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High energy scatterings in higher dimensional theories

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The high energy behavior of scattering amplitudes in spacetime dimensions, $D > 4$, is investigated. The bound on total cross sections, $\sigma_t \leq Constant\ (log s)^{D-2}$, $D \geq 4$ has been obtained in the past under usual assumptions. I derive new bounds on scattering amplitudes in the region $|t| < T_0$, t being momentum transfer squared and T_0 is a constant. The existence of a zero-free region for the amplitude, in complex t-plane, is proved. I prove stronger upper and lower bounds for the absorptive amplitude in the domain $0 < t < T_0$ under additional assumptions. © *2015 AIP Publishing LLC.* [http://dx.doi.org/10.1063/1.4933220]

The Froissart-Martin bound restricts the growth of hadronic total cross sections, $\sigma_t(s)$, at high energies. Heisenberg,[1] in 1952, argued that total cross sections may grow as fast as $(\ln s)^2$ at high energies. The arguments were based on his intuitions and he supported them by deriving the energy dependence of σ_t from the Born-Infeld action for a scalar field.

Let us summarize the essential ingredients for derivation of rigorous results in strong interactions. These results are especially important in the context of hadronic reactions at asymptotic energies. We refer to books and reviews[2–5] for detail discussions and references. We consider scattering of scalar particles throughout. The derivation of exact results rests on four pillars based on axiomatic field theory.

(A1) Unitarity of S-matrix.

(A2) The amplitude, for fixed s, is analytic in complex $\cos\theta$-plane inside the Lehmann-Martin ellipse; s and θ being the energy squared and scattering angles, respectively, in the center of mass frame. The foci of the ellipse lie at $(-1,+1)$ and its semi-major axis is $\cos\theta_0 = 1 + 2t_0/s$, t_0 being an s-independent constant. For many hadronic processes, $t_0 = 4m_\pi^2$. The partial wave expansion of the amplitude, $F(s,t)$, converges absolutely inside Lehmann-Martin ellipse,

$$F(s,t) = \frac{\sqrt{s}}{k}\sum_{l=0}^{\infty}(2l+1)f_l(s)P_l(\cos\theta). \tag{1}$$

$P_l(\cos\theta)$ is analytic in $\cos\theta$, so is $F(s,t)$ in t, $\cos\theta = 1 + 2t/s$; k being the momentum in the center of mass frame. Unitarity bounds on the partial-wave amplitudes are

$$0 \leq |f_l(s)|^2 \leq \mathrm{Im} f_l(s) \leq 1. \tag{2}$$

(A3) Polynomial boundedness inside Lehmann-Martin ellipse.[6] For $0 \leq t \leq t_0$,

$$|F(s,t)| <_{|s|\to\infty} |s|^N, \tag{3}$$

where N being a positive integer.

(A4) $F(s,t)$ is analytic in the complex s-plane. There are cuts in the s-plane as a consequence of s-channel unitarity and crossing symmetry-crossing is a requirement. The Froissart-Martin bound is[7,8]

$$\sigma_t(s) < \frac{4\pi}{t_0 - \epsilon}[\ln(s/s_0)]^2. \tag{4}$$

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ϵ is an arbitrary small constant and s_0 is an undetermined energy scale. However, recently, it has been shown[9] that ${s_0}^{-1} = 17\pi\sqrt{\pi/2}{m_\pi}^{-2}$ for $\pi\pi$ scattering. There are host of important, exact, and well tested bounds on[2,4,5]

$$\sigma_t,\ \sigma_{el},\ (\frac{d\sigma}{dt})_{el},\ b(s),\ \mathbf{h}_d = \frac{d\sigma}{dt}|_{t=0}, \tag{5}$$

where $b(s)$ is the slope of the elastic diffraction peak and $\mathbf{h}_b$ is its height at $t = 0$.

The purpose of this article is to derive some important bounds for amplitudes describing scattering in $D > 4$. The bounds on σ_t and slope of the diffraction peak associated with the imaginary part of the amplitude, etc., have been derived in the past.[10,11] I derive new bounds on the elastic amplitude $F^\lambda(s,t)$, defined below, in a region $|t| \le T_0$, T_0 being a constant. This result extends the previously obtained results of Refs. 10 and 11 to a region alluded to above. Moreover, a bound is derived on the distribution of zeros of $F^\lambda(s,t)$ in the complex t-plane. The radius of the zero-free circle shrinks as $(1/\ln s)^2$. I prove that the absorptive amplitude is bounded from above and below in the domain $0 < t < T_0$. There is a close relationship between the energy dependence of $b(s)$ and the zeros of $F^\lambda(s,t)$. The assumptions leading to these bounds are stated in sequel; although these are quite general, they do not follow from axiomatic field theory like (A1)-(A4) for $D = 4$ case.

We recall that, for $D > 4$, the elastic scattering amplitude of spinless particles admits partial wave expansion as demonstrated by Soldate.[12] The amplitude depends on s and $\cos\theta$. The ultraspherical Jacobi function, the Gegenbauer polynomials,[13,14] is the basis as these are the "spherical harmonics" associated with the $SO(D-1)$ rotation group. The amplitude, $F^\lambda(s,t)$, is expanded as

$$F^\lambda(s,t) = A_1 s^{-\lambda+1/2}\sum_{l=0}^{\infty}(l+\lambda)f_l^\lambda(s)C_l^\lambda(t)(1+2t/s), \tag{6}$$

where $\lambda = \frac{1}{2}(D-3)$ and (s,t) are the usual Mandelstam variables. $A_1 = 2^{4\lambda+3}\pi^\lambda\Gamma(\lambda)$, independent of s and t. $C_l^\lambda(x)$ are Gegenbauer polynomials satisfying orthogonality conditions with weight factor $(1-x^2)^{\frac{D}{2}-1}$, $-1 \le x \le +1$.[13] Expansion (6) converges in the domain $-1 \le \cos\theta \le +1$.[13] The partial-wave amplitudes, $\{f_l^\lambda(s)\}$, satisfy the unitarity constraints,

$$0 \le |f_l^\lambda(s)|^2 \le \mathrm{Im} f_l^\lambda(s) \le 1, \tag{7}$$

as derived in Ref. 12.

Additional assumptions are needed to derive analog of Froissart-Martin bound (4). The existence of extended domain of analyticity for (6) is not proven for the theories in $D > 4$, unlike the $D = 4$ case. Thus, two reasonable requirements were imposed in Refs. 10 and 11. These are the following: (I) the domain of convergence, $\mathbf{D}$, of $F^\lambda(s,t)$ is an extended ellipse with the semimajor axis $1 + 2T_0/s$ in the t-plane.[15,16] (II) The amplitude is polynomially bounded: $|F^\lambda(s,t)| < Cs^N$ in $\mathbf{D}$; C and N are undetermined constants, $N \in \mathbf{R}$. The domain of convergence of Gegenbauer polynomial is $[-1,+1]$ and a theorem (see Theorem 9.1.1 of Ref. 13) states the convergence properties of functions such as $F^\lambda(s,t)$. The partial wave amplitudes $f_l^\lambda(s)$ decay exponentially with l[10,11] under assumptions (I) and (II). Therefore, expansion (6) can be truncated at $L = \frac{1}{2}(N-1)\sqrt{s/T_0}\ln s$ and importantly, L is D-independent. Consequently, σ_t is bounded[10,11] as

$$\sigma_t \le C_0(\ln s)^{D-2}. \tag{8}$$

Note that $\ln s$ is to be understood as $\ln(s/\hat{s})$, $\hat{s}$ is like s_0 which scales s. Important bounds on $F^\lambda(s,t)$ and its absorptive part are derived by utilizing crucial properties of $C_l^\lambda(x)$ and assumptions (I) and (II). Let us summarize the useful inequalities as Ref. 13.

Lemma 1. For $1 < x_1 < x_2 < 1 + T_0/s$, $C_l^\lambda(x_1) < C_l^\lambda(x_2)$.

Proof. Define $z = \frac{1}{2}(1-x)$. For $x > 1$, $z < 0$. $C_l^\lambda(x)$, as a hypergeometric function is expanded as[14]

$$C_l^\lambda(x) = \frac{\Gamma(\frac{\lambda+1}{2})}{\Gamma(2\lambda)}\sum_{k=0}^{l}\frac{\Gamma(2\lambda+n+k)}{k!(l-k)!\Gamma(\lambda+k+1/2)}(-z)^k. \tag{9}$$

Notice: (i) Coefficients of $(-z)^k$ are positive. (ii) For $1 < x_1 < x_2$, each term in expansion (9), $B_k(z_1) < B_k(z_2)$ and therefore, $C_l^\lambda(x_1) < C_l^\lambda(x_2)$ in the range of interest. Moreover, $C_l^\lambda(x)$ oscillates in the physical region, $-1 \le x < +1$, $x = \cos\theta$, $t < 0$. We derive following bound on the modulus of the amplitude, $|F^\lambda(s,t)|$, $0 < |t| < T_0$, utilizing the Lemma 1 and noting that $C_l^\lambda(1+2t/s) \le C_l^\lambda(1+2|t|/s)$.

Theorem. The modulus $|F^\lambda(s,t)|$ is bounded as

$$|F^\lambda| \le A_2(\frac{1}{|t|})^{\frac{\lambda+1}{2}}(\frac{1}{T_0})^{\lambda/2} s^{1+(N-1)\sqrt{\frac{|t|}{T_0}}}(\ln s)^\lambda, \tag{10}$$

where $0 < |t| < T_0$ and $A_2 = 2\lambda A_1 2^{-2\lambda-1}\Gamma^{-2}(\lambda+1)$.

Proof. The proof proceeds through the following steps.
Decompose $F^\lambda(s,t)$ as

$$F^\lambda(s,t) = F^{(1)\lambda}(s,t) + F^{(2)\lambda}(s,t). \tag{11}$$

The expansion of $F^{(1)}(s,t)$ is terminated at $l = L$ and expansion of $F^{(2)}$ is from $L+1$ to ∞. Therefore,

$$F^{(1)\lambda}(s,t) \le A_1 s^{-\lambda+1/2}\sum_{l=0}^{L}(l+\lambda)|f_l^\lambda|C_l^\lambda(1+\frac{2|t|}{s}). \tag{12}$$

Invoking unitarity and setting $|f_l^\lambda(s)| = 1$, (12) reduces to a sum: $\sum(l+\lambda)C_l^\lambda(1+2|t|/s)$. For large s,

$$\sum_{l=0}^{L}(l+\lambda)C_l^\lambda(1+2|t|/s) \sim C_1(\sqrt{\frac{s}{|t|}})^{\frac{1+\lambda}{2}} L^\lambda \exp(2L\sqrt{\frac{s}{|t|}}). \tag{13}$$

Note $L = \frac{1}{2}(N-1)\sqrt{\frac{s}{T_0}}\ln s$ and $C_1 = 2\lambda 2^{-2(\lambda+1)}\Gamma(\lambda+1)$. Thus, the desired bound is

$$|F^{(1)\lambda}(s,t)| \le A_2(\frac{N-1}{2})^\lambda(\frac{1}{|t|})^{\frac{1+\lambda}{2}}(\frac{1}{T_0})^{\lambda/2}$$
$$s(\ln s)^\lambda \exp(2L\sqrt{\frac{|t|}{s}}) \tag{14}$$

with $A_2 = A_1C_1$. Let us turn the attention on $F^{(2)\lambda}$. We argue, as before, to convince the reader that

$$|F^{(2)\lambda}(s,t)| \le A_1 s^{-\lambda+1/2}\sum_{L+1}^{\infty}(l+\lambda)|f_l^\lambda|(s)C_l^\lambda(1+2|t|/s) \tag{15}$$

and re-express right hand side (15) as

$$|F^{(2)\lambda}(s,t)| \le A_1 s^{-\lambda+1/2}\sum_{L=1}^{\infty}\left[\sqrt{(l+\lambda)}\frac{C_l^\lambda(1+2|t|/s)}{\sqrt{C_l^\lambda(1+2T_1/s)}}\right]$$
$$\times\left[\sqrt{(l+\lambda)}\sqrt{C_l^\lambda(1+2T_1/s)}|f_l^\lambda(s)|\right], \tag{16}$$

where $|t| < T_1 = T_0 - \delta_1$; $\delta_1 > 0$ but infinitesimally small. Now, invoke the Schwarz inequality on the *r.h.s* of (16),

$$F^{(2)\lambda}(s,t) \le A_1 s^{-\lambda+1/2}\mathcal{S}_1\mathcal{S}_2, \tag{17}$$

where

$$\mathcal{S}_1 = \left[\sum_{L+1}^{\infty}(l+\lambda)\frac{\left(C_l^\lambda(1+2|t|/s)\right)^2}{C_l^\lambda(1+2T_1/s)}\right]^{1/2}, \tag{18}$$

$$\mathcal{S}_2 = \left[\sum_{L+1}^{\infty}(l+\lambda)|f_l^\lambda|^2 C_l^\lambda(1+2T_1/s)\right]^{1/2}. \tag{19}$$

In what follows, I outline the estimation of the upper bounds to $F^{(2)\lambda}(s,t)$ which is a generalization of the proof for the case $\lambda = 1/2$. The term $\mathcal{S}_1$ involves ratio of two Gegenbauer polynomials with arguments $(1 + 2|t|/s)$ and $(1 + 2T_1/s)$. In the large s, limit $\mathcal{S}_1 \sim (\ln s)^{\lambda-1}$. For $\mathcal{S}_2$, invoke unitarity, $|f_l^\lambda|^2 \leq 1$, and polynomial boundedness, (recall $|F^{(2)\lambda}(s,t)| < Cs^N$), to give suppressed energy dependence with appropriate choice of C (see Ref. 5 for detail arguments). Thus, $|F^{(2)\lambda}|$ has subdominant energy dependence: $(\ln s)^{\lambda-1}$ (compared to $|F^{(1)\lambda}| \sim (\ln s)^\lambda$). Therefore,

$$|F^\lambda(s,t)| \leq A_2(\frac{1}{|t|})^{\frac{1+\lambda}{2}}(\frac{1}{T_0})^{\lambda/2}s^{1+(N-1)\sqrt{\frac{|t|}{T_0}}}(\ln s)^\lambda. \tag{20}$$

Remarks: (R1.1) The bound holds in the domain $|t| < T_0$, including a part of the physical region $t < 0$. (R1.2) This is generalization of the bound for the $D = 4$; note the power dependence and on $|t|$. The known result[5] is recovered, setting $\lambda = 1/2$. (R1.3) An important and noteworthy point is the appearance of Regge-like power law in (20), in the domain $0 < |t| < T_0$ and the power of s is λ-independent. It is a reminiscence of open string tree amplitude in the Regge limit. (R1.4) The bound is a substantial improvement over that of Refs. 10 and 11 since theirs was only for $|F^\lambda(s,0)|$.

A bound on number of zeros of $F^\lambda(s,t)$ follows from (20).

Lemma 2. The number, $n_r(s)$, of zeros of $F^\lambda(s,t)$ within the disk $|t| < r < T_0$ is bounded from above by

$$n_r(s) \leq \frac{e\sqrt{r}}{2\sqrt{T_0}}\ln s. \tag{21}$$

The terms of $o((\ln s)^{-1})$ in the $r.h.s$ of (21) are ignored.

Proof. According to Jensen's theorem,[17] $n_r(s)$ satisfies the inequality

$$n_r(s) \leq \frac{1}{\ln 1/\delta}\text{Max }\ln|\frac{F^\lambda(s,r/\delta)}{F^\lambda(s,0)}|, \quad r/T_0 < \delta < 1. \tag{22}$$

Note that $|F^\lambda(s,0)| = \sqrt{(ReF^\lambda(s,0))^2 + (ImF^\lambda(s,0))^2}$ and $|F^\lambda(s,0)| \geq s\sigma_t$ as per our normalization: $ImF^\lambda(s,0) = s\sigma_t$. Therefore,

$$n_r(s) \leq \frac{1}{\ln 1/\delta}\text{Max}\Big(F^\lambda(s,\frac{r}{\delta})\frac{1}{s\sigma_t}\Big) \tag{23}$$

by the virtue of (20); on the $r.h.s$, we are left with

$$\ln\Big[C_2\exp(\{(N-1)\sqrt{\frac{r/\delta}{T_0}}\ln s\})(\frac{1}{r/\delta})^{\frac{1+\lambda}{2}}(\frac{1}{T_0})^{\lambda/2}\frac{1}{\sigma_t}\Big]. \tag{24}$$

Retaining the leading order term in $\ln s$,

$$\ln|\frac{F^\lambda(s,r/\delta)}{F^\lambda(s,o)}| \leq \ln s\Big[(N-1)\sqrt{\frac{r}{\delta}} + \frac{1}{(\ln s)}\{\ln C_1 + \lambda\ln\ln s - (\frac{(\lambda+1)}{2})\frac{r}{\delta} - \frac{\lambda}{2}T_0 - \ln(\sigma_t)\}\Big].$$

No rigorous lower bound exists on σ_t for $D > 4$ like the Jin-Martin[18] bound: $\sigma_t \geq s^{-6}$ in $D = 4$. Even if it existed, its contribution to (25) will be $(\ln s)^{-1}\ln\sigma_t \sim o(1)$. Thus, retaining the $\ln s$ term, in large s limit, is justified in estimation of the upper bound on $\ln|F^\lambda(s,r/\delta)/F^\lambda(s,0)|$; consequently,

$$n_r(s) \leq \frac{(N-1)\sqrt{\frac{r}{\delta}}}{(\ln 1/\delta)\sqrt{T_0}}\ln s \tag{25}$$

follows from (22). Optimizing (25) with respect to δ leads to desired bound (21).

Corollary: There are no zeros inside a disk of radius r_0 in the t-plane and

$$r_0 < \frac{C_2}{(\ln s)^2}. \tag{26}$$

Remarks: (R2.1) $C_2 = T_0[e(N-1)]^{-2}$ is independent of s. The correction to (25) is order $o(1)$ if Jin-Martin exists for $D > 4$. (R2.2) There is a zero-free disk with shrinking radius. More importantly, to leading order in $\ln s$, the upper bound on r_0 is D-independent. (R2.3) There is a single zero in the $|t|$-plane in an annular region,

$$r_0(s) < \frac{C_2}{(\ln s)^2} < r_1(s) < \frac{C_3}{(\ln s)^2}, \tag{27}$$

where C_3 being another s-independent constant. We recall that the study of the zeros of amplitudes has played a crucial role in deriving many rigorous results in the past.[19–21]

The bound on $A^\lambda(s,t) = \text{Im}\, F^\lambda(s,t)$ can be improved substantially in the unphysical region $0 < t \le T_0$, utilizing unitarity, properties of Gegenbauer polynomials, and bound (20). The important point to note is that $A^\lambda(s,t) > 0$ for $t > 0$ as evident from (6) since $0 \le \text{Im}\, f_l^\lambda \le 1$ and $C_l^\lambda(x) > 0$ for $x > 1$. We arrive at the final result through following steps. The zeros of Jacobi polynomial $P_l^{(\alpha,\beta)}(\cos\theta)$, for $\alpha > -1,\ \beta > -1$ satisfy following properties: (Theorem 8.9.1 of Ref. 13), let $0 < \theta_1 < \theta_2, \ldots, < \theta_l < \pi$ be zeros of $P_l^{(\alpha,\beta)}(\cos\theta)$ (it applies to $C_l^\lambda(\cos\theta)$ since $\alpha = \beta = \lambda > 1/2$). Then,

$$\theta_\nu = \frac{1}{l}(\nu\pi + O(1)) \tag{28}$$

with $O(1)$ being uniformly bounded constant for all values of $\nu = 1,2,\ldots,l;\ l = 1,2,\ldots$. Thus, two consecutive zeros are separated[13] as $\theta_{\nu+1} - \theta_\nu = \frac{\pi}{l}$. The distribution of zeros of Legendre polynomial (a special case of Gegenbauer polynomial with $\lambda = 1/2$) was crucially used by Kinoshita to derive an important bound[22] on absorptive amplitude, for $D = 4$, in the region $0 < t < t_0$. We recall that the Gegenbauer polynomials are the ultraspherical Jacobi polynomials and the distribution of the zeros of the Gegebauer polynomials is governed by the theorem stated above. Therefore, for the case at hand, the spacing of zeros (28) is also quite analogous. Factorizing $C_l^\lambda(z) = \mathbf{\Pi}_{\nu=1}^{l}[\frac{(z-z_\nu)}{1-z_\nu}]$, $z_\nu = \cos\theta_\nu$ are the location of the zeros. $A^\lambda(s,t)$ becomes a complex function when t is complex, i.e., when $C_l^\lambda(z)$ becomes complex. Thus, if $\mathbf{\Phi}$ is the phase of $C_l^\lambda(z)$ for $z = 1 + a + ib$, we look for the curve in z-plane which intersects the *ellipse* where $C_l^\lambda(z)$ hits its first zero, i.e., $\mathbf{\Phi} = \pi/2$. For complex t, decompose $t = u + iv$, u and v being the real and imaginary parts of t. There is a small domain, $\mathcal{D}$, in the upper half t plane, which is intersection of this curve with the *ellipse* such that $\text{Re}\, A^\lambda(s,t)$ is positive inside $\mathcal{D}$. Recall that as we go off real t axis into this domain, $A^\lambda(s,t)$ becomes complex. The arguments of Ref. 22 can be adopted to show that $\mathcal{D}$ is given by

$$|v| \le \frac{\pi\sqrt{u}}{2C_4 \ln s}. \tag{29}$$

C_4 is s-independent constant. Thus, $\text{Re}\, A^\lambda(s,t)$ is a positive, harmonic function inside $\mathcal{D}$. Consequently, a powerful theorem on positive harmonic functions holds.[23] Now define R_0, $0 < R_0 < T_0 - \delta$, such that it is within $\mathcal{D}$. Consider a disk $|t - R_0 < \pi(\sqrt{R_0})/(2C_4 \ln s)$ which is shrinking as $\ln s$, note that $t > 0$.

The *Harnack's theorem*[23] states that for any t in the smaller disk,

$$|t - R_0| < \frac{\pi r\sqrt{R_0}}{2C_4 \ln s}, \quad 0 < r < 1 \tag{30}$$

with r inside the disk, the positive, harmonic function, $\text{Re}\, A^\lambda(s,t)$ is bounded from above and below as

$$\frac{1-r}{1+r} A^\lambda(s,R_0) < \text{Re}\, A^\lambda(s,t) < \frac{1+r}{1-r} A^\lambda(s,R_0). \tag{31}$$

$A^\lambda(s,R_0)$ is real and positive as argued before. $A^\lambda(s,t)$ is defined inside the circle which lies in $\mathcal{D}$ and hence $\text{Re}\, A^\lambda(s,t) > 0$. Note that the radius is order $\frac{1}{\ln s}$; therefore, inequality (31) conveys that

Re $A^{\lambda}(s,t)$ will not increase by more than a finite factor when t is increased by $\frac{1}{\ln s}$. We can derive various bounds on $A^{\lambda}(s,t)$ and its derivatives through applications of theorem (31) inside disk (30).

An upper bound on $b(s)$ is derived utilizing bound (20) and the inequality $|F^{\lambda}(s,0)| \geq s\sigma_t$. It follows from Cauchy's inequality and suitable adaptation of the arguments of Refs. 21 and 24. If the zero-free radius shrinks less rapidly than (26), i.e., $r_0(s) > (\ln s)^{-2+\eta}$, $\eta > 0$, then bounds on $b(s)$ and σ_t are improved.[24]

We conclude this note with following remarks and observations in the context of higher dimensional theories. There are scenarios where higher dimensional theories admit a low compactification scale, $\sqrt{\hat{s}} \sim$ a few TeV. The phenomenological consequences of such models have been explored extensively.[25–28] Petrov[29] has argued that the cross section of a $D > 4$ theory will have same behavior as that of a $D = 4$ theory below the compactification scale. He supported this claim through model calculations. We mention in passing that the S-matrix has interesting analyticity properties in the energy plane in case of potential scattering when some spatial dimensions are compactified.[30,31] These issues have not been thoroughly investigated in field theoretic frame work.

Let us consider a higher dimensional theory with a low energy scale of compactification, $\hat{s}$. When the theory is probed with energies below the compactification scale, it will behave like a $D = 4$ theory as argued by Petrov. However, above the scale $\hat{s}$, σ_t of a $D > 4$ theory might behave as if (4) is violated, although σ_t does *not necessarily have to violate* $D = 4$ bound (4). Thus, it might be worth while to explore qualitatively whether σ_t admits an s-dependence: $(\ln \frac{s}{\hat{s}})^{\beta}$, $\beta > 2$ at the extremely high energies accessed by Large Hadron Collider (LHC) and in cosmic ray experiments. If the total cross section data for extreme high energies show such an energy dependence, then one might get a hint of decompactification at lower scale as alluded to earlier. Moreover, there are other avenues to test unitarity bounds on quantities listed in (5) for $D = 4$. They will be measured with accuracy at LHC from $s = 36\ \mathrm{TeV}^2$ to $s = 196\ \mathrm{TeV}^2$ at LHC. Moreover, $b(s)$ and $\mathbf{h}_d$ are bounded by $(\ln s)^2$; as $\mathbf{h}_d$ grows with energy, the width of $\frac{d\sigma}{dt}$ shrinks.[5,21] Thus, precision measurements of $b(s)$, $\mathbf{h}_d$, and other measurable quantities listed in (5) will stringently test unitarity constraints at LHC. An unambiguous deviation from unitarity bounds, derived for $D = 4$ theories, might provide an indirect evidence for existence of higher dimensional theories.

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