

Parallel Metric Tree Embedding based on an Algebraic View on Moore-Bellman-Ford

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Abstract

A *metric tree embedding* of expected *stretch* α maps a weighted n -node graph $G = (V, E, \omega)$ to a weighted tree $T = (V_T, E_T, \omega_T)$ with $V \subseteq V_T$ such that $\text{dist}(v, w, G) \leq \text{dist}(v, w, T)$ and $\mathbb{E}[\text{dist}(v, w, T)] \leq \alpha \text{dist}(v, w, G)$ for all $v, w \in V$. Such embeddings are highly useful for designing fast approximation algorithms, as many hard problems are easy to solve on tree instances. However, to date the best parallel polylog n depth algorithm that achieves an asymptotically optimal expected stretch of $\alpha \in O(\log n)$ requires $\Omega(n^2)$ work and requires a metric as input.

In this paper, we show how to achieve the same guarantees using $\tilde{O}(m^{1+\varepsilon})$ work, where m is the number of edges of G and $\varepsilon > 0$ is an arbitrarily small constant. Moreover, one may reduce the work further to $\tilde{O}(m + n^{1+\varepsilon})$, at the expense of increasing the expected stretch α to $O(\varepsilon^{-1} \log n)$. Our main tool in deriving these parallel algorithms is an algebraic characterization of a generalization of the classic Moore-Bellman-Ford algorithm. We consider this framework, which subsumes a variety of previous “Moore-Bellman-Ford-flavored” algorithms, to be of independent interest.

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1 Introduction

In many fundamental graph problems the objective is closely related to distances in the graph. Prominent examples are shortest path problems, minimum weight spanning trees, a plethora of Steiner-type problems [22], the traveling salesman, finding a longest simple path, and many more. The hardness of such problems ranges from P over NP-complete to APX-hard.

If approximation is viable or mandatory, a successful strategy is to approximate the distance structure of the weighted graph $G = (V, E, \omega)$ by a simpler graph G' , where “simpler” can mean fewer edges, smaller degrees, being from a specific family of graphs, or any other constraint making the considered problem easier to solve. One then proceeds to solve a related instance of the problem on the simpler graph and maps the solution back, yielding an approximate solution to the original instance. Naturally, this requires a mapping between solutions in G and G' with bounded impact on the objective value.

A standard tool are *metric embeddings*, mapping G to $G' = (V', E', \omega')$, such that $V \subseteq V'$ and $\text{dist}(v, w, G) \leq \text{dist}(v, w, G') \leq \alpha \text{dist}(v, w, G)$ for some $\alpha \geq 1$ referred to as *stretch*.¹ An especially convenient class of metric embeddings are *metric tree embeddings*, plainly because very few problems are hard to solve on trees. The utility of tree embeddings originates in the fact that, despite their extremely simple topology, it is possible to randomly construct an embedding of any graph G into a tree T so that the *expected stretch* $\alpha = \max\{\mathbb{E}_T[\text{dist}(v, w, T)] / \text{dist}(v, w, G) \mid v, w \in V\}$ satisfies $\alpha \in O(\log n)$ [18]. By linearity of expectation, for most problems this ensures an expected approximation ratio of $O(\log n)$, and by repeating the process $\log(\varepsilon^{-1})$ times and taking the best result, one obtains an $O(\log n)$ -approximation with probability at least $1 - \varepsilon$.

A substantial advantage lies in the simplicity of applying the machinery once the tree embedding is computed: Translating the instance on G to one on T , solving the instance on T , and translating the solution back tends to be extremely efficient and highly parallelizable, compare Appendix B. Note also that the embedding can be computed as a preprocessing step, which is highly useful for online approximation algorithms [18]. Hence, a low-depth small-work parallel algorithm constructing a tree embedding in the vein of Fakcharoenphol et al. [18], henceforth *FRT embedding*, would give rise to fast and efficient parallel approximations for a large class of graph problems. Unfortunately, the degree of concurrent parallelism and efficiency, i.e., trade-off between depth and work, that is achieved by state-of-the-art parallel algorithms for this purpose is suboptimal. Concretely, all algorithms of polylog n depth use $\Omega(n^2)$ work, whereas we are not aware of any stronger lower bound than the trivial $\Omega(m)$ work bound.²

Our Contribution Reducing the amount of work to obtain an FRT embedding in graphs with $m \ll n^2$ without sacrificing the depth bound of polylog n is the primary motivation of this paper. We present the following results.

- For any constant $\varepsilon > 0$, there is a randomized parallel algorithm of depth polylog n and work $\tilde{O}(m^{1+\varepsilon})$ that computes a metric tree embedding of expected stretch $O(\log n)$ w.h.p.
- The work can be reduced to $\tilde{O}(m + n^{1+O(\varepsilon)})$ when stretch $O(\varepsilon^{-1} \log n)$ is acceptable, by applying the spanner construction of Baswana and Sen [8] as a preprocessing step.

¹By $\text{dist}(v, w, G)$ we refer to the distance between v and w in G ; $\text{dist}(\cdot, \cdot, G)$ gives rise to a metric space.

²Partition $V = A \dot{\cup} B$ evenly, and add spanning trees of A and B consisting of edges of weight 1. Connect A and B with $m - n - 2$ edges, either all of weight $W \gg n$ (w.p. $1/2$), or all of weight W but one uniformly chosen weight-1 edge. To approximate the distance between $a \in A$ and $b \in B$ better than factor $W/n \gg \log n$ with probability substantially larger than $1/2$, any algorithm must examine $\Omega(m)$ edges in expectation.

- A key tool in achieving this goal is an algebraic interpretation of “Moore-Bellman-Ford-like” (MBF-like) algorithms. Since our framework subsumes a large class of previous algorithms and explains them from a different perspective, we consider it to be of independent interest.
- We illustrate the utility of our main results by providing efficient approximation algorithms for k -median and buy-at-bulk network design in Appendix B.
- Furthermore, our techniques allow to improve over previous distributed algorithms computing tree embeddings in the Congest³ model [33]; this is detailed in Appendix C.

Our Approach Khan et al. show how to compute an FRT embedding by a distributed algorithm [25] in the Congest model [33]. Their algorithm gives rise to an $\tilde{O}(\text{SPD}(G))$ depth parallel algorithm, where the *shortest-path diameter (SPD)* is the maximum over all pairs of nodes $v, w \in V$ of the minimum hop length of a path p from v to w with $\omega(p) = \text{dist}(v, w, G)$. Intuitively, $\text{SPD}(G)$ captures the number of iterations of MBF-like algorithms: Each iteration updates distances (see below), so the $(\text{SPD}(G) + 1)$ -th iteration does not yield new information. Unfortunately, $\text{SPD}(G) = n - 1$ is possible, so applying this algorithm naively results in poor performance.

A natural idea is to reduce the number of iterations the algorithm by Khan et al. requires by adding “shortcuts” to the graph. Cohen [13] provides an algorithm of depth $\text{polylog } n$ and work $m^{1+O(\varepsilon)}$ that computes a (d, ε) -hop set with $d \in \text{polylog } n$: a set of $n^{1+O(\varepsilon)}$ additional edges E' such that $\text{dist}(v, w, G) \leq \text{dist}(v, w, G') \leq \text{dist}^d(v, w, G') \leq (1 + \varepsilon) \text{dist}(v, w, G)$ for all $v, w \in V$, where G' is G augmented with E' and $\text{dist}^d(v, w, G')$ is the minimum weight of a v - w -path with at most d edges in G' . In other words, Cohen computes a metric embedding with the additional property that polylogarithmically many MBF-like iterations suffice to determine $(1 + \varepsilon)$ -approximate distances. The course of action might now seem obvious: Run Cohen’s algorithm, then run the algorithm by Khan et al. on the resulting graph for $d \in \text{polylog } n$ rounds, and conclude that the resulting output corresponds to a tree embedding of the original graph G of stretch $O((1 + \varepsilon) \log n) = O(\log n)$.

Alas, this reasoning is flawed: The FRT construction crucially relies on the fact that the distances form a metric, i.e., satisfy the triangle inequality. An approximate triangle inequality for approximate distances is insufficient, since the FRT construction relies on the *subtractive* form of the triangle inequality, i.e., $\text{dist}(v, w, G') - \text{dist}(v, u, G') \leq \text{dist}(w, u, G')$ for arbitrary $v, w, u \in V$. We overcome this obstacle by embedding G' into a complete graph H on the same node set that $(1 + o(1))$ -approximates distances in G but fulfills $\text{SPD}(H) \in \text{polylog } n$. In other words, where Cohen preserves distances *exactly* and ensures existence of *approximately* shortest paths with few hops, we preserve *distances* approximately, but guarantee that we obtain *exact* shortest paths with few hops. Unfortunately, this introduces a new obstacle: As H is complete, we cannot explicitly compute H without incurring $\Omega(n^2)$ work.

MBF-like Algorithms This is where our novel perspective on MBF-like algorithms comes into play. It is well-known [2, 35] that distance computations can be performed by multiplication with the (weighted) adjacency matrix A over the min-plus semiring⁴ $\mathcal{S}_{\min,+} = (\mathbb{R}_{\geq 0} \cup \{\infty\}, \min, +)$. For instance, if $B = A^h$ with $h \geq \text{SPD}(G)$, then $b_{vw} = \text{dist}(v, w, G)$.

In an iteration of an MBF-like algorithm, (1) the information stored at each node is *propagated* to its neighbors, (2) each node *aggregates* the received information, and (3) optionally *filters* out

³In the Congest model, the nodes of the graph perform the computation, communicating over the graphs edges in synchronous rounds. Nodes initially know their neighborhood in the graph and need to determine the local part of the output only. Local computation is free; the main limitation is that messages are of size $O(\log n)$.

⁴A semiring is a ring without inverse elements w.r.t. the semiring’s “addition” (here \min).

uninteresting parts. For example, in order for each node to determine the k nodes closest to it, each node stores vertex/distance pairs (initially only themselves at distance 0), (1) communicates them to its neighbors (distances uniformly increased by the corresponding edge weight), (2) aggregates the received values by picking the vertex-wise minimum, and (3) discards all but the pairs but the k corresponding to the closest sources.

In terms of the semiring, propagation is the “multiplication” with an edge weight, and aggregation is “summation.” The $(i + 1)$ -st iteration results in $x^{(i+1)} = r^V A x^{(i)}$, where r^V is the (vertex-wise) filter, A as above, and $x \in M^V$ the node values. Both M and M^V form semimodules⁵ over $\mathcal{S}_{\min,+}$.

In other words, in an h -iteration MBF-like algorithm each node determines its part of the output based on its h -hop distances to all other nodes. However, for efficiency reasons, various algorithms [3, 6, 7, 24, 27, 28, 29] compute only a subset of these distances. The role of the filter is to remove the remaining values to allow for better efficiency. The core feature of an MBF-like algorithm is that filtering is *compatible* with propagation and aggregation: If a node discards information and then propagates it, the discarded parts must be “uninteresting” at the receiving node as well. We model this as *equivalence classes* of node states; filters simply pick suitable (efficiently encodable) representatives of such classes, and the equivalence relation has to be a *congruence relation* on the semimodule of possible node states.

This helps us to determine an FRT embedding as follows. First, we observe that acquiring the information required to represent an FRT embedding can be done by an MBF-like algorithm. Second, we can *simulate* any MBF-like algorithm on H , without explicitly storing H , using polylogarithmic overhead and MBF-iterations on G' : Due to the way H is constructed, we can rewrite its adjacency matrix as $A_H = \bigoplus_{\lambda=0}^{\Lambda} P_{\lambda} A_{\lambda}^d P_{\lambda}$, where \bigoplus is the “addition” of functions induced by the semimodule, P_{λ} is a simple projection, A_{λ} is essentially the adjacency matrix of G' , and $\Lambda \in O(\log n)$. Because we are interested in $r^V A_H^h x^{(0)}$, we may compute $(r^V \bigoplus_{\lambda=0}^{\Lambda} P_{\lambda} (r^V A_{\lambda})^d P_{\lambda})^h x^{(0)}$ instead: r^V is a projection that preserves equivalence classes, so we may apply it to intermediate results. Note that applying $(r^V A_{\lambda})^d$ basically means to iterate d times on G' , and the repeated application of r^V keeps the intermediate results small, ensuring that we can perform multiplication with A_{λ} with $\tilde{O}(|E| + |E'|) \subseteq \tilde{O}(m^{1+\epsilon})$ work. As $d, h \in \text{polylog } n$, this yields a highly efficient parallel algorithm of depth $\text{polylog } n$ and work $\tilde{O}(m^{1+\epsilon})$.

1.1 Related Work

We confine the discussion to undirected graphs.

Classic Algorithms for Distance Computation The earliest—and possibly also most basic—algorithms for single-source shortest paths (SSSP) computation are Dijkstra’s algorithm [17] and the Moore-Bellman-Ford (henceforth MBF) algorithm [9, 19, 32]. From the perspective of parallel algorithms, Dijkstra’s algorithm performs excellent in terms of work, requiring $\tilde{O}(m)$ computational steps, but suffers from being inherently sequential, processing one vertex at a time.

While the MBF algorithm also has depth $\Omega(n)$ in the worst case and may require $\Omega(nm)$ work, it can perform substantially better. Concretely, the MBF algorithm can be interpreted as a fixpoint iteration $Ax^{(i+1)} = Ax^{(i)}$, where A is the adjacency matrix of the graph G and “multiplication” and “addition” are replaced by \cdot and \min , respectively (compare Section 1.2). From this point of view, the *shortest path diameter (SPD)* of G , $\text{SPD}(G)$, is the number of iterations until a fixpoint

⁵A semimodule supports scalar multiplication (propagation) and provides a semigroup (representing aggregation).

is reached. More precisely, MBF thus has depth $\tilde{O}(\text{SPD}(G))$ and work $\tilde{O}(m \text{SPD}(G))$, and it is possible that $\text{SPD}(G) \in O(1)$.

One may overcome the issue of large depth entirely by performing the fixpoint iteration on the matrix instead: setting $A^{(0)} := A$ and iterating $A^{(i+1)} := A^{(i)}A^{(i)}$, after $\lceil \log \text{SPD} \rceil \leq \lceil \log n \rceil$ iterations a fixpoint is reached [15]. The final matrix then, in fact, has as entries exactly the pairwise node distances, and the computation has polylogarithmic depth. This comes at the cost of $\Omega(n^3)$ work (even if $m \ll n^2$), but in dense graphs it is as efficient in terms of work as (n instances of) Dijkstra’s algorithm for solving APSP, without having depth $\Omega(n)$.

Approximate Distances As metric embeddings reproduce distances only approximately, we may base them on approximate distance computation in the original graph. Using rounding techniques and embedding the min-plus semiring $\mathcal{S}_{\min,+} = (\mathbb{R}_{\geq 0} \cup \{\infty\}, \min, +)$, see Section 1.2, into a polynomial ring, this enables to use fast matrix multiplication to speed up the aforementioned fixpoint iteration $A^{(i+1)} := A^{(i)}A^{(i)}$ [35]. This reduces the work to $\tilde{O}(n^\omega)$ at the expense of approximating distances up to factor $(1 + o(1))$ only, where $\omega < 2.3729$ [20] denotes the fast matrix multiplication exponent. However, even if the conjecture that $\omega = 2$ holds true, this technique must result in $\Omega(n^2)$ work, simply because $\Omega(n^2)$ pairwise distances are computed.

For SSSP, for a long time there was no work-efficient low-depth parallel algorithm, even when allowing for approximation. This was referred to as the “sequential bottleneck:” matrix-matrix multiplication was inefficient in terms of work, while exploring (shortest) paths sequentially resulted in depth $\Omega(\text{SPD}(G))$. Klein and Subramanian [26] showed that depth $\tilde{O}(\sqrt{n})$ can be achieved with $\tilde{O}(m\sqrt{n})$ work, beating the n^2 work barrier with sublinear depth in sparse graphs. As an aside, similar bounds were later achieved for exact SSSP computation by Shi and Spencer [34].

In a seminal paper, Cohen [13] proved that SSSP can be approximately solved with depth polylog n and near-optimal $\tilde{O}(m^{1+\varepsilon})$ work, for any constant choice of $\varepsilon > 0$. Her approach is based on an efficient randomized parallel hop set construction, adding $\tilde{O}(m^{1+\varepsilon})$ edges so that, for some $d \in \text{polylog } n$, it holds that d -hop distances are at most by factor $1 + 1/\text{polylog } n$ larger (and never smaller) than the original distances. As d MBF iterations in the augmented graph compute these d -hop distances, this yields an efficient SSSP approximation algorithm. Perhaps surprisingly, similar guarantees can be achieved deterministically. In a recent result, Henzinger et al. [24] show that a hop set yielding $(1 + 1/\text{polylog } n)$ -approximate distances for $d \in 2^{O(\sqrt{\log n})} \subset n^{o(1)}$ can be computed with depth $2^{O(\sqrt{\log n})} \subset n^{o(1)}$ and work $m2^{O(\sqrt{\log n})} \subset m^{1+o(1)}$.⁶

Metric Tree Embeddings When metrically embedding into a tree, in general it is impossible to guarantee a small stretch. For instance, when the graph is a cycle with unit edge weights, it is impossible to embed it into a tree without having at least one edge with stretch $\Omega(n)$. However, *on average* the edges in this example are stretched by a constant factor only, justifying the hope that one may be able to randomly embed into a tree such that, for each pair of nodes, the *expected* stretch is small. A number of elegant algorithms [3, 6, 7, 18] compute tree embeddings, culminating in the one by Fakcharoenphol, Rao, and Talwar (FRT) [18] that achieves stretch $O(\log n)$ in expectation. This stretch bound is optimal in the worst case, as illustrated by expander graphs [7]. Mendel and Schwob show how to sample from the FRT distribution in $\tilde{O}(m)$ steps [30], matching the trivial $\Omega(m)$ lower bound up to polylogarithmic factors. However, their approach relies on a pruned version of Dijkstra’s algorithm for distance computations and hence does not lead to a low-depth parallel algorithm.

⁶The authors focus on Congest algorithms, which can be interpreted in our framework to infer the stated bounds.

Several parallel and distributed algorithms compute FRT embeddings [10, 21, 25]. These algorithms and ours have in common that they (essentially) represent the embedding by *least element (LE) lists*, which were first introduced in [12, 14]. In the parallel case, the state-of-the-art solution due to Blelloch et al. [10] achieves $O(\log^2 n)$ depth and $O(n^2 \log n)$ work. However, Blelloch et al. assume the input to be given as an n -point *metric*, where the distance between two points can be queried at constant cost. We note that our approach is more general, as we could interpret the metric as a complete weighted graph (using the query access to determine edge weights) of shortest path diameter 1 and then solve in a single MBF iteration; this reproduces the result by Blelloch et al. Moreover, this point of view shows that it is necessary that the input is a (sparse) graph to achieve subquadratic work: in case of a metric, the trivial work lower bound of $\Omega(m)$ always translates to $\Omega(n^2)$. For graph inputs, we are not aware of any algorithms achieving polylog n depth at a non-trivial work bound, i.e., not incurring the $\Omega(n^3)$ work caused by relying on matrix-matrix multiplication.

In the distributed setting, Khan et al. [25] show how to compute LE lists in $O(\text{SPD}(G) \log n)$ rounds in the Congest model [33]. On the lower bound side, trivially $\Omega(D)$ rounds are required, where D is the maximum hop distance (i.e., ignoring weights) between nodes. However, even if $D \in O(\log n)$, $\tilde{\Omega}(\sqrt{n})$ rounds are necessary [16, 21]. Extending the algorithm by Khan et al., in [21] it is shown how to obtain a round complexity of $\tilde{O}(\min\{n^{1/2+\varepsilon}, \text{SPD}(G)\} + D)$ for any $\varepsilon > 0$, at the expense of increasing the stretch to $O(\varepsilon^{-1} \log n)$. We partly build on these ideas; specifically, the construction in Section 3 can be seen as a generalization of the key technique from [21]. As detailed in Appendix C, our framework subsumes these algorithms and can be used to improve on the result from [21]: Leveraging further recent results [24, 29], we obtain a metric tree embedding with expected stretch $O(\log n)$ that is computed in $\min\{n^{1/2+o(1)} + D^{1+o(1)}, \tilde{O}(\text{SPD}(G))\}$ rounds.

1.2 Notation and Preliminaries

We consider weighted, undirected graphs $G = (V, E, \omega)$ without loops or parallel edges. Unless specified otherwise, we set $n := |V|$, $m := |E|$, and $V = \{v_1, \dots, v_n\}$. For an edge $e = \{v, w\} \in E$, we write $\omega(v, w) := \omega(e)$, and fix $\omega(v, v) := 0$ for $v \in V$ and $\omega(v, w) := \infty$ for $\{v, w\} \notin E$. We assume that the ratio between maximum and minimum edge weight is polynomially bounded in n and that each edge weight and each constant can be stored with sufficient precision in a single register.⁷ We assume that G is given in the form of an adjacency list. W.l.o.g., we assume that there are no isolated nodes; in particular, $n \in O(m)$.

Let $p \subseteq E$ be a path. p has $|p|$ hops, and *weight* $\omega(p) := \sum_{e \in p} \omega(e)$. For the nodes $v, w \in V$ let $P(v, w, G)$ denote the set of paths from v to w , and $P^h(v, w, G)$ the set of such paths using at most h hops. We denote by $\text{dist}^h(v, w, G) := \min\{\omega(p) \mid p \in P^h(v, w, G)\}$ the minimum weight of an h -hop path from v to w , where (by slight abuse of notation) $\min \emptyset := \infty$; the *distance* between v and w is $\text{dist}(v, w, G) := \text{dist}^n(v, w, G)$. The *shortest path hop distance* between v and w is $\text{hop}(v, w, G) := \min\{|p| \mid p \in P(v, w, G) \wedge \omega(p) = \text{dist}(v, w, G)\}$; $\text{MHSP}(v, w, G) := \{p \in P^{\text{hop}(v, w, G)}(v, w, G) \mid \omega(p) = \text{dist}(v, w, G)\}$ denotes all *min-hop shortest paths* from v to w . Finally, the *shortest path diameter (SPD)* of G is $\text{SPD}(G) := \max\{\text{hop}(v, w, G) \mid v, w \in V\}$.

We assume $0 \in \mathbb{N}$ and denote for $k \in \mathbb{N}$ by $[k] := \{0, \dots, k\}$. For a set N and $k \in \mathbb{N}$ we define $\binom{N}{k} := \{M \subseteq N \mid |M| = k\}$ and denote by $\text{id}: N \rightarrow N$ the identity function. We use weak asymptotic notation hiding polylogarithmic factors in n : $O(f(n) \text{ polylog}(n)) = \tilde{O}(f(n))$, etc.

⁷As we are interested in approximation algorithms, $O(\log n)$ bits suffice to encode values with sufficient precision.

Model of Computation We use an abstract model of parallel computation similar to those used in circuit complexity.⁸ The computation is represented by a directed acyclic graph (DAG) with constantly bounded maximum indegree, where nodes represent words of memory that are given as input (indegree 0) or computed out of previously determined memory contents (non-zero indegree). When a word is computed, this happens with a constant number of basic instructions, e.g., addition, multiplication, comparison, etc.; here, we also allow for the use of (independent) randomness. For simplicity, a memory word may hold any number computed throughout the algorithm. As pointed out earlier, for our purposes $O(\log n)$ -bit words suffice in all cases.

An algorithm defines, given the input, the DAG and how the nodes' content is computed, as well as which nodes represent the output. Given an instance of the problem, the *work* is the number of nodes of the corresponding DAG and the *depth* is the longest path in the graph. Assuming that there are no read or write conflicts, the work is thus (proportional to) the time required by a single processor (of uniform speed) to complete the computation, whereas the depth is the time required by an infinite number of processors. Note that the DAG may be a random graph, as the algorithm may use randomness, implying that work and depth may be random variables. When making probabilistic statements, we require that they hold for all instances, i.e., the respective probability bounds are satisfied after fixing an arbitrary instance.

Probability A claim holds *with high probability (w.h.p.)*, if it occurs with a probability of at least $1 - n^{-c}$ for any fixed choice of $c \in \mathbb{R}_{\geq 1}$; c is a constant in terms of the O -notation. We use the following basic statement frequently and implicitly throughout this paper.

Lemma 1.1. *Let $\mathcal{E}_1, \dots, \mathcal{E}_k$ be events occurring w.h.p., and $k \in \text{poly } n$. $\mathcal{E}_1 \cap \dots \cap \mathcal{E}_k$ occurs w.h.p.*

Proof. We have $k \leq an^b$ for fixed $a, b \in \mathbb{R}_{>0}$, and choose that all \mathcal{E}_i occur with probability $1 - n^{-c'}$ with $c' = c + b + \log_n a$ for some fixed $c \geq 1$. The union bound yields

$$\mathbb{P}[\overline{\mathcal{E}_1 \cap \dots \cap \mathcal{E}_k}] \leq \sum_{i=1}^k \mathbb{P}[\overline{\mathcal{E}_i}] \leq kn^{-c'} = an^b n^{-c-b-\log_n a} = n^{-c}, \quad (1.1)$$

and $\mathcal{E}_1 \cap \dots \cap \mathcal{E}_k$ occurs w.h.p. as claimed. \square

Hop Sets A graph $G = (V, E, \omega)$, contains a (d, ε) -hop set if

$$\forall v, w \in V: \quad \text{dist}^d(v, w, G) \leq (1 + \varepsilon) \text{dist}(v, w, G), \quad (1.2)$$

i.e., if its d -hop distances are a $(1 + \varepsilon)$ -approximation of its distances. This definition is based on Cohen [13], who describes how to efficiently add edges to G to establish this property.

Distance Metrics The min-plus algebra $\mathcal{S}_{\min,+} = (\mathbb{R}_{\geq 0} \cup \{\infty\}, \min, +)$ forms a semiring, i.e., a ring without additive inverses (see Definition A.2 in Appendix A). For the sake of presentation, we associate \oplus and \odot with the addition and multiplication of the underlying ring throughout the paper; in this case we use $a \oplus b := \min\{a, b\}$ and $a \odot b := a + b$. Observe that ∞ and 0 are the neutral elements w.r.t. \oplus and \odot , respectively. We follow the standard convention to occasionally leave out \odot and give it priority over \oplus , e.g., interpret $ab \oplus c$ as $(a \odot b) \oplus c$ for all $a, b, c \in \mathbb{R}_{\geq 0} \cup \{\infty\}$.

⁸The goal here is to avoid being distracted by details such as read or write collisions or load balancing issues typical to PRAM models, noting that these can be resolved with (at most) logarithmic overheads.

The min-plus algebra is a well-established tool to determine pairwise distances in a graph via the distance product, see e.g. [2, 35]. Let $G = (V, E, \omega)$ be a weighted graph and let $A \in (\mathbb{R}_{\geq 0} \cup \{\infty\})^{V \times V}$ be its *adjacency matrix* A , given by

$$a_{vw} = \omega(v, w). \quad (1.3)$$

Throughout this paper, the operations involved in matrix multiplication are induced by the ring operations of $\mathcal{S}_{\min,+}$, i.e., for square matrices A, B with row and column index set V , we have

$$(A \oplus B)_{vw} = \min\{a_{vw}, b_{vw}\}, \text{ and} \quad (1.4)$$

$$(AB)_{vw} = \min_{u \in V} \{a_{vu} + b_{uw}\}. \quad (1.5)$$

The *distance product* A^h corresponds to h -hop distances, i.e., $(A^h)_{vw} = \text{dist}^h(v, w, G)$ [2]. In particular, this corresponds to the exact distances between all pairs of nodes for $h \geq \text{SPD}(G)$.

2 MBF-like Algorithms

The Moore-Bellman-Ford (MBF) algorithm [9, 19, 32] is both fundamental and elegant. In its classical form, it solves the single-source shortest path (SSSP) problem: In each iteration, each node communicates its current upper bound on its distance to the source node s (initially ∞ at all nodes but s) plus the corresponding edge weight to its neighbors, which then keep the minimum of the received values and their previously stored one. Iterating h times determines all nodes' h -hop distances to s .

Over the years, numerous algorithms emerged that use similar iterative schemes for distributing information [3, 6, 7, 18, 24, 27, 28, 29]. It is natural to ask for a characterization that captures all these algorithms. In this section, we propose such a characterization: the class of *MBF-like algorithms*. The common denominator of these algorithms is the following:

- The initial state vector $x^{(0)}$ contains information initially known to the nodes.
- In each *iteration*, information is first *propagated* along all edges.
- All nodes then *aggregate* the received information. This and the previous step are precisely the same as updating the state vector $x^{(i)}$ by the matrix-vector product $x^{(i+1)} = Ax^{(i)}$ over the min-plus semiring.
- Finally, redundancies are *filtered* out before moving on to the next iteration.

As a concrete example, consider the task of determining for each node the list of its k closest nodes. To this end, one needs to consider all nodes as sources, i.e., run the multi-source variant of the classic MBF algorithm with all nodes as sources. Nodes store values in $(\mathbb{R}_{\geq 0} \cup \{\infty\})^V$, so that in iteration i each node $v \in V$ knows $\text{dist}^i(v, w, G) \in \mathbb{R}_{\geq 0} \cup \{\infty\}$ for all $w \in V$. Initially, $x_{vw}^{(0)}$ is 0 if $v = w$ and ∞ everywhere else (the 0-hop distances). *Propagating* these distances over an edge of weight $\omega(e)$ means increasing all these distances by $\omega(e)$. During *aggregation*, the vertex-wise minimal distance is picked. This is costly due to possibly maintaining at each node non- ∞ distance values for *all* other nodes. To increase efficiency, we *filter* out, in each iteration and at each node, all source/distance pairs but the k pairs with smallest distance. This reduces the amount of work by a factor of $\Theta(n/k)$ compared to the naive approach.

The filtering step thus generalizes from classic MBF to an “MBF-like” algorithm, with the goal of reducing work. The crucial characteristics exploited by this idea are the following.

- Propagation and aggregation are interchangeable. It makes no difference whether two pieces of information are propagated separately or as a single aggregated piece of information.
- Filtering or not filtering after aggregation has no impact on the correctness (i.e., the output), only on efficiency.

In this section, we formalize this approach for later use in more advanced algorithms. To this end, we develop a characterization of MBF-like algorithms in Sections 2.1 – 2.3. Sections 2.4 and 2.5 establish basic properties and apply our characterization to existing algorithms.

2.1 Propagation and Aggregation

Let M be the set of possible values that an MBF-like algorithm can store at a vertex. We represent propagation of $x \in M$ over an edge of weight $s \in \mathbb{R}_{\geq 0} \cup \{\infty\}$ by $s \odot x$ and aggregation of $x, y \in M$ at some node by $x \oplus y$; filtering is discussed in Section 2.2. Concerning aggregation of information, we demand that $\oplus: M \times M \rightarrow M$ is associative and has a neutral element $\perp \in M$ encoding no available information, hence (M, \oplus) is a semigroup with neutral element \perp .⁹ For propagation with $\odot: \mathbb{R}_{\geq 0} \cup \{\infty\} \times M \rightarrow M$, we require for all $s, t \in \mathbb{R}_{\geq 0} \cup \{\infty\}$ and $x, y \in M$:

$$0 \odot x = x \tag{2.1}$$

$$\infty \odot x = \perp \tag{2.2}$$

$$s \odot (x \oplus y) = (s \odot x) \oplus (s \odot y) \tag{2.3}$$

$$(s \oplus t) \odot x = (s \odot x) \oplus (t \odot x) \tag{2.4}$$

$$(s \odot t) \odot x = s \odot (t \odot x). \tag{2.5}$$

Our requirements are quite natural: Equations (2.1) and (2.2) state that propagating information over zero distance does not alter it, and that propagating it infinitely far away means losing it, respectively. Note that 0 and ∞ are the neutral elements w.r.t. \odot and \oplus in $\mathcal{S}_{\min,+}$. Equation (2.3) says that propagating aggregated information is equivalent to aggregating propagated information, Equation (2.4) means that propagating information over a shorter of two edges is equivalent to moving it along both edges and then aggregating it, and Equation (2.5) states that propagating propagated information can be combined in one step.

Altogether, this is equivalent to demanding that $\mathcal{M} = (M, \oplus, \odot)$ is a zero-preserving semimodule over $\mathcal{S}_{\min,+}$ (see Definition A.3 in Appendix A). For simplicity, in the following we fix the semiring to always be $\mathcal{S}_{\min,+}$,¹⁰ i.e., whenever referring to a semimodule, it is over $\mathcal{S}_{\min,+}$. Moreover, we write $x \in \mathcal{M}$ to indicate that $x \in M$ and \oplus and \odot operations involving x are those of \mathcal{M} .

The straightforward choice of \mathcal{M} being the direct product of $|V|$ copies of $\mathbb{R}_{\geq 0} \cup \{\infty\}$ is suitable for most applications we consider.

Definition 2.1 (Distance Map). *The distance map semimodule $\mathcal{D} := ((\mathbb{R}_{\geq 0} \cup \{\infty\})^V, \oplus, \odot)$ is given by, for all $s \in \mathbb{R}_{\geq 0} \cup \{\infty\}$ and $x, y \in (\mathbb{R}_{\geq 0} \cup \{\infty\})^V$,*

$$(x \oplus y)_v = \min\{x_v, y_v\} \tag{2.6}$$

$$(s \odot x)_v = s + x_v \tag{2.7}$$

where $\perp := (\infty, \dots, \infty) \in (\mathbb{R}_{\geq 0} \cup \{\infty\})^V$ is the neutral element w.r.t. \oplus .

⁹Note that we “overload” the symbols for \oplus and \odot , as it is clear from context which operation is meant.

¹⁰In principle, it is possible use different semirings. The generalization is straightforward.

Observation 2.2. \mathcal{D} is a zero-preserving semimodule with zero $\perp = (\infty, \dots, \infty)$.

Distance maps can be represented by only storing those distances (and their indices from V) that are not ∞ . This is of interest when there are few non- ∞ entries, which can be ensured by filtering (see below). In the following, we denote by $|x|$ the number of non- ∞ entries of $x \in \mathcal{D}$. The following lemma shows that it is possible to efficiently aggregate distance maps represented like this.

Lemma 2.3. *Suppose $x_1, \dots, x_n \in \mathcal{D}$ are stored in lists of index/distance pairs as above. Then $\bigoplus_{i=1}^n x_i$ can be computed with $O(\log n)$ depth and $O(\sum_{i=1}^n |x_i| \log n)$ work.*

Proof. We sort $\bigcup_{i=1}^n x_i$ in ascending lexicographical order. This can be done in parallel with $O(\log(\sum_{i=1}^n |x_i|)) \subseteq O(\log n)$ depth and $O(\sum_{i=1}^n |x_i| \log n)$ work [1]. Then we delete each pair for which the next smaller pair has the same index; the resulting list hence contains, for each $v \in V$ for which there is a non- ∞ value in some list x_i , the minimum such value. As this operation is easy to implement with $O(\log n)$ depth and $O(\sum_{i=1}^n |x_i| \log n)$ work, the claim follows. \square

The remaining statements of this section are not specific to \mathcal{D} , so we make them with respect to a generic semimodule \mathcal{M} .

2.2 Filters

MBF-like algorithms maintain efficiency by propagating only a filtered (small) representative of the information they obtained. Our next goal is to capture the properties a suitable filter must satisfy for the intuition of filtering being optional and not affecting output correctness. We start from a congruence relation on \mathcal{M} , i.e., an equivalence relation that is compatible with propagation and aggregation. The filter $r: M \rightarrow M$ is a function that projects all members of an equivalence class to an appropriate representative within that class, compare Definition 2.6.

Definition 2.4 (Congruence Relation). *Let $\mathcal{M} = (M, \oplus, \odot)$ be a semimodule and \sim an equivalence relation on M . We call \sim a congruence relation on \mathcal{M} iff*

$$\forall s \in \mathbb{R}_{\geq 0} \cup \{\infty\}, \forall x, x' \in M: \quad x \sim x' \Rightarrow sx \sim sx' \quad (2.8)$$

$$\forall x, x', y, y' \in M: \quad x \sim x' \wedge y \sim y' \Rightarrow x \oplus y \sim x' \oplus y'. \quad (2.9)$$

A congruence relation induces a quotient semimodule.

Observation 2.5. *Denote by $[x]$ the equivalence class of $x \in M$ under the congruence relation \sim of semimodule $\mathcal{M} = (M, \oplus, \odot)$. Set $M/\sim := \{[x] \mid x \in M\}$. Then $\mathcal{M}/\sim = (M/\sim, \oplus, \odot)$ is a semimodule with the operations $[x] \oplus [y] := [x \oplus y]$ and $s \odot [x] = [sx]$.*

An MBF-like algorithm implicitly operates on this quotient semimodule, meaning that it operates on suitable representatives of the equivalence classes that allow efficient computations. Obtaining such representatives is the task of the filtering step.

Definition 2.6 (Representative Projection). *Let $\mathcal{M} = (M, \oplus, \odot)$ be a semimodule and \sim a congruence relation on \mathcal{M} . The function $r: M \rightarrow M$ is a representative projection w.r.t. \sim iff*

$$\forall x \in M: \quad x \sim r(x) \quad (2.10)$$

$$\forall x, y \in M: \quad x \sim y \Rightarrow r(x) = r(y). \quad (2.11)$$

Observation 2.7. *A representative projection is a projection, i.e., $r^2 = r$.*

The following lemma is useful for showing whether equivalence classes defined by a projection yields a congruence relation, i.e., is suitable for MBF-like algorithms.

Lemma 2.8. *Let $r: M \rightarrow M$ be a projection, and for all $x, y \in M$, $x \sim y \Leftrightarrow r(x) = r(y)$. Then \sim is a congruence relation with representative projection r if:*

$$\forall s \in \mathbb{R}_{\geq 0} \cup \{\infty\}, \forall x, x' \in M: \quad r(x) = r(x') \Rightarrow r(sx) = r(sx'), \text{ and} \quad (2.12)$$

$$\forall x, x', y, y' \in M: \quad r(x) = r(x') \wedge r(y) = r(y') \Rightarrow r(x \oplus y) = r(x' \oplus y'). \quad (2.13)$$

Proof. Obviously, \sim is an equivalence relation, and r fulfills (2.10) and (2.11). Conditions (2.8) and (2.9) follow from the preconditions of the lemma. \square

An MBF-like algorithm has to behave in a compatible way for all vertices in that each vertex follows the same propagation, aggregation, and filtering rules. This induces a semimodule structure on the (possible) state vectors of the algorithm in a natural way.

Definition 2.9 (Power Semimodule). *Given a node set V and a zero-preserving semimodule $\mathcal{M} = (M, \oplus, \odot)$, we define $\mathcal{M}^V = (M^V, \oplus, \odot)$ by applying the operations of \mathcal{M} coordinatewise, i.e., $\forall v, w \in M^V, s \in \mathbb{R}_{\geq 0} \cup \{\infty\}$:*

$$(x \oplus y)_v = x_v \oplus y_v, \text{ and} \quad (2.14)$$

$$(s \odot x)_v = s \odot x_v. \quad (2.15)$$

Furthermore, by r^V we denote the componentwise application of a representative projection r of \mathcal{M} , i.e.,

$$(r^V x)_v := r(x_v), \quad (2.16)$$

and define the equivalence relation \sim via $x \sim y$ iff $x_v \sim y_v$ for all $v \in V$.

Observation 2.10. \mathcal{M}^V is a zero-preserving semimodule with neutral element $\perp^V := (\perp, \dots, \perp) \in M^V$, where \perp is the neutral element of \mathcal{M} . The equivalence relation \sim induced by r^V is a congruence relation on \mathcal{M}^V with representative projection r^V .

2.3 The Class of MBF-like Algorithms

The following definition connects the properties introduced and motivated above.

Definition 2.11 (MBF-like Algorithm). *An MBF-like algorithm \mathcal{A} is determined by*

1. a zero-preserving semimodule $\mathcal{M} = (M, \oplus, \odot)$ over $\mathcal{S}_{\min,+} = (\mathbb{R}_{\geq 0} \cup \{\infty\}, \min, +)$,
2. a congruence relation on \mathcal{M} with representative projection r , and
3. initial values $x^{(0)} \in M^V$ for the nodes,

where $x^{(0)}$ depends on the input graph. On a weighted graph G , h iterations of \mathcal{A} determine

$$\mathcal{A}^h(G) := x^{(h)} := r^V A^h x^{(0)}, \quad (2.17)$$

where A is the adjacency matrix of G as defined in Equation (1.3).

The i -th iteration of an MBF-like algorithm determines $x^{(i)} := r^V A x^{(i-1)}$ (propagate, aggregate, and filter). Thus, h iterations yield $(r^V A)^h x^{(0)}$, which we show to be identical to $r^V A^h x^{(0)}$ in Section 2.4. We remark that after at most $\text{SPD}(G) < n$ iterations, a fixpoint is reached, i.e., $x^{(i+1)} = x^{(i)}$. Hence, we abbreviate $\mathcal{A}(G) := \mathcal{A}^n(G)$ for convenience.

2.4 Manipulating the State of MBF-like Algorithms

As motivated above, an MBF-like algorithm \mathcal{A} efficiently represents intermediate results by choosing a compact representation by applying r^V . We show the feasibility of this approach by interpreting the performed operations on a semiring of functions compatible with these equivalence classes. These functions are a subset of the functions on the aforementioned quotient semimodule M/\sim (which we do not introduce explicitly), and the function semiring is a subset of the semiring of linear functions¹¹ on M/\sim .

The functions f of interest are “simple” in the sense that they map $x \in \mathcal{M}^V$ such that $(f(x))_v$ is a linear combination of the coordinates x_w , $w \in V$, of x .

Definition 2.12 (Simple Linear Function). *Let $\mathcal{M} = (M, \oplus, \odot)$ be a semimodule over $\mathcal{S}_{\min,+}$. A matrix $A \in (\mathbb{R}_{\geq 0} \cup \{\infty\})^{V \times V}$ defines a simple linear function $A: M^V \rightarrow M^V$ (and vice versa) by*

$$(A(x))_v := (Ax)_v = \bigoplus_{w \in V} a_{vw} x_w. \quad (2.18)$$

Thus, each iteration of an MBF-like algorithm is an application of the simple linear function given by the adjacency matrix, followed by application of r^V .

Example 2.13 (Non-simple Linear Function). *We remark that not all linear functions on \mathcal{M}^V are simple. For a counter-example, suppose $V = \{1, 2\}$, $\mathcal{M} = \mathcal{S}_{\min,+}^V$, and consider $f: \mathcal{M}^V \rightarrow \mathcal{M}^V$ given by $((f(x))_1)_1 = (x_1)_1 \oplus (x_1)_2$ and $((f(x))_v)_w = \infty$ for $v \neq 1$ or $w \neq 1$. Clearly f is linear, but since it acts on x_1 in a non-trivial way, it is not simple.*

We require an intermediate lemma stating that equivalence is preserved by linear combinations.

Lemma 2.14. *Let $\mathcal{M} = (M, \oplus, \odot)$ be a semimodule and \sim a congruence relation on \mathcal{M} . Furthermore, for some $k \in \mathbb{N}$, let $x_1, \dots, x_k, x'_1, \dots, x'_k \in \mathcal{M}$ such that $x_i \sim x'_i$ for all $1 \leq i \leq k$. Then for all $s_1, \dots, s_k \in \mathbb{R}_{\geq 0} \cup \{\infty\}$, it holds that*

$$\bigoplus_{i=1}^k s_i x_i \sim \bigoplus_{i=1}^k s_i x'_i. \quad (2.19)$$

Proof. We show the claim by induction over k . For $k = 1$, the claim follows from Equation (2.8). Regarding $k \geq 2$, suppose the claim holds for $k - 1$. Since $x_k \sim x'_k$, by (2.8) we have that $s_k x_k \sim s_k x'_k$. The induction hypothesis yields $\bigoplus_{i=1}^{k-1} s_i x_i \sim \bigoplus_{i=1}^{k-1} s_i x'_i$. Hence,

$$\bigoplus_{i=1}^k s_i x_k = \left(\bigoplus_{i=1}^{k-1} s_i x_i \right) \oplus s_k x_k \stackrel{(2.9)}{\sim} \left(\bigoplus_{i=1}^{k-1} s_i x'_i \right) \oplus s_k x'_k = \bigoplus_{i=1}^k s_i x'_k. \quad (2.20) \quad \square$$

We proceed to show important properties of simple linear functions. First of all, they map equivalent parameters to equivalent results, which is crucial when switching between dealing with filtered and unfiltered states. Furthermore, we show that matrix addition and multiplication are equivalent to the addition and concatenation of simple linear function matrices. In the following, for simple linear functions A and B , we write $(A \oplus B)(x) \mapsto A(x) \oplus B(x)$, and $(A \circ B)(x) \mapsto A(B(x))$. We denote by Ax and AB matrix-vector and matrix-matrix multiplication, respectively, and by $A \oplus B$ componentwise “addition” of matrices; as addition and concatenation of functions are associative, the equivalence we show now justifies this notation without parenthesis.

¹¹A linear function f on semimodule \mathcal{M} satisfies for all $x, y \in \mathcal{M}$ and $s \in \mathbb{R}_{\geq 0} \cup \{\infty\}$ that $f(x \oplus y) = f(x) \oplus f(y)$ and $f(s \odot x) = s \odot f(x)$.

Lemma 2.15. *Let A and B be simple linear functions and $x \sim x' \in M^V$. Then*

$$Ax \sim Ax', \quad (2.21)$$

$$(A \oplus B)(x) = (A \oplus B)x, \text{ and} \quad (2.22)$$

$$(A \circ B)(x) = ABx. \quad (2.23)$$

Proof. Regarding (2.21), we have for all $v \in V$

$$(Ax)_v = \bigoplus_{w \in V} a_{vw} x_w \stackrel{(2.19)}{\sim} \bigoplus_{w \in V} a_{vw} x'_w = (Ax')_v. \quad (2.24)$$

As for (2.22) and (2.23), observe that for all $v \in V$,

$$((A \oplus B)(x))_v = \bigoplus_{w \in V} (a_{vw} \oplus b_{vw}) x_w = \bigoplus_{w \in V} (A \oplus B)_{vw} x_w = ((A \oplus B)x)_v, \text{ as well as} \quad (2.25)$$

$$((A \circ B)(x))_v = \bigoplus_{u \in V} a_{vu} \left(\bigoplus_{w \in V} b_{uw} x_w \right) = \bigoplus_{w \in V} \left(\bigoplus_{u \in V} a_{vu} b_{uw} \right) x_w = \bigoplus_{w \in V} (AB)_{vw} x_w = (ABx)_v. \square$$

Due to this lemma, in the following we identify simple linear functions and their matrices. We proceed to formalize in what manner an MBF-like algorithm operates on \mathcal{M}/\sim .

Lemma 2.16 (Simple Linear Function Semirings). *Denote by F the set of simple linear functions on semimodule \mathcal{M} and let \sim be a congruence relation on \mathcal{M} with representative projection r .*

1. $\mathcal{F} = (F, \oplus, \circ)$ is a semiring.
2. For $A \in F$, $[A][x] := [Ax]$ defines a simple linear function on \mathcal{M}/\sim .
3. For $A, B \in F$, $[A] \oplus [B] = [A \oplus B]$ and $[A] \circ [B] = [AB]$.
4. Setting $F/\sim := \{[A] \mid A \in F\}$, $\mathcal{F}/\sim := (F/\sim, \oplus, \circ)$ is a semiring.
5. $[r^V]x := [r^V x]$ defines a simple linear function on F/\sim , which in fact is the multiplicative identity of the semiring.

Proof. We first argue that (F, \oplus, \circ) is a semiring. This readily follows from Lemma 2.15, which states that addition and concatenation of simple linear functions is equivalent to addition and multiplication of their matrices, respectively. In other words, (F, \oplus, \circ) is isomorphic to the matrix semiring $((\mathbb{R}_{\geq 0} \cup \{\infty\})^{V \times V}, \oplus, \cdot)$.

To show the second statement, observe that for any $x \sim y$ and all $v \in V$, we have

$$(Ax)_v = \bigoplus_{w \in V} a_{vw} x_w \stackrel{(2.19)}{\sim} \bigoplus_{w \in V} a_{vw} y_w = (Ay)_v, \quad (2.26)$$

implying that $[A]$ is well-defined as a function on \mathcal{M}/\sim (i.e., $[Ax]$ does not depend on the choice of the representant $x \in [y]$). To see that it is linear, we compute

$$[A][x] \oplus [A][y] = [Ax] \oplus [Ay] = [Ax \oplus Ay] = [A(x \oplus y)] = [A](x \oplus y), \text{ and} \quad (2.27)$$

$$[A](sx) = [A(sx)] = [s(Ax)] = s[Ax] = s[A]x. \quad (2.28)$$

To see that it is also simple, note that

$$([A][x])_v = [Ax]_v = \left[\bigoplus_{w \in V} a_{vw} x_w \right] = \bigoplus_{w \in V} a_{vw} [x_w]. \quad (2.29)$$

The third statement follows from Lemma 2.15, immediately also implying the fourth. Finally, by definition $x \sim r^V x$, i.e., $[x] = [r^V x]$ for all $x \in \mathcal{M}$. Therefore, $[r^V]$ is the identity function, which is the multiplicative identity of \mathcal{F}/\sim . \square

The following corollary is a key property used throughout the paper. It allows us to apply filter steps, whenever convenient. We use this to simulate MBF-like iterations on an implicitly represented graph whose edges correspond to entire paths in the original graph. This works because intermediate filtering steps are allowed.

Corollary 2.17 ($r^V \sim \text{id}$). *For any simple linear function A and representative projection r on \mathcal{M} , it holds that*

$$r^V A \sim A r^V \sim A, \quad (2.30)$$

where \sim extends to functions by $A \sim B$ iff $A(x) \sim B(x)$ for all $x \in \mathcal{M}^V$. In particular, for any MBF-like algorithm \mathcal{A} , we have that

$$\mathcal{A}^h(G) = (r^V A)^h x^{(0)} = r^V A^h x^{(0)}. \quad (2.31)$$

We stress that both the restriction to simple linear functions and the componentwise application of r to determine r^V are crucial for Lemma 2.16 and Corollary 2.17 to hold.

Example 2.18 (Non-simple linear functions break Corollary 2.17). *Consider the case where $V = \{1, 2\}$, $\mathcal{M} = \mathcal{S}_{\min,+}^V$, and the linear function $f: \mathcal{M}^V \rightarrow \mathcal{M}^V$ given by $((f(x))_1)_1 = (x_1)_1 \oplus (x_1)_2$ and $((f(x))_v)_w = \perp$ for $v \neq 1$ or $w \neq 1$. If $r(x) = (x_1, \infty)$ for all $x \in \mathcal{M}$, we have that*

$$r^V f \left(\begin{pmatrix} (2, 1) \\ \perp \end{pmatrix} \right) = \left(\begin{pmatrix} (1, \infty) \\ \perp \end{pmatrix} \right) \not\sim \left(\begin{pmatrix} (2, \infty) \\ \perp \end{pmatrix} \right) = f r^V \left(\begin{pmatrix} (2, 1) \\ \perp \end{pmatrix} \right), \quad (2.32)$$

implying that $r^V f \not\sim f r^V$.

Example 2.19 (Non-component-wise filtering breaks Corollary 2.17). *Suppose f is the simple linear function given by $f x = \begin{pmatrix} x_1 \oplus x_2 \\ \perp \end{pmatrix}$ and $r^V(x) = \begin{pmatrix} x_1 \\ \perp \end{pmatrix}$, i.e., r^V is not componentwise application of some representative projection r on \mathcal{M} , but still a representative projection on \mathcal{M}^V . Then we have that*

$$r^V f \left(\begin{pmatrix} (2, \infty) \\ (1, \infty) \end{pmatrix} \right) = \left(\begin{pmatrix} (1, \infty) \\ \perp \end{pmatrix} \right) \not\sim \left(\begin{pmatrix} (2, \infty) \\ \perp \end{pmatrix} \right) = f r^V \left(\begin{pmatrix} (2, \infty) \\ (1, \infty) \end{pmatrix} \right), \quad (2.33)$$

again implying that $r^V f \not\sim f r^V$.

2.5 Algorithms

For the purpose of illustration, we give some basic examples in this section. Recall that in order to specify an MBF-like algorithm, we have to provide a zero-preserving $\mathcal{S}_{\min,+}$ -semimodule \mathcal{M} , a representative projection of a congruence relation on \mathcal{M} , initial states $x^{(0)}$, and the number of iterations h (compare Definition 2.11). $\mathcal{D} = ((\mathbb{R}_{\geq 0} \cup \{\infty\})^V, \min, +)$ is a valid choice of \mathcal{M} by Lemma 2.2.

Example 2.20 (SSSP). *Single-source shortest paths (SSSP) requires to determine the h -hop distance to $s \in V$ for all $v \in V$. It is solved by an MBF-like algorithm with $\mathcal{M} = \mathcal{S}_{\min,+}$, $r = \text{id}$, and $x_s^{(0)} = 0$, $x_v^{(0)} = \infty$ for all $v \neq s$. Alternatively, one may use $\mathcal{M} = \mathcal{D}$, $(r(x))_v = x_v$ if $v = s$ and $(r(x))_v = \infty$ otherwise, and $(x_v^{(0)})_w = 0$ if $v = w = s$ and $(x_v^{(0)})_w = \infty$ otherwise.*

Example 2.21 (k -SP). *k -shortest paths (k -SP) requires to determine, for each node, the k closest nodes in terms of the h -hop distance $\text{dist}^h(\cdot, \cdot, G)$. It is solved by an MBF-like algorithm, as can be seen by choosing $\mathcal{M} = \mathcal{D}$,*

$$r(x)_v \mapsto \begin{cases} x_v & \text{if } x_v \text{ is among the } k \text{ smallest entries of } x \text{ (ties broken by index)} \\ \infty & \text{otherwise,} \end{cases} \quad (2.34)$$

and $x_v^{(0)} = e_v = (\infty, \dots, \infty, 0, \infty, \dots, \infty)$.

Example 2.21 is slightly more involved, as it may not be immediately obvious that r is a representative projection. However, by Lemma 2.8, it suffices to show that $r^2 = r$, $r(x \oplus y) \sim r(x) \oplus r(y)$, and $r(sx) \sim sr(x)$; each of these properties is easy to verify.

Example 2.22 (APSP). *All-pairs shortest paths (APSP) is the task of determining the h -hop distance between all pairs of nodes. It is solved by an MBF-like algorithm, because we can choose $k = n$ in k -SSP (resulting in $r = \text{id}$).*

Example 2.23 (Forest Fires). *The goal is to detect, for each node, if there is a forest fire within distance d , where every node initially knows if it is on fire. For an MBF-like algorithm solving it, pick, e.g., $\mathcal{M} = \mathcal{S}_{\min,+}$,*

$$r(x) = \begin{cases} x & \text{if } x \leq d \\ \infty & \text{otherwise,} \end{cases} \quad (2.35)$$

and $x_v^{(0)} = 0$ if there is a fire at v and $x_v^{(0)} = \infty$ otherwise.

Example 2.23 can also be handled in different ways. We could use $\mathcal{M} = \mathcal{D}$, $(r(x))_v = x_v$ if $x_v < x_w$ for all $w < v$, $x_v \leq x_w$ for all $w > v$ and $x_v \leq d$, and $x_v = \infty$ in all other cases, and $x_v^{(0)} = e_v$ as in Example 2.21. This would also reveal the closest node on fire, whereas the solution from Example 2.23 may operate in anonymous networks. It should also be noted that one can interpret both of these solutions as instances of SSSP with a virtual source $s \notin V$ that is connected to all nodes on fire by an edge of weight 0. However, this requires a simulation argument and, if the closest node on fire is to be determined, additional reasoning. Therefore, we believe that our framework provides a cleaner perspective on the problem.

Finally, we point out that more advanced distributed algorithms computing tree embeddings based on the FRT construction have straightforward interpretations in our framework. This is discussed in Appendix C, alongside an improved distributed algorithm based on the other results of this work.

3 The Simulated Graph

In order to determine the tree into which G is embedded, we need to determine LE lists (compare Section 5). These are the result of an MBF-like algorithm, where r ensures that $|r(x^{(i)})_v| \in O(\log n)$ w.h.p. for all $i \in \{0, \dots, h\}$. This allows for performing an iteration with $\tilde{O}(m)$ work. However, doing so on G might require $h \geq \text{SPD}(G)$, which in general can be as large as $n - 1$.

To resolve this problem, we seek to reduce the number of iterations of the algorithm, accepting a slight increase in stretch. Cohen [13] shows how to efficiently compute a (d, ε) -hop set, i.e., a small number of additional (weighted) edges for G , such that for all $v, w \in V$, $\text{dist}^d(v, w, G') \leq (1 + \varepsilon) \text{dist}(v, w, G)$, where G' is G plus the additional edges.

Her algorithm is sufficiently efficient in terms of depth, work, and number of additional edges to suit our needs. Yet, our problem is not solved: the FRT construction critically depends on the triangle inequality and thus the use of *exact* distances. In this section, we resolve this issue by embedding $G = (V, E, \omega)$ with small stretch into a complete weighted graph H on node set V so that $\text{SPD}(H) \in \text{polylog } n$. Note that constructing H explicitly would cause $\Omega(n^2)$ work; we navigate this obstacle in Section 4 with the help of the machinery developed in Section 2. Our construction requires to first add a hop set to G . For the sake of presentation, we assume G to already contain a (d, ε) -hop set throughout this section.

We begin our construction of H by sampling levels for the vertices V : Every vertex starts at level 0. In step $\lambda \geq 1$, each vertex in level $\lambda - 1$ is raised to level λ with probability $1/2$. We continue until the first step $\Lambda + 1$ where no node is sampled. $\lambda(v)$ refers to the level of $v \in V$, and $\lambda(e)$ to the level of $e \in E$, the minimum of its endpoints' levels.

Lemma 3.1. *W.h.p., $\Lambda \in O(\log n)$.*

Proof. For $c \in \mathbb{R}_{\geq 1}$, $v \in V$ has $\lambda(v) < c \log n$ with probability $1 - (\frac{1}{2})^{c \log n} = 1 - n^{-c}$, i.e., w.h.p. Lemma 1.1 yields that all nodes have a level of less than $c \log n$ w.h.p., and the claim follows. \square

The idea is to use the levels in the following way. We devise a complete graph H on V . An edge of H with level λ is weighted with the d -hop distance between its endpoints in G (which is a $(1 + \varepsilon)$ -approximation of their exact distance by assumption), but multiplied with a penalty of $(1 + \varepsilon)^{\Lambda - \lambda}$. This way, high-level edges are “more attractive” for shortest paths, because they receive smaller penalties.

Definition 3.2 (Simulated graph H). *Let $G = (V, E, \omega)$ be a graph that contains a (d, ε) -hop set with levels sampled as above. We define the complete graph H as*

$$H := \left(V, \binom{V}{2}, \omega_\Lambda \right) \quad (3.1)$$

$$\omega_\Lambda(\{v, w\}) \mapsto (1 + \varepsilon)^{\Lambda - \lambda(v, w)} \text{dist}^d(v, w, G). \quad (3.2)$$

We formalize the notion of high-level edges being “more attractive” than low-level paths: Any min-hop shortest path between to nodes of level λ is exclusively comprised of edges of level λ or higher. Therefore, all min-hop shortest paths decompose into two subpaths, the first of increasing and the second of decreasing level.

Lemma 3.3. *Consider $v, w \in V$, $\lambda = \lambda(v, w)$, and $p \in \text{MHSP}(v, w, H)$. Then all edges of p have level at least λ .*

Proof. The case $\lambda = 0$ is trivial. Consider $1 \leq \lambda \leq \Lambda$, and, for the sake of contradiction, let q be a (non-trivial) maximal subpath of p containing edges of level strictly less than λ only. Observe that $q \in \text{MHSP}(v', w', H)$ for some $v', w' \in V$ with $\lambda(v'), \lambda(w') \geq \lambda$. We have

$$\omega_\Lambda(q) \geq (1 + \varepsilon)^{\Lambda - (\lambda - 1)} \text{dist}(v', w', G). \quad (3.3)$$

However, the edge $e = \{v', w'\}$ has level $\lambda(v', w') \geq \lambda$ and weight

$$\omega_\Lambda(e) \leq (1 + \varepsilon)^{\Lambda - \lambda} \text{dist}^d(v', w', G) \leq (1 + \varepsilon)^{\Lambda - (\lambda - 1)} \text{dist}(v', w', G) \leq \omega_\Lambda(q) \quad (3.4)$$

by construction. Since $|q|$ is minimal and $\lambda(v'), \lambda(w') \geq \lambda$, q can only be a single edge of level λ or higher, contradicting the assumption. \square

Since edge levels in a min-hop shortest path are first increasing and then decreasing, the next step is to limit the number of hops spent on each level.

Lemma 3.4. *Consider vertices v and w of H with $\lambda(v) = \lambda(w) \geq \lambda$. Then w.h.p. one of the following statements holds:*

$$\text{hop}(v, w, H) \in O(\log n), \text{ or} \quad (3.5)$$

$$\forall p \in \text{MHSP}(v, w, H) \exists e \in p: \lambda(e) \geq \lambda + 1. \quad (3.6)$$

Proof. Condition on the event \mathcal{E}_{V_λ} that $V_\lambda \subseteq V$ is the set of nodes with level λ or higher (with level $\lambda + 1$ not yet sampled). Let $H_\lambda := (V_\lambda, \binom{V_\lambda}{2}, \omega_\lambda)$ with $\omega_\lambda(\{v, w\}) \mapsto (1 + \varepsilon)^{\Lambda - \lambda} \text{dist}^d(v, w, G)$ denote the subgraph of H spanned by V_λ and capped at level λ .

Consider $p \in \text{MHSP}(v, w, H_\lambda)$. Observe that $\mathbb{P}[\lambda(u) \geq \lambda + 1 \mid \mathcal{E}_{V_\lambda}] = \frac{1}{2}$ independently for all $u \in V_\lambda$, and hence $\mathbb{P}[\lambda(e) \geq \lambda + 1 \mid \mathcal{E}_{V_\lambda}] = \frac{1}{4}$ for all $e \in p$. This probability holds independently for every other edge of p . If $|p| \geq 2c \log_{4/3} n$ for some choice of $c \in \mathbb{R}_{\geq 1}$, the probability that p contains no edge of level $\lambda + 1$ or higher is bounded from above by $(\frac{3}{4})^{|p|/2} \leq (\frac{3}{4})^{c \log_{4/3} n} = n^{-c}$, so p contains such an edge w.h.p.

We show that for all $q \in \text{MHSP}(v, w, H)$, w.h.p. (3.5) or (3.6) holds. Observe that $\omega_\Lambda(q) \leq \omega_\Lambda(p)$. Furthermore, q uses only edges of level λ or higher by Lemma 3.3. If q contains an edge of level $\lambda + 1$ or higher, (3.6) holds for q . Otherwise, we have $\omega_\lambda(q) = \omega_\Lambda(q)$, and distinguish two cases:

Case $|p| \geq 2c \log_{4/3} n$: Recall that w.h.p. p has an edge of level $\lambda + 1$ or higher. This yields $\omega_\Lambda(p) < \omega_\lambda(p)$, implying

$$\omega_\Lambda(p) \stackrel{\text{w.h.p.}}{<} \omega_\lambda(p) \leq \omega_\lambda(q) = \omega_\Lambda(q), \quad (3.7)$$

which contradicts $q \in \text{MHSP}(v, w, H)$.

Case $|p| < 2c \log_{4/3} n$: Not unlike above, we have

$$\omega_\Lambda(p) \leq \omega_\lambda(p) \leq \omega_\lambda(q) = \omega_\Lambda(q), \quad (3.8)$$

so $\omega_\Lambda(q) = \omega_\Lambda(p)$ and $|q| \leq |p| \in O(\log n)$ follows.

Let \mathcal{E}_{vw} denote the event that (3.5) or (3.6) holds for $v, w \in V$. We conclude that

$$\mathbb{P}[\mathcal{E}_{vw} \mid \lambda(v, w) \geq \lambda] = \sum_{V_\lambda \subseteq V} \mathbb{P}[\mathcal{E}_{V_\lambda} \mid \lambda(v, w) \geq \lambda] \mathbb{P}[\mathcal{E}_{vw} \mid \mathcal{E}_{V_\lambda}] \quad (3.9)$$

$$= \sum_{\{v, w\} \subseteq V_\lambda \subseteq V} \mathbb{P}[\mathcal{E}_{V_\lambda} \mid \lambda(v, w) \geq \lambda] \mathbb{P}[\mathcal{E}_{vw} \mid \mathcal{E}_{V_\lambda}] \quad (3.10)$$

$$\geq \sum_{\{v, w\} \subseteq V_\lambda \subseteq V} \mathbb{P}[\mathcal{E}_{V_\lambda} \mid \lambda(v, w) \geq \lambda] (1 - n^{-c}) \quad (3.11)$$

$$= (1 - n^{-c}) \sum_{V_\lambda \subseteq V} \mathbb{P}[\mathcal{E}_{V_\lambda} \mid \lambda(v, w) \geq \lambda] \quad (3.12)$$

$$= 1 - n^{-c}, \quad (3.13)$$

which is precisely the statement of the lemma. \square

We argue above that any min-hop shortest path in H traverses every level at most twice. Furthermore, Lemma 3.4 states that w.h.p. each such traversal only has a logarithmic number of hops, and Lemma 3.1 asserts that there are only logarithmic ally many levels. The following theorem concludes this section by making the connection to the shortest path diameter of H .

Theorem 3.5. *W.h.p., $\text{SPD}(H) \in O(\log^2 n)$. Furthermore, w.h.p. for all $v, w \in V$,*

$$\text{dist}(v, w, G) \leq \text{dist}(v, w, H) \leq (1 + \varepsilon)^{O(\log n)} \text{dist}(v, w, G). \quad (3.14)$$

Proof. Fix a level λ . Any fixed pair of vertices of level λ or higher fulfills, w.h.p., (3.5) or (3.6) by Lemma 3.4. Since there are at most $\binom{n}{2}$ such pairs, w.h.p., all of them fulfill (3.5) or (3.6) by Lemma 1.1.

Let \mathcal{E}_{\log} denote the event that there is no higher level than $\Lambda \in O(\log n)$, which holds w.h.p. by Lemma 3.1. Furthermore, let \mathcal{E}_λ denote the event that all pairs of vertices of level λ or higher fulfill (3.5) or (3.6), which holds w.h.p. as argued above. Then $\mathcal{E} = \mathcal{E}_{\log} \cap \mathcal{E}_0 \cap \dots \cap \mathcal{E}_\Lambda$ holds w.h.p. by Lemma 1.1.

Condition on \mathcal{E} ; in particular, no min-hop shortest path whose edges all have the same level has more than $O(\log n)$ hops. Consider some min-hop shortest path p in H . By Lemma 3.3, p has two parts; in the first part, the edge level increases, and in the second part, it decreases. Hence, p can be split up into at most $2\Lambda - 1$ segments, in each of which all edges have the same level. As this holds for all min-hop shortest paths, we conclude that $\text{SPD}(H) \in O(\Lambda \log n) \subseteq O(\log^2 n)$ w.h.p., as claimed.

As for Inequality (3.14), recall that H is constructed from $G = (V, E, \omega)$, and that G contains a (d, ε) -hop set. For all $v, w \in V$, we have

$$\text{dist}(v, w, H) \leq \omega_\Lambda(v, w) \leq (1 + \varepsilon)^\Lambda \text{dist}^d(v, w, G) \leq (1 + \varepsilon)^{\Lambda+1} \text{dist}(v, w, G) \quad (3.15)$$

by construction of H . Recalling that $\Lambda \in O(\log n)$ w.h.p., this completes the proof. \square

To wrap things up: Given an arbitrary weighted graph G , we can efficiently augment G with a (d, ε) -hop set using the result of Cohen [13]. After that, the d -hop distances in G approximate the actual distances in G , but these approximations may violate the triangle inequality. We fix this by constructing H , which uses geometrically sampled node levels and an exponential penalty on the edge lengths with decreasing levels.

H is a complete graph, so explicitly constructing it and running even a single iteration of an MBF-like algorithm on it is prohibitively costly in terms of work. The next section shows how to avoid this issue by efficiently simulating MBF-like algorithms on H .

4 The Oracle

Given a weighed graph G and any $\varepsilon > 0$, Section 3 introduces a complete graph H that $(1 + \varepsilon)$ -approximates the distances of G and has, w.h.p., a polylogarithmic shortest path diameter; here we assume that G already contains a (d, ε') -hop set for suitable $\varepsilon \in \Theta(\varepsilon'/\log n)$. Throughout this section, we denote by A_G and A_H the adjacency matrices of G and H , respectively.

We do not explicitly write H into memory, as this requires an unacceptable $\Omega(n^2)$ work. Instead, we dedicate this section to an *oracle* that takes an MBF-like algorithm \mathcal{A} , a weighted graph G , a number of iterations h , and returns $\mathcal{A}^h(H)$. The properties of MBF-like algorithms discussed in Section 2 allow the oracle to internally work on G and simulate iterations of \mathcal{A} on H using a polylogarithmic number of matrix-vector multiplications with A_G .

We establish this section's results in two steps: Section 4.1 derives a representation of A_H in terms of A_G , which is then used to efficiently implement the oracle in Section 4.2. The oracle is used to construct an FRT embedding with polylogarithmic depth in Section 5.

4.1 Decomposing H

The idea is to simulate one iteration of an MBF-like algorithm \mathcal{A} on H using d iterations on G . This is done for each level $\lambda \in [\Lambda]$ in parallel. For level λ , we run \mathcal{A} for d iterations on G with edge weights scaled up by $(1 + \varepsilon)^{\Lambda - \lambda}$, where the initial vector is obtained by discarding all information at nodes of level smaller than λ . Afterwards, we again discard everything stored at vertices not of level λ or higher. Since $(A_G^d)_{vw} = \text{dist}^d(v, w, G)$, this ensures that we propagate information between nodes $v, w \in V$ with $\lambda(v, w) = \lambda$ with the corresponding edge weight, while discarding any exchange between nodes with $\lambda(v, w) < \lambda$ (which is handled by the respective parallel run). While we also propagate information between v and w if $\lambda(v, w) > \lambda$ —over a too long distance, because edge weights are scaled by $(1 + \varepsilon)^{\Lambda - \lambda} > (1 + \varepsilon)^{\Lambda - \lambda(v, w)}$ —the parallel run for $\lambda(v, w)$ correctly propagates values. Therefore, aggregating the results of all levels (i.e., applying \oplus , the source-wise minimum) and applying r^V completes the simulation of an iteration of \mathcal{A} .

This approach resolves two complexity issues. First, we multiply (polylogarithmically often) with A_G , which—as opposed to the dense A_H —has $O(m)$ non- ∞ entries only. Second, Corollary 2.17 shows that we are free to filter using r^V at any time, keeping the entries in state vectors small.

We formalize the above intuition. Recall that

$$(A_H)_{vw} = \omega_\Lambda(v, w) = (1 + \varepsilon)^{\Lambda - \lambda(v, w)} \text{dist}^d(v, w, G) = (1 + \varepsilon)^{\Lambda - \lambda(v, w)} (A_G^d)_{vw}. \quad (4.1)$$

For $\lambda \in [\Lambda]$, denote by P_λ the \mathcal{M}^V -projection to coordinates $V_\lambda := \{v \in V \mid \lambda(v) \geq \lambda\}$:

$$(P_\lambda x)_v = \begin{cases} x_v & \text{if } \lambda(v) \geq \lambda \\ \perp & \text{otherwise.} \end{cases} \quad (4.2)$$

Observe that P_λ is a simple linear function on \mathcal{M}^V , where $(P_\lambda)_{vv} = 0$ if $v = w \in V_\lambda$ and $(P_\lambda)_{vw} = \infty$ otherwise. This gives us the tools to decompose A_H as motivated above.

Lemma 4.1. *With $(A_\lambda)_{vw} := (1 + \varepsilon)^{\Lambda - \lambda} (A_G)_{vw}$, we have*

$$A_H = \bigoplus_{\lambda=0}^{\Lambda} P_\lambda A_\lambda^d P_\lambda. \quad (4.3)$$

Proof. Since $(A_G^d)_{vw} = \text{dist}^d(v, w, G)$, it holds that $(A_\lambda^d)_{vw} = (1 + \varepsilon)^{\Lambda - \lambda} \text{dist}^d(v, w, G)$. Therefore,

$$(A_\lambda^d P_\lambda)_{vw} = \min_{u \in V} \left\{ (A_\lambda^d)_{vu} + (P_\lambda)_{uw} \right\} = \begin{cases} (1 + \varepsilon)^{\Lambda - \lambda} \text{dist}^d(v, w, G) & \text{if } w \in V_\lambda \\ \infty & \text{otherwise,} \end{cases} \quad (4.4)$$

and hence

$$(P_\lambda A_\lambda^d P_\lambda)_{vw} = \min_{u \in V} \left\{ (P_\lambda)_{vu} + (A_\lambda^d P_\lambda)_{uw} \right\} = \begin{cases} (1 + \varepsilon)^{\Lambda - \lambda} \text{dist}^d(v, w, G) & \text{if } v, w \in V_\lambda \\ \infty & \text{otherwise.} \end{cases} \quad (4.5)$$

We conclude that

$$\left(\bigoplus_{\lambda=0}^{\Lambda} P_{\lambda} A_{\lambda}^d P_{\lambda} \right)_{vw} = \min_{\lambda \in [\lambda(v,w)]} \left\{ (1 + \varepsilon)^{\Lambda - \lambda} \text{dist}^d(v, w, G) \right\} \quad (4.6)$$

$$\begin{aligned} &= (1 + \varepsilon)^{\Lambda - \lambda(v,w)} \text{dist}^d(v, w, G) \\ &= (A_H)_{vw}. \quad \square \end{aligned} \quad (4.7)$$

Having decomposed A_H , we proceed with decomposing $r^V A_H^h$, where h is the shortest path diameter of H , taking the freedom to apply filters intermediately. For all $h \in \mathbb{N}$, we have

$$A_H^h \stackrel{(4.3)}{=} \left(\bigoplus_{\lambda=0}^{\Lambda} P_{\lambda} A_{\lambda}^d P_{\lambda} \right)^h \stackrel{(2.30)}{\approx} \left(r^V \left(\bigoplus_{\lambda=0}^{\Lambda} P_{\lambda} (r^V A_{\lambda})^d P_{\lambda} \right) \right)^h r^V, \quad (4.8)$$

and hence, for all $h \geq \text{SPD}(H)$

$$\mathcal{A}(H) = \mathcal{A}^h(H) = r^V A_H^h x^{(0)} \stackrel{(2.11), (4.8)}{=} \left(r^V \left(\bigoplus_{\lambda=0}^{\Lambda} P_{\lambda} (r^V A_{\lambda})^d P_{\lambda} \right) \right)^h r^V x^{(0)}. \quad (4.9)$$

Observe that we can choose $h \in O(\log^2 n)$ so that $h \geq \text{SPD}(H)$ w.h.p. by Theorem 3.5, and recall that $d \in \text{polylog } n$. Overall, this allows us to execute $\mathcal{A}(H)$ with polylogarithmic depth and $\tilde{O}(m)$ work, provided we can implement the individual steps, see below, at this complexity.

4.2 Implementing the Oracle

With the decomposition at hand, the oracle can be implemented as follows. Simulate one iteration of \mathcal{A} on H for edges of level λ , i.e., determine $y_{\lambda} := P_{\lambda} (r^V A_{\lambda})^d P_{\lambda} x^{(i)}$ as follows: (1) Discard entries at nodes of a level smaller than λ , (2) run d iterations of \mathcal{A} with distances stretched by $(1 + \varepsilon)^{\Lambda - \lambda}$ on G , and (3) again discard at levels smaller than λ . After this procedure is run in parallel for all $\lambda \in [\Lambda]$, we perform the \bigoplus operation and apply the filter, i.e., each node $v \in V$ determines $x_v^{(i+1)} = r(\bigoplus_{\lambda \in [\Lambda]} y_{\lambda})$.

To be able to reason about the complexity of the individual operations, the following theorem fixes the semimodule to be the distance map $\mathcal{D} = (\mathcal{S}_{\min,+}^V, \oplus, \odot)$ given in Definition 2.1. Given that all our examples and applications are covered by this special case, this appears to be no substantial restriction.

Theorem 4.2. *Suppose that $\mathcal{M} = \mathcal{D}$, where we represent $x \in \mathcal{D}$ as list of index/distance pairs (dropping all ∞ values). Assume that for each intermediate state vector $y = (r^V A_{\lambda})^f x^{(i)}$ for some $f \in [d]$, $i \in [h]$, and $\lambda \in [\Lambda]$, we can compute $r^V A_{\lambda} x$ with depth D and work W . Then, w.h.p., $\mathcal{A}(H)$ can be executed with $O((dD + \log n) \log^2 n)$ depth and $O((d + \log n)W \log^3 n)$ work.*

Proof. By Lemma 3.1 and Theorem 3.5, w.h.p. $\Lambda \in O(\log n)$ and $\text{SPD}(H) \leq h \in O(\log^2 n)$. By (4.9), we have to compute

$$\mathcal{A}(H) = r^V \left(\bigoplus_{\lambda=0}^{\Lambda} P_{\lambda} (r^V A_{\lambda})^d P_{\lambda} \right)^h r^V x^{(0)}. \quad (4.10)$$

The claim readily follows if we show that the applications of P_{λ} and aggregation over λ do not affect depth and work too much.

Concerning P_λ , note that we can evaluate $(P_\lambda x)_{v \in V}$ lazily, i.e., determine whether $(P_\lambda x)_v$ evaluates to \perp or to x_v only if it is accessed. Thus, work and depth can increase by at most a constant factor due to all applications of P_λ , $\lambda \in [\Lambda]$.

By Lemma 2.3, aggregation can be performed with $O(\log n)$ depth and an overhead of factor $O(\log n)$ in work as compared to writing the lists. The latter takes at most $W\Lambda \in O(W \log n)$ w.h.p. in each iteration of $\mathcal{A}(H)$. Since $h \in O(\log^2 n)$, w.h.p. the \oplus operation thus causes an additive $O(\log^3 n)$ depth and $O(W \log^4 n)$ work in total. \square

5 FRT Construction

We start with a weighted graph $G = (V, E, \omega)$. Section 3 demonstrates how to construct a complete graph H on V with a polylogarithmic shortest path diameter, and Section 4 shows how to simulate $\mathcal{A}(H)$ on G using polylogarithmic overhead compared to iterating on G (provided the hop parameter d of the hop set satisfies that $d \in \text{polylog } n$). In this section, we leverage these results to construct a metric tree embedding of H with expected logarithmic stretch, automatically yielding an according embedding of G .

We start with a formal definition of metric tree embeddings in Section 5.1, proceed to show that the underlying algorithm is MBF-like (Section 5.2) and that all intermediate steps are efficient in terms of depth and work (Section 5.3), and present our main results in Section 5.4. Section 5.5 describes how to retrieve the paths in G that correspond to the edges of the FRT embedding.

5.1 Metric Tree Embeddings

We use this section to introduce the (distribution over) metric tree embeddings of Fakcharoenphol, Rao, and Talwar, referred to as FRT embedding, which has expected stretch $O(\log n)$ [18].

Definition 5.1 (Metric Embedding). *Let $G = (V, E, \omega)$ be a graph. A metric embedding of stretch α of G is a graph $G' = (V', E', \omega')$, such that $V \subseteq V'$, and*

$$\forall v, w \in V: \quad \text{dist}(v, w, G) \leq \text{dist}(v, w, G') \leq \alpha \text{dist}(v, w, G), \quad (5.1)$$

where $\alpha \in (\mathbb{R}_{\geq 1} \cup \{\infty\})$. If G' is a tree, we refer to it as metric tree embedding. For a random distribution of metric embeddings G' , we require $\text{dist}(v, w, G) \leq \text{dist}(v, w, G')$ and define the expected stretch as

$$\alpha = \max_{v \neq w \in V} \mathbb{E} \left[\frac{\text{dist}(v, w, G')}{\text{dist}(v, w, G)} \right]. \quad (5.2)$$

We show how to efficiently sample from the FRT distribution for the graph H introduced in Section 3. As H has, up to a factor of $1 + \varepsilon$ for an $\varepsilon > 0$ under our control, the same distances as G , this results in a good tree embedding of G . Khan et al. [25] show that a suitable representation of (a tree sampled from the distribution of) the FRT embedding [18] can be constructed as follows.

1. Choose $\beta \in [1, 2)$ uniformly at random.
2. Choose uniformly at random a total order of the nodes (i.e., a uniformly random permutation). In the following, $v < w$ means that v is smaller than w w.r.t. to this order.
3. Determine for each node $v \in V$ its *least element (LE) list*: This is the list obtained by deleting from the list $\{(\text{dist}(v, w, H), w) \mid w \in V\}$ all pairs $(\text{dist}(v, w, H), w)$ for which there is some $u \in V$ with $\text{dist}(v, u, H) \leq \text{dist}(v, w, H)$ and $u < w$. Essentially, v learns, for every distance d , the smallest node within distance d or less, i.e., $\min\{w \in V \mid \text{dist}(v, w, G) \leq d\}$.

4. Denote by $\omega_{\min} := \min_{e \in E} \{\omega(e)\}$ and $\omega_{\max} := \max_{e \in E} \{\omega(e)\}$ the minimum and maximum edge weight, respectively; recall that $\omega_{\max} / \omega_{\min} \in \text{poly } n$ by assumption. From the LE lists, determine for each $v \in V$ and distance $\beta 2^i \in [\omega_{\min}/2, 2\omega_{\max}]$, $i \in \mathbb{Z}$, the node $v_i := \min\{w \in V \mid \text{dist}(v, w, H) \leq \beta 2^i\}$. W.l.o.g., we assume that $i \in [k]$ for $k \in O(\log n)$ (otherwise, we shift the indices of the nodes v_i accordingly). Hence, for each $v \in V$, we obtain a sequence of nodes (v_0, v_1, \dots, v_k) . (v_0, v_1, \dots, v_k) is the leaf corresponding to $v = v_0$ of the tree embedding, (v_1, \dots, v_k) is its parent, and so on; the root is (v_k) . The edge from (v_i, \dots, v_k) to (v_{i+1}, \dots, v_k) has weight $\beta 2^i$, and $k \in O(\log n)$.

For a more detailed summary on how this works, we refer to [21]. The next lemma shows that the last step of the construction can be executed easily.

Lemma 5.2. *Given LE lists of length $O(\log n)$ for all vertices $v \in V$, the corresponding FRT embedding can be determined using $O(n \log^2 n \log \log n)$ work and $O(\log n \log \log n)$ depth.*

Proof. Determining ω_{\max} , ω_{\min} , and the range of indices i is straightforward at this complexity. For each $v \in V$ and each i , compute v_i in parallel. This is a simple binary search on the LE list of v , requiring $O(\log n)$ work and depth, for a total of $O(kn \log n) \subseteq O(n \log^2 n)$ work. Note that the results are already sorted w.r.t. the index i , i.e., we may readily read each suffix (v_i, \dots, v_k) of the LE list of v from memory.

Next, we sort all $O(n \log n)$ suffixes, which are ordered lexicographically. A comparison between two suffixes requires $O(\log \log n)$ depth and work using simple 3-valued comparisons and binary search for the first distinct elements. Hence, this step has depth $O(\log n \log \log n)$ and work $O(n \log^2 n \log \log n)$ [1]. Now it is trivial to remove duplicate entries with this complexity; the remaining suffixes encode the tree, as they reveal both the parent node and the edge weight. If desired, it is also straightforward to extract a standard adjacency list representation of the tree with the stated depth and work bounds. \square

5.2 Determining LE Lists is MBF-like

As choosing a random order of the nodes can be easily done w.h.p. by assigning to each node a string of $O(\log n)$ uniformly and independently random bits and picking β is trivial, it remains to compute the LE lists efficiently. In the following, we use the distance map module \mathcal{D} from Definition 2.1.

We establish that LE lists are the image of a representative projection r , which by the results from Section 2, especially Definition 2.11, means that they can be computed by the corresponding MBF-like algorithm.

Definition 5.3. *For $x \in \mathcal{D}$, define*

$$r(x)_v := \begin{cases} \infty & \exists w < v: x_w \leq x_v \\ x_v & \text{otherwise, and} \end{cases} \quad (5.3)$$

$$x \sim y \iff r(x) = r(y). \quad (5.4)$$

In other words, $r(x)$ is the LE list of $v \in V$ if $x_w = \text{dist}(v, w, H)$, and we consider two lists equivalent iff they result in the same LE list.

The following lemma prepares the proof that retrieving LE lists is an MBF-like algorithm. It first states that filtering keeps the interesting information: If a node/distance pair is dominated by an entry in a distance map (smaller vertex ID and at least as close), then the filtered distance map also contains a (possibly different) dominating entry; in this sense, filtering keeps the relevant information. Similar statements hold when filtering is applied before aggregation.

Lemma 5.4. Consider arbitrary $x, y \in \mathcal{D}$, $v \in V$, and $s \in \mathbb{R}_{\geq 0} \cup \{\infty\}$. Then

$$\exists w < v: x_w \leq s \quad \Leftrightarrow \quad \exists w < v: r(x)_w \leq s, \quad (5.5)$$

$$\exists w < v: (x \oplus y)_w \leq (x \oplus y)_v \quad \Leftrightarrow \quad \exists w < v: (r(x) \oplus r(y))_w \leq (r(x) \oplus r(y))_v, \text{ and} \quad (5.6)$$

$$\nexists w < v: (x \oplus y)_w \leq (x \oplus y)_v \quad \Rightarrow \quad (x \oplus y)_v = (r(x) \oplus r(y))_v. \quad (5.7)$$

Proof. Observe that the necessity “ \Leftarrow ” in (5.5) is trivial. As for sufficiency, suppose that for given s and v there is $w < v$ such that $x_w \leq s$. If $r(x)_w = x_w$, we are done. Otherwise, there must be some $u < w < v$ satisfying $x_u < x_w < x_v$. Since $|V|$ is finite, an inductive repetition of the argument yields that there is some $w' < v$ with $r(x)_{w'} = x_{w'} \leq s$.

As for (5.6) and (5.7), we have

$$\nexists w < v: (x \oplus y)_w \leq (x \oplus y)_v \quad (5.8)$$

$$\Leftrightarrow (\nexists w < v: x_w \leq \min\{x_v, y_v\}) \wedge (\nexists w < v: y_w \leq \min\{x_v, y_v\}) \quad (5.9)$$

$$\stackrel{(5.5)}{\Leftrightarrow} (\nexists w < v: r(x)_w \leq \min\{x_v, y_v\}) \wedge (\nexists w < v: r(y)_w \leq \min\{x_v, y_v\}). \quad (5.10)$$

W.l.o.g. assume $x_v = \min\{x_v, y_v\}$; the other case is symmetric. Under this condition it holds that

$$\nexists w < v: r(x)_w \leq \min\{x_v, y_v\} \quad (5.11)$$

$$\Leftrightarrow \nexists w < v: r(x)_w \leq x_v \quad (5.12)$$

$$\stackrel{(5.5)}{\Leftrightarrow} \nexists w < v: x_w \leq x_v \quad (5.13)$$

$$\Leftrightarrow r(x)_v = x_v, \quad (5.14)$$

which due to $r(y)_v \geq y_v \geq x_v$ entails that $\min\{x_v, y_v\} = \min\{r(x)_v, r(y)_v\}$; note that, in particular, this proves (5.7). Therefore, we conclude that (5.10) is equivalent to

$$(\nexists w < v: r(x)_w \leq \min\{r(x)_v, r(y)_v\}) \wedge (\nexists w < v: r(y)_w \leq \min\{r(x)_v, r(y)_v\}) \quad (5.15)$$

$$\Leftrightarrow \nexists w < v: (r(x) \oplus r(y))_w \leq (r(x) \oplus r(y))_v. \quad \square$$

Equipped with this lemma, we now can prove that \sim is a congruence relation on \mathcal{D} with representative projection r .

Lemma 5.5. The equivalence relation \sim from Definition 5.3 is a congruence relation on the zero-preserving semimodule \mathcal{D} . The function r from Definition 5.3 is a representative projection w.r.t. \sim .

Proof. \mathcal{D} is a zero-preserving semimodule by Lemma 2.2. Clearly, r is a projection, i.e., $r^2(x) = r(x)$ for all $x \in \mathcal{D}$. By Lemma 2.8, it hence suffices to show that (2.12) and (2.13) hold. As for all $s \in \mathbb{R}_{\geq 0} \cup \{\infty\}$ we have $x_v \leq x_w \Rightarrow s + x_v \leq s + x_w$, (2.12) holds. Regarding (2.13), we show that for all $x, y \in \mathcal{D}$, it holds that

$$r(x \oplus y) = r(r(x) \oplus r(y)), \quad (5.16)$$

which readily implies (2.13). Let $v \in V$ be an arbitrary vertex. In case $x_v = y_v = \infty$, we have $r(x)_v = r(y)_v = \infty$ and thus

$$r(x \oplus y)_v = r(r(x) \oplus r(y))_v = \infty. \quad (5.17)$$

Otherwise, we have $(x \oplus y)_v < \infty$ and distinguish two cases.

Case $r(x \oplus y)_v = \infty$: As $(x \oplus y)_v \neq r(x \oplus y)_v$, by (5.3) we have that

$$\exists w < v: (x \oplus y)_w < (x \oplus y)_v. \quad (5.18)$$

Using (5.6) yields

$$\exists w < v: (x \oplus y)_w < (x \oplus y)_v \quad \Rightarrow \quad \exists w < v: (r(x) \oplus r(y))_w \leq (r(x) \oplus r(y))_v \quad (5.19)$$

$$\stackrel{(5.3)}{\Rightarrow} \quad r(r(x) \oplus r(y))_v = \infty. \quad (5.20)$$

Case $r(x \oplus y)_v < \infty$: By (5.3), this necessitates $(x \oplus y)_v = r(x \oplus y)_v$ and

$$\nexists w < v: (x \oplus y)_w < (x \oplus y)_v. \quad (5.21)$$

By (5.7), this implies that $(x \oplus y)_v = (r(x) \oplus r(y))_v$, and, again by (5.6), we get

$$\nexists w < v: (x \oplus y)_w \leq (x \oplus y)_v \quad \Rightarrow \quad \nexists w < v: (r(x) \oplus r(y))_w \leq (r(x) \oplus r(y))_v \quad (5.22)$$

$$\stackrel{(5.3)}{\Rightarrow} \quad r(r(x) \oplus r(y))_v = (r(x) \oplus r(y))_v. \quad (5.23)$$

Altogether, we obtain

$$r(x \oplus y)_v = (x \oplus y)_v = (r(x) \oplus r(y))_v = r(r(x) \oplus r(y))_v. \quad (5.24)$$

□

Lemma 5.5 establishes that determining LE lists can be done by an MBF-like algorithm, where $\mathcal{M} = \mathcal{D}$ and r as in Definition 5.3. This allows us to invoke the full force of the machinery developed in Sections 2 – 4.

5.3 Efficient LE List Computation

To show that the computation is efficient, we make use of Theorem 4.2. To this end, it remains to establish that we can perform the matrix-vector multiplication with A_λ (a scaled version of A_G , the adjacency matrix of G) and apply r^V efficiently.

Recall that by $|x|$ we denote the number of non- ∞ entries of $x \in \mathcal{D}$. Our first step is Lemma 5.6, which shows that any LE list $r(x) \in \mathcal{D}$ has length $|x| \in O(\log n)$ w.h.p., provided that x does not depend on the random node ordering.¹² Observe that, in fact, the lemma is quite powerful, as it suffices that there is *any* $y \in [x]$ that does not depend on the random node ordering.

Lemma 5.6. *Suppose that $x \in \mathcal{D}$ is arbitrary, but independent of the random order on the nodes. Then $|r(x)| \in O(\log n)$ w.h.p.*

Proof. Order the non- ∞ values of x according to ascending distance (ties broken arbitrarily, but independently of the random node order). Denote for $i \in \{1, \dots, |x|\}$ by $v_i \in V$ the node satisfying that the i -th entry x_i of x w.r.t. this order is x_{v_i} . For $i \in \{1, \dots, |x|\}$, denote by X_i the indicator variable which is 1 iff $v_i < v_j$ for all $j \in \{1, \dots, i-1\}$. Clearly, $\mathbb{E}[X_i] = 1/i$, implying for $X := \sum_{i=1}^{|x|} X_i$ that

$$\mathbb{E}[X] = \sum_{i=1}^{|x|} \frac{1}{i} \leq \sum_{i=1}^n \frac{1}{i} \in \Theta(\log n). \quad (5.25)$$

Observe that X_i is independent of $\{X_1, \dots, X_{i-1}\}$, as whether $v_i < v_j$ for all $j < i$ or not is independent of the internal order of the set $\{v_1, \dots, v_{i-1}\}$. This is sufficient to apply Chernoff's

¹²We remark that it is well-known that LE lists have length $O(\log n)$ w.h.p. throughout intermediate computations [21, 25], and the key argument is identical to the corresponding lemmas in [21] and [25]. However, these lemmas assume the LE lists are assembled using k -hop distances (where k increases in each iteration). Lemma 5.6 is more general in that it makes no assumption about x except for its independence of the random node order; we need the more general statement due to our decomposition of A_H .

bound,¹³ yielding that $X \in \Theta(\log n)$ w.h.p. As $\mathbb{P}[X = k] = \mathbb{P}[|r(x)| = k]$, this concludes the proof. \square

Lemma 5.6 shows that intermediate lists are short w.h.p. The next lemma shows that such a short list can be computed efficiently.

Lemma 5.7. *For $x \in \mathcal{D}$, $r(x)$ can be computed with depth $O(|r(x)| \log n)$ and work $O(|x|(|r(x)| + \log n))$.*

Proof. Construct a tournament tree on the non- ∞ elements of x and identify its leaves with their indices $v \in V$ ($O(\log n)$ depth and $O(|x|)$ work). Iterate the following process until all leaves are discarded. In iteration i , find the element with the smallest node index v w.r.t. the random node order whose corresponding leaf remains in the tree ($O(\log n)$ depth and $O(|x|)$ work). Store that $r(x)_v = x_v$. Then, mark each leaf w for which $x_v \leq x_w$ as discarded, including v ($O(1)$ depth and $O(|x|)$ work).

Note that for each $w \neq v$ for which the corresponding node is discarded, we have $r(x)_w = \infty$. On the other hand, by construction we have for all v for which we stored $r(x)_v = x_v$ that there is no $w \in V$ satisfying both $x_w \leq x_v$ and $w < v$. Thus, the computed list is indeed $r(x)$.

The depth and work bounds follow from the above bounds on the complexities of the individual steps and by observing that in each iteration, we add a distinct index/value pair (with non- ∞ value) to the list that after termination equals $r(x)$. \square

Based on the previous two lemmas, Lemma 5.8 now establishes that each of the intermediate results can be computed efficiently. Any such intermediate result $y \in \mathcal{D}$ is of the form

$$y = (r^V A_\mu)^f P_\mu \underbrace{\left(r^V \left(\bigoplus_{\lambda=0}^{\Lambda} P_\lambda (r^V A_\lambda)^d P_\lambda \right) \right)^h}_{=: x^{(h)}} r^V x^{(0)}, \quad (5.26)$$

where $x^{(h)} = (r^V A_H^h) x^{(0)}$ is the intermediate result of h iterations on H , and $(r^V A_\mu)^f P_\mu$ represents another f iterations on G used by the oracle to simulate the $(h+1)$ -th iteration on H .

Lemma 5.8. *Suppose $x^{(0)} \in \mathcal{D}^V$ is given by $(x_v)_w = 0$ for $v = w$ and $(x_v)_w = \infty$ everywhere else ($x_v^{(0)}$ is the v -th unit vector w.r.t. \mathcal{D}). For arbitrary $d, h \in \mathbb{N}$, $f \in [d]$, and $\mu \in [\Lambda]$, suppose that*

$$y = (r^V A_\mu)^f P_\mu \left(r^V \left(\bigoplus_{\lambda=0}^{\Lambda} P_\lambda (r^V A_\lambda)^d P_\lambda \right) \right)^h r^V x^{(0)}. \quad (5.27)$$

Then $A_\mu y$ and $r^V A_\mu y$ can be computed using $O(m \log^2 n)$ work and depth $O(\log^2 n)$.

Proof. By (2.30) and (4.8), we have

$$y = (r^V A_\mu)^f P_\mu \left(r^V \left(\bigoplus_{\lambda=0}^{\Lambda} P_\lambda (r^V A_\lambda)^d P_\lambda \right) \right)^h r^V x^{(0)} = r^V A_\mu^f P_\mu \left(\bigoplus_{\lambda=0}^{\Lambda} P_\lambda A_\lambda^d P_\lambda \right)^h x^{(0)} =: r^V y'. \quad (5.28)$$

¹³Commonly the assumption that the variables $\{X_1, \dots, X_{|x|}\}$ are independent is made. In the standard derivation this is used to show for fixed $t > 0$ that $\mathbb{E}[e^{tX}] = \prod_{j=1}^i \mathbb{E}[e^{tX_j}]$, which follows from the weaker condition as well.

Since y' does not depend on the random order of V , we know from Lemma 5.6 that, for all $v \in V$, $|y_v| \in O(\log n)$ w.h.p. We compute $(A_\mu y)_v$ in parallel for each $v \in V$. By Lemma 2.3, this can be done with depth $O(\log n)$ and work

$$O\left(\sum_{v \in V} \sum_{\substack{w \in V \\ \{v,w\} \in E}} |y_w| \log n\right) = O\left(\sum_{\{v,w\} \in E} |y_w| \log n\right) \subseteq O(m \log^2 n) \quad (5.29)$$

w.h.p., where we note that the cost of propagation (i.e., addition of the edge weights) is asymptotically dominated by the aggregation (i.e., the min operations) with respect to both depth and work. To bound the cost of computing $r^V A_\mu y$, we again invoke (4.8), showing that

$$r^V A_\mu y = r^V A_\mu r^V y' = r^V A_\mu y', \quad (5.30)$$

and Lemma 5.6, showing that $|(r^V A_\mu y)_v| \in O(\log n)$ w.h.p. for all $v \in V$. Applying Lemma 5.7, we deduce that $r^V A_\mu y$ can be computed from $A_\mu y$ with depth $O(\log^2 n)$ and work

$$O\left(\sum_{v \in V} |(A_\mu y)_v| \log n\right) \subseteq O\left(\sum_{v \in V} \sum_{\substack{w \in V \\ \{v,w\} \in E}} |y_w| \log n\right) = O\left(\sum_{\{v,w\} \in E} |y_w| \log n\right) \subseteq O(m \log^2 n) \quad (5.31)$$

w.h.p. □

5.4 Metric Tree Embedding in Polylogarithmic Time and Near-Linear Work

Determining LE lists on H yields a probabilistic tree embedding for G with expected stretch $O(\log n)$, see Theorem 3.5 and Section 5.1, is the result of an MBF-like algorithm (Section 5.2), and this algorithm is efficient (Theorem 4.2 and Section 5.3). We sum this up in Theorem 5.9, which relies on the graph containing a suitable hop set. Corollaries 5.10 and 5.11 remove this assumption by invoking known algorithms to establish this property first; note that improved hop set constructions also result in improved tree embedding algorithms based on our machinery.

Theorem 5.9. *Suppose we are given the incidence list (including weights) of a graph $G = (V, E, \omega)$ satisfying for some $\alpha \geq 1$ and $d \in \mathbb{N}$ that $\text{dist}(v, w, G) \leq \alpha \text{dist}^d(v, w, G)$ for all $v, w \in V$. Then, w.h.p., we can compute a tree embedding of G of expected stretch $O(\alpha^{O(\log n)} \log n)$ with depth $O(d \log^4 n) \subset \tilde{O}(d)$ and work $O(m(d + \log n) \log^5 n) \subset \tilde{O}(md)$.*

Proof. By Lemma 5.8, we can apply Theorem 4.2 with $D \in O(\log^2 n)$ and $W \in O(m \log^2 n)$, showing that we can compute LE lists for H with depth $O(d \log^4 n)$ and work $O(m(d + \log n) \log^5 n)$. As shown in [18], the tree T represented by these lists has expected stretch $O(\log n)$ w.r.t. the distance metric of H . By Theorem 3.5, this implies also that $\text{dist}(v, w, G) \leq \text{dist}(v, w, T)$ and $\text{dist}(v, w, T) \in O(\alpha^{O(\log n)} \log n \text{dist}(v, w, G))$ in expectation. Thus, it remains to discuss how to map an edge in T to a path of at most the same weight in G . Recall that we compute the LE lists for H by repeated application of the operations r^V , \oplus , P_λ , and A_λ (where $\lambda \in [\Lambda]$). Except for A_λ , all of these operations eliminate redundant information, i.e., distance values to nodes that do not end up in the final lists and are therefore not relevant to routing. A_λ , on the other hand, is simply a (classic) Moore-Bellman-Ford step. Thus, we may simply store the necessary information for backtracing the induced $\tilde{O}(d)$ -hop paths at each node. □

As mentioned above, we require G to contain a (d, ε) -hop set, and we need $d \in \text{polylog } n$ in order to achieve polylogarithmic depth. We also need to determine such a hop set using polylog n depth and $\tilde{O}(m)$ work, and that it does not increase the problem size significantly by adding too many edges. Cohen’s hop sets [13] meet all these requirements, yielding the following corollary.

Corollary 5.10. *Suppose we are given the incidence list (including weights) of a graph G . Then, for any constant $\varepsilon > 0$, w.h.p. we can compute a tree embedding of expected stretch $O(\varepsilon^{-1} \log n)$ with depth polylog n and work $\tilde{O}(m + n^{1+\varepsilon})$.*

Proof. We apply the hop set construction by Cohen [13] to determine an intermediate graph G' , which adds $\tilde{O}(m^{1+\varepsilon})$ edges to the graph¹⁴ for a constant $\varepsilon > 0$ that can be chosen freely. The algorithm ensures that $\text{dist}(v, w, G) \leq \alpha \text{dist}^d(v, w, G)$ for $d \in \text{polylog } n$ and $\alpha \in 1 + \Omega(1/\text{polylog } n)$; it has depth polylog n and work $\tilde{O}(m^{1+\varepsilon})$. Choosing $\alpha \in 1 + o(1/\log n)$ and applying Theorem 5.9, the claim of the corollary follows. \square

We stress that, in terms of stretch, the embedding is essentially as good as directly constructing and FRT embedding of G : The difference is a factor of $1 + o(1)$. If we are willing to sacrifice more on this front, we can decrease the overhead in terms of work further.

Corollary 5.11. *Suppose we are given the incidence list (including weights) of a graph G . Then, for any constant $\varepsilon > 0$ and any $k \in \mathbb{N}$, w.h.p. we can compute a tree embedding of G of expected stretch $O(k \log n)$ with depth polylog n and work $\tilde{O}(m + n^{1+\varepsilon})$.*

Proof. We apply the same strategy as before, except that we first reduce the number of edges by computing a $(2k - 1)$ -spanner of $G = (V, E, \omega)$, i.e., a subgraph $G' = (V, E', \omega)$ satisfying for all $v, w \in V$ that $\text{dist}(v, w, G') \leq (2k - 1) \text{dist}(v, w, G)$. Note that $\text{dist}(v, w, G) \leq \text{dist}(v, w, G')$ holds because G' is a subgraph of G . Baswana and Sen [8] give an algorithm of depth polylog n and work $\tilde{O}(m)$ that computes a $(2k - 1)$ -spanner with $\tilde{O}(n^{1+1/k})$ edges w.h.p.¹⁵ Applying Corollary 5.10 to G' hence yields the stated result. \square

5.5 The Data Structure Representing the Tree

There are further concerns for the data structure representing the embedding, like its size and, e.g., the depth and work of determining all edges on the path in G corresponding to a given tree edge. While our purpose is not to provide specifically tailored data structures, note that it is straightforward to obtain reasonably efficient ones. As our computation has depth polylog n and we know that the local lists after each application of r^V have size $O(\log n)$, we need to store polylog n bits per node. This entails that if we compute for each node $v \in V$ (in parallel) a routing tree comprising each node $w \in V$ for which $(y_w)_v \neq \infty$ in some reduced list y during the course of the computation, each node is involved in polylog n such trees only. As trees allow for highly efficient computations and data structures, the embeddings provided by Corollaries 5.10 and 5.11 can not only be computed, but also used efficiently.

¹⁴Although Cohen does not discuss this in her article, it is not hard to see that the new edges can be efficiently mapped to paths in the original graph.

¹⁵The authors bound the expected number of edges as $O(kn^{1+1/k})$. Noting that w.l.o.g. $k \in O(\log n)$ and applying Chernoff’s bound, one obtains the bound that holds w.h.p.

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A Algebraic Foundations

Definitions A.1, A.2, and A.3 are slightly adapted from Chapters 1 and 5 of [23]. In this section, we refer to the neutral elements of addition and multiplication as 0 and 1. Note, however, that in the min-plus semiring $\mathcal{S}_{\min,+}$ the neutral element of “addition” (min) is ∞ , and that of “multiplication” (+) is 1.

Definition A.1 (Semigroup). *Let $M \neq \emptyset$ be a set and $\circ: M \times M \rightarrow M$ a binary operation. (M, \circ) is a semigroup if and only if \circ is associative, i.e.,*

$$\forall x, y, z \in M: \quad x \circ (y \circ z) = (x \circ y) \circ z. \quad (\text{A.1})$$

A semigroup (M, \circ) is commutative if and only if

$$\forall x, y \in M: \quad x \circ y = y \circ x. \quad (\text{A.2})$$

$e \in M$ is a neutral element of (M, \circ) if and only if

$$\forall x \in M: \quad e \circ x = x \circ e = x. \quad (\text{A.3})$$

Some authors do not require semirings to have neutral elements. We, however, need them and work on $\mathcal{S}_{\min,+}$ which provides them, anyway.

Definition A.2 (Semiring). *Let $M \neq \emptyset$ be a set, and $\oplus, \odot: M \times M \rightarrow M$ binary operations. Then (M, \oplus, \odot) is a semiring if and only if*

1. (M, \oplus) is a commutative semigroup, with neutral element 0
2. (M, \odot) is a semigroup with neutral element 1, and
3. the distributive laws hold:

$$\forall x, y, z \in M: \quad x \odot (y \oplus z) = (x \odot y) \oplus (x \odot z) \quad (\text{A.4})$$

$$\forall x, y, z \in M: \quad (y \oplus z) \odot x = (y \odot x) \oplus (z \odot x). \quad (\text{A.5})$$

Definition A.3 (Semimodule). *Let $\mathcal{S} = (S, \oplus, \odot)$ be a semiring. $\mathcal{M} = (M, \oplus, \odot)$ with binary operations $\oplus: M \times M \rightarrow M$ and $\odot: S \times M \rightarrow M$ is a semimodule over \mathcal{S} if and only if*

1. (M, \oplus) is a semigroup, and
2. for all $s, t \in S$ and all $x, y \in M$:

$$1 \odot x = x \quad (\text{A.6})$$

$$s \odot (x \oplus y) = (s \odot x) \oplus (s \odot y) \quad (\text{A.7})$$

$$(s \oplus t) \odot x = (s \odot x) \oplus (t \odot x) \quad (\text{A.8})$$

$$(s \odot t) \odot x = s \odot (t \odot x). \quad (\text{A.9})$$

\mathcal{M} is zero-preserving if and only if

1. (M, \oplus) has the neutral element 0, and
- 2.

$$\forall x \in M: \quad 0 \odot x = 0. \quad (\text{A.10})$$

B Sample Applications

In this section, we consider two applications of the embedding that were given by Blelloch et al. [10] and show how their results are improved by applying our techniques. The main difference worth emphasizing again is that Blelloch et al. assume that the metric is given by query access, whereas we assume that the metric is given in the form of a graph. This is more general, as the metric can be interpreted as a complete graph of shortest path diameter 1. We remark, however, that the use of hop sets restricts us to polynomially bounded edge ratios (or our solution loses efficiency).

B.1 k -Median

In the k -median problem, the task is to determine $F \subseteq V$, $|F| \leq k$, such that $\sum_{v \in V} \text{dist}(v, F, G)$ is minimized, where $\text{dist}(v, F, G) := \min_{f \in F} \{\text{dist}(v, f, G)\}$ is the distance of v to the set F . In [10], an $O(\log k)$ -approximation with depth $O(\log^2 n)$ and work $\tilde{O}(nk + k^3)$ is given for $k \geq \log n$; the special case $k < \log n$ admits an $\tilde{O}(n)$ -work solution with the same depth [11].

The algorithm essentially consists of three steps:

1. Use a parallel version of a sampling technique due to Mettu and Plexton [31] to reduce the number of candidates for the set F to a set of $O(k)$ nodes.
2. Sample an FRT tree for the (sub)metric on these $O(k)$ nodes.
3. Run an $\tilde{O}(k^3)$ work dynamic programming algorithm to solve the tree instance optimally.

We leave the third step as is and replace the second with our approach. To compute the embedding only on $O(k)$ nodes, one simply sets $x_{vv}^{(0)} = 0$ if v was sampled into the candidate set and $x_{vv}^{(0)} = \infty$ in all other cases, and considers only the LE lists at sampled nodes when constructing the tree.

The first step requires (for $O(\log n/k)$ iterations) to sample $O(k)$ nodes from the current candidate set (initially all nodes) and determine for each of them the distance to this set. In [10], this is simply done by querying the metric $O(nk)$ times, for each pair of a candidate and a sampled node. Since the size of the candidate set decreases by a constant factor in each iteration, this results in $O(nk)$ total work for this step.

We do not have this option, as we cannot access the metric directly. However, we can bound the work for this operation as follows. We perform the sampling *after* embedding into the graph H from Section 3 (which costs only factor $(1 + o(1))$ in approximation, regardless of k). As it suffices to determine for each (sampled) node the closest candidate, we can, by Theorem 3.5, compute the required information with $\text{SPD}(H) \in O(\log^2 n)$ iterations of the MBF-like algorithm given in Example 2.23 (for $d = \infty$), ensuring that the “lists” kept by the nodes always have at most one entry. Using Cohen’s hop sets in the construction of H , we can simulate an iteration on H using $\text{polylog } n$ iterations on the original graph (augmented by the hop set), and arrive at the following corollary.

Corollary B.1. *For any fixed constant $\varepsilon > 0$, an expected $O(\log k)$ -approximation to k -median can be computed with depth $\text{polylog } n$ and work $\tilde{O}(m^{1+\varepsilon} + k^3)$.*

B.2 Buy-at-Bulk Network Design

In this problem, one is given a weighted graph G , demands $(s_i, t_i, d_i) \in \binom{V}{2} \times \mathbb{R}_{>0}$, $i \in [k]$, and a set of cable types, where the cable of type j has capacity u_j and costs $c_j \omega(e)$ when purchased for edge e . The goal is to find an assignment of cable types to edges minimizing the total cost such

that the resulting edge capacities allow to simultaneously route d_i units of (distinct) flow from s_i to t_i (for all $i \in [k]$). Andrews shows that this problem is hard to approximate better than factor $\log^{1/2-o(1)} n$ [4].

Blelloch et al. [10] give an expected $O(\log n)$ -approximation with polylog n depth and $O(n^3 \log n)$ work for this problem, which is a straightforward parallelization of the algorithm by Awerbuch and Azar [5].¹⁶ Our tools allow for a more efficient parallelization of this algorithm, as the work of the implementation by Blelloch et al. is dominated by solving APSP to determine the distance metric of the graph. The approach can be summarized as follows.

1. Metrically embed G into a tree T with expected stretch $O(\log n)$. As the objective is linear in the edge weights, an optimal solution in G induces a solution in T whose expected cost is by at most a factor $O(\log n)$ larger.
2. $O(1)$ -approximate on the tree.
3. Map the tree solution back to G of by at most a constant factor larger cost.

Using our machinery (with Cohen’s hop set), the first step has polylog n depth and $\tilde{O}(m^{1+\varepsilon})$ work. For the second step, Blelloch et al. discuss a polylog n depth solution with $\tilde{O}(n+k)$ work.

Concerning the third step, we note that each tree edge $\{v, w\}$ maps back to a path of $O(\text{SPD}(H))$ hops in H as follows. First, to keep the notation simple, identify each tree node (given as tuple (v_1, \dots, v_j)) with its “leading” node (i.e., v_1); in particular, each leaf is identified with the node in V that is mapped to it. If one of the endpoints of $\{v, w\}$, say v , is a leaf, then v has an entry $(\text{dist}(v, w, H), w)$ in its LE list, and we can trace the shortest path from v to w based on the LE lists. Moreover, $\text{dist}(v, w, H) \leq \omega_T(v, w)$, i.e., we may map the tree edge back to the path without incurring larger cost than in T . If v and w are inner nodes, choose an arbitrary leaf u that is a common descendant.¹⁷ We then can trace shortest paths from u to v and from u to w in H , respectively. The cost of their concatenation is $\text{dist}(u, v, H) + \text{dist}(u, w, H) \leq 3/2 \cdot \omega_T(v, w)$, because the cost of tree edges decreases by factor 2 for each additional hop away from the root.

Using this observation, we can map the solution on T back to one in H whose cost is at most by factor $3/2$ larger. Assuming suitable data structures are used, this operation has depth polylog n and requires $\tilde{O}(\min\{k, n\})$ work w.h.p., where we exploit that $\text{SPD}(H) \in O(\log^2 n)$ w.h.p. by Theorem 3.5 and T has depth $O(\log n)$, implying that the number of edges in T with non-zero flow is bounded by $O(\min\{k, n\} \log n)$.

Finally, we map back from H to G' (G augmented with hop set edges) and then to G . The specifics depend on the hop set used and, again, we assume that suitable data structures are in place, see Section 5.5. This can be handled with depth polylog n and $\tilde{O}(\text{SPD}(G))$ work for a single edge in H . Since we deal with $\tilde{O}(\min\{k, n\})$ edges in H , this yields a total work of $\tilde{O}(\min\{k, n\} \text{SPD}(G)) \subseteq \tilde{O}(n^2)$.

Corollary B.2. *For any constant $\varepsilon > 0$, an $O(\log n)$ -approximation to buy-at-bulk network design can be computed with depth polylog n and work $\tilde{O}(\min\{m^{1+\varepsilon}, n^2\} + \min\{k, n\} \text{SPD}(G)) \subseteq \tilde{O}(n^2)$.*

C Distributed FRT Constructions in our Framework

Throughout this section we fix \sim to be the congruence relation given in Definition 5.3. Furthermore, let $G = (V, E, \omega)$ be a weighted graph with adjacency matrix $A_G \in (\mathbb{R}_{\geq 0} \cup \{\infty\})^{V \times V}$. Distributed

¹⁶Awerbuch and Azar also note that their approach gives rise to an online algorithm of the same asymptotic approximation ratio. The adaption is straightforward, so for the sake of compactness we do not discuss it here.

¹⁷This choice can, e.g., be fixed when constructing the tree from the LE list, without increasing the asymptotic bounds on depth or work.

algorithms for constructing FRT-type tree embeddings in the Congest model [33] are covered by our framework as well. For the sake of brevity, we do not formally introduce this model of computation here. The key aspects are the following:

- The computation proceeds by nodes exchanging $O(\log n)$ -sized messages over the edges of the graph in synchronous rounds. Recall that edge weights, and by extension all path weights relevant to our computations, can be encoded using $O(\log n)$ bits, i.e., one message can basically carry one index/distance pair.
- Nodes initially know their neighbors and the weight of their incident edges. They need to compute only “their” part of the output. As pointed out in Section 5, sampling a tree from the distribution boils down to computing LE lists, so each node needs to compute its LE list.
- Local computations are free. The goal is to minimize the number of communication rounds.

Sections C.1 and C.2 briefly summarize the distributed FRT algorithms by Kahn et al. [25], and Ghaffari and Lenzen [21]. Section C.3 demonstrates how to eliminate an n^ε overhead of [21], using our machinery and a distributed hop set construction due to Henzinger et al. [24].

C.1 The Algorithm by Khan et al.

In our terminology, the algorithm due to Khan et al. [25] computes $r^V A_G^h x^{(0)} = (r^V A_G)^h x^{(0)}$, where $h = \text{SPD}(G)$ and $x^{(0)} \in \mathcal{D}^V$ (see Definitions 2.1 and 2.9) given by $x_v^{(0)} = e_v$, i.e., $x_{vw}^{(0)} = 0$ iff $v = w$ and ∞ otherwise. It does so in the straightforward way: One initializes $x^{(0)}$ as above and iteratively computes $x^{(i+1)} := r^V A_G x^{(i)}$ until a fixpoint is reached, i.e., $x^{(i+1)} = r^V A_G x^{(i)}$. As $(r^V A_G)^i x^{(0)} = r^V A_G^i x^{(0)}$ by (2.30), Lemma 5.6 shows that, w.h.p., $|x_v^{(i)}| \in O(\log n)$ for all $i \in [h]$ and all $v \in V$. Therefore, $v \in V$ can transmit $x_v^{(i)}$ to all of its neighbors using $O(\log n)$ messages, and upon reception of its neighbors’ lists compute $x_v^{(i+1)}$ locally. Thus, each iteration takes $O(\log n)$ rounds w.h.p., readily implying the total round complexity of $O(\text{SPD}(G) \log n)$ w.h.p. shown in [25].

C.2 The Algorithm by Ghaffari and Lenzen

The strongest lower bound for constructing a (low-stretch) metric tree embedding of G in the Congest model is $\tilde{\Omega}(\sqrt{n} + D)$ [16, 21], where D is the hop diameter of the graph. If $\text{SPD}(G) \gg D + \sqrt{n}$, one may thus hope for a solution that runs in $\tilde{o}(\text{SPD}(G))$ rounds. For $\varepsilon > 0$, in [21] a running time of $\tilde{O}(n^{1/2+\varepsilon} + D)$ is achieved by the following strategy.

1. Abbreviate $\ell := \lceil \sqrt{n} \rceil$. For a sufficiently large constant c , sample $\lceil c\ell \log n \rceil$ nodes uniform at random; call this set S . Define the *skeleton graph* $G_S = (S, E_S, \omega_S)$, where for $s, t \in S$, we set $\omega_S(s, t) = \text{dist}^\ell(s, t, G)$, i.e., $\{s, t\} \in E_S$ iff $\text{dist}^\ell(s, t, G) \neq \infty$. Then, w.h.p., $\text{dist}(s, t, G_S) = \text{dist}(s, t, G)$ for all $s, t \in S$ (Lemma 4.6 of [27]).
2. For $k \in \Theta(1/\varepsilon)$, construct a $(2k - 1)$ -spanner G'_S of the skeleton that has $\tilde{O}(\ell^{1+1/k}) \subseteq \tilde{O}(n^{1/2+\varepsilon})$ edges (Lemma 4.9 of [27]). Make the spanner known to all nodes by broadcasting it via a BFS tree of G ($\tilde{O}(n^{1/2+\varepsilon} + D)$ rounds).
3. Define $H = (V, E_H, \omega_H)$ by E_H being the union of the skeleton spanner edges with E . Edge $e \in E_H$ has weight $\omega_H(e) := \omega_S(e)$ if e is in the skeleton spanner and $\omega_H(e) := (2k - 1)\omega(e)$ otherwise. Note that by construction G thus embeds into H with stretch $2k - 1$, i.e., $\text{dist}(v, w, G) \leq \text{dist}(v, w, H) \leq (2k - 1) \text{dist}(v, w, G)$.

4. Choose uniform permutations of S and $V \setminus S$. We extend these permutations to a permutation of all nodes by ruling that for all $s \in S$ and $v \in V \setminus S$, we have $s < v$. The LE lists of H corresponding to this permutation induce an embedding of H into a tree T with expected stretch $O(\log n)$, see Lemma 4.9 of [21]. Accordingly, G embeds into T with expected stretch $O(k \log n) = O(\varepsilon^{-1} \log n)$.

It is then shown that LE lists of H can be computed fast in the Congest model. To this end, one exploits that (min-hop) shortest paths in H contain only a single (maximal) subpath consisting of spanner edges, where the maximal subpaths of non-spanner edges have at most $\ell \in \Theta(\sqrt{n})$ hops w.h.p.¹⁸ Given that $s < v$ for all $s \in S$ and $v \in V \setminus S$, it follows that for each $v \in V$ and each entry $(w, \text{dist}(v, w, H))$ of its LE list, w.h.p. there is a min-hop shortest path from v to w that consists of a prefix of at most ℓ non-spanner edges followed by a shortest path in G'_S . This entails that

$$r^V A_H^{\text{SPD}(H)} x^{(0)} = r^V A_{G,2k-1}^\ell A_{G'_S}^{|S|} x^{(0)} = r^V A_{G,2k-1} \left(r^V A_{G'_S}^{|S|} x^{(0)} \right), \quad (\text{C.1})$$

where $x^{(0)}$ is as above, A_G is the adjacency matrix of a graph G , and $A_{G,s}$ is A_G with entries multiplied by factor $s \in \mathbb{R}_{\geq 0} \cup \{\infty\}$. Here, $A_{G'_S}$ is interpreted as an adjacency matrix for node set V by setting $(A_{G'_S})_{vw} = \infty$ if $\{v, w\} \not\subseteq S$. Clearly, $A_H x \leq A_{G,2k-1} x$ and $A_H x \leq A_{G'_S} x$ for all x (where \leq is to be read componentwise), while $A_{G'_S}^{|S|} x^{(0)}$ yields distances in G'_S and $A_{G,2k-1}^\ell$ propagates information via non-spanner edges for ℓ hops.

Because G'_S is known to all nodes, each $v \in V$ can compute $\bar{x}_v^{(0)} := (A_{G'_S}^{|S|} x^{(0)})_v$ locally. Subsequently, nodes determine their component of $r^V A_{G,2k-1}^\ell \bar{x}^{(0)} = (r^V A_{G,2k-1}^\ell)^{\ell} \bar{x}^{(0)}$ via ℓ iterations $\bar{x}^{(i+1)} := r^V A(G, k) \bar{x}^{(i)}$. Here, one exploits that, for all i , $|\bar{x}_v^{(i)}| \in O(\log n)$ w.h.p. by Lemma 5.6,¹⁹ and thus each iteration can be performed by sending $O(\log n)$ messages over each edge. The time complexity of $\tilde{O}(n^{1/2+\varepsilon} + D)$ for an embedding of expected stretch $O(\varepsilon^{-1} \log n)$ shown in [21] follows.

C.3 Achieving Stretch $O(\log n)$ in Near-Optimal Time

The factor n^ε overhead is due to constructing and broadcasting the skeleton spanner. We can improve on this by relying on hop sets, just as we do in our parallel construction. In recent work, Henzinger et al. [24] show how to compute an $(n^{o(1)}, o(1))$ -hop set of the skeleton (in the sense that nodes learn their incident edges and their weights) in $n^{1/2+o(1)} + D^{1+o(1)}$ rounds.²⁰

The prerequisite of the algorithm is that the skeleton nodes learn their skeleton edges and their weights. Unfortunately, this computation would require $\tilde{\Omega}(n)$ rounds in the Congest model [29]. Instead, we $(1 + o(1/\log n))$ -approximate $\text{dist}^\ell(s, t, G)$ for all $s, t \in S$, which can be done in $\tilde{O}(\sqrt{n})$ rounds [29]. Here, $s \in S$ learns the approximate ℓ -hop distance to each $t \in S$. This can be seen as metrically embedding the skeleton graph with stretch $1 + o(1/\log n)$, and the representation of the resulting graph allows to apply the algorithm from [24]. Overall, we obtain a graph G'_S satisfying for all $s, t \in G$ that $\text{dist}(s, t, G) \leq \text{dist}^{n^{o(1)}}(s, t, G'_S) \in (1 + o(1/\log n)) \text{dist}(s, t, G)$ w.h.p., where nodes in S know their incident edges in this graph.

¹⁸This follows analogously to Lemma 3.4, with 2 levels and a sampling probability of $\tilde{\Theta}(1/\sqrt{n})$ instead of $1/2$.

¹⁹More precisely, one applies the lemma twice, as it requires the considered module elements to be independent of the randomness of the permutation. First, one considers the computation starting from $(y_v)_w = 0$ if $v = w \in S$ and $(y_v)_w = \infty$ else. Since $s < v$ for all $s \in S$ and $v \in V \setminus S$, this shows that the (sub)lists concerning nodes from S have $O(\log n)$ non- ∞ entries w.h.p. Then one applies the lemma to $(r^V A(G, k))^i z = r^V A(G, k)^i z$, $i \in \{1, \dots, \ell\}$, with $(z_v)_w = 0$ if $v = w \in V \setminus S$ and $(z_v)_w = \infty$ else, and argues that the sublists concerning nodes from $V \setminus S$ have length $O(\log n)$ w.h.p., too, because they are sublists of $(r^V A(G, k))^i z = r^V A(G, k)^i z$ for some i .

²⁰Cohen's approach [13] may be of use here as well, but requires translating her construction to this setting.

Now we can apply the machinery established for the parallel setting. First, we embed G'_S into H_S as in Section 3, where $d \in n^{o(1)}$. To this end, nodes determine their level locally and broadcast it over a BFS tree, which takes $O(|S| + D) \subset \tilde{O}(\sqrt{n} + D)$ rounds; thus, $s \in S$ knows the level of $\{s, t\} \in E_{H_S}$ for each $t \in S$.

We define H as in Section C.2, but with H_S taking the role of the skeleton spanner. Since, by Theorem 3.5 for $\varepsilon \in o(1/\log n)$, for all $s, t \in V$ we have that $\text{dist}(s, t, H_S) \in (1 + o(1)) \text{dist}(s, t, G)$ w.h.p., it suffices to increase the weight of edges from G by factor $1 + o(1)$, i.e., G embeds into H with stretch $1 + o(1)$.

To determine the LE lists for H , we must compute

$$r^V A_H^{\text{SPD}(H)} x^{(0)} = (r^V A_{G, 1+o(1)})^\ell (r^V A_{H_S})^{\text{SPD}(H_S)} x^{(0)}, \quad (\text{C.2})$$

where $A_{G, 1+o(1)}$ is given by multiplying each entry of A_G by a sufficiently large factor $f \in 1+o(1)$ and A_{H_S} is extended to an adjacency matrix on the node set V as above. Applying $r^V A_{G, 1+o(1)}$ is the same as before, taking in total $\tilde{O}(\ell) = \tilde{O}(\sqrt{n})$ rounds. Concerning A_{H_S} , we follow the same strategy as in Theorem 4.2, i.e., it suffices to show that we can efficiently perform a matrix-vector multiplication $A_{G'_S} x$ for any x that may occur during the computation (applying r^V is a local operation and thus free), assuming that each node $v \in V$ knows x_v and its row of the matrix. As multiplications with $A_{G'_S}$ affect only lists at nodes $s \in S$, this can be done by local computations once all nodes know x_s for each $s \in S$. As before, $|x_s| \in O(\log n)$ w.h.p., so $\sum_{s \in S} |x_s| \in O(|S| \log n) \subset \tilde{O}(\sqrt{n})$ w.h.p. We broadcast these lists over a BFS tree (of G) to all nodes, taking $\tilde{O}(\sqrt{n} + D)$ rounds per matrix-vector multiplication. Overall, this results in round complexity $n^{1/2+o(1)} + D^{1+o(1)}$.

Theorem C.1. *There is a randomized distributed algorithm computing a metric tree embedding of expected stretch $O(\log n)$ in $(\sqrt{n} + D)2^{O(\sqrt{\log n})}$ rounds of the Congest model.*

Proof sketch. Follows from the above discussion, noting that the $n^{o(1)}$ and $D^{o(1)}$ factors from [24] are, more precisely, $2^{O(\sqrt{\log n})}$ factors, $\text{SPD}(H_S) \in O(\log^2 n)$ w.h.p. by Theorem 3.5, and multiplication with A_{H_S} requires $\tilde{O}(d) \subset 2^{O(\sqrt{\log n})}$ multiplications with $A_{G'_S}$. \square

Note that by combining this result with the algorithm by Khan et al., the round complexity can be improved to $\min\{n^{1/2+o(1)} + D^{1+o(1)}, \tilde{O}(\text{SPD})\}$.