ϕ to be a constant. It gives nice applications that will be illustrated very briefly below.

Theorem IV.6. *Let $(\Sigma; g, N, \phi)$ be a geodesically complete solution of the static Einstein-RealScalarField system with potential $V(\phi)$. If V is bounded below and*

$$V''(x) + \frac{V(x)}{n-1} \geq 0, \tag{73}$$

for all x, then $\phi = \phi_0$, (constant), and ϕ_0 is a critical point of $V(\phi)$.

Proof. Make $f = -\ln N$. Then, using (27) with $\chi = \phi$ we obtain

$$\frac{1}{2}\Delta_f |\nabla\phi|^2 \geq \big(V''(\phi) + \frac{V(\phi)}{n-1}\big)|\nabla\phi|^2 + |\nabla\phi|^4 \tag{74}$$

If (73) holds an Σ is compact then $\nabla\phi = 0$ by integrating (74) over Σ. On the other hand if Σ is non-compact and (73) holds then $\nabla\phi = 0$ from Corollary III.3.

Finally if $\phi = \phi_0$ then equation (6) shows that ϕ_0 is a critical point of $V(\phi)$. □

To illustrate the relevance of this Theorem let us consider a set of simple and (more or less) natural potentials and let us enumerate, without entering into further discussion, the strong conclusions that can be deduced in each case.

1. $V(\phi) = \lambda\phi^{2n}$, $\lambda > 0$, $n = 1, 2, 3, \ldots$. In this case (73) is verified and therefore any geodesically complete solution must have $\phi = 0$.

2. $V(\phi) = \lambda\cosh\phi$, $\lambda > 0$. In this case (73) is verified and therefore any geodesically complete solution must have $\phi = 0$.

3. $V(\phi) = \lambda e^{\phi}$, $\lambda > 0$. In this case (73) is verified but there cannot be geodesically complete solutions at all because V has no critical points.

4. $V(\phi) = \lambda \sin\sqrt{(n-1)}\phi$ (a type of Sine-Gordon potential). In this case the l.h.s of (73) is identically zero and thus any geodesically complete solution

must have $\phi = (-\pi/2 + 2j\pi)/\sqrt{n-1}, j \in \mathbb{Z}$ (the other critical points make V strictly positive). This example is interesting because it shows that strong conclusions can be obtained even when V is not a non-negative potential.

5. $V(\phi) = \lambda(\phi^2 - \phi_0^2)^2$, $\lambda > 0$, (a type of Higgs potential). In this case one can show that if $\phi_0^2 > 6(n-1)$ then any geodesically complete solution must have $|\phi| = |\phi_0|$. To see this observe that, in this case, (73) is equivalent to

$$12(\phi^2 - \phi_0^2) + 8\phi_0^2 + \frac{(\phi^2 - \phi_0^2)^2}{n-1} \geq 0 \tag{75}$$

Making $z = \phi^2 - \phi_0^2$, the previous equation is equivalent to $12z + 8\phi_0^2 + z^2/(n-1) \geq 0$ for all $z \geq -\phi_0^2$. But if $\phi_0^2 \geq 6(n-1)$ then the polynomial $12z + 8\phi_0^2 + z^2/(n-1)$ is non-negative.

D. The energy of Klein-Gordon geons

Following [6], a Klein-Gordon geon is a solution of the Einstein-(Complex)ScalarField equations where the spacetime metric is static but the scalar field oscillates in time with frequency ω. Accordingly, we assume the spacetime dependence $\phi(x,t) = e^{i\omega t}\phi_0(x)$, $(x \in \Sigma)$. If the scalar field has a self interacting potential field $V(|\phi|)$, then the static equations of a geon are the same as the static Einstein-ScalarField equations (4)-(6) with the potential $\hat{V}(\phi_0) = -\omega^2|\phi_0|^2 + V(\phi_0)$. In particular, if $V(|\phi_0|) = m^2|\phi_0|^2$, then the geon equations are equivalent to that of the static Einstein-ScalarField equations with 'mass', $\hat{m}^2 = -\omega^2 + m^2$ (of course if $\omega^2 > m^2$ then $\hat{m}$, as defined, is imaginary). The direct conclusion from Theorem IV.1 is that geons do not exist unless the frequency $|\omega|$ is greater than the mass, i.e. $\omega^2 > m^2$. As mentioned in the introduction, if this condition holds, then spherically symmetric solutions are known to exist and area called Mini Boson stars [4].

[1] Michael T. Anderson. On stationary vacuum solutions to the Einstein equations. *Ann. Henri Poincaré*, 1(5):977–994, 2000.

[2] MichaelT. Anderson. On the structure of conformally compact einstein metrics. *Calculus of Variations and Partial Differential Equations*, 39(3-4):459–489, 2010.

[3] Piotr Bizon. Gravitating solitons and hairy black holes. *Acta Phys.Polon.*, B25:877–898, 1994.

[4] Piotr Bizoń and Arthur Wasserman. On existence of mini-boson stars. *Comm. Math. Phys.*, 215(2):357–373, 2000.

[5] Jeffrey S. Case. The nonexistence of quasi-Einstein metrics. *Pacific J. Math.*, 248(2):277–284, 2010.

[6] David J. Kaup. Klein-gordon geon. *Phys. Rev.*, 172:1331–1342, Aug 1968.

[7] Ulises Nucamendi and Marcelo Salgado. Scalar hairy black holes and solitons in asymptotically flat spacetimes. *Phys.Rev.*, D68:044026, 2003.

[8] Peter Petersen. *Riemannian geometry*, volume 171 of *Graduate Texts in Mathematics*. Springer, New York, second edition, 2006.

[9] M S Volkov. Gravitating non-Abelian solitons and hairy black holes in higher dimensions. (hep-th/0612219):18 p, Dec 2006.

[10] Guofang Wei and Will Wylie. Comparison geometry for the Bakry-Emery Ricci tensor. *J. Differential Geom.*, 83(2):377–405, 2009.