Instantons in String Theory

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Abstract. These proceedings from the second Caesar Lattes meeting in Rio de Janeiro 2015 are a brief introduction to how automorphic forms appear in the low energy effective action of maximally supersymmetric string theory. The explicit example of the R^4-interaction of type IIB string theory in ten dimensions is discussed. Its Fourier expansion is interpreted in terms of perturbative and non-perturbative contributions to the four graviton amplitude.

HIGHER DERIVATIVE CORRECTIONS TO GENERAL RELATIVITY

String theory offers an ultraviolet completion to general relativity. A good way of thinking about this is from the perspective of the low energy effective action.

Low energy effective actions

An *n*th-order effective action of a theory is an action which produces the same amplitudes at tree level (i.e. classically) as the original action at *n* loops. One can construct such an action by pre-computing all the scattering amplitudes of the physical processes up to loop order *n* and creating corresponding interactions for the concerned fields whose coupling constants are proportional to the computed scattering amplitudes. This works because at tree level, the scattering amplitude as computed from the effective action are simply proportional its coupling constants, which in turn (by construction) are proportional to the *n*-loop amplitudes of the original theory, as desired.

Recall that becasue of Lorentz invariance, only scalar fields are allowed vacuum expectation values[1] (vevs). The vevs of the scalar fields therefore have the potential of playing the roles of coupling constants and are referred to as the moduli of the theory. The set of all allowed vevs for all moduli makes up the moduli space. Type IIB string theroy contains two scalar fields, the axion χ and the dilaton ϕ. It also has two coupling constants denoted α' and g_s where

$$g_s = e^{\phi} \tag{1}$$

is related to the dilaton[2] and measures the strength of the string interaction. Furthermore,

$$\alpha' = l_s^2 \tag{2}$$

is related to the string length l_s, the natural length scale present in the theory, and thereby also the string tension. Type IIB string theory also comes with a two-dimensional moduli space that will be discussed below. Any closed string amplitude $\mathcal{A}$ computed in perturbation theory should therefore take the form

$$\mathcal{A} = \sum_{n=0}^{\infty} \sum_{g=0}^{\infty} (\alpha')^{n-4}\, g_s^{2(g-1)} \mathcal{A}_{(n,g)} \tag{3}$$

[1]A vector field, for example, with a vev would single out a distinguished direction in spacetime which would depend on the observer.
[2]The vev of the axion plays a role analogous to that of the θ-angle in Yang-Mills theory.

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with partial amplitudes $\mathcal{A}_{(n,g)}$. The powers in the coupling constant g_s count the genus of the string worldsheet in the perturbation expansion. Type IIB string theory can therefore be defined perturbatively as an expansion in α' and g_s. Singling out the purely gravitational part of the action, i.e. the terms whole field dependence is only the metric, one has an expansion of the form

$$S = \sum_{n=0}^{\infty}\sum_{g=0}^{\infty} S_{(n,g)} = \sum_{n=0}^{\infty}\sum_{g=0}^{\infty} (\alpha')^{n-4}\, g_s^{2(g-1)} \int \mathrm{d}^{10}x\,\sqrt{G}e^{-2\phi}\mathcal{R}_{(n,g)} \tag{4}$$

where $\mathcal{R}_{(n,g)}$ is some combination of the metric tensor alone and

$$S_{(0,0)} = (\alpha')^{-4}\int \mathrm{d}^{10}x\,\sqrt{G}e^{-2\phi}R \tag{5}$$

is the Einstein-Hilbert action written in string frame[3].

The classical action (of which (5) is the purely gravitational part) has a symmetry group $G = \mathrm{SL}(2,\mathbb{R})$. Futhermore, type IIB string theory is a theory with maximal amount of supersymmetry, and Cremmer and Julia showed that the moduli space for a maximally supersymmetric theory is $G(\mathbb{R})/K(\mathbb{R})$ where $K(\mathbb{R})$ is the maximal compact subgroup of $G(\mathbb{R})$. For type IIB string theory, the moduli space is $\mathrm{SL}(2,\mathbb{R})/\mathrm{SO}(2)$ and can be seen to be isomorphic to the upper half plane $\mathbb{H}$. It is therefore convenient to group the scalar fields ϕ and χ into a complex valued scalar field

$$\tau = \tau_1 + i\tau_2 := \chi + ie^{-\phi} \in \mathbb{H} \tag{6}$$

called the axidilaton. An element g of the symmetry group $\mathrm{SL}(2,\mathbb{R})$ then acts on τ via a modular transformation[4]

$$\tau' = g\cdot\tau = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\cdot\tau = \frac{a\tau+b}{c\tau+d} \in \mathbb{H}. \tag{7}$$

Considering the action (4) order by order in α' we make the ansatz

$$S = \sum_{n=0}^{\infty} S_{(n)} = \sum_{n=0}^{\infty} (\alpha')^{n-4}\int \mathrm{d}^{10}x\,\sqrt{G}e^{-2\phi} f_n(\tau)\mathcal{R}_{(n)} \tag{8}$$

where the $g_s = e^{\phi}$ dependence is represented by the τ-dependence of the functions f_n. Hull and Townsend [1] argued based on the Dirac-Schwinger-Zwanziger quantization condition that the classical symmetry group $G(\mathbb{R})$ will be broken by quantum effects into a discrete group $G(\mathbb{Z})$ known as the U-duality group. One furthermore requires that U-duality works order by order in α'. We also note that since α' has the dimension of length squared, for each extra order of α' one needs to additional spacetime derivatives. The concerned Lagrangian is therefore a higher derivative expansion of the metric with the Einstein-Hilbert term at its lowest order.

Automorphic forms

Green and Sethi [2] showed that imposing supersymmetry puts additional constraints on the f_n in the form of PDEs, (10) to (12). Taking all this into account and passing to Einstein frame, one ends up with the expression

$$S = (\alpha')^{-4}\int \mathrm{d}^{10}x\,\sqrt{G}\left(R + (\alpha')^3 f_3(\tau)R^4 + (\alpha')^5 f_5(\tau)D^4R^4 + (\alpha')^6 f_6(\tau)D^6R^4 + \dots\right) \tag{9}$$

where the results of Green and Sethi imply the vanishing of f_1, f_2 and f_4 as well as the conditions

$$\left(\Delta - \frac{3}{4}\right) f_3 = 0 \tag{10}$$

$$\left(\Delta - \frac{15}{4}\right) f_5 = 0 \tag{11}$$

$$(\Delta - 12)\, f_6 = -f_3^2 \tag{12}$$

$$\vdots$$

[3] An acion in string frame vs. Einstein frame is a matter of performing a Weyl-rescaling of the metric to leave manifest or absorb the factor $e^{-2\phi}$.

[4] This symmetry is made manifest by passing to Einstein frame.

The notation R^4, D^4R^4 and D^6R^4 is shorthand for some combination of Riemann tensors, Ricci tensors and Ricci scalars and derivatives thereof to obtain the correct combination of metrics and derivatives of the metric. The exact form of these won't concern us here, but it was worked out by Gross and Witten [3].

The f_n are hence functions from $\mathrm{SL}(2,\mathbb{R})/\mathrm{SO}(2)$ into the reals, invariant under $\mathrm{SL}(2,\mathbb{Z})$ and obey some PDEs. Functions satisfying these conditions are called automorphic forms, and these constraints are sometimes enough to determine them completely.

Taking f_3 as an example, the solution was found by Green and Gutperle [4] to be

$$f_3(\tau) = \sum_{(m,n)\neq(0,0)} \frac{\tau_2^2}{|m+n\tau|}. \tag{13}$$

Since the action (9) is the purely gravitational sector of the low energy effective action of type IIB string theory, we know that the various terms correspond to physical processes of various loop orders and the couplings are proportional to the corresponding amplitudes. In this case, the physical process in question is four graviton scattering, so somehow the functions f_n must encode amplitudes for four graviton scattering. This encoding lies in the Fourier expansion of the respective f_n. Heuristically, it makes sense to speak of the fourier transform of automorphic forms, since invariance under a discrete group $G(\mathbb{Z})$ plays the role of periodicity. In the case at hand, $\mathrm{SL}(2,\mathbb{Z})$ is generated by two elements S and T according to

$$S\cdot\tau = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\cdot\tau = \frac{-1}{\tau}, \quad T\cdot\tau = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\cdot\tau = \tau+1. \tag{14}$$

In particular, the T-transformation reveals that the f_n are periodic in τ_1 with period 1.

Finding fourier coefficients of automorphic forms is tricky business and is a central theme of what is called the Langlands program. Nevertheless, for f_3 it can be done with relative ease and the Fourier expansion (in τ_1) is

$$f_3(\tau) = 2\zeta(3)\tau_2^{3/2} + 4\zeta(2)\tau_2^{-1/2} + 8\pi\sqrt{\tau_2}\sum_{N\neq 0}\mu_{-2}(N)NK_1(2\pi|N|\tau_2)e^{-2\pi i N\tau_1}. \tag{15}$$

K_1 is a modified Bessel function of the second kind and

$$\mu_{-2}(N) = \sum_{m|N,m>0} m^{-2} \tag{16}$$

is called the instanton measure and will be discussed below. The notation $\sum_{m|N}$ means summing over all divisors m of N, i.e. all way's that N can be written as a product of two numbers.

Now one may start attempting to interpret this result and learn about four graviton scattering in type IIB string theory. Recall that $\tau_2 = e^{-\phi} = g_s^{-1}$ is the coupling constant relating to the topology (in fact the genus) of the string worldsheet of the perturbation expansion. The correct interpretation is therefore that the two τ_2-power behaved terms correspond to two perturbative effects, namely tree level ($\tau_2^{3/2}$) and one loop ($\tau_2^{-1/2}$) with amplitudes $2\zeta(3)$ and $4\zeta(2)$ respectively. This marvelously agrees with amplitudes already calculated perturbatively. The power of automorphic forms in this context therefore also predicts the strong result that no higher perturbative processes exist. Lastly one must reflect upon the physical meaning of the infinite series in (16). The meaning of these terms will be given in the following interlude, but just to ruin the suspense: they correspond to non-perturbative processes known as instantons.

INTERLUDE: INSTANTONS IN IIB SUPERGRAVITY

Instantons are solutions with a finite value of the action to the equations of motion of a theory. Furthermore they are non-perturbative, which means that they are inaccessible to perturbation theory. The first example of instantons in type IIB string theory was found by Gibbons, Green and Perry [5]. Their calculation is very briefly outlined here. More details can be found in Gustafsson [6].

The starting point is the Euclidean version of the purely scalar part of the type IIB action. It reads

$$S = \int_{\mathbb{R}^{10}} \mathrm{d}^{10}x\,\sqrt{G}\left(-R + \frac{1}{2}(\partial\phi)^2 - \frac{1}{2}e^{2\phi}(\partial\chi)^2\right) \tag{17}$$

which gives the equations of motion

$$G_{\mu\nu}: \qquad R_{\mu\nu} - \tfrac{1}{2}\left(\partial_\mu\phi\partial_\nu\phi - e^{2\phi}\partial_\mu\chi\partial_\nu\chi\right) = 0 \tag{18}$$

$$\phi: \qquad \nabla^2\phi + e^{2\phi}(\partial\chi)^2 = 0 \tag{19}$$

$$\chi: \qquad \nabla_\mu\left(e^{2\phi}\partial^\mu\chi\right) = 0. \tag{20}$$

The ansatz that produces the instaton solution is

$$G_{\mu\nu} = \delta_{\mu\nu} \quad \text{(flat space)} \tag{21}$$

$$\partial_\mu\chi = \pm e^{-\phi}\partial_\mu\phi. \tag{22}$$

The two signs are to be kept track of and will give two different solutions called instantons and anti-instantons. Assuming that ϕ is spherical with radial coordinate r, we are led to the solutions

$$e^\phi = e^{\phi_\infty} + \frac{c}{r^8} \tag{23}$$

$$\chi - \chi_\infty = \mp\left(e^{-\phi} - e^{-\phi_\infty}\right) \tag{24}$$

where c is a constant to be determined and ϕ_∞ and χ_∞ are the values of ϕ and χ at infinity.

Next, we notice that the action (17) enjoys the symmetry $\chi \to \chi + \text{const}$. It is straightforward to compute the associated Nther current $J^\mu = \pm\partial^\mu e^\phi$ and the conserved charge

$$Q = \int_{S^9} J^\mu\, \mathrm{d}\Sigma_\mu = \ldots = \mp 8c\,\mathrm{Vol}\left(S^9\right), \tag{25}$$

so the constant c is related to the conserved charge Q. It is now tempting to plug in the solutions (23) and (24) into (17) to find the value of the action. One finds however that because of (22), the action vanishes, so a bit more care must be taken. One notices that the singular behavior of (23) at $r = 0$ is such that when inserted into the equations of motion (19) and (20) (after taking $G_{\mu\nu} = \delta_{\mu\nu}$), one picks up a delta-function from

$$\partial^2 e^\phi = \frac{1}{r^9}\partial_r\left(r^9\partial_r e^\phi\right) = \frac{1}{r^9}\partial_r\left(r^9\partial_r\frac{c}{r^8}\right) = A\delta^{(10)}(x) \tag{26}$$

with the constant A given by

$$A = \int_{\mathbb{R}^{10}} \mathrm{d}^{10}\partial^2 e^\phi = \int_{S^9_\infty} \partial^\mu e^\phi\, \mathrm{d}\Sigma_\mu = \int_{S^9_\infty} \partial^\mu e^\phi\, \mathrm{d}\Sigma_\mu = -8c\,\mathrm{Vol}\left(S^9\right) = -|Q|. \tag{27}$$

In order to reflect these changes to the equations of motion, one needs to add an appropriate source term to the action. Furthermore, one also needs to add a boundary term in order to restore the symmetry $\chi \to \chi + \text{const}$. The full action is then

$$S = \int_{\mathbb{R}^{10}} \mathrm{d}^{10}x\left(\frac{1}{2}(\partial\phi)^2 - \frac{1}{2}e^{2\phi}(\partial\chi)^2\right) + \int_{\mathbb{R}^{10}} \mathrm{d}^{10}x|Q|\delta^{(10)}(x)\left(e^{-\phi} \pm \chi\right) + \int_{\mathbb{R}^{10}} \partial^{10}x\partial_\mu\left(\chi e^{2\phi}\partial^\mu\chi\right). \tag{28}$$

Plugging in the instanton solutions (23) and (24) in this action gives the value

$$S = \frac{|Q|}{g_s}. \tag{29}$$

As mentioned in [4], the instanton will also get an imaginary contribution coming from the possibility of adding yet another boundary term to the action, which leaves the equations of motion and the instanton solution intact. The final instanton and anti-instanton action is then

$$S_{\text{inst}} = -i|Q|\tau_\infty \quad \text{and} \quad S_{\text{anti}} = i|Q|\bar{\tau}_\infty \tag{30}$$

where $\tau_\infty = \chi_\infty + ie^{-\phi_\infty}$.

Lastly, the actions S_{inst} and S_{anti} will turn out to be quantized. This is seen by considering euclidean type IIA string theory. One starts by compactifying the euclidean time direction on a circle S_R of radius R. Next, one may consider a D0-brane with its one dimensional world volume wrapped around the circle. The momentum of the D0-brane must then be quantized. Assuming a D0-brane with a winding number of m and n units of momentum, its action may be computed (see [4]). Under T-duality, type IIA string theory goes to type IIB compactified on a circle $S_{1/R}$ of radius $1/R$ and Dp-branes go to D$(p-1)$-branes[5]. By taking the limit $R \to \infty$, the circle decompactifies and one is left with the action for D(-1)-branes, also known as D-instantons, in type IIB string theory. The action for D0-branes in type IIA, which is easily calculated, is hence related to the action for D-instantons in type IIB. The calculation gives

$$S_{\text{inst}}(\tau, mn) = -2\pi i |mn| \tau \quad \text{and} \quad S_{\text{anti}}(\tau, mn) = 2\pi i |mn| \overline{\tau} \tag{31}$$

and the actions are thus quantized by the integers m and n.

NON-PERTURBATIVE TERMS

After this interlude, the infinite series in the R^4 coefficient f_3 may now be interpreted as follows. For large argument, the asymptotic behavior of the Bessel function K_1 is given by

$$K_1(x) = \sqrt{\frac{\pi}{2x}} e^{-x} \left(1 + O\left(x^{-1}\right)\right). \tag{32}$$

The limit $\tau_2 = e^{-\phi} = g_s^{-1} \to \infty$ is the weak-coupling limit and gives

$$\begin{aligned} f_3(\tau) &= 2\zeta(3)\tau_2^{3/2} + 4\zeta(2)\tau_2^{-1/2} + 4\pi \sum_{N\neq 0} \sqrt{|N|}\mu_{-2}(N) e^{2\pi(iN\tau_1 - |N|\tau_2)} \left(1 + O\left(\tau_2^{-1}\right)\right) = &(33)\\ &= 2\zeta(3)\tau_2^{3/2} + 4\zeta(2)\tau_2^{-1/2} + 4\pi \sum_{N>0} \sqrt{|N|}\mu_{-2}(N) \left(e^{-S_{\text{inst}}(\tau,N)} + e^{-S_{\text{anti}}(\tau,N)}\right)\left(1 + O\left(\tau_2^{-1}\right)\right). &(34)\end{aligned}$$

So, for each N, one obtains two infinite series $\left(1 + O\left(\tau_2^{-1}\right)\right)$ in $\tau_2^{-1} = g_s$ exponentially suppressed by the value of a charge N instanton and anti-instanton respectively. This corresponds to physical processes taking place in the backgrounds of instantons and anti-instantons.

Calling these terms non-perturbative is motivated by the fact that

$$e^{-S_{\text{inst}}(\tau,N)} = e^{-2\pi i |N| \chi} e^{2\pi |N|/g_s} \tag{35}$$

has no power expansion around $g_s = 0$ (likewise of course for $e^{-S_{\text{anti}}}$). If one has a machinery which produces terms order by order in g_s (which in this case is perturbation theory), there is then no hope of producing the terms corresponding to perturbative processes in the instanton and anti-instanton backgrounds.

OUTLOOK

The moral of the story told thus far is that U-duality in string theory strongly constrains the effective action and is a powerful tool for finding explicit expressions in the action encoding physical amplitudes. The objects encoding these amplitudes are automorphic forms and the physics is revealed through their fourier expansion.

Of further interest are the cases of dimensional reduction $\mathbb{R}^{10} \to \mathbb{R}^{10-d} \times T^d$ on a d-dimensional torus. These cases are similar to the analysis outlined above for uncompactified type IIB string theory but with some important differences. Firstly, one gets additional moduli from the Kaluza-Klein scalars, which gives a higher dimensional moduli space. The moduli space still has the form $G(\mathbb{R})/K(\mathbb{R})$ but with a larger symmetry group $G(\mathbb{R})$, which in turn gives a larger U-duality group $G(\mathbb{Z})$. The PDEs (10) to (12) take the same form but with different eigenvalues[6].

Given that one manages to find solutions to the PDEs, the problem of Fourier expanding the automorphic forms thus obtained is often very nontrivial. Fortunately, there is a representation theoretic viewpoint of this problem that

[5] Or D$(p+1)$-branes depending on which direction one dualizes in.

[6] For some special values of d they also pick up a source term in the right hand side, see Green, Miller and Vanhove.

can be of great help. By viewing the function space of $G(\mathbb{Z})$-invariant functions as a representation space for $G(\mathbb{R})$, the automorphic forms at hand can be seen to be attached to representations of different complexity (such as the minimal or next-to-minimal representation). This restricts some Fourier coefficients to be zero. Furthermore, one can find that the Fourier coefficients fall into orbits under the action of $G(\mathbb{Z})$ which effectively means that one only needs to compute a few coefficients (some orbit representatives) explicitly and the rest can be found by $G(\mathbb{Z})$-transformations of these representatives. This fact also has a physical interpretation in terms of BPS-bounds on the corresponding non-perturbative processes that the automorphic forms contain.

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REFERENCES

[1] C. Hull and P. Townsend, Nuclear Physics B **438**, 109–137 (1995).
[2] M. B. Green and S. Sethi, Physical Review D **59**, p. 046006 (1999).
[3] D. J. Gross and E. Witten, Nuclear Physics B **277**, 1–10 (1986).
[4] M. B. Green and M. Gutperle, Nuclear Physics B **498**, 195–227 (1997).
[5] G. W. Gibbons, M. B. Green, and M. J. Perry, Physics Letters B **370**, 37–44 (1996).
[6] H. P. A. Gustafsson, *Eisenstein Series and Instantons in String Theory*, Master's thesis (2013).